Transmission and Reflection of Electromagnetic Waves

Normally Incident on a Warm Plasma

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Abstract

The solution for an electric field normally incident onto a warm, semi-infinite plasma is obtained by means of a modified Wiener-Hopf technique. The warm plasma is taken into account by means of the relativistic Vlasov equation. It is found that the previously obtained solution of Taylor gives the correct term for the wave number in the plasma, but not the correct answer for the field. It is shown that the method of considering an "equivalent" fully infinite plasma corresponds to a physically unrealistic matching of the plasma to the vacuum. The field inside the plasma is found. The field just inside the plasma is discussed and the non-uniform limit from warm to cold plasma found by Taylor is not found in our solution.

## I. Introduction

In a recent article[1] Taylor has presented a solution to the transmission and reflection of an electromagnetic wave normally incident on a semi-infinite warm plasma. His approach, and the approach of several others to similar problems,[2-4] involves converting the given half-space problem into a full-space problem and assuming that the solutions to the two problems are equivalent. We will attack the same problem as Taylor without this assumption, by use of a modified Wiener-Hopf technique. Taylor[1] obtains three results: the warm temperature correction to the wave number of wave propagation in the plasma; the amount of the incident wave transmitted and reflected at the interface; and the fields just inside the interface. We shall show that his first result is valid sufficiently far inside the plasma, but not near the interface. We shall show that his other two results are not valid. In particular Taylor's conclusion about a non-uniform transition from warm plasma to zero temperature plasma does not appear.

## II. Formulation of the Equations

Following Taylor[1] we consider a rectangular coordinate system such that the plasma is contained in the half space $z > 0$. An incident transverse wave with frequency $\omega$ propagates in the vacuum in the z direction, with its electric field aligned in the positive x direction. It is no additional complication to consider the plasma to have a current sheet in the plane $z = 0$. Setting $E = \hat{e}_x E(z)e^{+i\omega t}$, then the relevant

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1. Edward C. Taylor, Radio Science 69, 735, (1965).
2. E. C. Taylor, Phys. Fluids 6, 1305, (1963).
3. V. D. Shafranov, JETP (USSR) 6, 1010 (1958).
4. G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. A, 195, 336, (1948).

equation is

$$\frac{\partial^2 E}{\partial z^2} + \frac{\omega^2}{c^2} E + i\,\omega\,\mu_o\, j = - A\, \delta\,(z) \qquad (1)$$

where j(z) is the current density at the point (x, y, z) and A is the strength of the current density sheet. If a relativistic collisionless plasma is considered, then the plasma current is given by

$$j = e \int \beta\, c\, u_x\, f\, d^3 u$$

where $\beta = 1 + u^2$,

and f is the solution of the relativistic, linearized, collisionless Vlasov equation[5]

$$-i\,\omega\, f + \beta\, c\, u_z \frac{\partial f}{\partial z} = - \frac{e}{mc}\, E\, \frac{\partial f_o}{\partial u_x}\ . \qquad (2)$$

Assuming specular reflection of the plasma particles at the boundary we obtain the integral equation for E

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right) E - \int_o^\infty K_\tau\,(\,|z - z'|\,)\, E(z')\, d\,z' \qquad (3)$$

$$- \int_o^\infty K_\tau\,(\,|z + z'|\,)\, E(z')\, d\,z' = - A\,\delta\,(z)$$

where

$$K_\tau(\,|\,\tau\,|\,) = \frac{i\omega^2\ p^{\omega}}{c^2} \int_{-\infty}^{\infty} d\,u_x \int_{-\infty}^{\infty} d\,u_y \int_C d\,u_z\, \frac{u_x}{u_z}\, \frac{\partial f_o}{\partial u_x}$$

$$\exp\left(\frac{i\,\omega\,|\,\tau\,|}{\beta c\, u_z}\right), \qquad (4)$$

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5. R. C. Clemmow and A. J. Wilson, Proc. Roy. Soc. (London) A237, 117, (1956).

which equations are Taylor's (4) and (5) respectively.

To solve equation (3) transforms seem an obvious choice. Unfortunately the range of integration of the integrals in (3) is only semi infinite; in any case equation (3) is valid only for $z > 0$. Taylor[1] gets around this problem by the assumption that $E(z) = E(-z)$ (and the implicit assumption that $f_o(z) = f_o(-z)$) thus combining the integrals. He thus considers the different problem where the medium is infinite and the fields are symmetrical. He then intends to match the solution for $z > 0$ to this problem to the solution for an incoming wave. It is not at all clear that the symmetry which is inherent in this different problem is relevant to the original problem.

We note that equation (3) is really valid only for $z > 0$. Thus the given problem seems a logical candidate for a Wiener-Hopf approach. Because of the $z + z'$ in the last integral, the solution is not a straight forward application of the ordinary Fourier transform. We need to use the generalized concept of a Fourier transform.[6] We then define the four integral transforms

$$\left.\begin{aligned} \bar{E}_+(k) &= \int_0^\infty e^{ikz} E(z)\, dz \\ \bar{E}_-(k) &= \int_{-\infty}^0 e^{ikz} E(z)\, dz \\ \bar{E}_+^*(k) &= \int_0^\infty e^{-ikz} E(z)\, dz \\ \bar{E}_-^*(k) &= \int_{-\infty}^0 e^{-ikz} E(z)\, dz \end{aligned}\right\} \qquad (5)$$

6. See, for example, J. Irving and N. Mullineux, Mathematics in Physics and Engineering, Academic Press, 1959.

Let

$$H_1(k) = \int_0^\infty e^{ikz'} E(z') \int_{-z'}^{o} K_T(|u|)\, du\, dz'$$

and

$$H_2(k) = \int_0^\infty e^{-ikz'} E(z') \int_{o}^{z'} K_T(|u|)\, du\, dz'$$

Then equation (3) becomes the 4 equations

$$(\omega^2/c^2 - k^2 - \overline{\kappa}_+)\overline{E}_+ - E'_+(o) + i\,k\,E(o) - \overline{\kappa}_+ \overline{E}_+^* - H_1 + H_2 = -A/2$$

$$(\omega^2/c^2 - k^2)\overline{E}_- - \overline{\kappa}_- \overline{E}_+ + E'_-(o) - i\,k\,E(o) - \overline{\kappa}_- \overline{E}_+^* + H_1 - H_2 = -A/2$$

$$(\omega^2/c^2 - k^2 - \overline{\kappa}_-)\overline{E}_+^*(o) - i\,k\,E(o) - \overline{\kappa}_- \overline{E}_+ - H_2 + H_1 = -A/2$$

$$(\omega^2/c^2 - k^2)\overline{E}_-^* - \overline{E}_+^* \overline{\kappa}_+ + E'_-(o) + i\,k\,E(o) - \overline{\kappa}_+ \overline{E}_+ + H_2 - H_1 = -A/2$$

where we have used the symmetry of $K_T$ (and $\overline{\kappa}_\pm$ is the appropriate transform of $K_T$) to eliminate $\overline{\kappa}_\pm^*$ and $E'_\pm(o) = \frac{\partial E}{\partial z}(z = o^\pm)$, while $E(o)$ is $E(z = o)$.

These equations form a linear set for the four quantities $\overline{E}_\pm$, $\overline{E}_\pm^*$. It is convenient to eliminate the difficult quantities $H_1$ and $H_2$. We obtain, after some algebra,

$$\overline{E}_+ + \overline{E}_- = -\frac{2A}{(\omega^2/c^2 - k^2 - \overline{K})} + \frac{2\,\overline{K}\,E'_+(o)}{(\omega^2/c^2 - k^2)(\omega^2/c^2 - k^2 - \overline{K})} \qquad (6)$$

where $$\overline{K} = \overline{\kappa}_+ + \overline{\kappa}_- = \int_{-\infty}^{\infty} K_T(|z|)\, e^{ikz}\, dz \qquad (7)$$

This simplifies to

$$E_+ + E_- = \frac{-(A + 2\,E'_+(o))}{\omega^2/c^2 - k^2 - \overline{K}} - \frac{2\,E'_+(o)}{\omega^2/c^2 - k^2} \qquad (8)$$

Now $E_+ + \overline{E}_-$ is not really the ordinary Fourier transform of E, for the entire space, since the paths of integration involved in the integrals are different. $\overline{E}_+$ is the Fourier transform of a function

$$f(z) = \begin{cases} E(z) & z>o \\ o & z<o \end{cases} \tag{9}$$

and conversely for $\overline{E}_-$. Thus the analytic function $\overline{E}_+$ (k) is analytic in the upper half k plane. If E(z) vanished for z sufficiently large, then, as can be readily seen from the transform definitions (5) $\overline{E}_+$ (k) will actually be analytic for the imaginary part of k slightly negative. On the other hand $\overline{E}_-(k)$, which is the Fourier transform of a function

$$g(z) = \begin{cases} o & z>o \\ E(z) & z<o\ , \end{cases} \tag{10}$$

is analytic, from its definition (5), in the entire lower half plane. The boundary conditions which we will impose on the solution E(z) to our problem will be the following: E(z) is to be oscillatory, or perhaps damped, but not exponentially growing, in both half planes. Then <u>both</u> $\overline{E}_+(k)$ and $\overline{E}_-(k)$ will be analytic up to and including the real k axis, except perhaps at isolated singularities on the real axis, or branch lines which extend into the lower or upper half planes respectively. Thus there is at least a common line, namely most of the real axis, with an uncountable number of points, along which both $\overline{E}_+(k)$ and $\overline{E}_-(k)$ are analytic.

We now will apply a Wiener-Hopf argument. That is, we try to separate equation (18) into plus functions, functions analytic in the upper half plane, and minus functions, analytic in the lower half plane.

The term $\dfrac{2\ \ E_+'(o)}{\omega^2/c^2 - k^2}$ has poles on the real axis. It is not immediately obvious whether it is a + function or a - function. We employ the boundary condition that a wave of the free space wavelength cannot propagate in the plasma. Thus it must be assigned as a - function. The term

$$- \frac{A - 2E'_+(o)}{\omega^2/c^2 - k^2 - \overline{K}(k)} \equiv I(k) \tag{11}$$

has both poles and two infinite lines of singularities, so it is not obviously + or -. Its singularities are depicted below in figure 1. Consider the possible cases. As long as the poles and branch lines are split, one can draw two lines, depicted as 1 and 2. (If the poles and "branch points" are on the real axis one can still draw such a pair of lines, only around the poles and branch points in an obvious way.) In the cross hatched region, I (k) is analytic. Thus we can write, using Cauchy's integral formula,

$$I(k) = \frac{1}{2\pi i} \oint \frac{I(\xi)}{\xi - k}\, d\xi$$

$$= \frac{1}{2\pi i} \int_1 \frac{I(\xi)}{\xi - k}\, d\xi + \frac{1}{2\pi i} \int_2 \frac{I(\xi)}{\xi - k}\, d\xi . \tag{12}$$

We now have I (k) written as the sum of two functions. Looking at the first integral, as a function of k, we see that its only singularity is a pole, and that the curve 1 passes below the pole. Thus the first integral

$$I_+(k) = \frac{1}{2\pi i} \int_1 \frac{I(\xi)}{(\xi - k)}\, d\xi$$

can be differentiated under the integral sign, and the resultant integral will converge for all k lying above that pole. Thus $I_+(k)$ is actually a plus function. Similarly

$$I_-(k) = \frac{1}{2\pi i} \int_2 \frac{I(\xi)}{(\xi - k)} \, d\xi$$

is a minus function. Thus we have written

$$I(k) = I_-(k) + I_+(k). \tag{12a}$$

To evaluate these integrals we complete a closed contour for each, and chose to close up for each integral; then use Jordan's Lemma. Then

$$\left.\begin{aligned} I_-(k) &= \frac{-C}{k - k_p} + \frac{C\,k_p}{\pi i} \int_3 \frac{d\xi}{(\xi - k)(\xi^2 + \overline{K}(\xi) - \omega^2/c^2)} \\ I_+(k) &= I(k) + \frac{C}{k - k_p} - \frac{i\,C\,k_p}{\pi} \int_3 \frac{d\xi}{(\xi - k)(\xi^2 + \overline{K}(\xi) - \omega^2/c^2} \end{aligned}\right\} \tag{13}$$

where $k_p$ is the positive zero of

$$k^2 + \overline{K}(k) - \omega^2/c^2$$

(It is anticipated that, to the first approximation, there is only one),

and $$C = \frac{2E'_+(o) - A}{2\,k_p}.$$

Thus

$$\overline{E}_+ - I_+ = -\overline{E}_-(k) + I_-(k) - \left( \frac{2E'_+(o)}{\omega^2/c^2 - k^2} \right) \tag{14}$$

The right hand side of (14) is analytic in the lower half plane, and the left hand side analytic in the upper half. Thus the twosides are different representations of a single function, which is analytic everywhere. If $E(z)$ is integrable at the origin, then by the Riemann-Lebesgue theorem its Fourier transform vanishes for $|k| \to \infty$. By inspection, the other terms in (13) vanish at infinity. The only entire function which vanishes everywhere at infinity is zero. So

$$\left.\begin{aligned} \overline{E}_+ &= I_+(k) \\ \overline{E}_- &= I_-(k) - \frac{2\,E'_+(o)}{\omega^2/c^2 - k^2} \end{aligned}\right\} \tag{15}$$

Using the inverse transforms for equations (19) and (20), and adding them we obtain

$$E(z) = \frac{1}{2\pi}\int_{c_1} e^{-ikz}\,\overline{E}_+(k)\,d\,k + \frac{1}{2\pi}\int_{c_2} e^{-ikz}\,\overline{E}_-(k)\,d\,k \tag{16}$$

where the path $c_1$ is necessarily above all the singularities of $\overline{E}_+(k)$ and the path $c_2$ is below all those of $\overline{E}_-(k)$. For $z > 0$ the integrals in (31) can be evaluated, using Cauchy's formula, by closing a contour downwards. Since the second integral has no singularities there, it is sufficient to write

$$E(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikz}\,\overline{E}_+(k)\,d\,k\ ,\ z > 0. \tag{16a}$$

The integral for $E_-(z)$ is not meaningful for our problem since equation (3) is valid for $z \geqslant o$ only.

## III. Solution to the Equations

The problem is then reduced to evaluating the integral (16a) for the electric field in the plasma. It is useful, before evaluating this integral, to analyze the approximate nature of the results.

From (11), (13) and (15)

$$E_+ (k) = \frac{2k_p C}{\omega^2/c^2 - k^2 - \overline{K}(k)} + C\left\{ \frac{1}{k-k_p} - \frac{ik_p}{\pi} \int_3 \frac{d\xi}{(\xi-k)(\xi^2 + \overline{K}(\xi) - \omega^2/c^2)} \right\} \tag{17}$$

Equation (17) has poles at the zeros of

$$k^2 + \overline{K}(k) - \omega^2/c^2 \tag{18}$$

Anticipating, as we already have done in the previous section, that there are only two such poles, symmetrically placed, we see that the second term in (17) cancels the pole at $k = k_p$. Thus there is only one wave, corresponding to $k = -k_p$, which is propagating into the plasma. In the above statement we have tacitly assumed the zero of (18) is real. If it is complex, then the wave is, of course, damped. There is also a contribution to the field from the branch line type singularities. In general, this would mean a decaying contribution to the solution, whose exact nature will have to be determined.

We note that the field in the plasma is linearly dependent on the zero field $E'_+(0)$ as well as (contrary to Taylor's formulation[1],) on the current sheet strength.

We see then that the branch line situation must be that shown in Figure 1. Any other location of the poles and branch lines relative to the path of integration, while possible, would result in either an incoming wave due to the other pole, or a growing exponential due to the other branch line lying in the upper half plane.

The solution to (17) will be matched to the solution to equation (1) for $z < o$, namely an incoming and reflected wave.

We must now turn to the task of evaluating the integral above. We are first faced with the calculation of $\overline{K}(k)$, which is given by equation (7). Using the definition (4) it is easy to see that

$$\overline{K}(k) = \frac{2\omega_p^2}{c^2} \int_{-\infty}^{\infty} du_x \int_{-\infty}^{\infty} du_y \int_{o}^{\infty} du_z \ \beta\, c\, u \frac{\partial f_o}{\partial u_x} \frac{}{\left(k \frac{c\, u_z}{\omega}\right)^2 - 1} \tag{19}$$

Thus it is easily seen that $\overline{K}(k)$ has singularities due to the vanishing denominator of the integrand. As will be shown later, these singularities appear at all k such that $\pm k > \omega/c$, which curves are, effectively, branch lines.

If $f_o$ is taken to be Maxwellian in velocity space, then the integral (19) is easily converted to the triply infinite integral

$$\overline{K}(k) = \frac{\omega_p^2}{/c^2} \int d\, u \frac{c\, \beta\, u_x \frac{\partial f_o}{\partial u_x}}{\frac{k\, c}{\omega} \quad \beta\, u_z - 1}$$

If we let

$$n = \frac{kc}{\omega},$$

it is shown in the appendix that this can be written

$$\overline{K}(k) = 2\pi \frac{\omega_p^2}{c^2} \int_o^{\infty} \frac{2}{\lambda} \ f_o \frac{du}{(1+u^2)^{3/2} \ ([1-n^2]\, u^2 + 1)} \tag{20}$$

where $\lambda = \dfrac{mc^2}{kT}$.

This integral shows clearly that $\overline{K}(k)$ has branch lines extending outward on the real axis from $k = \pm\, {}^{\omega}/c$. The integral (20) can not be evaluated explicitly in closed form. However, if $f_o$ is Maxwellian, and the temperature of the plasma is fairly low (ie: $\lambda >> 1$), then the integral can be evaluated asymptotically for $\lambda \to \infty$. This is done in the appendix and the first order (in ${}^{kT}/mc^2$) term calculated explicitly, as a function of n and $\lambda$. The answer is

$$\overline{K}(k) \sim \frac{\omega_p^2}{c^2} \left\{ \frac{3}{2(n^2-1)} + \frac{\sqrt{\pi\lambda}}{i\sqrt{2(n^2-1)}} \left[ 1 - \frac{3}{2(n^2-1)} \right] \exp\left( \frac{-\lambda}{2(n^2-1)} \right) \right. \tag{21}$$

$$\left. \operatorname{erfc}\left( \frac{-i\sqrt{\lambda}}{\sqrt{2(n^2-1)}} \right) \right\}$$

Equation (21) is too complicated to be useful. One can simplify (21) for either of two cases, $\sqrt{\lambda}/n >> 1$ and $\sqrt{\lambda}/n << 1$. The former corresponds to low temperatures at some distance from the plane z = 0, while the latter corresponds to low temperatures very close to the plane z = 0 (from Tauberian theorems about Fourier transforms). One obtains

$$\lim_{\sqrt{\lambda}/n \to \infty} \overline{K}(k) \sim \left[ 1 + (n^2 - {}^{5}/2) \frac{kT}{mc^2} \right] \omega_p^2/c^2 \tag{21a}$$

which is the answer obtained by Taylor[1]. That is, at some distance from the interface z = 0, $\overline{K}(k)$ is essentially quadratic in k, so that

$$\frac{\omega^2}{c^2} = k^2 - \overline{K}(k)$$

$$\approx \frac{\omega^2}{c^2} \left\{ 1 - n^2 - \frac{\omega_p^2}{\omega^2} \left[ 1 + (n^2 - {}^{5}/_{2}) \frac{kT}{mc^2} \right] \right\}.$$

Thus we obtain

$$k_p = \frac{\omega}{c} \left( \frac{1 - \left(\frac{\omega_p^2}{\omega^2}\right)\left(1 - 5/2 \ \frac{kT}{mc^2}\right)}{1 + \left(\frac{\omega_p^2}{\omega^2}\right) \ \frac{kT}{mc^2}} \right)^{\frac{1}{2}} \tag{22}$$

Thus, far from the interface there exists, for $\omega^2 > \omega_p^2 \ (1 - 5/2\,\lambda)$ a propagating, unattenuating wave with wave number given by (22). For lesser frequencies this "wave" is damped with a characteristic length given by the reciprocal of the absolute value of (22).

The limit (21a) is equivalent to an expansion of $\overline{K}(k)$ for small k. As seen from the definition of the inverse transform (16a), and the usual ideas involved in the method of stationary phase, for z large compared to $1/\sqrt{\lambda}$, it is those values of $\overline{E}(k)$ for small (compared to $1/\sqrt{\lambda}$) k which contribute to the integral. On the other hand, for z small compared to $1/\sqrt{\lambda}$, the entire range of values of E(k) and in particular the values for large n are significant. We obtain

$$\lim_{\sqrt{\lambda/n} \to 0} \overline{K}(k) \sim - \frac{\omega_p^2}{c^2} \ i \sqrt{\frac{\lambda \pi}{2(n^2-1)}}$$

so that

$$\frac{\omega^2}{c^2} - k^2 - \overline{K}(k) \sim \left(\frac{\omega^2}{c^2}\right) \left(1 - n^2\right) \left(1 - i \frac{\omega_p^2}{\omega^2} \sqrt{\frac{\lambda \pi}{2(n^2-1)^3}} \right)$$

Thus, for large enough k, the branch line of $\overline{K}(\kappa)$ is the most significant part and the term $\omega^2/c^2 - k^2 - \overline{K}(k)$ does not appear to have a pole. Thus, the first two terms of the inverse transform (17) become

$$E(z) \sim (A - 2E'_{+}(o)) \frac{c}{\omega} \int_{-\infty}^{\infty} \frac{\epsilon^{-inz\omega/c}\, dn}{(n^2-1)\left(1 - i\frac{\omega_p^2}{\omega^2}\sqrt{\frac{\pi\lambda}{2(n^2-1)^3}}\right)}$$

For sufficiently large n the denominator may be expanded, and then integrated term by term.

$$E(z) \sim \frac{(A-2E'_{+}(o)c}{4\pi\omega}\left\{ e^{-i\omega z/c} - \frac{1}{3}\frac{\omega_p^2}{c^2}\sqrt{\frac{\lambda\pi}{2}}\; z^2 H_2^{(2)}\left(\frac{\omega z}{c}\right)\right\} \qquad (23)$$

where $H_\nu^{(2)}(x)$ is the Hankel function of order $\nu$. The solution (23) is an asymptotic solution and is clearly not convergent unless $\lambda < 1$. Quantitative answers cannot be obtained from this expression for the value of E. We shall return to this point later. We note, however, that both terms of (23) are essentially waves of the free space wavelength. Thus, the incident wave appears to penetrate the plasma with its wavelength essentially unchanged for a sufficiently small distance.

We have yet to consider the branchline integral in (17). This is

$$I_3(k) \equiv \int_3 \frac{\xi d\xi}{(\xi - k)(\xi^2 + \overline{K}(\xi) - \omega^2/c^2)}$$

Let $c_4$ be the curve on one side of curve 3 and $c_5$ the curve on the other (see Figure 1). Then

$$I_3(k) = \int_{\xi_o}^{\infty} \frac{\xi d\xi}{(\xi - k)(\xi^2 + \overline{K}_4(\xi) - \omega^2/c^2)}$$

$$- \int_{\xi_o}^{\infty} \frac{\xi d\xi}{(\xi - k)(\xi^2 + \overline{K}_5(\xi) - \omega^2/c^2)}$$

$$= \int_{\xi_o}^{\infty} \frac{\xi \{\overline{K}_5(\xi) - \overline{K}_4(\xi)\} \, d\xi}{(\xi - k)(\xi^2 + \overline{K}_4(\xi) - \omega^2/c^2)(\xi^2 + \overline{K}_5(\xi) - \omega^2/c^2)} \quad . \qquad (25)$$

Using the integral representation for $\overline{K}$ we obtain, following Taylor,[2]

$$\overline{K}_5(\xi) - \overline{K}_4(\xi) = \frac{-4\pi\omega_p^2}{\omega^2} \int_o^{\infty} x \text{ Residue} \left[ \left(\frac{\partial f_o}{\partial u_z}\right) \frac{c\beta u_z \omega}{(\xi c \beta u_z - \omega)} \right]_{\beta u_z = \frac{\omega}{\xi c}} dx \qquad (26)$$

$$= \frac{\omega^2 pc}{2\pi\omega} (\lambda)^{3/2} \left\{ 1 - \frac{2}{\lambda} \left(1 - \frac{\omega^2}{\xi^2 c^2}\right) \right\} \exp \left[ -\frac{\lambda}{2} \frac{\omega^2}{c^2 \xi^2 - \omega^2} \right]$$

The important feature in (26) is the presence of the exponential $\exp \left\{ -\frac{\lambda}{2} \frac{\omega^2}{c^2 \xi^2 - \omega^2} \right\}$. For $\sqrt{\lambda}/\xi \rightarrow \infty$ then this is a very rapidly decreasing function. Since $\xi_o$ is $\omega/c$ then the integral (25) is a LaPlace type integral in the variable

$$x = \frac{1}{\frac{c^2 \xi^2}{\omega^2} - 1} \, ;$$

thus

$$I_3(n) = B \int_o^{\infty} \frac{\left\{ 1 - \frac{2}{\lambda}(1 - x) \right\} \exp \left[ -\lambda/2x \right] \, dx}{\left[ \frac{1}{x} + \frac{c^2}{\omega^2} \overline{K}_4(x) \right] \left[ \frac{1}{x} + \frac{c^2}{\omega^2} \overline{K}_5(x) \right] \left( \sqrt{1 + 1/x} - n \right)} \qquad (27)$$

where $B = \frac{1}{4\pi} \frac{\omega_p^{\,2}}{\omega^{2}} \lambda^{3/2} \frac{c^{3}}{\omega^{3}}$, and again $n = \frac{kc}{\omega}$ .

The evaluation of (27) is not available for arbitrary values of n. However, since what is desired is the inverse transform of (25), we will compute

$$E_2(z) = \frac{(2E_+'(0) - A)}{2\pi} \frac{\omega}{c} B \int_{-\infty}^{\infty} e^{-inz\frac{\omega}{c}} I_3(n)\, dn,$$

by inverting the order of integration. For $z > 0$,

$$E_2(z) = \frac{B(2E_+'(0) - A)}{2} \int_0^{\infty} \frac{x^2 \left\{1 - \frac{2}{\lambda}(1-x)\right\} \exp\left\{ -\frac{\lambda}{2} x - i\frac{\sqrt{1+x}}{\sqrt{x}} \frac{z\omega}{c}\right\} dx}{(1 + x \frac{\omega_p^{\,2}}{\omega^2} \overline{K_1}(x))(1 + x\frac{\omega_p^{\,2}}{\omega^2} \overline{K_2}(x))} \tag{28}$$

This is evaluated by steepest descent (for large $\lambda$ ) in the appendix. We obtain

$$E_2(z) \approx -D \left(\frac{\lambda c}{\omega z}\right)^{3/2} \left[1 - \frac{2}{\lambda}\left(1 - \sqrt{\frac{ic\lambda}{\omega z}}\right)\right] \exp\left\{-\sqrt{\frac{ic\lambda^3}{\omega z}} + 2\sqrt{\frac{\omega\lambda z}{ic}}\right\}, \tag{29}$$

where $D = \frac{1-i}{2\pi^{3/2}} \frac{c^3 (2E_+'-A)}{\omega_p^{\,2}}$ and $\lambda = \frac{mc^2}{kT}$ as before. The exponential factor decays so rapidly for small temperature (large $\lambda$ ) that the multiplicative factor of $\lambda^{3/2}$ has no effect. Likewise for small z, $E_2$ decays exponentially as $z \to 0$. As in the previous calculations there is a difference in detailed behavior depending on whether the lim ( $\lambda \to \infty$ and $z \to 0$) is taken as $\frac{\sqrt{\lambda}}{z} \to 0$ or $\frac{\sqrt{\lambda}}{z} \to \infty$. However, both limits have an exponential decay to zero for $z \to 0$.

Thus in the evaluation of the two portions of the electric field on the plasma we find several distinct differences from Taylor's results,[1] and one identical result. The wave number for the wave propagating far enough in the plasma is the same. The value of the amplitude of this wave differs from Taylor. Our result gives

$$|E_T(z >> o)| = |2E'_+(0) - A| / 2k_p = |C|$$

and thus depends on the as yet undetermined $E'_+(0)$ as well as A.

The field just inside the plasma boundary is not that obtained by Taylor. First we do not obtain a contribution from the branch line integral, which gives Taylor his non-uniform limit. We find this contribution vanishes. We find that the solution several Debye lengths into the plasma can not be continued to the boundary. We are, however, unable to obtain the warm plasma solution near the boundary in a form which allows a limit $T > o$ and $z \rightarrow o$ to be taken. This non-uniformity is not surprising since the form of the distribution function for the plasma (A3) is a singular function of T. To obtain the correct expression for the field near the boundary, the full expression (21) must be used in the inverse transform. This is a task which the author has been unable to accomplish. We can only state that the wave appears to penetrate at the free space wavelength and then, in a distance of the order of a Debye length, alter to the plasma wavelength. However we can state that Taylor's non-uniformity is not valid.

## IV. Matching to Free Space

We show how we might match our solution to the incident wave in free space, which can only be in the form

$$E(z < o) = E_o \exp - i(\omega t - kz) + R E_o \exp - i(\omega t + kz), \qquad (30)$$

where R is the reflection coefficient. Let $2E'_+(0) - A = 2i\,k_p\,TE_o$, where T is thus the transmission coefficient. The matching conditions are the appropriate continuity equations for the fields. From (1) it follows that E must be continuous across $z = 0$. However, $\frac{\partial E}{\partial z}$ is clearly not, and thus B is not continuous. It follows from the presence of the current sheet, or from integrating (1) from $-\varepsilon$ to $+\varepsilon$, that

$$\frac{\partial E}{\partial z}(z > 0) - \frac{\partial E}{\partial z}(z < 0) = -A. \tag{31}$$

Then

$$\begin{aligned} E_o(1 + R) &= E(0^+) \\ i\,k\,E_o(1 - R) &= \frac{\partial E(0^+)}{\partial z} + A \end{aligned} \tag{32}$$

are the matching conditions, and the amplitude of the wave sufficiently far inside the plasma is given by

$$E(z >> o) = \frac{2E'_+(0) - A}{2i\,k_p} \exp -i(\omega t - k_p z). \tag{33}$$

For $T > 0$ we have been unable to evaluate $E'_+(0)$ further.

In the zero temperature limit where the solution (33) can be extrapolated to the plane $z = 0$, then one obtains

$$\begin{aligned} T &= \frac{2}{1+\alpha} - \frac{A}{iE_o(1+\alpha)} \\ R &= \frac{1-\alpha}{1+\alpha} - \frac{A}{ikE_o(1+\alpha)} \end{aligned} \tag{34}$$

where $\alpha = \frac{k_p}{k}$. Taylor's expressions for transmission and reflection are obtained by taking $A = 0$. We note that the transmission can be blocked by taking $A = 2ikE_o$. A solution for non-zero temperature awaits an inversion of the transform valid for small z.

## V. Summary and Conclusions

The usual approach to waves incident on a semi-infinite plasma has been to convert the problem to an infinite domain problem as Taylor has done. To do so,[3] an artificial sheet current source must be introduced at the origin, of strength $A \neq 0$. We have here solved the problem by considering the semi-infinite problem directly. The current source sheet has been retained, although there is no need for it. The field inside the plasma has been calculated. The wave number $k_p$ in the plasma, except near the interface, is given by (22) and the amplitude by (33) in terms of the field at the boundary and the strength A of the current sheet. The zero temperature transmission and reflection coefficients have been explicitly calculated. We obtain Taylor's zero temperature results by taking $A = 0$. Thus it would appear that we have successfully avoided the necessity of creating a current sheet, and we can investigate what a real current sheet will do. As we point out, such a non-zero sheet will block some of the transmission, and increase the reflection.

For non-zero temperatures the wave number sufficiently far inside the plasma agrees with Taylor.[1] However, the calculations for the field do not agree. We have a solution for the field strength, which Taylor does not have.

The behavior near the boundary is quite different. The term which gives rise to a non-uniform limit in Taylor's calculations vanishes in ours. We have an integral representation of the field near the boundary, on the Fourier transform (21). The detailed study of the field near the boundary requires the inversion of this integral. We have been unable to do so in a form suitable for computation. Thus a numerical value for the field at the boundary is not yet available, except for the zero temperature limit. We conjecture from one study of this transform that the wave enters the plasma with its wavelength unchanged, but generates other waves which then alter the wavelength to $1/k_p$. We note, however, that in the derivation of $\overline{E}$ (k) we have assumed that the distribution function $f_o$ is independent of z. This is certainly not so near z = 0. Thus even when we obtain an inversion of (21) near z = 0, it must be suspect. We can conclude that the previously obtained answers of Taylor appear to be wrong.

Acknowledgement

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## Appendix I

Following Taylor[1] we write the integral (19) in spherical velocity coordinates. Then

$$\overline{K(k)} = \frac{\omega_p^2}{c^2} \int_0^\infty du \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, \frac{u^3 \sin^3\theta \cos^2\varphi}{n\beta u \cos\theta - 1} \, \beta \, \frac{\partial f_o}{\partial u} \tag{A1}$$

Integrating on $\varphi$ and changing variables $\xi = \cos\theta$, we obtain

$$\overline{K} = \frac{\omega_p^2}{c^2} \pi \int_0^\infty du \, u^2 \frac{\partial f_o}{\partial u} \int_{-1}^1 d\xi \, \frac{1 - \xi^2}{n\xi - \frac{1}{\beta u}} \tag{A2}$$

We define

$$2h(u) = \int_{-1}^1 \frac{d\xi}{n\xi - \frac{1}{\beta u}}$$

$$2g(u) = \beta^2 \; u^2 \int_{-1}^1 d\xi \, \frac{\xi^2}{n\xi - \frac{1}{\beta u}}$$

$$\frac{d}{du} H(u) = u^2 \frac{df_o}{du}$$

and $$\frac{d}{du} G(u) = (1 + u^2) \frac{df_o}{du}$$

Using these four in (A2) and integrating once by parts, using some algebra, we obtain

$$\overline{K(k)} = \frac{2\pi\omega_p^2}{c^2} \int_0^\infty du \, \frac{(1 + u^2) H(u) - u^2 G(u)}{(1 + u^2)^{3/2} [ (1-n^2) u^2 + 1 ]}$$

Then, if $f_o = \left( \frac{\lambda}{2\pi} \right)^{3/2} \exp\left( - \frac{\lambda u^2}{2} \right), \lambda = \frac{mc^2}{kT}$, (A3)

then

$$\overline{K}(k) = \frac{2\pi\omega_p^2}{c^2} \int_o^\infty \frac{2}{\lambda} f_o \frac{du}{(1+u^2)^{3/2} [\ (1-n^2)u^2+1\ ]}$$

$$= \frac{2\omega_p^2}{c^2} \left( \frac{\lambda}{2\pi} \right)^{1/2} \int_o^\infty \exp\left( - \frac{\lambda u^2}{2} \right) \frac{du}{(1+u^2)^{3/2} [\ (1-n^2)u +1\ ]} \qquad \text{(A4)}$$

Let $\rho^2 = (n^2 - 1)\, u^2$

Then $$\overline{K} = \frac{\omega_p^2}{c^2} \frac{2}{\sqrt{\pi}} \int_o^\infty \sqrt{\frac{\lambda}{2(n^2-1)}} \exp\left( - \frac{\lambda \rho^2}{2(n^2-1)} \right) \frac{d\rho}{(1-\rho^2)\left(1 + \frac{\rho^2}{n^2-1}\right)^{3/2}}$$

Let $t^2 = \frac{\lambda \rho^2}{2(n^2-1)}$ so that

$$\overline{K}(k) = \frac{\omega_p^2}{c^2} \frac{\lambda}{\sqrt{\pi}\sqrt{(n^2-1)}} \int_o^\infty \frac{e^{-t^2}}{\alpha^2 - t^2} \left(1 - \frac{3}{\lambda} t^2 + \ldots.\right) dt \qquad \text{(A5)}$$

where $\alpha^2 = \frac{\lambda}{2(n^2-1)}$ and we have expanded $\frac{1}{\left(1+\frac{t^2}{\lambda}\right)^{3/2}}$ for large $\lambda$.

The integrals in (A5) are tabulated in Abramowitz and Stegun (Handbook of Mathematical Functions) and elsewhere.

Thus $\overline{K}(k) = \dfrac{\lambda\omega_p^{\ 2}}{c^2\sqrt{\pi}(n^2-1)} \left\{ \dfrac{-i\pi}{\sqrt{2}} \sqrt{\dfrac{n^2-1}{\lambda}} \exp\left(\dfrac{-\lambda}{2(n^2-1)}\right) \mathrm{erfc}\left(\dfrac{-i\sqrt{\lambda}}{\sqrt{2(n^2-1)}}\right)\right.$

$$\left. + \frac{3}{2\lambda}\left[\sqrt{\pi} + \pi i\sqrt{\frac{\lambda}{2(n^2-1)}} \exp\left(\frac{-\lambda}{2(n^2-1)}\right) \mathrm{erfc}\left(\frac{-i\sqrt{\lambda}}{\sqrt{2(n^2-1)}}\right)\right]\right\},$$

so $\dfrac{c^2\overline{K}(k)}{\omega_p^{\ 2}} = \dfrac{3}{2(n^2-1)} + \dfrac{\sqrt{\pi\lambda}}{i\sqrt{2(n^2-1)}} \left\{1 - \dfrac{3}{2(n^2-1)}\right\} \exp\left(\dfrac{-\lambda}{2(n^2-1)}\right) \mathrm{erfc}\left(\dfrac{-i\sqrt{\lambda}}{\sqrt{2(n^2-1)}}\right)$

For n finite, the limit for large $\lambda$ (small temperature) is obtained by the asymptotic expansion of the complementary error function

Thus $\lim\limits_{\lambda\to\infty} \dfrac{c^2\overline{K}}{\omega_p^{\ 2}}(\lambda, n) = 1 + (n^2 - \dfrac{5}{2})\dfrac{1}{\lambda}$ (A7)

However, for $n \to \infty$, for fixed $\lambda$, one needs the series expansion of the error function. We obtain

$$\lim_{n\to\infty} \frac{c^2}{\omega_p^{\ 2}} \overline{K}(\lambda, n) = \frac{3}{2(n^2-1)} + \frac{\sqrt{\pi\lambda}}{i\sqrt{2(n^2-1)}}\left\{1 - \frac{\lambda+3}{2(n^2-1)} + \dots\right\}$$

$$+ \frac{\lambda}{n^2-1}\left\{1 - \frac{9+2\lambda}{6(n^2-1)} + \frac{\lambda}{2(n^2-1)}\right\} \qquad \text{(A8)}$$

## Appendix II

We wish to evaluate equation (28) by the method of steepest descent. Let

$$I = \int_0^\infty \frac{x^2\left\{1 - \frac{2}{\lambda}(1-x)\right\} \exp\left\{-\frac{\lambda}{2}x - i\frac{\sqrt{1+x}}{\sqrt{x}}\frac{z\omega}{c}\right\}}{\left\{1 + x\frac{\omega_p^2}{\omega^2}\overline{K}_4(x)\right\}\left\{1 + x\frac{\omega_p^2}{\omega^2}\overline{K}_5(x)\right\}} dx \qquad \text{(B1)}$$

and let

$$h(x) = x + \frac{i\sqrt{1+x}}{\sqrt{x}} \; \frac{2 z \omega}{c \lambda} \tag{B2}$$

Then $\frac{dh}{dx} = o = 1 - \frac{i z \omega}{\lambda c} \; \frac{1}{x^{3/2}\sqrt{1+x}}$

Thus the saddle points are at the solutions of

$$x_o^3 \; (1 + x_o) = - \frac{\lambda^2 c^2}{\omega^2 z^2} \; . \tag{B3}$$

At the saddle points

$$\frac{d^2 h}{d x^2} = \frac{1}{2} \left(\frac{\omega z}{\lambda c}\right)^4 (4 + 3/x_o)$$

The quartic equation (B3) is very difficult to solve. However, for large $\lambda$ or small z, it is easy to see that $x_o$ is large, and thus the solutions to (B3) are approximately the four fourth roots of a negative number. And for large $\lambda$ the function h(x) is approximately x, so that the paths of steepest descent, that is the paths of Im h(x) = constant, are the horizontal lines through $x_o$. Thus the original path of integration from $o \rightarrow \infty$ <u>can</u> be deformed to the path of steepest descent. Thus for sufficiently large $\lambda$ (or sufficiently small z) we approximate our answers by

$$x_o \approx \left(\frac{1 + i}{2}\right) \sqrt{\frac{\lambda c}{\omega z}} \tag{B4}$$

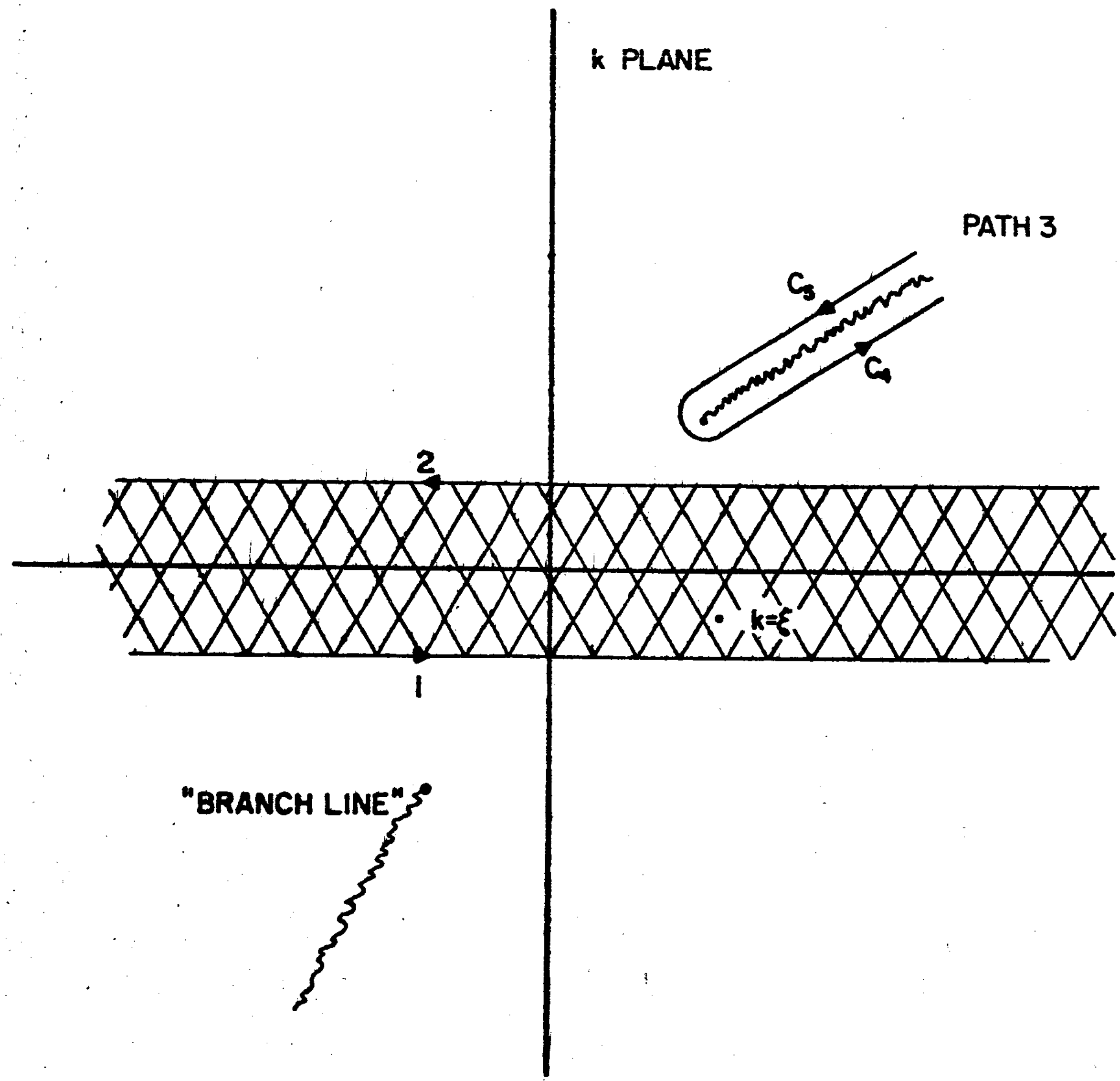
$$h''(x_o) \approx 2 \left(\frac{\omega z}{\lambda c}\right)^4$$

Then

$$I \approx \frac{x_o^2 \left\{1 - \frac{2}{\lambda}(1 - x_o)\right\} \exp\left\{\left[\frac{\lambda}{2} \; x_o + 2/x_o\right]\right\} \sqrt{\frac{\pi}{h_o'' \lambda}}}{\left\{1 + x_o \frac{\omega_p^2}{\omega^2} \overline{K}_4 (x_o)\right\} \left\{1 + x_o \frac{\omega_p^2}{\omega^2} \overline{K_5} (x_o)\right\}} \tag{B5}$$

Since $x_o$ is large, then the denominator should be simplified, using the expansion of $\overline{K}(k)$ for large value of the argument. Then

$$I \approx \frac{-\omega^4}{\omega_p^4} \frac{2}{x_o \pi \lambda} \sqrt{\frac{\pi}{\lambda h_o''}} \left\{1 - \frac{2}{\lambda}(1 - x_o)\right\} \exp\left\{\frac{\lambda}{2}\left[x_o + \frac{2}{x_o}\right]\right\} \tag{B6}$$




INTEGRATION PATHS IN THE COMPLEX k PLANE

FIGURE I