# The Greenhouse Effect in a Gray Planetary Atmosphere

Rupert Wildt

Yale University Observatory

and

Institute for Space Studies
Goddard Space Flight Center, NASA

New York, New York

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## ABSTRACT

Hopf's analytical solution is illustrated for several values of the greenhouse parameter, i.e., the ratio of gray absorption coefficients for insolating and escaping radiation, assumed to be constant at all depths.

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In a classical memoir Emden (1913) formulated the problem of strict radiative equilibrium in a gray atmosphere of infinite depth, and Milne (1922) put it into the form of the non-homogeneous integral equation now bearing his name. The Neumann solution of this equation was reduced by Hopf (1934) to the product of an integral over his $q$ -function times what later on came to be called Chandrasekhar's H-function. Many years after precise values of these functions had become available, attention was called to Hopf's solution, and the temperature distribution in absence of a greenhouse effect was determined for several angles of incidence of the insolating flux (Wildt 1961). Moreover, Hopf's solution comprises, by a certain scale transformation, the case of an atmosphere in which both the absorption coefficients for incident solar radiation and escaping planetary radiation are gray, provided that their ratio, $n$ , here referred to as greenhouse parameter, is independent of depth. Illustration of the greenhouse effect had to await preparation of tables of the H-function with arguments greater than unity (Stibbs 1962)

[redacted]

and their extension during the work reported here. As the familiar planetary atmospheres are non-gray in the extreme, this model of the greenhouse effect does not contribute much to understanding their temperature regime. Nevertheless, it deserves to be known more widely; for to date it is the only problem in planetary radiative equilibrium that has been solved rigorously.

A parallel insolating flux,

$$\pi S\,[\mathrm{erg\ cm^{-2}\ sec^{-1}}] = \sigma T_e^4 \qquad (1)$$

incident at an angle $\theta = \cos^{-1}\mu$ with the normal to the surface of an infinitely deep, plane parallel atmosphere, whose gray absorption coefficients are $\kappa_p$ for planetary radiation and $\kappa_s$ for solar radiation; and a local rate of isotropic emission,

$$4\pi\kappa_p B(\tau,\mu)\,[\mathrm{erg\ cm^{-3}\ sec^{-1}}] = 4\kappa_p\sigma T(\tau)^4 \qquad (2)$$

at the optical depth

$$\tau = \int_x^\infty \kappa_p\,dx \qquad (3)$$

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below the boundary of the atmosphere, jointly imply the local energy balance in strict radiative equilibrium

$$B(\tau,\mu) = \Lambda_\tau\{B(t,\mu)\} + S\frac{n}{4}e^{-\tau n/\mu} \qquad (4)$$

where $\kappa_s/\kappa_p = n < 1$ is a constant independent of depth and $\Lambda$ denotes the Hopf operator,

$$\Lambda_\tau\{\phi(t)\} = \frac{1}{2}\int_0^\infty \phi(t)\, E_1(|t-\tau|)\, dt. \qquad (5)$$

The general solution of (4) is the sum of the solution of the homogeneous Milne equation and of the Neumann solution for the exponential term. Hence the planetary source function is

$$B(\tau,\mu) = \frac{3}{4}f(\tau)F + n\,g(\tau, n/\mu)\,S, \qquad (6)$$

with the following notation:

$\pi F$ emergent flux of planetary heat generated in the remote interior,

$f(\tau) = \tau + q(\tau)$ normalized solution of the homogeneous Milne equation,

$\pi S$ insolating flux

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$g(\tau, n/\mu_0)$ normalized Neumann solution of the non-homogeneous

Milne equation, sc. (Hopf 1934)

$$g(\tau,s) = \frac{3}{4}\int_0^\infty f(t)e^{-st}\,st\left[f(\tau) - \int_0^\tau f(t)e^{s(t-\tau)}\,s\,dt\right], \qquad (7)$$

If the planetary heat source is neglected $(\pi F = 0)$, the temperature distribution in local thermodynamic equilibrium becomes

$$T(\tau, n/\mu)/T_e = [ng(\tau, n/\mu)]^{1/4}, \qquad (8)$$

where $T_e = 392\,°K\,/\sqrt{R}$, with $R$ in astronomical units, is the effective temperature of the insolating flux, e.g.

| Planet | Venus | Jupiter | Saturn | Uranus | Neptune |
|---|---|---|---|---|---|
| $T$ °K | 464 | 173 | 128 | 89 | 78 |

The behavior of the right-hand side of (8) as function of the greenhouse parameter is shown on Fig. 1-8. A recent paper on the greenhouse effect in a gray atmosphere (King 1963) neither makes reference to Hopf, nor does it provide extensive illustration of the form of the solution.

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# LEGENDS

Fig. 1. Temperature distribution as a function of optical depth in the absence of a greenhouse effect (n=1) for flux incident at various angles.

Fig. 2. Temperature distribution as a function of optical depth with a moderate greenhouse effect (n=1/10) for flux incident at various angles.

Fig. 3. Temperature distribution as a function of optical depth with a strong greenhouse effect (n=1/100) for flux incident at various angles.

Fig. 4. Temperature as a function of $\log^{10} n$ for $\mu = 0.05$ at various optical depths.

Fig. 5. Temperature as a function of $\log^{10} n$ for $\mu = 0.25$ at various optical depths.

Fig. 6. Temperature as a function of $\log^{10} n$ for $\mu = 0.50$ at various optical depths.

Fig. 7. Temperature as a function of $\log^{10} n$ for $\mu = 0.75$ at various optical depths.

Fig. 8. Temperature as a function of $\log^{10} n$ for $\mu = 1$ at various optical depths.

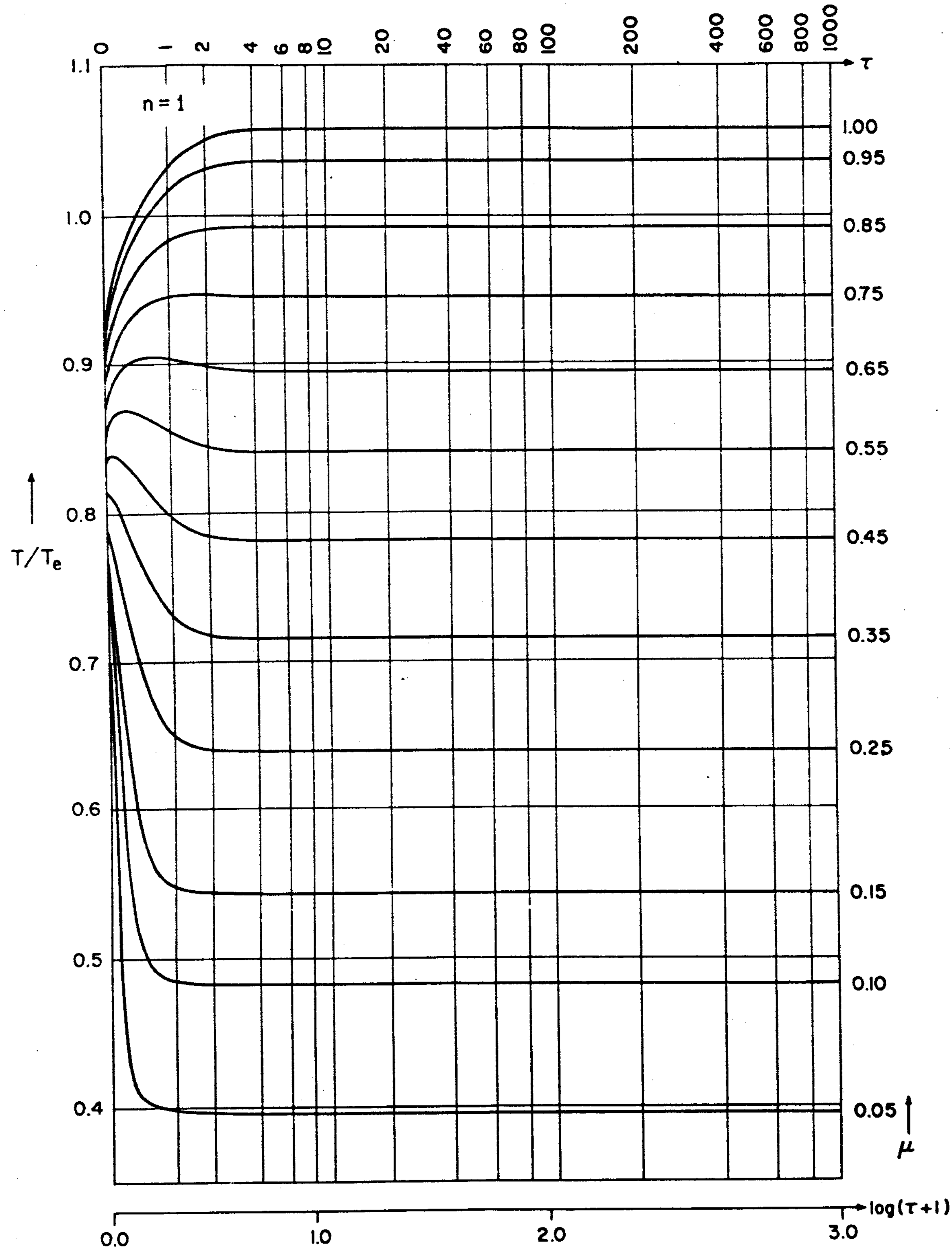


Fig. 1

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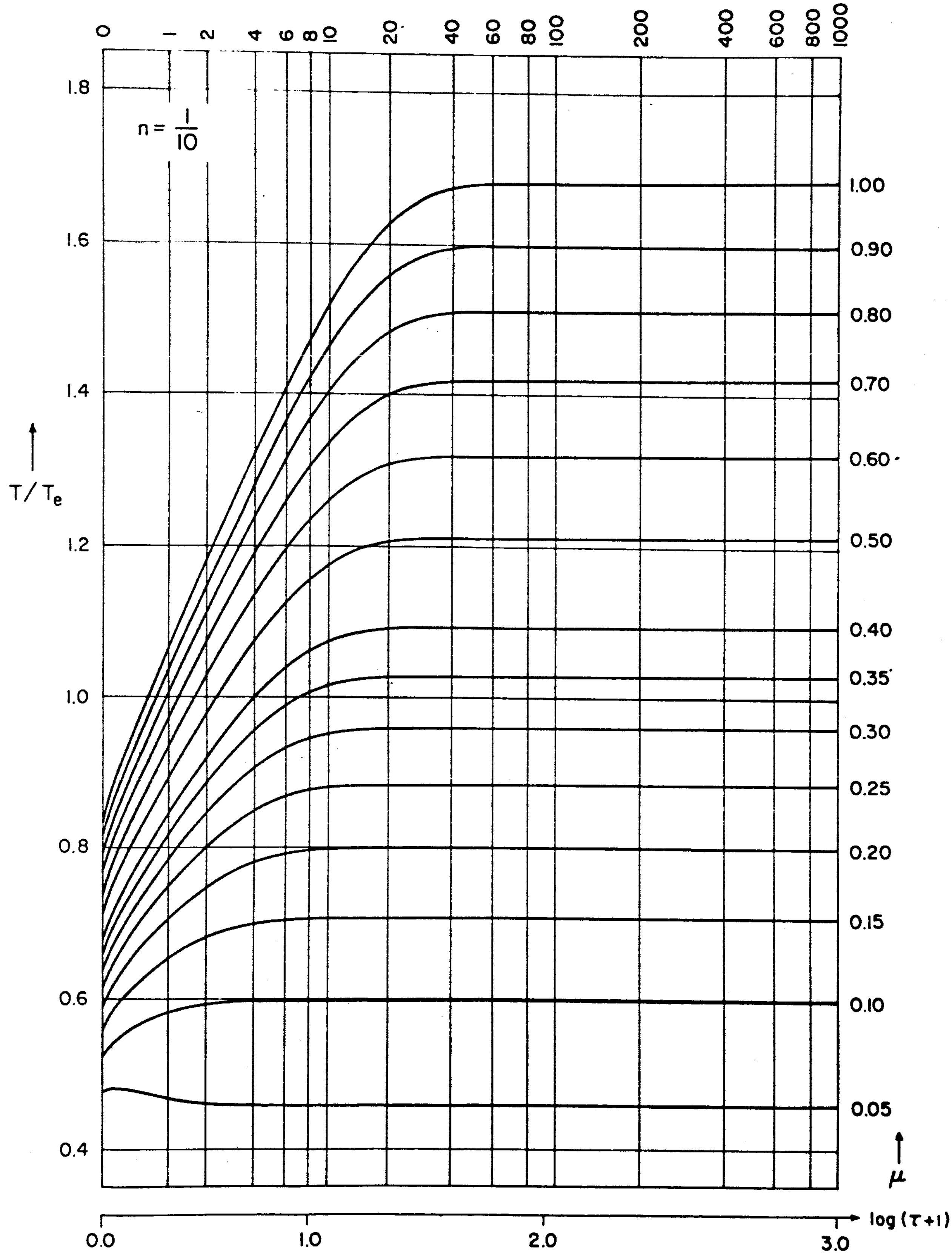

0
1
2
4
6
8
10
20
40
60
80
100
200
400
600
800
1000
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
$n = \frac{1}{10}$
$T/T_e$
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
$\mu$
0.0
1.0
2.0
3.0
log (τ+1)


Fig. 2

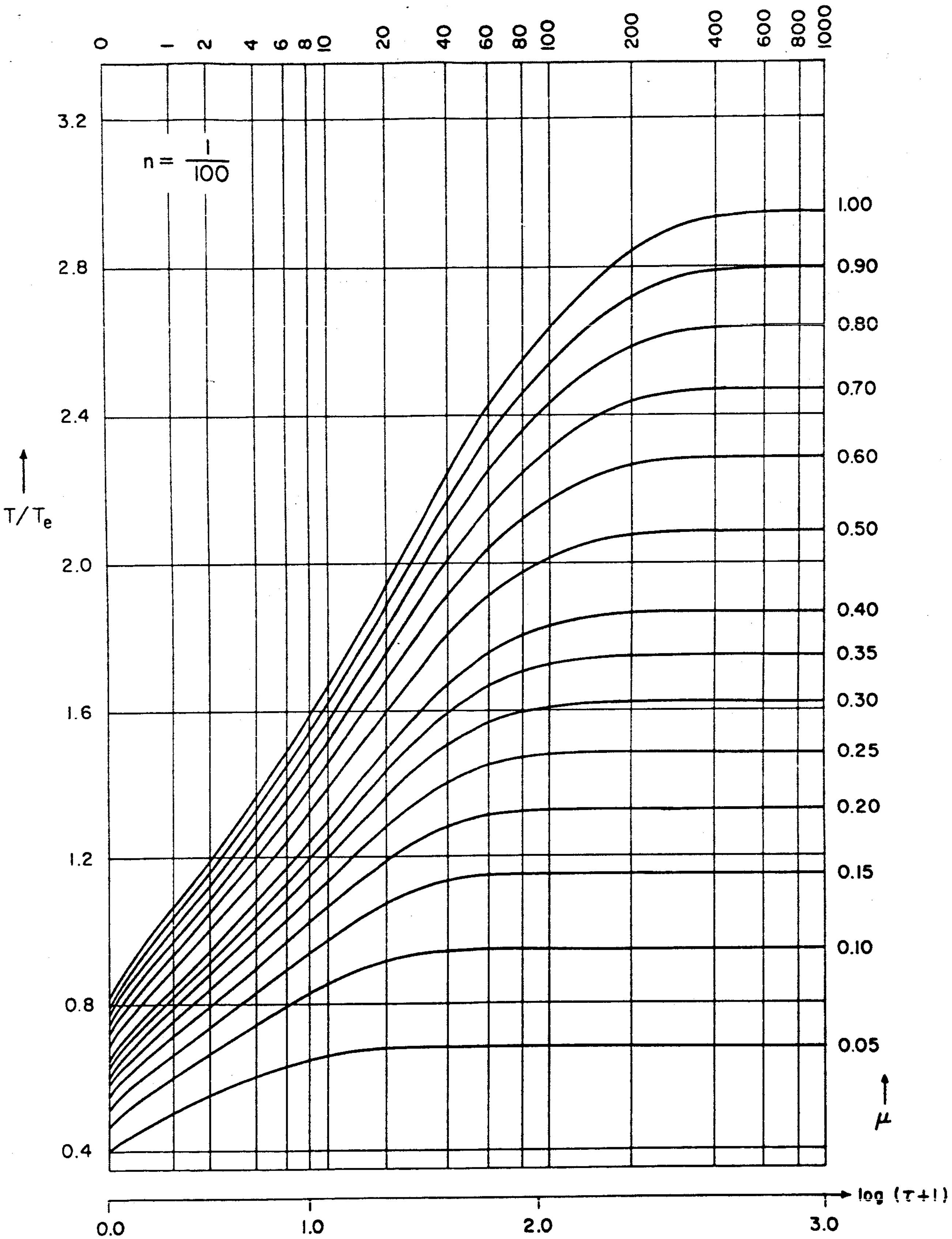


Fig. 3

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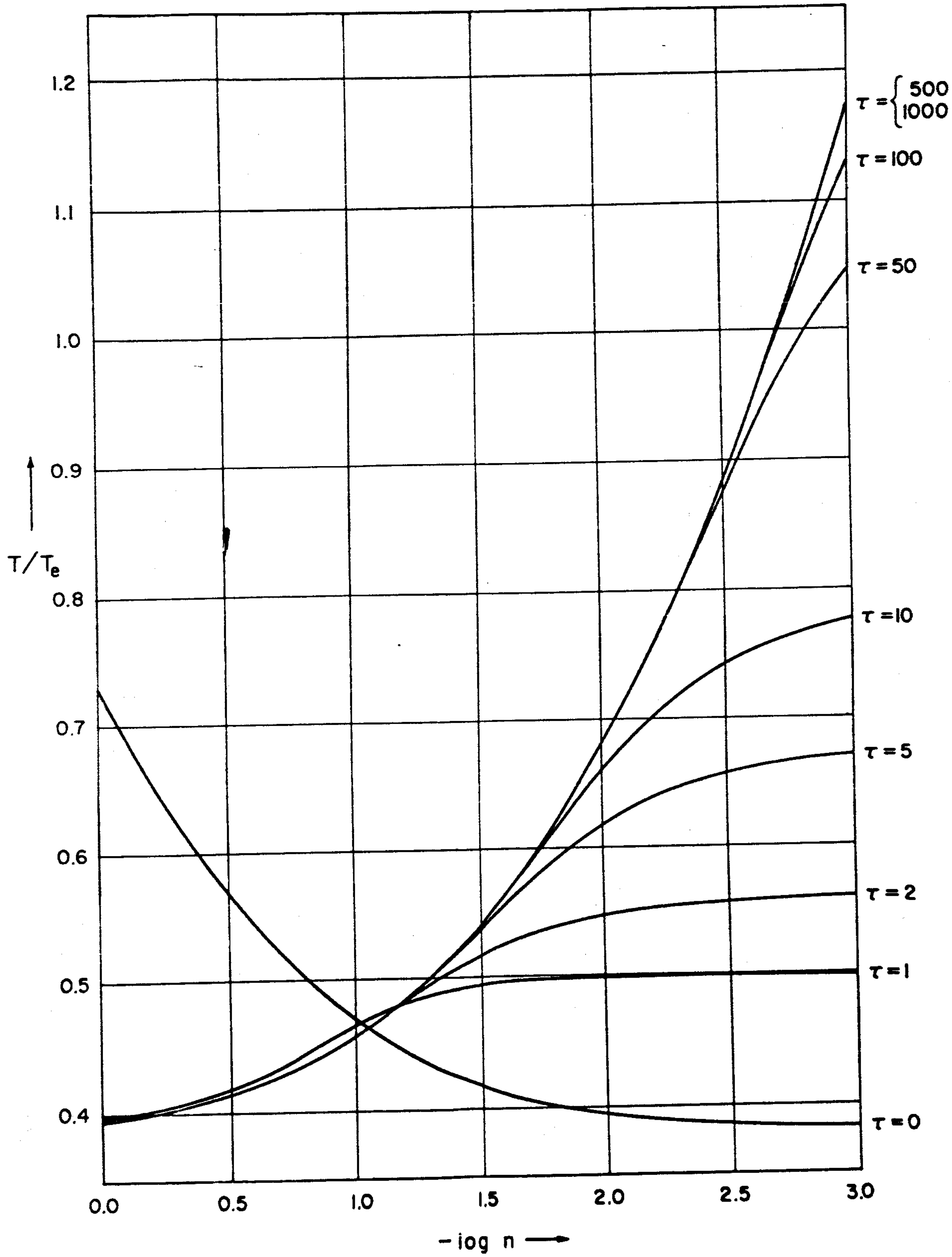


Fig. 4

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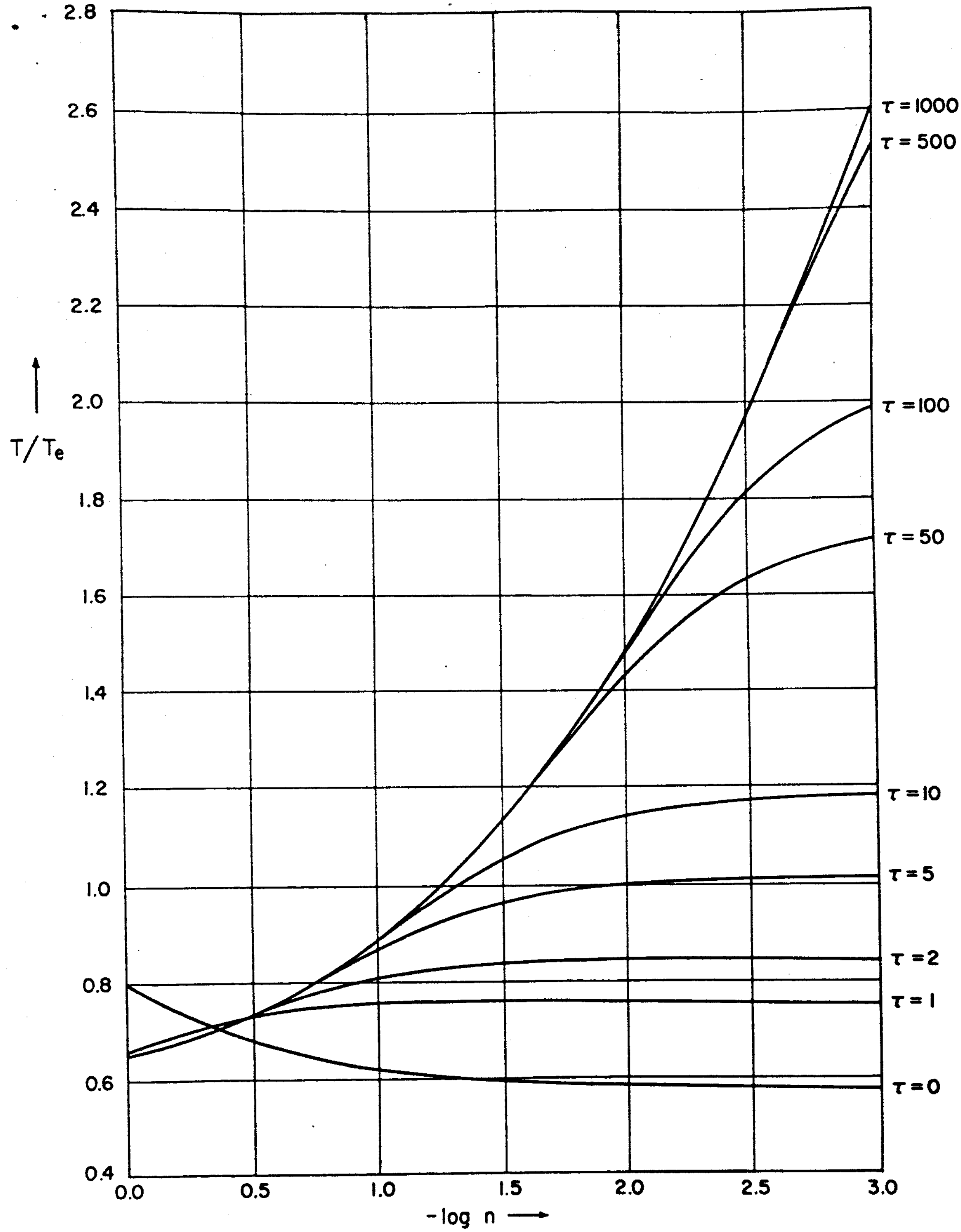


Fig. 5

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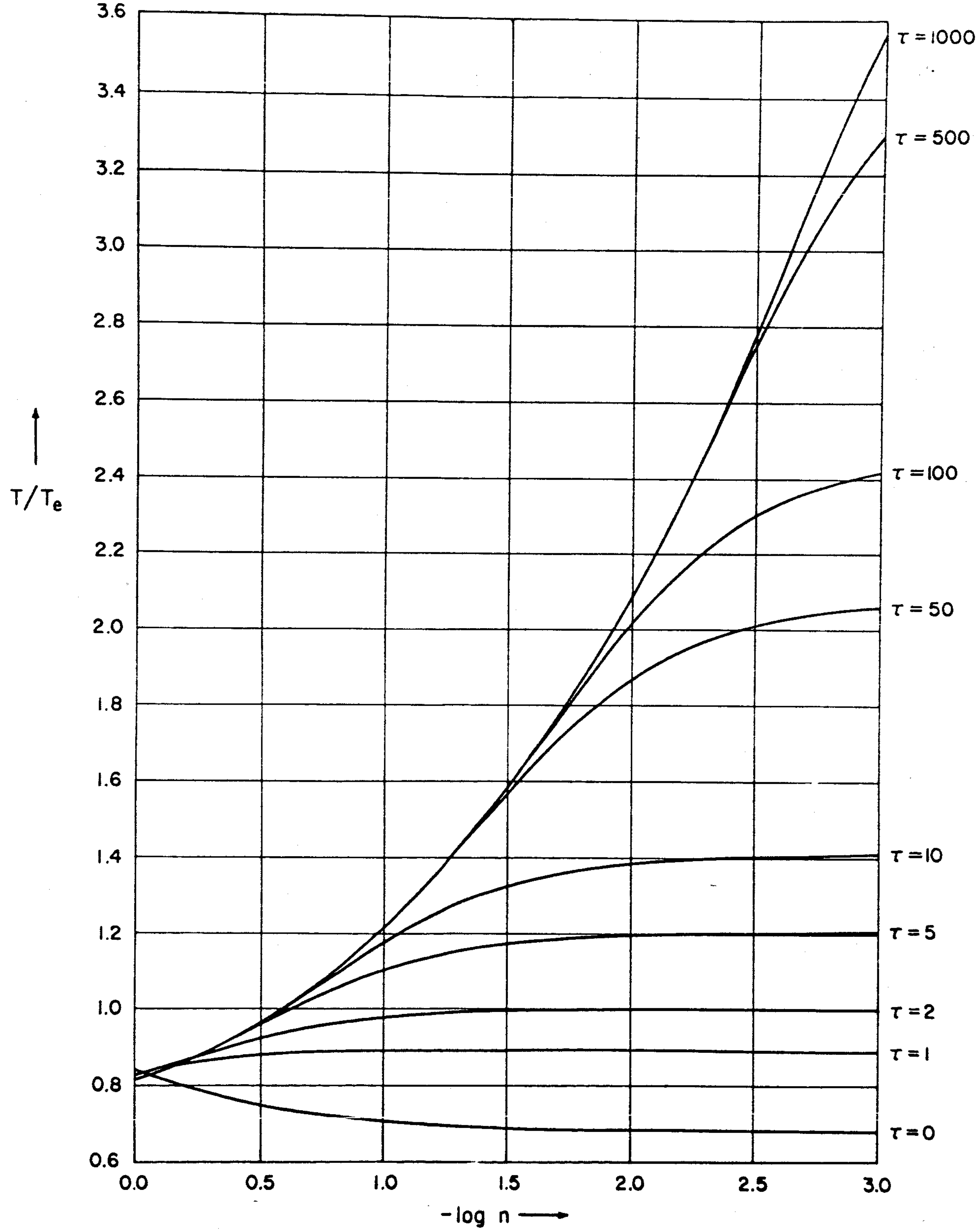
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
T/T_e
0.0
0.5
1.0
1.5
2.0
2.5
3.0
-log n
τ = 1000
τ = 500
τ = 100
τ = 50
τ = 10
τ = 5
τ = 2
τ = 1
τ = 0

Fig. 6

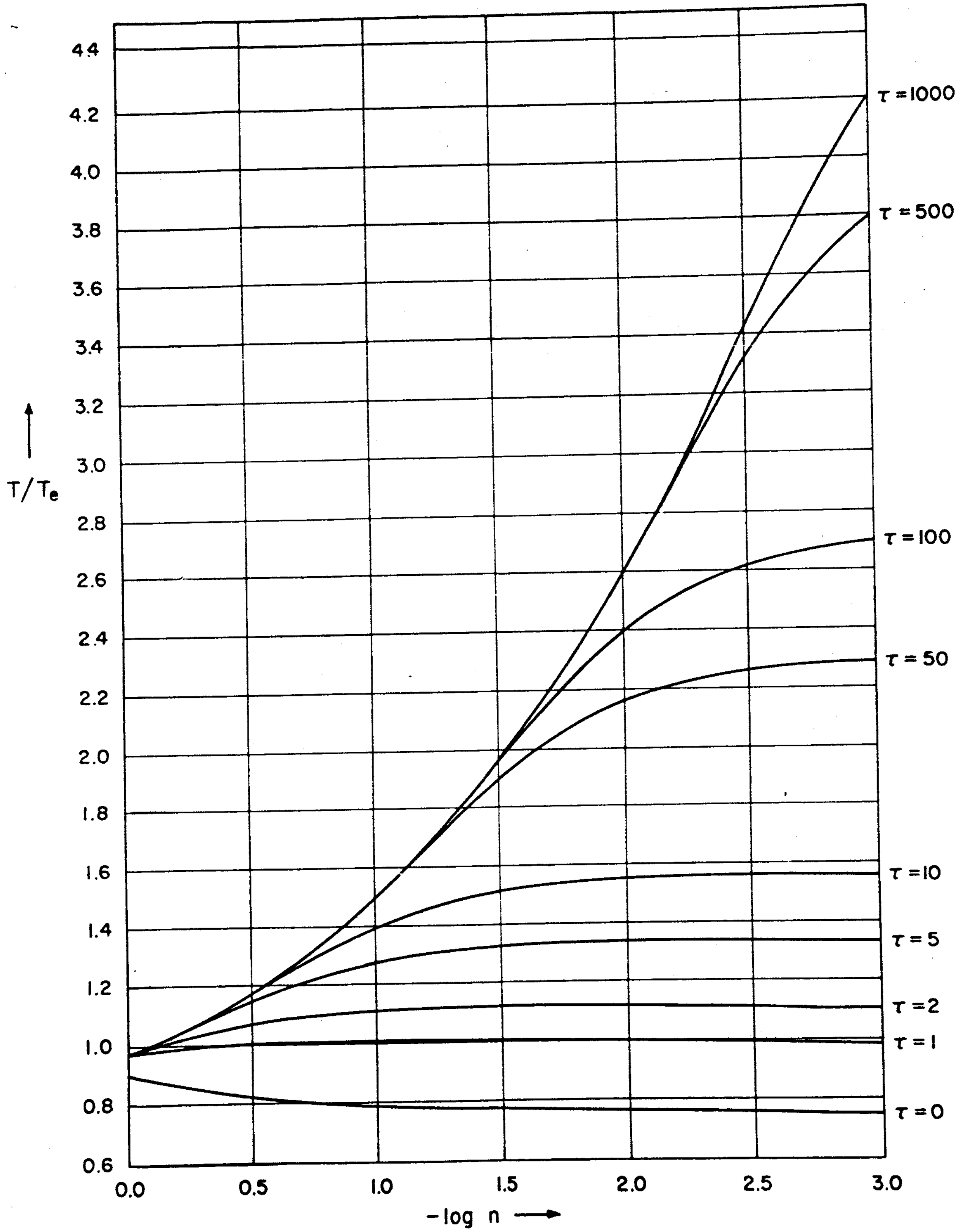


Fig. 7

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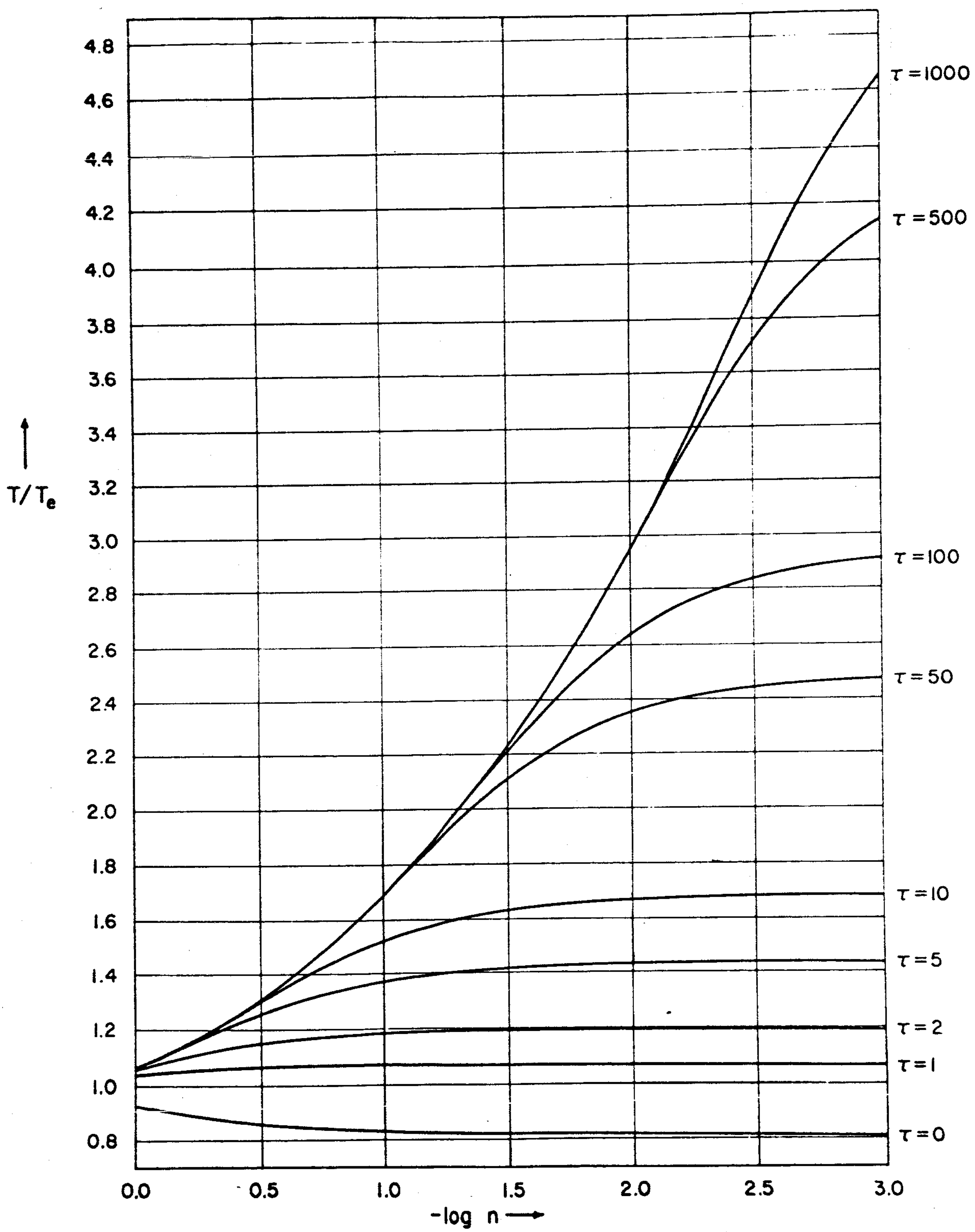

4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
T/T_e
0.0
0.5
1.0
1.5
2.0
2.5
3.0
-log n
τ = 1000
τ = 500
τ = 100
τ = 50
τ = 10
τ = 5
τ = 2
τ = 1
τ = 0


Fig. 8