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LOWEST AXISYMMETRIC BODY MODES OF VIBRATION OF THE APOLLO VEHICLE

By

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SUMMARY

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The lowest axisymmetric body modes of vibrations of the ring stiffened Apollo type vehicle are studied by means of a Rayleigh-Ritz procedure. A method is developed for converting the ring stiffened structure into an equivalent orthotropic, truncated, conical shell.

A simplified theory in the manner of Love's first approximation in the classical thin shell theory is used. Results are given in the form of dimensionless frequency parameters for various values of length to mean radius ratio, thickness to mean ratio, and the ratios of elastic constants of the equivalent orthotropic shell. It is found that stiffening the circumferential rings has the same effect on the natural frequency of vibration of the conical shell structure as that of increasing its thickness whereas increasing the cross stiffness has very little effect on the natural frequency. Results for the equivalent isotropic shell are determined as a special case of the equivalent orthotropic shell and are found to be in good agreement with those predicted by Love.

In an appendix, a method is developed to estimate the vibration frequencies of the panels between shell stiffeners by assuming them to be equivalent to simply supported orthotropic rectangular plates with no curvature.

The results of these analyses provide a basic working tool for investigations of coupled vibration modes of the Apollo structure in the intermediate and critical frequency range between low frequency launch vehicle modes and high frequency, high-density modes of the primary and secondary structure.

author

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LIST OF SYMBOLS

E'_x	elastic modulus in the meridional direction
E'_θ	elastic modulus in the tangential direction
E''	cross stiffness elastic modulus
g	acceleration due to gravity
h	thickness of the shell
L	length of the shell measured along the cone axis
M_x	bending moment per unit length in the meridional direction
M_θ	bending moment per unit length in the tangential direction
N_x	vertical shear stress along the meridian
N_θ	vertical shear stress along the tangential direction
p	frequency of vibration of the shell
r_1, r_2	radii of the end sections of the conical shell
R	mean radius
R_2	principal radius of curvature in the z direction
T	total kinetic energy in the shell
u	displacement of the median surface along the x axis
$\dot{u}$	velocity of the median surface along the x axis
U	total displacement along the x axis

V	total energy in the shell
w	displacement of the median surface along the z axis
$\dot{w}$	velocity of the median surface along the z axis
W	total displacement along the z axis
x, θ, z	coordinate system
x'	ratio $\frac{x_2}{x_1}$
y	a parameter $= \ln_e \frac{x}{x_1}$

Greek Symbols

α	semicone angle
$\gamma_{x\theta}, \gamma_{xz}$	shear strains
ϵ_1, ϵ_2	resultant total strain in the meridianal and tangential directions, respectively
$\epsilon_x, \epsilon_\theta$	strains in the median surface in the meridianal and tangential directions, respectively

$\lambda = \frac{p^2 \rho x_1^2}{g E'_x}$ = frequency parameter

$\xi = \frac{2\pi}{y_2}$, a parameter

σ_x, σ_θ	stresses in the meridianal and tangential directions respectively

$\tau_{x\theta}$	shear stress
ρ	weight per unit volume of the shell element

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1.0 INTRODUCTION

The analysis of environmental vibration of a complex space vehicle such as the Apollo will involve consideration of three broad frequency regions:

a) Low frequencies involving bending and longitudinal modes of the entire launch vehicle where individual sections, such as the payload, tend to vibrate very nearly as rigid bodies.

b) Intermediate frequencies involving low order coupled modes of the local basic airframe.

c) High frequencies involving high order modes of the airframe and secondary structure which are grouped so closely that statistical analysis is generally required for analysis of vibration levels.

It is the intermediate frequency range which presents one of the more difficult problems in defining vibration environments and it is to this frequency range that this report is directed.

A method is developed for general application to the Apollo to define the lowest axisymmetric modes of the basic airframe. These modes constitute the lower bound for this intermediate range of coupled modes and are generally responsible for major dynamic loads on secondary structure and major equipment packages. This report represents a significant step forward in dynamic analysis of the vibration characteristics of a complex vehicle such as the Apollo. Although the analysis is based on vibration of the basic airframe, without loading by equipment, it is considered a necessary preliminary step for developing a full understanding of the vibration loads developed in the flight system.

The structural frame work of the Apollo space vehicle consists essentially of a truncated cone-shaped surface built up of panels reinforced with rigid stiffness in the longitudinal and circumferential directions. A detailed analysis of the vibration characteristics of the vehicle should take into consideration the exact location and spacing of the stiffeners and the mass distribution of the entire structure and its pay load. However, to estimate the fundamental frequencies of the body modes of vibration of the structure, it should be sufficient to assume it to be equivalent to an orthotropic truncated conical shell with difficult elastic properties in the meridional and tangential directions. The thickness-to-radius ratio is assumed to be small so that the elastic properties in the radial direction do not enter into the study.

Vibration characteristics of isotropic conical shells have been studied by Federhofer[1*], Goldberg[2], Herman and Mirsky[3], and Garnet and Kempner[4]. In this report the lowest frequency of axisymmetric free vibrations of a simply supported truncated orthotropic shell is studied by means of a Rayliegh-Ritz procedure.

* Superscripts indicate references at the end of this report.

A method is also developed in Appendix A for estimating the lowest modes of the individual panels between stiffeners which make up the conical shell.

The results of these analyses are given in the form of non-dimensional frequency parameters which depend on material properties and structural geometry. Specific resonant frequencies can then be determined for a wide range of structural materials and structural designs which include the Apollo configuration.

2.0 THEORY

Figure 1 shows a conical shell and the coordinate system used. As shown, α is the projected semi-cone angle, x_1 and x_2 define the edges of the truncated cone, R is the mean radius of the cone, L is the length measured along the longitudinal axis, and r_1 and r_2 are the radii of the end surfaces of the cone. It can be shown that

$$\frac{L}{R} = \frac{2}{1 + \frac{x_2}{x_1}} \left(\frac{x_2}{x_1} - 1 \right) \cot \alpha \tag{1}$$

and

$$\frac{h}{x_1} = \frac{h}{2R} \left(1 + x' \right) \sin \alpha \tag{2}$$

It is assumed that the cone is simply supported at the edges $x = x_1$, $x = x_2$. For small deflections, Hooke's law for an orthotropic elastic body gives

$$\begin{aligned} \sigma_x &= E'_x \, \epsilon_x + E'' \, \epsilon_\theta \\ \sigma_\theta &= E'_\theta \, \epsilon_\theta + E'' \, \epsilon_x \\ \tau_{x\theta} &= 2G \, \gamma_{x\theta} \end{aligned} \tag{3}$$

The median fiber strains are given by

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_\theta &= \frac{u}{x} + \frac{w}{R_2} \end{aligned} \tag{4}$$

where the radius of curvature R_2 is related to x by

$$R_2 = x \tan \alpha \tag{5}$$

so that

$$\epsilon_\theta = \frac{u}{x} + \frac{w}{x \tan \alpha} \tag{6}$$

The displacements U and W in the meridianal and normal directions are

$$U = u - z \frac{\partial w}{\partial x}$$
$$W = w \tag{7}$$

when $u = u(x,t)$ and $w = w(x,t)$ are the displacements of the median surface

Following Herrman and Mirsky[3], the strains corresponding to these displacements are

$$\epsilon_1 = \epsilon_x - z \frac{\partial^2 w}{\partial^2 x} \tag{8}$$

in the meridianal direction and

$$\epsilon_2 = \epsilon_\theta - \frac{z}{x} \frac{\partial w}{\partial x} \tag{9}$$

in the tangential direction.

Shear effects are not taken into consideration in this analysis.

The stress resultant in the x direction is

$$N_x = \int_{-h/2}^{+h/2} \sigma_x \left(1 + \frac{z}{R_2}\right) dz \tag{10}$$

and that in the tangential direction is

$$N_\theta = \int_{-h/2}^{+h/2} \sigma_\theta \, dz \tag{11}$$

The moment resultant in the x-direction is

$$M_x = \int_{-h/2}^{+h/2} \sigma_x z \left(1 + \frac{z}{R_2}\right) dz \tag{12}$$

and that in the tangential direction is

$$M_\theta = \int_{-h/2}^{+h/2} \sigma_\theta z \, d z \tag{13}$$

Substituting from (3) to (9) the stress and moment resultants (10) to (13) become

$$N_x = E'_x h \frac{\partial u}{\partial x} + E'' h \left(\frac{u}{x} + \frac{w}{x \tan \alpha}\right) \tag{14}$$

$$N_\theta = E'_x h \left(\frac{u}{x} + \frac{w}{x \tan \alpha}\right) + E'' h \frac{\partial u}{\partial x} \tag{15}$$

$$M_x = - E'_x \frac{h^3}{12} \frac{\partial^2 w}{\partial x^2} - E'' \frac{h^3}{12} \frac{1}{x} \frac{\partial w}{\partial x} \tag{16}$$

$$M_\theta = - E'_x \frac{h^3}{12} \frac{1}{x} \frac{\partial w}{\partial x} - E'' \frac{h^3}{12} \frac{\partial^2 w}{\partial x^2} \tag{17}$$

The strain energy in the conical shell is given by

$$V = \frac{1}{2} \int_{x_1}^{x_2} \int_0^{2\pi} \left(N_x \frac{\partial u}{\partial x} + N_\theta \left(\frac{u}{x} + \frac{w}{x \tan \alpha}\right) - M_x \frac{\partial^2 w}{\partial x^2} - \right.$$
$$\left. - M_\theta \frac{1}{x} \frac{\partial w}{\partial x}\right) x \sin \alpha \, d\theta \, d x \tag{18}$$

and the kinetic energy is

$$T = \frac{\rho h}{g} \int_{x_1}^{x_2} \int_0^{2\pi} \left[\frac{\dot{u}^2}{2} + \frac{\dot{w}^2}{2}\right] x \sin \alpha \, d\theta \, d x \tag{19}$$

where $\rho h/g$ is the mass per unit area of the shell.

Following Shulman, a change of coordinates

$$x = x_1 \, e^{y} \tag{20}$$

is used to reduce computational work. To use the Rayleigh-Ritz method the displacements are assumed to be of the form

$$\begin{aligned} u &= A_1 \cos \frac{\pi y}{y_2} \cos p t \\ w &= C_1 \sin \frac{\pi y}{y_2} \cos p t \end{aligned} \tag{21}$$

where $y_2 = \ln_e \frac{x_2}{x_1}$.

Then displacements satisfy the boundary conditions

$$w = 0 \quad \text{at} \quad x = x_1 , \quad x_2 \tag{22}$$

Substituting (21) into (19) the kinetic energy becomes

$$T = \frac{1}{2} \frac{p^2 \rho h}{g} \int_{x_1}^{x_2} \int_{0}^{2\pi} (u^2 + w^2) \; x \; \sin \alpha \; \sin^2 p t \; d\theta \; dx \tag{23}$$

Its maximum occurs when the shell is in its middle position, i.e., when $\sin p t = 1$

$$T_{max} = \frac{1}{2} \frac{p^2 \rho h}{g} \int_{x_1}^{x_2} \int_{0}^{2\pi} (u^2 + w^2) \; x \sin \alpha \; d\theta \; dx \tag{24}$$

The potential energy attains its maximum when the shell is in its extreme position, i.e., when $\cos p t = 1$. Since it is a conservative system, the maximum potential energy is equal to the maximum kinetic energy. From (24) and (18) using (21) with $\cos p t = 1$,

$$p^2 = \frac{2 g}{\rho h} \frac{V \Big/_{\cos \omega t = 1}}{\int_{x_1}^{x_2} \int_{0}^{2\pi} (u^2 + w^2) \; x \; \sin \alpha \; d\theta \; dx} \tag{25}$$

This expression for the frequency is minimized with respect to the constants A_1 and C_1 which gives the following two equations for the determination of the frequencies

$$\frac{\partial}{\partial A_1}\left[V\Big/_{\cos p t = 1} - \frac{p^2 \rho h}{2g}\int_{x_1}^{x_2}\int_{0}^{2\pi} (u^2 + w^2)\, x \sin \alpha \; d\theta\, dx \right] = 0 \tag{26}$$

$$\frac{\partial}{\partial C_1}\left[V\Big/_{\cos p t = 1} - \frac{p^2 \rho h}{2g}\int_{x_1}^{x_2}\int_{0}^{2\pi} (u^2 + w^2)\, x \sin \alpha \; d\theta\, dx \right] = 0 \tag{27}$$

After substitution and considerable simplification, equation (26) gives

$$A_1\left[\left(\frac{\xi}{2}\right)^2 \ln x' + \frac{E'_\theta}{E'_x} \ln x' - \lambda \left\{ \frac{1}{2}\,(x'^2 - 1) + \frac{x'^2 - 1}{\xi^2 (1 + \xi^2)} \right\}\right]$$

$$+ \; C_1 \left[- \frac{E''}{E'_x}\left(\frac{\xi}{2}\right)\frac{1}{\tan \alpha} \ln x' \right] = 0 \tag{28}$$

where

$$\xi = \frac{2\pi}{y_2}$$

$$\frac{x_2}{x_1} = x'$$

and

$$\lambda = \frac{p^2 \rho x_1^2}{g E'_x}$$

is the frequency parameter. Similarly, equation (27) gives

$$-A_1 \frac{E''}{E'_x}\left(\frac{\xi}{2}\right)\frac{\ln x'}{\tan\alpha}$$

$$+ C_1\left[\frac{E'_\theta}{E'_x}\,\frac{\ln x}{\tan^2\alpha} + \frac{1}{24}\left(\frac{h}{x_1}\right)^2\left\{\left(\frac{\xi}{2}\right)^2\left(1-\frac{1}{x^2}\right)\left(2+\left(\frac{\xi}{2}\right)^2\right)\right\}\right.$$

$$+\frac{1}{24}\,\frac{E'_\theta}{E'_x}\left(\frac{h}{x_1}\right)^2\left(1-\frac{1}{x'^2}\right)\left\{\frac{2+\left(\frac{\xi}{2}\right)^2}{1+\left(\frac{\xi}{2}\right)^2}\right\} \tag{29}$$

$$\left.-\frac{1}{6}\,\frac{E''}{E'_x}\left(\frac{h}{x_1}\right)^2\left(\frac{\xi}{2}\right)^2\left(1-\frac{1}{x'^2}\right) - \lambda\left\{\frac{x'^2-1}{2} - \frac{x'^2-1}{\xi^2(1+\xi^2)}\right\}\right] = 0$$

For a nontrivial solution, the determinant of the coefficients of A_1 and C_1 in equations (28) and (29) should be zero. This gives a quadratic in λ for the determination of the frequencies p. This quadratic has been solved with the Wyle CDC 3200 computer for various values of the ratios

$$\frac{x_2}{x_1}\,,\quad \frac{L}{R}\,,\quad \frac{E'_\theta}{E'_x}\,,\quad \frac{E''}{E'_x}\,,\quad \text{and}\ \alpha\,,$$

and the results are tabulated in Tables 1 - 4.

3.0 RESULTS

It is seen that the effect on frequency of increasing the stiffness of the circumferential rings is the same as that of increasing the thickness of the shell all other dimensions and elastic properties remaining the same. Thus, the frequency parameter λ is approximately 95 percent higher at $E'_\theta/E'_x = 5$ than that at $E'_\theta/E'_x = 0.2$.

Variations in the cross stiffness parameter E''/E'_x have very little effect on the vibration frequency.

As shown in Table 4 by the special case of an isotropic truncated conical shell, the results obtained by the one term approximation of the present theory are within 1 percent of those predicted by Love[5] and are within 3 percent of those predicted by Garnet and Kempner[4]. For engineering applications, therefore, the results of this report are accurate enough to be incorporated into any test or design specifications.

To apply the results obtained in this report to a ring stiffened shell vehicle such as the Apollo it is necessary to devise a method of converting a ring stiffened shell to an equivalent orthotropic shell. Such methods have been used by Bodner[6] for stability problems and by Hoppmann[7] for vibration problems. Bodner's method is the simpler one and involves the main assumptions that the ring stiffness has very little effect on the axial extensional, bending rigidity, and shear rigidity of the isotropic shell.

According to this theory, the ratios of the elastic constants E'_θ/E'_x and E''/E'_x of an equivalent orthotropic shell in terms of the various dimensions of a reinforced shell are given by, (see Figure 2),

$$\frac{E'_\theta}{E'_x} = \frac{12(1-\nu^2) I_\phi}{\ell_s h^3} \quad \text{and} \quad \frac{E''}{E'_x} = -\nu \tag{30}$$

where ν = Poisson's ratio of the material of the longitudinal stiffener.

ℓ_s = Spacing between longitudinal stiffeners.

h = Thickness of the shell between the stiffeners.

$$I_\phi = I_0 + A_s (z_s - z_c)^2 + \left[\frac{\ell_s h}{1-\nu^2}\right] \left[\frac{h^2}{12} + Z_c^2\right] \tag{31}$$

I_0 = Stiffener moment of inertia about its own centroid.

A_s = Area of cross section of the stiffener.

z_s = Distance of the centroid of the stiffener from the middle surface of the shell.

z_c = Distance of the overall centroid of the ring shell combination from the middle surface.

For details of Bodner's[6] method, the reader is referred to the cited references.

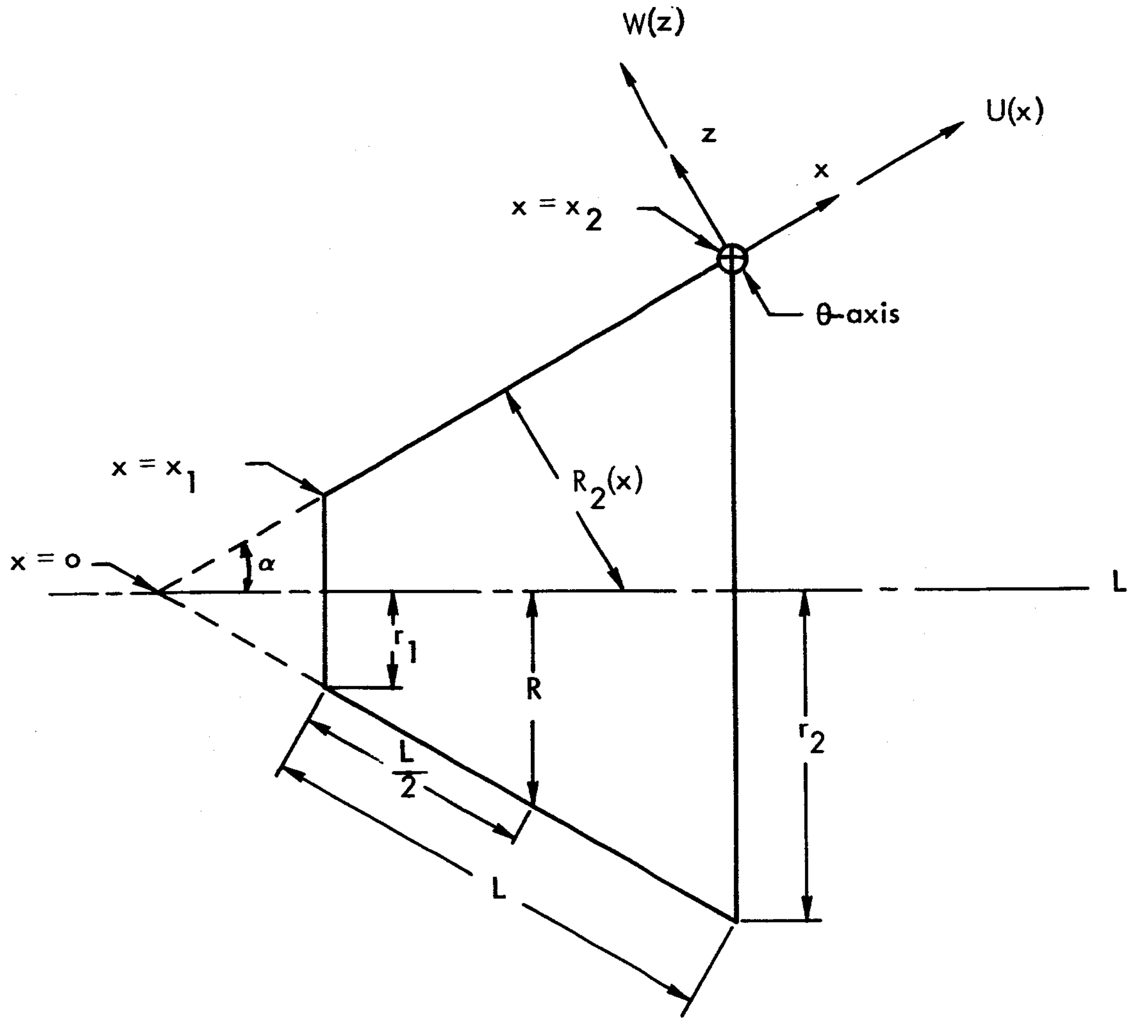

Figure 1. Coordinate System and Parameters of a Conical Shell

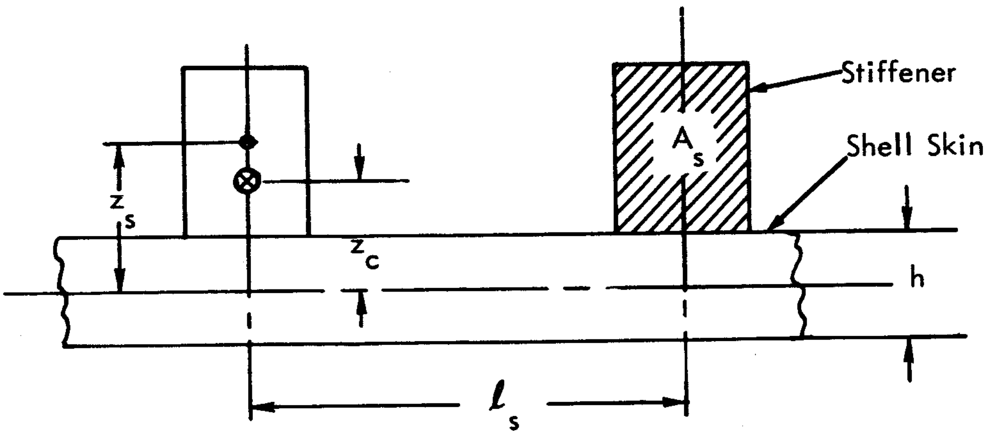

Figure 2. Geometry of a Reinforced Shell Element Between Stiffener

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TABLE 1

Values of the Frequency Parameter λ for Fundamental Axisymmetric Mode of Orthotropic, Truncated, Conical Shells

$$E''/E_x' = -0.2$$

			E_θ' / E_x'						
			0.2	0.6	1.0	1.5	2	4	5
α	$\frac{h}{R}$	$\frac{L}{R}$	λ						
5°	0.05	0.25	679.33	730.46	781.58	845.49	909.39	1165	1292
5°	0.05	0.375	148.92	199.46	250.00	313.17	376.34	629.0	755
5°	0.05	0.50	60.25	110.21	160.16	222.60	285.05	534.79	659.65
10°	0.15	0.30	672.10	684.36	696.63	711.96	727.29	788.62	319.28
10°	0.15	0.50	88.21	99.99	111.76	126.48	141.20	200.07	229.50
10°	0.15	1.0	9.05	19.74	30.42	43.76	57.09	110.20	136.78
15°	0.20	0.375	195.26	200.36	205.46	211.84	218.22	243.74	256.5
15°	0.20	1.0	4.93	9.13	13.32	18.56	23.79	44.69	55.09
20°	0.10	0.375	24.99	27.64	30.28	33.58	36.88	50.09	56.70
20°	0.10	0.50	8.26	10.76	13.28	16.41	19.55	32.09	38.36

TABLE 2

Values of the Frequency Parameter λ for Fundamental Axisymmetric Mode of Orthotropic, Truncated, Conical Shells

$E'' / E_x' = -0.3$

			E_θ' / E_x'						
			0.2	0.6	1.0	1.5	2	4	5
α	$\frac{h}{R}$	$\frac{L}{R}$	λ						
5°	0.05	0.25	672.74	723.85	774.95	838.83	902.71	1158	1285
5°	0.05	0.375	142.5	193.0	243.51	306.63	369.76	622.24	748.47
5°	0.05	0.50	53.94	103.83	153.7	216.08	278.44	527.81	652.47
10°	0.15	0.30	670.27	682.53	694.78	710.10	725.42	786.70	817.34
10°	0.15	0.50	86.65	98.41	110.17	124.86	139.56	198.33	227.71
10°	0.15	1.0	7.69	18.21	28.93	42.19	55.43	108.1	134.17
15°	0.20	0.375	194.5	199.60	204.72	211.1	217.47	242.9	255.7
15°	0.20	1.0	4.39	8.57	12.74	17.94	23.14	43.84	54.09
20°	0.10	0.375	24.67	27.31	39.95	33.25	36.55	49.75	56.35
20°	0.10	0.50	7.94	10.45	12.96	16.10	19.22	31.75	38.01

TABLE 3

Values of the Frequency Parameter λ for Fundamental Axisymmetric Mode of Orthotropic, Truncated, Conical Shells

$$E'' / E_x' = -0.4$$

			E_θ' / E_x'						
			0.2	0.6	1.0	1.5	2	4	5
α	$\frac{h}{R}$	$\frac{L}{R}$	λ						
5°	0.05	0.25	663.51	714.59	765.67	829.52	893.37	1148	1276
5°	0.05	0.375	133.54	183.99	234.44	297.50	360.55	612.76	738.84
5°	0.05	0.50	45.14	94.94	144.74	206.98	269.21	518.07	642.45
10°	0.15	0.30	667.68	679.92	692.17	707.48	722.78	784.00	814.61
10°	0.15	0.50	844.62	961.98	107.93	122.59	137.26	195.90	225.21
10°	0.15	1.0	5.79	16.35	26.88	40.02	53.14	105.13	130.69
15°	0.20	0.375	193.48	198.57	203.66	210.03	216.39	241.83	254.56
15°	0.20	1.0	3.65	7.79	11.93	17.09	22.24	42.68	52.73
20°	0.10	0.375	24.20	26.84	29.48	32.78	36.07	49.26	55.85
20°	0.10	0.50	7.50	10.06	12.51	15.63	18.76	31.27	37.52

TABLE 4

Comparison of Fundamental Frequency Parameter λ Predicted for Isotropic Conical Shell by Present Theory and by Classical Theory (Love Formulation)

$E'' / E_x' = -0.3$			$E_\theta' / E_x' = 1.0$ λ	
α	$\frac{h}{R}$	$\frac{L}{R}$	Present Theory	Classical Theory
5°	0.05	0.25	774.95	770.63
5°	0.05	0.375	243.51	242.73
5°	0.05	0.50	153.7	153.51
10°	0.15	0.30	694.78	693.8
10°	0.15	0.50	110.17	110.02
10°	0.15	1.00	28.93	28.73
15°	0.20	0.375	204.72	203.63
15°	0.20	1.00	12.74	11.53
20°	0.10	0.375	29.95	29.70
20°	0.10	0.50	12.96	12.82

APPENDIX A

A METHOD FOR ESTIMATING THE FREQUENCIES AND MODE SHAPES OF SIMPLY SUPPORTED ORTHOTROPIC PLATES

The vibration characteristics of the individual panel sections which form the shell of the Apollo vehicle may be estimated, for a first approximation, by accounting for non-uniform bending stiffness in the two directions on the panel but neglecting the curvature of the panel. This method also neglects any coupling between panels induced by torsional motion of the stiffeners. The latter effect has been shown by Lin to cause coupled panel modes to group in frequency bands.[8] However, the lower frequency of the n_{th} band, for pinned supports, coincides with the n_{th} resonant mode for a single span. Thus, analysis of a single panel provides a reasonable estimate of the lower bound for individual panel modes. The major reason for presenting this analysis, however, is to focus attention on an analysis method suitable for orthotropic plates representative of Apollo skin structure.

NOMENCLATURE

a, b	sides of the rectangular plate
h	plate thickness
E'_x, E'_y, E'', G	elastic constants of the orthotropic material
$D_x = \dfrac{E'_x h^3}{12}$	bending stiffness in the x-direction
$D_y = \dfrac{E'_y h^3}{12}$	bending stiffness in the y-direction
$H = \dfrac{E'' h^3}{12} + \dfrac{G h^3}{6}$	rigidity in torsion of the orthotropic plate
ρ	weight density of the plate
ν	Poisson's ratio
w	deflection of plate
p	frequency of vibration
x_m, y_n	characteristic functions of the vibrating beam problem
ϵ_x, ϵ_y	tensile strain in x and y direction
γ_{xy}	shear strain
σ_x, σ_y	tensile stress in x and y direction
τ_{xy}	shear stress

The stress-displacement relationships for an orthotropic plate are

$$\sigma_x = -z\left(E'_x \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = -z\left(E'_y \frac{\partial^2 w}{\partial y^2} + E'' \frac{\partial^2 w}{\partial x^2}\right) \tag{1}$$

$$\tau_{xy} = -2Gz \frac{\partial^2 w}{\partial x \partial y}$$

The stress–strain relationships are

$$\begin{aligned} \sigma_x &= E'_x \, \epsilon_x + E'' \, \epsilon_y \\ \sigma_y &= E'_y \, \epsilon_y + E'' \, \epsilon_x \\ \tau_{xy} &= G \, \gamma_{xy} \end{aligned} \tag{2}$$

The strain energy in the plate is

$$V = \int_0^a \int_0^b \left(\frac{\epsilon_x \, \sigma_x}{2} + \frac{\epsilon_y \, \sigma_y}{2} + \frac{\gamma_{xy} \, \tau_{xy}}{2} \right) dx \; dy \; dz \tag{3}$$

For simply supported orthotropic plates the deflection defined by

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \tag{4}$$

satisfies the boundary conditions at both edges.

Substituting (4) into (3) and making use of (1) and (2) the strain energy in the plate can be written as

$$\begin{aligned} V = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi^4 ab}{8} D_x \, \phi_{mn}^2 \Bigg\{ & \left(\frac{m}{a}\right)^4 + \frac{E'_y}{E'_x} \left(\frac{n}{b}\right)^4 \\ & + 2 \frac{E''}{E'_x} \left(\frac{m}{a}\right)^2 \left(\frac{n}{b}\right)^2 + 4 \frac{G}{E'_x} \left(\frac{m}{a}\right)^2 \left(\frac{n}{b}\right)^2 \Bigg\} \end{aligned} \tag{5}$$

Choosing ϕ_{mn} as the generalized coordinates, the kinetic energy in the plate can be written as

$$T = \frac{\rho h}{2g} \frac{ab}{4} \sum \sum \dot{\phi}_{mn}^2 \tag{6}$$

Using Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_{mn}}\right) - \frac{\partial V}{\partial \phi_{mn}} = 0 \tag{7}$$

ϕ_{mn} are found to satisfy the equation

$$\frac{\rho h}{g}\ddot{\phi}_{mn} + \pi^4 D_x \phi_{mn}\left\{\left(\frac{m}{a}\right)^4 + \frac{E'_y}{E'_x}\left(\frac{n}{b}\right)^4 + 2\frac{E''}{E'_x}\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + 4\frac{G}{E'_x}\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2\right\} = 0 \tag{8}$$

a solution of which is

$$\phi_{mn} = C_1 \cos pt + C_2 \sin pt \tag{9}$$

where

$$p = \pi^2\sqrt{\frac{g D_x}{\rho h}}\left\{\left(\frac{m}{a}\right)^4 + \frac{E'_y}{E'_x}\left(\frac{n}{b}\right)^4 + 2\frac{E''}{E'_x}\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + 4\frac{G}{E'_x}\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2\right\}^{1/2} \tag{10}$$

From (4) and (10) all the frequencies and modes of vibrations of a simply supported orthotropic plate can be determined.

For an isotropic (or uniform) plate, $\frac{E''}{E'_x} = \nu$, $\frac{G}{E'_x} = \frac{1-\nu}{2}$, $E'_x = E'_y = \frac{E}{1-\nu^2}$,

and $D_x = \frac{E h^3}{12(1-\nu^2)} = D$ and equation (10) reduces to the usual expression for modes of a simply supported plate.

$$P = \pi^2\sqrt{\frac{gD}{\rho h}}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \tag{11}$$