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THE PATTERN SYNTHESIS PROBLEM FOR A SLOTTED INFINITE CYLINDER

by O. Einarsson, F. B. Sleator, and P. L. E. Uslenghi

Prepared under Grant No. NsG-444 *by*
THE UNIVERSITY OF MICHIGAN
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I

INTRODUCTION

Any attempt to classify or organize the tremendous body of existing theory of antennas is certain to lead almost immediately to the distinction between the analysis problem, i. e. the determination of field patterns or radiation characteristics of given antenna forms, and the synthesis problem, i. e. that of determining the form and excitation of an antenna which will produce a prescribed field pattern. The two problems are of roughly equal practical importance, but the latter, being inherently less well defined and straightforward and therefore more difficult, has received a rather meager share of the attention. A number of significant contributions have appeared recently, however, some of which are noted below, and the present report is an attempt to extend and clarify some of the results and conclusions reached.

The choice of the particular questions considered here was motivated initially by a problem in satellite communication. Simply stated, the requirement is for a flush-mounted antenna on an essentially cylindrical body which will produce a far-field pattern of sufficient uniformity so that radio contact can be maintained at any orientation within a certain angular region. Disregarding the questions of implementation, which are by no means trivial, we are left with considerable leeway in the formulation of an appropriate analytical problem. Further restrictions, however, are afforded by considerations of simplicity and feasibility of solution, and we accordingly limit ourselves here to the problem of determining the excitations required in slots of various types in the surface of a conducting infinite cylinder in order to produce the best approximation in a certain sense, to certain prescribed far-field radiation patterns.

To the best of our knowledge the treatment presented here is essentially new. However, there are numerous recent papers whose substance is related in some degree to that of the present report and which deserve some mention as antecedents and sources of inspiration. Results obtained for the radiated fields of slotted cylinders with specified excitation are legion. A comprehensive bibliography of these is contained in the book by Wait (1959), and among the more recent contributions are papers by Knudsen (1959), Nishida (1960), Logan, Mason and Yee (1962) and Hasserjian and Ishimaru (1962a, b). Solutions of synthesis problems for such structures are, however, few. Notable among these are the results of Wait and Householder (1959), who developed a procedure for synthesizing a given radiation pattern by means of a cylinder excited by a circumferential array of axial slots, and DuHamel (1952), who considered antenna arrays on circular, elliptical, and spherical surfaces, and proved for the circular case that the radiation pattern obtained with a certain minimum number of antennas differs by only a few percent from that produced when the antennas are replaced by a continuous current distribution.

The literature on the synthesis problem for a single aperture on a conducting cylinder is even more sparse. The authors are currently aware of no other investigations which treat such a problem explicitly. There are a number of papers, however, which deal with single apertures of various shapes in infinite conducting plane screens, employing formulations and techniques similar to some of those used here for the cylindrical case. Among these we note the following. Various extremal problems, with the common stipulation of a fixed number of spherical wave function in the field representations, have been considered by Chu (1948) and Harrington (1957), the variational quantities being the gain, quality factor, the ratio of these, and the side lobe level for given main beam width. The problem of finding a pattern function which takes given values at a certain number of specified points and which minimizes the square integral of its corresponding aperture function was treated by Woodward and Lawson (1948) and by Yen (1957). Determination of the aperture function specifiable with a given number of harmonics which minimizes the sidelobe level for given width of the main beam was carried out by Taylor (1955), Mittra (1959) and Fel'd and Bakhrakh (1963). The latter paper, along with one by Kovács and Solymár (1956), deals also with the question of the best mean-square approximation over the whole space (visible and invisible) to a function which equals some given function in the visible region and vanishes outside it. Solymár (1958) and Collin and Rothschild (1963) consider the maximization of the directivity with a given number of harmonics in the aperture function and a specified value of the supergain ratio or quality factor. In a paper by Ling, Lefferts, Lee and Potenza (1964) the normalized second moment of the far-field power pattern is minimized for various plane aperture shapes, including the rectangle, circle, annulus and ellipse. Finally, in a mathematically elegant analysis, Rhodes (1963) has exhibited the optimum mean-square approximation to a given pattern function with a fixed number of terms in the aperture field representation and a given value of the supergain ratio, as defined by Taylor (1955).

It might be observed here that all of the above analyses concern themselves with field strength rather than power patterns. The only valid example of power pattern synthesis known to the authors at present is a paper by Caprioli, Scheggi and Toraldo di Francia (1961) which makes use of a technique of interpolation between sampling points.

As remarked above, some of the formulations and techniques developed in the treatment of plane apertures have a direct bearing on the cylindrical problem. However, there are several important differences here which limit their applicability and necessitate certain modifications. If one attempts, for example, to follow the procedure of Rhodes (1963), it develops immediately that the kernel of the integral equation relating the aperture and far fields in the cylindrical case is not symmetric, and thus possesses no orthogonal set of eigenfunctions. One of the principal features of Rhodes' method, namely the orthogonality of the set of pattern functions corresponding to an orthogonal set of aperture functions, is therefore not available here, and the solution of a system of linear equations apparently cannot be avoided. Certain extremal properties of the eigenfunctions for the plane case are also lacking in the cylindrical case. Even more important, perhaps, is the fact that whereas in the plane case all the known pattern synthesis procedures yield an approximating pattern which is real if the prescribed pattern is, this does not hold for the cylindrical case. As a consequence, it turns out that here the best admissible mean-square approximation to a given real pattern may be an extremely poor approximation in amplitude.

These are among the principal considerations which governed the formulation of the problems treated in the present report. The choice of the mean-square deviation as the measure of the degree of approximation is more or less mandatory from an algebraic standpoint. Since it appears that there is little to be gained in the cylindrical case through the use of special basis functions, we have employed only exponential or trigonometric functions, with the inclusion of a weight factor in some cases which satisfies an edge condition at the slot boundaries. Because the amplitude of the radiation pattern far outweighs the phase in practical importance, an iteration scheme was developed for the case of a single slot, in which the phase of the prescribed pattern is sacrificed for the sake of substantially improving the amplitude approximation. This scheme has not been used for numerical computations in cases with multiple slots such as those considered in Section III, though there seems to be no reason why it could not be.

The authors wish to acknowledge the considerable and sustained efforts of certain colleagues, in particular D.R. Hodgins, T.L. Boynton, J.A. Rodnite, J.A. Ducmanis and Miss Austra Maldups, who programmed the numerical work reported here. Credit is also due to the University of Michigan Computing Facility, which actually produced the numbers.

II

THE SYNTHESIS PROBLEM WITH CONTINUOUS SOURCE DISTRIBUTIONS

2.1 Plane Aperture

Consider a finite aperture S in an infinite conducting screen lying in the yz plane of a rectangular coordinate system and introduce polar coordinates

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta \quad .$$

It is well known that the far field can be expressed in terms of the distribution over the aperture of the tangential components of the electric field strength (Silver, 1949). Thus it is found that

$$E_\theta \underset{r\to\infty}{\sim} \frac{ik}{2\pi}\frac{e^{ikr}}{r}\cos\phi\iint_S E_z(0,y,z)\exp\left[-ik(y\sin\theta\sin\phi+z\cos\theta)\right]dydz \tag{2.1}$$

$$E_\phi \underset{r\to\infty}{\sim} \frac{-ik}{2\pi}\frac{e^{ikr}}{r}\sin\theta\iint_S\left(E_y(0,y,z)+\frac{\cos\theta\sin\phi}{\sin\theta}E_z(0,y,z)\right)\exp\left[-ik(y\sin\theta\sin\phi+z\cos\theta)\right]dydz \quad , \tag{2.2}$$

where the time dependence $e^{-i\omega t}$ is everywhere suppressed.

By virtue of Babinet's principle the substitution $\underline{E}\to\underline{H}$, $\underline{H}\to-\underline{E}$ makes (2.1) and (2.2) valid also for the complementary problems where the fields are caused by surface currents H_z, H_y on a conducting plane disk of the same shape as the aperture.

If we assume the aperture to be rectangular and aperture field linearly polarized (say $E_y=0$) and separable, that is $E_z(0,y,z)=e_1(y)\cdot e_2(z)$, the radiation pattern in the xy and xz planes depends solely on $e_1(y)$ and $e_2(z)$ respectively. Thus the problem of synthesizing the radiation patterns in the main planes in this case simplifies to finding two independent one-dimensional aperture distributions.

The synthesis problem is usually formulated as such a one-dimensional problem. For a general rectangular aperture the assumptions of a separable and linearly polarized aperture field seem quite questionable (c.f. Collin, 1964) but for a narrow slot (width $< \lambda / 10$) they are certainly accurate enough. One other case in which a one-dimensional aperture distribution can be used is when the aperture is infinite in one direction and the derivatives of all field components with respect to that direction vanish.

We can now write the relation between the radiation pattern and the aperture field as

$$g(\xi) = \int_{-1}^{1} e^{-i \frac{kL}{2} \xi \eta} f(\eta) d\eta \quad , \qquad |\xi| \leqslant 1 \quad , \tag{2.3}$$

where the physical significance of the functions f, g in the cases of an infinite slot and a line current is given in Table II-1. The notation corresponds to Fig. 2-1.

It is natural to restrict the functions g and f in eq. (2.3) to be complex valued functions, square integrable over the interval (-1, 1) (notations: f, g, $\in L_2$). It was early recognized that the synthesis problem as expressed by (2.3) has the following properties (see Bouwkamp and DeBruijn, 1946).

a) For an arbitrary function $h(\xi) \in L_2$, there is in general no aperture function $f(\eta) \in L_2$ such that the corresponding pattern function $g(\xi) = h(\xi)$.

b) We can obtain, however, for every positive quantity ϵ an aperture function $f(\eta) \in L_2$ whose corresponding $g(\xi)$ satisfies

$$\int_{-1}^{1} |h(\xi) - g(\xi)|^2 d\xi < \epsilon \, .$$

PLANE APERTURE		LINE SOURCE
TM FIELD	TE FIELD	
$E_z \underset{\rho\to\infty}{\sim} E_0 \frac{kL}{2} \frac{e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{2\pi k\rho}} \cos\phi \, g(\sin\phi)$	$E_\phi \underset{\rho\to\infty}{\sim} E_0 \frac{kL}{2} \frac{e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{2\pi k\rho}} g(\sin\phi)$	$H_\phi \underset{r\to\infty}{\sim} H_0 \frac{kL}{2} \frac{e^{ikr}}{4\pi ikr} \sin\theta \, g(\cos\theta)$
$E_z(0,y) = E_0 f(\frac{2y}{L})$	$E_y(0,y) = E_0 f(\frac{2y}{L})$	$I(z) = \frac{H_0}{k} f(\frac{2z}{L})$

TABLE II-1: RELATION OF PHYSICAL FIELD QUANTITIES TO THE APERTURE AND PATTERN FUNCTIONS OF Eq. (2.3).

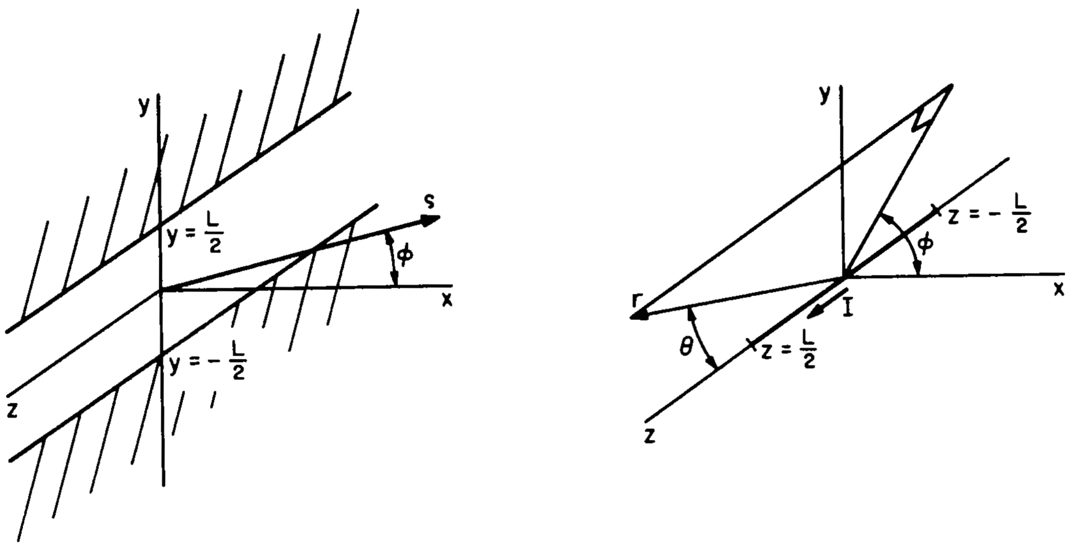


FIG. 2-1: APERTURE IN AN INFINITE SCREEN AND LINE SOURCE

There is, thus, in general no 'exact solution' of the synthesis problem but we can approximate any prescribed pattern arbitrarily closely in the mean-square sense. It is consequently possible, for every aperture however small, to find an aperture function $f \in L_2$ which delivers a radiation pattern with arbitrarily high directivity. Any attempt to obtain 'supergain' from practical antennas will, however, result in an unrealizable aperture function with high amplitude and rapidly varying phase.

As a measure of the 'realizability' of the aperture function, Taylor (1955) introduced the supergain ratio, which in our notation is defined as

$$\gamma = \frac{4\pi}{kL} \frac{\int_{-1}^{1} |f(\eta)|^2 d\eta}{\int_{-1}^{1} |g(\xi)|^2 d\xi} \tag{2.4}$$

Other 'quality factors' have been introduced by Chu (1948) and Collin and Rothschild (1963, 1964). We will discuss these different factors in some detail in Section 2.3.

In order to achieve the pattern synthesis we write the aperture function $f(\eta)$ as

$$f(\eta) = \sum_{0}^{\infty} a_n f_n(\eta) \quad , \tag{2.5}$$

where $\{f_n\}$ is given as a set of functions defined on the interval (-1, 1). We denote the pattern functions corresponding to $f_n(\eta)$ as $g_n(\xi)$. There is a large degree of freedom in the choice of $\{f_n\}$; the only necessary property is that this set be complete in the subspace of L_2 where the solution of our problem is to be found. We will consider two examples of functions which have been used in different synthesis procedures.

Example 1:

$$\left.\begin{array}{ll} f_n(\eta) = e^{iu_n\eta} & |\eta| \leqslant 1 \\ g_n(\xi) = \dfrac{2\sin\left(\frac{kL}{2}\xi - u_n\right)}{\frac{kL}{2}\xi - u_n} & |\xi| \leqslant 1 \end{array}\right\} \tag{2.6}$$

If we choose the parameters $u_n = n\pi$ and allow n to take negative as well as positive values the expression (2.5) is an ordinary Fourier series. In this case the radiation pattern in certain directions is related to the coefficients a_n by the expression

$$g_n\left(\frac{2n\pi}{kL}\right) = 2a_n, \qquad |n| \leqslant \frac{kL}{2\pi} = \frac{L}{\lambda} \quad ,$$

and an approximating pattern which coincides with the prescribed pattern at these points is readily obtained.

Example 2:

$$f_n(\eta) = S_{0n}\left(\frac{kL}{2}, \eta\right) \qquad |\eta| \leqslant 1$$

$$g_n(\eta) = \frac{2}{i^n} R^{(1)}_{0n}\left(\frac{kL}{2}, 1\right) S_{0n}\left(\frac{kL}{2}, \xi\right) \qquad |\xi| \leqslant 1 \quad . \tag{2.7}$$

R_{0n} is a radial and S_{0n} an angular prolate spheroidal function in the notations of Flammer (1957). The interesting properties of these functions are that they are the eigenfunctions of Eq. (2. 3) and that they bear a close relationship to the supergain ratio. They have been studied by Slepian and Pollak (1961) and Landau and Pollak (1961, 1962) and from their work we take the following results.

i. $\left\{S_{0n}\right\}$ is orthogonal and complete in the interval (-1, 1). It is also orthogonal in the interval $(-\infty, \infty)$ (but not, as Rhodes (1963) claims, complete there).

ii. The smallest possible value of the supergain ratio γ is

$$\gamma_{min} = \frac{\frac{\pi}{kL}}{\left[R^{(1)}_{00}\left(\frac{kL}{2}, 1\right)\right]^2} \tag{2.8}$$

and is obtained for the aperture function $f(\eta) = S_{00}\left(\frac{kL}{2}, \eta\right)$

iii. Consider the class of aperture functions which corresponds to a given value of the supergain ratio γ_0 (notation: $f \epsilon E(\gamma_0)$) and which are normalized such that

$$\int_{-1}^{1} |f(\eta)|^2 d\eta = 1 \ .$$

If we want to approximate such a function with a finite number of given functions $\left\{f_n\right\}_0^{N-1}$, then $\left\{S_{0n}\right\}_0^{N-1}$ are the best possible choice in the following sense: They are the functions which achieve

$$\min_{\{f_n\}_0^{N-1}} \max_{f \in E(\gamma_0)} \min_{\{a_n\}_0^{N-1}} \int_{-1}^{1} \left| f(\eta) - \sum_{n=0}^{N-1} a_n f_n(\eta) \right|^2 d\eta \tag{2.9}$$

Observe that iii ensures the best mean-square approximation of the aperture function, which does not necessarily imply the best fit of the pattern function. The set of pattern functions $\{g_n\}$ is here orthogonal and is consequently suited for a mean-square approximation of the prescribed pattern.

2.2 Aperture on an Infinite Circular Cylinder

Consider an infinite conducting circular cylinder of radius a with a finite aperture S. We introduce a cylindrical coordinate system (ρ, ϕ, z) such that the generating surface is given by $\rho = a$, and also polar coordinates (r, θ, ϕ) $(\rho = r \sin\theta,\ z = r\cos\theta)$. In analogy with the plane case the far field can be expressed as

$$E_\theta \underset{r\to\infty}{\sim} \frac{i}{2\pi^2} \frac{e^{ikr}}{r} \sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{\sin\theta\, i^n H_n^{(1)}(ka\sin\theta)} \iint_S E_z(a, \phi', z) \exp\left[-i(n\phi' + kz\cos\theta)\right] d\phi'\, dz \tag{2.10}$$

$$E_\phi \underset{r\to\infty}{\sim} \frac{1}{2\pi^2} \frac{e^{ikr}}{r} \sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{i^n H_n^{(1)\prime}(ka\sin\theta)} \iint_S \left(E_\phi(a, \phi', z) + \frac{n\cos\theta}{ka\sin^2\theta} E_z(a, \phi', z) \right) \exp\left[-i(n\phi' + kz\cos\theta)\right] d\phi' dz \quad , \tag{2.11}$$

where $H_n^{(1)}$ and $H_n^{(1)\prime}$ are the Hankel function of the first kind and its derivative with respect to the argument (Silver and Saunders, 1950).

Just as in the plane case the infinite axial slot and the narrow circumferential slot can be treated as one-dimensional problems. We write the relation between the radiation field and the aperture field as

$$P(\phi)=\int_{-\alpha}^{\alpha} K(\phi-\phi')A(\phi')d\phi' \qquad -\pi < \phi \leq \pi \ . \tag{2.12}$$

The physical significance of P and A is given in Table II-2 where the notations correspond to Fig. 2-2.

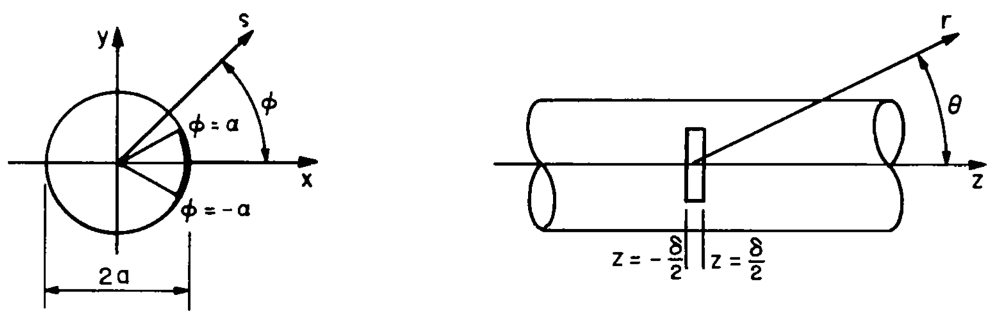


FIG. 2-2: INFINITE AXIAL AND NARROW CIRCUMFERENTIAL SLOT ON INFINITE CYLINDER

It turns out that the fundamental properties of the plane synthesis problem as expressed by a) and b) on page 6 are still valid in the cylindrical case. Thus;

a) For an arbitrary function $F(\phi)\in L_2^{\pi}$, there is in general no aperture function $A(\phi')\in L_2^{\alpha}$ such that the corresponding pattern function $P(\phi)=F(\phi)$.

b) We can obtain, however, for every positive quantity ϵ, an $A(\phi')\in L_2^{\alpha}$ whose corresponding $P(\phi)$ satisfies

$$\int_{-\pi}^{\pi} |F(\phi)-P(\phi)|^2 \, d\phi < \epsilon \quad .$$

NARROW CIRCUMFERENTIAL SLOT	INFINITE AXIAL SLOT	
	TM FIELD	TE FIELD
$(E_z)_{\theta=\frac{\pi}{2}} \underset{r\to\infty}{\sim} E_0 \frac{e^{ikr}}{kr} P(\phi)$	$E_z \underset{r\to\infty}{\sim} E_0\sqrt{2\pi}\, \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}}\, P(\phi)$	$E_\phi \underset{r\to\infty}{\sim} E_0\sqrt{2\pi}\, \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}} P(\phi)$
$V(\phi)=\int_{-\delta/2}^{\delta/2} E_z(a,\phi,z)dz=\frac{E_0}{k}A(\phi)$ (δ = width of slot)	$E_z(a,\phi)=E_0\ A(\phi)$	$E_\phi(a,\phi)=E_0A(\phi)$
$K(\phi)=\frac{1}{2\pi^2 i}\sum_{n=0}^{\infty}\frac{\epsilon_n \cos n\phi}{i^n H_n^{(1)}(ka)}$		$K(\phi)=\frac{1}{2\pi^2}\sum_{n=0}^{\infty}\frac{\epsilon_n \cos n\phi}{i^n H_n^{(1)\prime}(ka)}$
$(\epsilon_0=1,\ \epsilon_2=\epsilon_2=\cdots\cdots=2)$		

TABLE II-2: RELATION OF PHYSICAL FIELD QUANTITIES TO THE APERTURE AND PATTERN FUNCTIONS OF EQ (2.12)

A proof of these statements is given in the appendix. To obtain a meaningful synthesis problem, we must apparently, as in the plane case, put some constraint on the permissible aperture functions. The question of the most significant or appropriate constraint is discussed in the next section.

2.3 The Quality Factor

Integration of the complex Poynting vector over the aperture S in the infinite plane or on the cylinder yields

$$\frac{1}{2} \iint_S \underline{E} \wedge \underline{H}^* \cdot \hat{n} \, ds = P_r - 2i\omega(W_m - W_e) \tag{2.13}$$

where $\hat{n}$ is a unit vector normal to the aperture ($\hat{x}$ or $\hat{\rho}$ respectively). P_r is the radiated power and $W_e - W_m$ is the difference between the time-averages of the electric and magnetic energy densities $\frac{1}{4}(\epsilon_0 \underline{E}\cdot\underline{E}^* - \mu_0 \underline{H}\cdot\underline{H}^*)$ integrated over the half space or the region outside the cylinder respectively. If we try to calculate the stored electric and magnetic energies separately as the integrals of $\frac{1}{4}\epsilon_0 \underline{E}\cdot\underline{E}^*$ and $\frac{1}{4}\mu_0 \underline{H}\cdot\underline{H}^*$ respectively, the results will be infinite because of the slow decrease of the far fields. To overcome this, Collin and Rothschild (1963, 1964) defined the electric and magnetic energy densities as

$$\begin{aligned} w_e &= \frac{1}{4}\epsilon_0 \underline{E}\cdot\underline{E}^* - U_e \\ w_m &= \frac{1}{4}\mu_0 \underline{H}\cdot\underline{H}^* - U_m \end{aligned} \tag{2.14}$$

Here U_e and U_m are quantities equal to the energy densities in the far field when the distance from the source tends to infinity. In the cylindrical case we take

$$U_e = U_m = \frac{\sqrt{\mu_0 \varepsilon_0}}{4} \; R_e(\underline{E} \wedge \underline{H}^*)_\rho \quad . \tag{2.15}$$

The quanities given by Eq. (2.14) are finite if integrated over all space and can be considered as the remaining energies when a part corresponding to the power flow in the radial directions is subtracted.

A quantity which in the cylindrical case corresponds to the supergain ratio is apparently

$$\Gamma = \frac{1}{2\pi} \frac{\int_{-\alpha}^{\alpha} |A(\phi)|^2 d\phi}{\int_{-\pi}^{\pi} |P(\phi)|^2 d\phi} \quad . \tag{2.16}$$

It is often assumed that the supergain ratio is a measure of the reactive power, $2\omega(W_m - W_e)$, or the stored energy. This opinion has been critized by Collin and Rothschild (1963). The result of their investigation for the plane case is that a large value of the supergain ratio indicates a high amount of reactive power and consequently also a high amount of the stored energy, but the converse is not always true. Consider for example the TM-field in the infinite aperture (cf. Table II-1). The aperture field which pertains to the smallest possible value of the supergain ratio (Eq. 2.8) has a value different from zero at the end points of the aperture. This clearly violates the edge condition (cf. Meixner, 1949) and implies an energy density around the edges which tends to infinity in such a manner that it is not integrable. In spite of the fact that the supergain ratio takes its lowest possible value in this case both the reactive power and the stored energy are infinite. Observe that in the corresponding TE-case there is no violation of the edge condition. For a narrow slot the stored energy in the vicinity of the aperture tends to infinity as the width of the slot tends to zero. In the limit (i.e. a line source) the reactive power and stored energy are infinite for every aperture function. quite independently of the value of the supergain ratio.

Collin and Rothschild (1963 and 1964) have proposed a quality factor for radiation problems defined as

$$Q = \frac{2\omega W}{P_r} \tag{2.17}$$

where W is the larger of the time-averaged magnetic or electric energies stored in the "evanescent" field, as defined by (Eq. 2.14). P_r is the radiated power. The definition (2.17) is in accordance with the usual definition of quality factor for a network or microwave cavity and can be considered as characteristic of a radiating system which is tuned for resonance by the addition of a lossless reactive element.

If we express the fields as sums of cylindrical modes and thus expand the pattern function in a Fourier series

$$P(\phi) = \sum_{n=-\infty}^{\infty} \beta_n e^{in\phi} \tag{2.18}$$

there is no interaction energy between different modes and we can calculate the energy for each mode separately. The total quality factor is then obtained as

$$Q = \frac{\sum_{n=-\infty}^{\infty} |\beta_n|^2 Q_n}{\sum_{n=-\infty}^{\infty} |\beta_n|^2} \tag{2.19}$$

For the infinite axial slot the factors Q_n can be expressed explicitly. They are equal for the TM- and TE- cases and are calculated by Collin and Rothschild (1964) as

$$Q_n = \frac{\pi}{4}\left\{\frac{4ka}{\pi} + \left[n^2+1-(ka)^2\right](J_n^2+Y_n^2) - \left[(n+1)J_n - ka\, J_{n+1}\right]^2 - \left[(n+1)Y_n - ka\, Y_{n+1}\right]^2\right\} \tag{2.20}$$

The argument of the cylinder functions is everywhere ka.

The narrow circumferential slot delivers an expression for Q_n which depends on the distribution of the electrical field strength across the slot. It can be expressed as an infinite integral containing a combination of cylinder functions similar to Eq. (2.20).

Chu (1948) obtained a quality factor for the spherical case by using the recurrence relation for the spherical Bessel functions to define an equivalent RLC network for each mode. The quality factor was then defined as the ordinary Q related to this circuit. This procedure has been shown by Collin and Rothschild (1963) to be equivalent to the definition in Eq. (2.17). The method leads to tedious calculations for higher modes and Chu therefore introduced a simplified equivalent circuit and a slightly different quality factor which is not restricted to spherical modes and can be expressed as

$$Q' = \frac{\omega\frac{\partial}{\partial\omega}\omega(W_e - W_m) + \left|\omega(W_e - W_m)\right|}{P_r} \tag{2.21}$$

where the quantities involved are defined by Eq. (2.13). The derivative with respect to ω shall be taken with the tangential component of the electric field strength over the aperture kept constant. If the aperture is small compared to the wavelength this is equivalent to keeping the feeding voltage constant and we can write

$$\frac{1}{2}\int\!\!\int_S \underline{E}\wedge\underline{H}^{*}\cdot\hat{n}\, dS = V_e^2\,(G + iB) \tag{2.22}$$

and

$$Q' = \frac{\omega \frac{\partial B}{\partial \omega} + |B|}{2G} \tag{2.23}$$

If $B \neq 0$, we can tune the system for resonance by adding (connecting in parallel) a positive susceptance ωC or a negative one $-1/\omega L$. If we denote the resulting susceptance after tuning as B_o, we see that $\omega \frac{\partial B_o}{\partial \omega} = \omega \frac{\partial B}{\partial \omega} + |B|$. Thus Q' can be considered as a measure of the frequency sensitivity of the input susceptance if we tune the system to resonance. In the plane case $Q' = Q$ (Collin and Rothschild, 1963) but for cylindrical modes they are slightly different. A straightforward calculation yields, for example, Q' for a single cylindrical TM-mode as

$$Q'_n = Q_n + \frac{2}{\pi} \frac{1}{J_n^2(ka) + Y_n^2(ka)} - ka \quad . \tag{2.24}$$

The quantities used in the definition of Q' in Eq. (2.21) can all be obtained as surface integrals over the aperture. It is therefore possible to use this definition for a quite arbitrary conductive body with an aperture on its surface. In a general case, however, this Q' will have no connection with the energy stored in the vincinity of the body and it is, for example, possible for Q' to take negative values.

We can define the "supergain ratio" for a single cylindrical TM-mode as

$$\Gamma_n = \frac{\pi}{2} \left[J_n^2(ka) + Y_n^2(ka) \right] \qquad n = 0, 1, 2, \ldots \tag{2.25}$$

The factor Γ defined by Eq. (2.16) is then obtained as

$$\Gamma = \frac{\sum_{n=-\infty}^{\infty} |\beta_n|^2 \Gamma_n}{\sum_{n=-\infty}^{\infty} |\beta_n|^2} \tag{2.26}$$

The definition of Γ_n in Eq. (2.25) is such that $\Gamma_1 \rightarrow Q_1$ as $ka \rightarrow 0$. Q_n, Q'_n and Γ_n have been calculated from Eqs. (2.20), (2.24) and (2.25) for $n \leqslant 10$ and $ka \leqslant 15$ and are shown in Fig. 2-3 for some values of n.

The question of how high values of the quality factors are admissible in practical design of slot antennas is outside the scope of this report. However, it seems clear that there can be no close connection such that two different aperture distributions with equal quality factor are always equally easy (or difficult) to realize practically. All the quality factors considered here have the property that a high value implies impractical design. It is seen from Fig. 2-3 that Q is the most restrictive one and if we prescribe Q during the synthesis procedure the corresponding values of Q' and Γ will also be under control.

2.4 Comparison Between the Plane and Cylindrical Cases

As we have seen in Sections 2.1 and 2.2, the plane and cylindrical synthesis problems have the following main features in common: a) there is no "exact solution" and b) there is a theoretical possibility of obtaining supergain. Thus the approach for the cylindrical problem should be in general the same as in the plane case and in conformity with Eq. (2.5) we write the aperture distribution as

$$A(\phi) = \sum_{m=0}^{\infty} \gamma_m \psi_m(\phi), \qquad -\alpha \leqslant \phi \leqslant \alpha \qquad (2.27)$$

where $\{\psi_m\}$ is a given set of linearly independent functions, square integrable over the interval $-\alpha \leqslant \phi \leqslant \alpha$ (notation: $\psi_m \in L_2^{\alpha}$). We denote the pattern function that via Eq. (2.12) corresponds to $\psi_m(\phi)$ as $\Pi_m(\phi)$. As in the plane case, we have considerable freedom in the choice of $\{\psi_m\}$ as long as the set is complete in the subset of L_2^{α} where the solution of our synthesis problem is to be found.

The simplest type of meaningful restriction on the aperture function is to use a finite set of functions $\{\psi_m\}_0^N$ in the expression (2.27). We can then calculate the N+1 coefficients $\psi_o, \psi_1, \ldots \psi_N$ either in such a way that the corresponding pat-

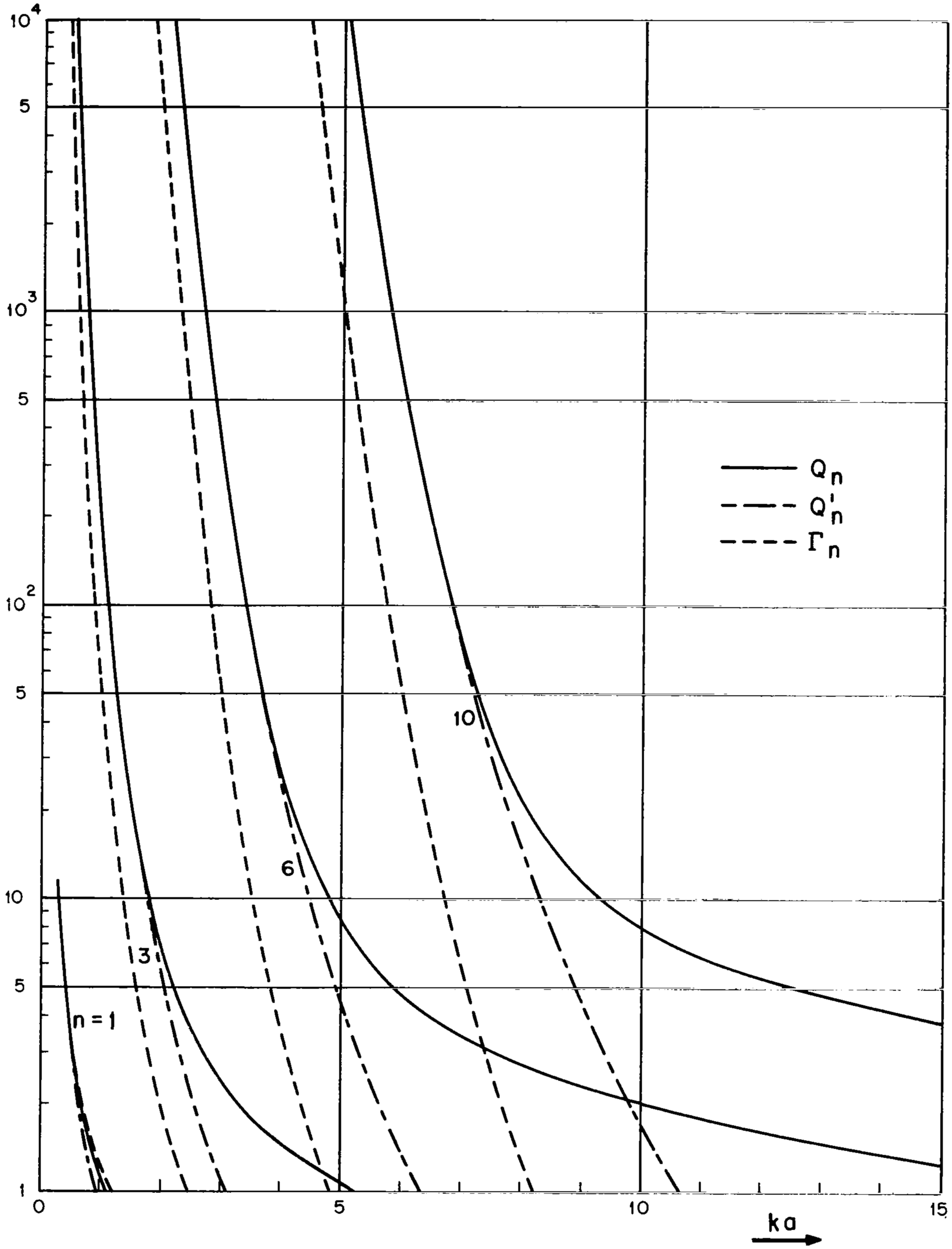


FIG. 2-3: DIFFERENT QUALITY FACTORS FOR CYLINDRICAL TM MODES

tern function is equal to the prescribed pattern for N+1 given values of ϕ in the interval $-\pi \leqslant \phi \leqslant \pi$, or so the approximating pattern is the best mean-square approximation of the prescribed pattern. The unknown coefficients are obtained in either case as the solution of a system of N+1 linear equations. In contrast to the plane case, there is no set of simple functions such as those of Eq. (2.6) which yield a direct relationship between the coefficients and the radiation pattern in certain directions. The mean-square approximation, on the other hand, would be simplified if one could find a set $\left\{\psi_m\right\}_0^N$ such that the corresponding pattern functions $\left\{\Pi_m\right\}_0^N$ were orthogonal, but there seem to be no well-known functions with this property either. A linearly independent set could be orthogonalized by the usual Schmidt process, but this is of course equivalent to solving the original system of linear equations.

A more satisfactory constraint on the aperture function than merely limiting the number of terms in the expansion (2.27) is to keep some quality factor constant during the synthesis procedure. If we use the mean-square approximation, such a scheme can be treated analytically by introducing a Lagrange multiplier. A process which for the plane case delivers the best mean-square approximation to a prescribed pattern for a given value of the supergain ratio has been proposed by Rhodes (1963). The functions used by Rhodes for expansion of the aperture function are the spheroidal functions of Eq. (2.7). Due to the orthogonality of the corresponding set of pattern functions and the special choice of constraint, the optimum pattern and aperture functions can be determined directly without solving any set of linear equations, i.e. the matrix of the system degenerates to a diagonal form. Rhodes uses a finite number of terms in the expansions (2.5) and thus the spheroidal functions are also the best set to use in approximating the aperture function in the sense of iii on p. 10. This, of course, does not mean that they are the best choice for a specific prescribed pattern, but only that they are the best for the worst possible pattern with the prescribed value of the supergain ratio. It may also be remarked that for the cases of a TM-field in an infinite aperture and a line source, the pattern function of Eq. (2.3) differs from the radiated field in the far zone by a cosine factor (cf. Table II-1). This means that in the procedure of Rhodes, deviation from the prescribed radiated field will be somewhat overemphasized in directions away from the normal to the aperture, to the detriment of the fit in directions close to the normal. If we should define instead a pattern function $h(\xi)$ directly proportional to the radiated far field in these cases, the integral relation corresponding to Eq. (2.3) would be

$$h(\xi) = \int_{-1}^{1} \sqrt{1-\xi^2}\, e^{-i\frac{kL}{2}\xi\eta} f(\eta)\, d\eta\,, \qquad -1 \leqslant \xi \leqslant 1\,. \tag{2.28}$$

The eigenfunctions of this equation are, as pointed out by Fel'd and Bakhrakh (1963), odd periodic Mathieu functions of argument $\arccos \xi$.

It may seem desirable to find a set of functions in the cylindrical case which have properties similar to i, ii and iii on p. 10 . If we write the integral relation in Eq. (2.12) so that the aperture function and the pattern function are defined in the same interval we get

$$P(\phi) = \frac{\alpha}{\pi} \int_{-\pi}^{\pi} K(\phi - \frac{\alpha}{\pi}\theta)\, A(\frac{\alpha}{\pi}\theta)\, d\theta, \qquad -\pi \leqslant \phi \leqslant \pi\,. \tag{2.29}$$

One important difference between this and the corresponding formula for the plane case, Eq. (2.3), is that the kernel is no longer symmetric. The eigenfunctions of Eq. (2.29) (if there are any) will consequently certainly not be orthogonal. The property that the spheroidal function S_{00} minimizes the supergain ratio is also a direct consequence of the symmetry of the kernel in the plane case. The aperture function in the cylindrical case for which the corresponding quantity Γ defined by Eq. (2.16) takes its lowest possible value is the eigenfunction which corresponds to the lowest eigenvalue of the kernel

$$K_L(\phi, \theta) = \int_{-\pi}^{\pi} K^*(\phi' - \frac{\alpha}{\pi}\phi)K(\phi' - \frac{\alpha}{\pi}\theta)d\phi' . \qquad (2.30)$$

This kernel is symmetric and consequently has orthogonal eigenfunctions. However, if we use them to express the aperture function, the corresponding set of pattern functions will not be orthogonal and not much is gained in the synthesizing of an arbitrary prescribed pattern.

The simplifications in the mean-square optimization procedure that can be achieved by a more sophisticated choice of $\{\psi_m\}$ and $\{\Pi_m\}$ must be weighed against the fact that we must deal numerically with more complicated functions. In the plane case these "best" functions turned out to be functions which were already tabulated and had suitable expansions available. However, even in this case it is not obvious that the expansion of the aperture function in a Fourier series, for example, instead of a series of spheroidal functions, would involve a significantly greater total amount of numerical calculations. In the cylindrical case where, due to the non-symmetry of the kernel, there appears to be no orthogonal set $\{\psi_m\}$ with $\{\Pi_m\}$ also orthogonal, it seems reasonable in the first instance to choose a set of aperture functions such that the corresponding set of pattern functions is easy to calculate. Only if it then turns out that this special choice delivers a numerically intractable optimization process, should there be any reason to look for a "better" set of functions. Such an approach gives no precedence to the "supergain ratio" over other possible quality factors in the formulation of the constraint on the aperture functions, and as indicated in Section 2.3, it may be appropriate to use a more restrictive quality factor.

Generally, only the amplitude of the prescribed pattern (i.e. the power pattern) is of interest in the synthesis process. If the phase of the prescribed pattern has any influence on the result of the synthesis, we have a possibility of improving the approximation by choice of a suitable phase function. A simple application of such a scheme for a plane aperture is given by Caprioli et al (1961).

There is one more (perhaps more basic) difference between the synthesis problems in the plane and cylindrical cases. Within the authors' knowledge, all proposed synthesis procedures in the plane case deliver a real approximating pattern if the prescribed pattern itself is a real function. In the cylindrical case, however, the approximating pattern related to a real prescribed pattern will, in general, be a complex-valued function. As we shall see in Section 2.6 this means that the best mean-square approximation to a real pattern can be a very poor approximation to the power pattern.

2.5 Formulation of the Synthesis Problem

The aim of our pattern synthesis is to achieve such an aperture function that the corresponding power pattern is the best possible approximation in some sense to a prescribed pattern. The realizability of the aperture function should be controlled during the process by limiting some factor connected with the energy stored in the near field around the aperture. This type of constraint is relatively tractable if we use a mean-square approximation, and a suitable definition of the optimum achievable pattern corresponding to a prescribed pattern $P_g(\phi)$ would be the function $P(\phi)$ which pertains to the quantity

$$\Lambda = \min_{P(\phi)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|P_g(\phi)| - |P(\phi)| \right)^2 d\phi \tag{2.31}$$

under an appropriate subsidiary condition. If the aperture function $A(\phi)$ is expressed as a linear combination of given functions as in Eq. (2.27) we can consider Λ as a function of the appropriate coefficients $\gamma_o, \gamma_1, \ldots$. Use of calculus to determine the minimum results in an infinite nonlinear system of equations for $\{\gamma_\nu\}$ and since this is extremely intractable we will modify the formulation somewhat. It is easy to see that Λ is also obtained as

$$\Lambda = \min_{\{\gamma_\nu\},\, \theta(\phi)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| |P_g(\phi)| e^{i\theta(\phi)} - P(\phi) \right|^2 d\phi \tag{2.32}$$

where the minimum of the right hand side shall be taken simultaneously with respect to the set of numbers $\{\gamma_\nu\}$ and the function $\theta(\phi)$. A necessary condition for this minimum is of course that we have a minimum of $\{\gamma_\nu\}$ alone if we keep $\theta(\phi)$ constant and vice versa. At least one such $\{\gamma_\nu\}$ and $\theta(\phi)$ together with the corresponding value of the integral in Eq. (2.32) can be constructed in the following way. Define

$$\Lambda_s \equiv \min_{\{\gamma_\nu\}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| |P_g(\phi)| \frac{P_{s-1}(\phi)}{|P_{s-1}(\phi)|} - P_s(\phi) \right|^2 d\phi \tag{2.33}$$

where $P_o(\phi)$ is a given function. The minimization is here an ordinary mean-square approximation of $P_s(\phi)$ with respect to the given function

$$|P_g(\phi)| \frac{P_{s-1}(\phi)}{|P_{s-1}(\phi)|} .$$

Thus the integral on the right hand side can be expressed as a positive definite quadratic form in $\{\gamma_\nu\}$ which has a single minimum obtained by solving a system of linear equations in $\gamma_o, \gamma_1 \ldots$. For a specified $P_o(\phi)$ the numbers $\Lambda_1, \Lambda_2, \ldots$ form a positive, monotonic, decreasing, and accordingly convergent sequence. The monotonicity is shown by the following reasoning: if we substitute

$$\frac{P_s(\phi)}{|P_s(\phi)|} \qquad \text{for} \qquad \frac{P_{s-1}(\phi)}{|P_{s-1}(\phi)|}$$

in Eq. (2.33), the value of the integral will clearly diminish. The succeeding minimization with respect to $P_{s+1}(\phi)$ to obtain Λ_{s+1} can then only result in a still smaller value. Now $\lim_{s \to \infty} \Lambda_s$ is apparently the desired stationary value, but as usual when we use necessary but not sufficient conditions for an extremum we have to check separately whether we have obtained the absolute minimum. There may exist several limit points of $\{\Lambda_s\}$, depending on the choice of $P_o(\phi)$, and there seems to be no simple rule which tells how to choose $P_o(\phi)$ so that the absolute minimum corresponding to Eqs. (2.31) and (2.32) is obtained as the limit.

For a prescribed real valued pattern function $P_g(\phi) \epsilon L_2^\pi$ and a given initial function $P_o(\phi) \epsilon L_2^\pi$ we now calculate the set of functions $\{P_s(\phi)\}$ consisting of those functions ϵL_2^π which minimize the corresponding quantities

$$\Delta_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| |P_g(\phi)| \frac{P_{s-1}(\phi)}{|P_{s-1}(\phi)|} - P_s(\phi) \right|^2 d\phi + \mu \frac{kZ_o}{2\pi^2 E_o^2} 2\omega W_s \qquad (2.34)$$

$$s = 1, 2, \ldots$$

where W_s is the larger of the time-averaged magnetic or electric energies stored in the evanescent field as defined by Eq.(2.14), connected with $P_s(\phi)$. The quantity μ is a given parameter which can be interpreted as the "weight" assigned to the stored energy compared to the deviation from the prescribed pattern. Since μ can also be considered as a Lagrange multiplier the functions $P_1(\phi), P_2(\phi), \ldots$ are also the ones which pertain to Λ_s in Eq. (2.33) under the constraint $2\omega W_s \leqslant G$, where G is a constant. If Λ_s is reasonably small the radiated power of the approximating pattern is close to that of the given pattern. In that case the above constraint is nearly equivalent to keeping Q as defined by Eq. (2.17) constant. It may seem more natural to prescribe a value of the stored energy or the quality factor than of μ during the optimization procedure. The reason for not doing so is that the problem then would contain an unknown Lagrange multiplier which would have to be determined by a "cut and try" procedure, i.e. we would have to guess a value of the multiplier and then solve the problem and check whether the solution satisfied the subsidiary condition. Even if we used the information from earlier trials to improve the subsequent guesses as much as possible the procedure would involve the solution of several times as many minimum problems as in our formulation. For a fixed prescribed pattern the parameter μ is a monotonic decreasing function of the stored energy and it may be assumed that it is as good (or bad) as this quantity as a measure of the realizability of an aperture function.

We will henceforth consider only a TM-field in an infinite axial slot. The aperture and pattern functions we obtain are thus valid also for the narrow circumferential slot. The corresponding procedure for a TE-field in an infinite slot is completely analogous.

It follows from Eq. (2.34) that the approximating pattern is an even or odd function of ϕ if the prescribed pattern has the same property, that is, if $P_o(\phi)$ is even, all $P_s(\phi)$ will be so and vice versa. As the kernel $K(\phi)$ in Eq. (2.12) is an even function, this means that the corresponding aperture function is also either even or odd, in accordance with parity of $P_s(\phi)$. Thus, if we divide a general prescribed pattern function into an even and an odd part the solution of the synthesis problem is the sum of the solutions for the even and odd parts separately. It is numerically advantageous to make this separation, and we express the aperture function as

$$A^N(\phi) = \sqrt{1-(\phi/\alpha)^2}\sum_{m=0}^{N}\gamma_m \begin{matrix}\cos\\ \sin\end{matrix} \frac{\pi m\phi}{\alpha}, \quad -\alpha \leqslant \phi \leqslant \alpha \qquad (2.35)$$

where the superscript N indicates the number of terms used in the expansion. In general, we will give the expressions for the even and odd case in the same formula, with the upper alternative pertaining to the even and the lower to the odd functions. The factor $\sqrt{1-(\phi/\alpha)^2}$ makes the aperture function fulfill the edge condition and thus ensures a finite amount of stored energy.

The kernel $K(\phi)$ in Eq. (2.12) is given as a trigonometric series with period $-\pi < \phi \leqslant \pi$ and we want to express $A^N(\phi)$ in the same way,

$$A^N(\phi) = \sum_{n=0}^{\infty} a_n^N \begin{matrix}\cos\\ \sin\end{matrix} n\phi \quad , \qquad -\pi < \phi \leqslant \pi \ . \tag{2.36}$$

We write the coefficients a_n^N as

$$a_n^N = \sum_{m=0}^{N} \gamma_m d_{nm} \tag{2.37}$$

where according to Eq. (2.35),

$$\begin{aligned} d_{nm} &= \frac{\epsilon_n}{2\pi} \int_{-\alpha}^{\alpha} \sqrt{1-(\phi/\alpha)^2} \begin{matrix}\cos\\ \sin\end{matrix} \frac{\pi m\phi}{\alpha} \begin{matrix}\cos\\ \sin\end{matrix} n\phi \, d\phi \ , \\ &= \frac{\epsilon_n \alpha}{4} \left(\frac{J_1(m\pi - n\alpha)}{m\pi - n\alpha} \pm \frac{J_1(m\pi + n\alpha)}{m\pi + n\alpha} \right) \\ & \qquad \epsilon_o = 1, \ \epsilon_1 = \epsilon_2 = \ldots = 2 \ . \end{aligned} \tag{2.38}$$

Since a Fourier series always can be integrated term by term we substitute $A^N(\phi)$ from Eq. (2.36) into Eq. (2.12) and obtain the following series for the corresponding pattern function

$$P^N(\phi) = \sum_{n=0}^{\infty} p_n^N \begin{matrix}\cos\\ \sin\end{matrix} n\phi \quad , \qquad -\pi < \phi \leqslant \pi \tag{2.39}$$

where

$$p_n^N = \frac{1}{\pi i} \frac{a_n^N}{i^n H_n^{(1)}(ka)} \ . \tag{2.40}$$

For a TM-field in an infinite axial slot the stored magnetic energy for each mode is always greater than the electric energy, and using Eqs. (2.17) and (2.19) we get

$$2\omega W_s^N = \frac{P_r^N}{\sum_{n=0}^{\infty} \frac{1}{\epsilon_n} \left| p_n^N \right|^2} \sum_{n=0}^{\infty} \frac{1}{\epsilon_n} \left| p_n^N \right|^2 Q_n = \frac{2\pi^2 E_o^2}{Z_o k} \sum_{n=0}^{\infty} \frac{1}{\epsilon_n} \left| p_n^N \right|^2 Q_n \tag{2.41}$$

where Q_n is given by Eq. (2.20).

Employing Parseval's relation we can now write Eq. (2.34) as

$$\Delta_s^N = \sum_{n=0}^{\infty} \frac{1}{\epsilon_n} \left(\left| p_n^g \right|^2 + \left| p_n^N \right|^2 - 2\,\mathrm{Re}(p_n^g\, p_n^{N*}) + \mu Q_n \left| p_n^N \right|^2 \right) \tag{2.42}$$

where

$$p_n^g = \frac{\epsilon_n}{2\pi} \int_{-\pi}^{\pi} \left| P_g(\phi) \right| \frac{P_{s-1}^N(\phi)}{\left| P_{s-1}^N(\phi) \right|} \begin{matrix} \cos \\ \sin \end{matrix} n\phi \, d\phi \tag{2.43}$$

is the nth Fourier coefficient of the prescribed pattern after s-1 steps of the iterative scheme. In the numerical computations a good approximation of p_n^g can be obtained without integration by constructing a finite trigonometric sum

$$\sum_{k=0}^{M} \alpha_k \begin{matrix} \cos \\ \sin \end{matrix} k\phi$$

whose value coincides with that of

$$P_g(\phi) \frac{P_{s-1}^N(\phi)}{\left| P_{s-1}^N(\phi) \right|}$$

at a sufficient number of equidistant values of ϕ.

If we use Eqs. (2.37) to express the coefficients p_n^N in Eq. (2.40) in terms of $\{\gamma_m\}$ we see that Δ_s^N is a (positive definite) quadratic form in these quantities and we obtain the minimum of Δ_s^N by putting

$$\frac{\partial \Delta_s^N}{\partial(\mathrm{Re}\,\gamma_m)} = 0 , \qquad \frac{\partial \Delta_s^N}{\partial(\mathrm{Im}\,\gamma_m)} = 0 .$$

This leads to a system of linear equations for the N+1 (even case) or N (odd case) unknowns γ_m,

$$\sum_{m=0}^{N} A_{\ell m}\gamma_m = C_\ell \qquad \ell = 0, 1, 2, \ldots N \tag{2.44}$$

where

$$A_{\ell m} = A_{m\ell} = \frac{\alpha^2}{16}\sum_{n=0}^{\infty}\epsilon_n \frac{1+\mu Q_n}{J_n^2(ka)+Y_n^2(ka)}\left[\left(\frac{J_1(m\pi - n\alpha)}{m\pi - n\alpha} \pm \frac{J_1(m\pi + n\alpha)}{m\pi + n\alpha}\right)\left(\frac{J_1(\ell\pi - n\alpha)}{\ell\pi - n\alpha} \pm \frac{J_1(\ell\pi - n\alpha)}{\ell\pi + n\alpha}\right)\right] \tag{2.45}$$

$$C_m = \pi i\,\frac{\alpha}{4}\sum_{n=0}^{\infty}\frac{p_n^g\, i^n}{H_n^{(2)}(ka)}\left(\frac{J_1(m\pi - n\alpha)}{m\pi - n\alpha} \pm \frac{J_1(m\pi + n\alpha)}{m\pi + n\alpha}\right) . \tag{2.46}$$

The advantage of separating the prescribed pattern in an even and an odd part is that we have only to solve two independent systems of equations in N and N+1 unknowns instead of one system with 2N+1 unknowns. As the set of functions we used in the expansion of $A^N(\phi)$ in Eq. (2.35) apparently is complete in the subset of L_2^α which consists of aperture functions with a finite amount of stored energy, we obtain the aperture function related to the stationary value of the integral in Eq. (2.32) as

$$A(\phi) = \lim_{N\to\infty}\,\lim_{s\to\infty} A^N(\phi) . \tag{2.47}$$

Thus the computational scheme is to iterate the phase of the prescribed pattern until there is no further improvement, for a fixed value of N so large that the resulting $A(\phi)$ is sufficiently close to the limit.

It may be noted that the system of equations (2.44) is intrinsically ill suited for numerical solution if N is large. If we compute the elements $A_{\ell m}$ of the coefficient matrix to a fixed number of significant figures we can replace the infinite sum in Eq. (2.45) by a summation up to $n = M$ if M is large enough. But for $n > M$ the column vectors of $\{A_{\ell m}\}$ are no longer linearly independent and the solutions of the system (2.44) do not minimize Δ_s^N. Thus the numerical stability of the problem is dependent on how rapidly the series in Eq. (2.45) converges. A simple calculation shows that due to the factor μQ_n the terms behave as $1/n^2$ when n is large and consequently, as could be expected, the stability increases with increasing value of μ.

When $\{\gamma_m\}_0^N$ is calculated (after a sufficient number of iterations of the phase of the prescribed pattern) the aperture function is obtained from Eq. (2.35) and the corresponding pattern function from Eqs. (2.37) - (2.4). We introduce the real quantities

$$X^N = \frac{1}{\pi^2} \sum_{\ell=0}^{N} \gamma_\ell^* C_\ell \tag{2.48}$$

$$Y^N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |P^N(\phi)|^2 d\phi = \frac{1}{\pi^2} \sum_{\ell=0}^{N} \sum_{m=0}^{N} \left[A_{\ell m}\right]_{\mu=0} \gamma_\ell^* \gamma_m \tag{2.49}$$

$$Z^N = \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} |A^N(\phi)|^2 d\phi = \frac{2}{3} |\gamma_o|^2 + \sum_{m=1}^{N} \left(\frac{1}{3} \mp \frac{1}{4\pi^2 m^2} \right) |\gamma_m|^2$$

$$- \frac{2}{\pi^2} \sum_{\ell=0}^{N} \sum_{m=\ell+1}^{N} (-1)^{m-\ell} \frac{(m+\ell)^2 \pm (m-\ell)^2}{(m^2-\ell^2)^2} \mathrm{Re}(\gamma_\ell^* \gamma_m). \tag{2.50}$$

Also we normalize the prescribed pattern as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P_g(\phi)\right|^2 d\phi = 1 \tag{2.51}$$

and obtain

$$\Delta^N_{min} = 1 - X^N \tag{2.52}$$

$$\epsilon^N = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\left|P_g(\phi)\right| - \left|P^N(\phi)\right|\right)^2 d\phi = 1 + Y^N - 2X^N. \tag{2.53}$$

The quality factor is then expressible as

$$Q^N \equiv \frac{2\omega W^N}{P_r^N} = \frac{X^N - Y^N}{\mu Y^N} \tag{2.54}$$

and the "supergain ratio" as

$$\Gamma^N \equiv \frac{\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left|A^N(\phi)\right|^2 d\phi}{\int_{-\pi}^{\pi}\left|P^N(\phi)\right|^2 d\phi} = \frac{\alpha}{2\pi^2}\,\frac{Z^N}{Y^N} \tag{2.55}$$

2.6 Numerical Results

The accompanying Figs. 2-4 through 2-8 have been taken from a large body of numerical data computed for the case of a single axial slot in an infinite cylinder under the sole constraint that the number of harmonics in the aperture field shall be fixed. The forms actually used in these computations were somewhat different from those described in the preceding sections, and a brief listing is perhaps desirable here.

The aperture function is first expanded in the two sets of exponentials corresponding to the two angular intervals $(-\alpha, \alpha)$ and $-\pi, \pi)$, with the expansion pertaining to the former limited to a fixed number of terms, thus,

$$A_N(\phi) = \sum_{n=-N}^{N} \gamma_n \, e^{i \frac{n\pi}{\alpha} \phi} = \sum_{n=-\infty}^{\infty} a_n^N e^{in\phi} \quad . \tag{2.56}$$

This yields at once the relation

$$a_n^N = \frac{\alpha}{\pi} \sum_{m=-N}^{N} \gamma_m \frac{\sin(m\pi - n\alpha)}{(m\pi - n\alpha)} \quad . \tag{2.57}$$

If the corresponding pattern function is

$$P_N(\phi) = \sum_{n=-\infty}^{\infty} P_n^N e^{in\phi} \tag{2.58}$$

the fundamental integral relation (2.12) provides that

$$P_n^N = \sqrt{\frac{2}{\pi i}} \frac{a_n^N}{i^n H_n^{(1)}(ka)} ,$$

and the application of the minimizing conditions results in the linear system of equations in the unknowns $\left\{ \gamma_m \right\}$

$$\sum_{m=-N}^{N} \gamma_m A_{mn} = B_n \qquad n=-N, \ldots, N \tag{2.59}$$

where

$$A_{mn} = \sum_{r=-\infty}^{\infty} \frac{\sin(m\pi - r\alpha)\sin(n\pi - r\alpha)}{H_r^{(1)}(ka)H_r^{(2)}(ka)(m\pi - r\alpha)(n\pi - r\alpha)}$$

$$B_n = \sum_{r=-\infty}^{\infty} \frac{i^r \sin(n\pi - r\alpha)\, P_r^g}{H_r^{(2)}(ka)\,(n\pi - r\alpha)}$$

$$P_r^g = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_g(\phi) e^{-in\phi} d\phi \ ,$$

$P_g(\phi)$ being the prescribed pattern function. If the inverse of the matrix $\{A_{mn}\}$ is denoted by $\{A_{mn}^{-1}\}$, then the actual pattern function $P_N(\phi)$ can be written

$$P_N(\phi) = \frac{\alpha}{\pi}\sqrt{\frac{2}{\pi i}} \sum_{m=-N}^{N} \sum_{n=-N}^{N} A_{mn}^{-1} B_n E_m(\phi) \tag{2.60}$$

where

$$E_m(\phi) = \sum_{n=-\infty}^{\infty} \frac{\sin(m\pi - n\alpha) e^{in\phi}}{i^n H_n^{(1)}(ka)(m\pi - n\alpha)}$$

and the mean-square error between the actual and prescribed patterns is finally

$$\epsilon_N \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| P_g(\phi) - P_N(\phi)\right|^2 d\phi = \sum_{n=-\infty}^{\infty} \left|P_n^g\right|^2 - \sum_{m=-N}^{N} \sum_{n=-N}^{N} A_{mn}^{-1} B_m B_n^* . \tag{2.61}$$

The infinite sums in the above forms were of course truncated at some point where the accuracy was found to be sufficient. The iteration procedure in this formulation was essentially the same as that described in Section 2.5.

The various parameters involved are defined as follows:

ka = 2π (cylinder radius/wavelength)

α = 1/2 angular width of slot

β = 1/2 angular width of sectoral prescribed pattern

N = highest order harmonic in aperture function.

The range of values of ka used in the computations was 12 - 21. In general the dependence of the phenomena of interest here on this parameter is not striking, and consequently only two values are treated in the results presented. Values of α ranged from .25 up to 2.6 radians, and those of β from .5 up to π radians. The maximum order N ranged from 1 to 5. The iteration procedure in general was continued until two successive values of ϵ_N were obtained which differed by less than 10 percent. In most cases this required only from two to four iterations.

Figure 2-4 shows values of ϵ_N plotted against α for a given value of ka and various values of N. The final iterated values shown for N = 3, 5 are not necessarily the minimum values obtainable by this process, but are very near these values, and the linearity of the behavior for N = 3 is perhaps noteworthy.

Figure 2-5 shows the effect of the iteration procedure for a relatively narrow slot and a uniform (omnidirectional) prescribed pattern. The values of ϵ_N obtained here ranged from .880 down to .325. The aperture field for the final iteration in the same case is shown in Fig. 2-6. It was found in general, as expected, that the combination of narrow slot and omnidirectional prescribed pattern resulted in the most widely fluctuating aperture fields.

Figure 2-7 shows the final iterated pattern functions for a prescribed pattern of approximately sectoral form, i.e. essentially a step function, of width 3 radians, with slot width 2 radians and various values of N. The corresponding aperture fields are shown in Fig. 2-8.

In Table II-3 are listed the values of ϵ_N at each iteration for the majority of the cases which have been computed with $A_N(\phi)$ given by Eq. (2.56) under the sole constraint that N shall be fixed.

TABLE II-3: VALUES OF ϵ_N

ka	α	β	N	s = 0	s = 1	s = 2	s = 3	s = 4
15	.25	π	3	.850	.417	.289	.266	
	1.0	.5	5	.0059	.0047	.0044		
		1.0	1	.027	.024			
			3	.021	.012	.010	.010	
			5	.0175	.0085	.0068	.0063	
		2.0	1	.300	.171	.168		
			3	.297	.100	.079	.073	
			5	.290	.070	.037	.030	.027
		π	1	.670	.507	.503		
			3	.668	.412	.378		
			5	.662	.312	.228	.217	
	1.4	π	3	.542	.327	.295		
	1.7			.447	.262	.242		
	2.0	1.0	3	.012	.012			
			5	.0066	.0066			
		π	3	.352	.195	.186		
			5	.351	.151	.119	.111	
	2.3	π	3	.257	.132	.130		
	2.6			.161	.076	.075		
21	.25	.5	1	.053	.016	.014	.014	
			3	.035	.022	.017	.014	.013
		1.5	1	.384	.130	.072	.063	.061
			3	.356	.084	.044	.034	.029
		π	1	.910	.620	.555	.547	
			3	.880	.465	.344	.325	
	1.0	1.5	1	.146	.076	.076		
			3	.144	.047	.031	.029	
			5	.138	.055	.029	.022	.019
	2.0	1.0	3	.013	.013			
			5	.0078	.0078			
		π	3	.355	.220	.216		
			5	.355	.183	.160	.151	

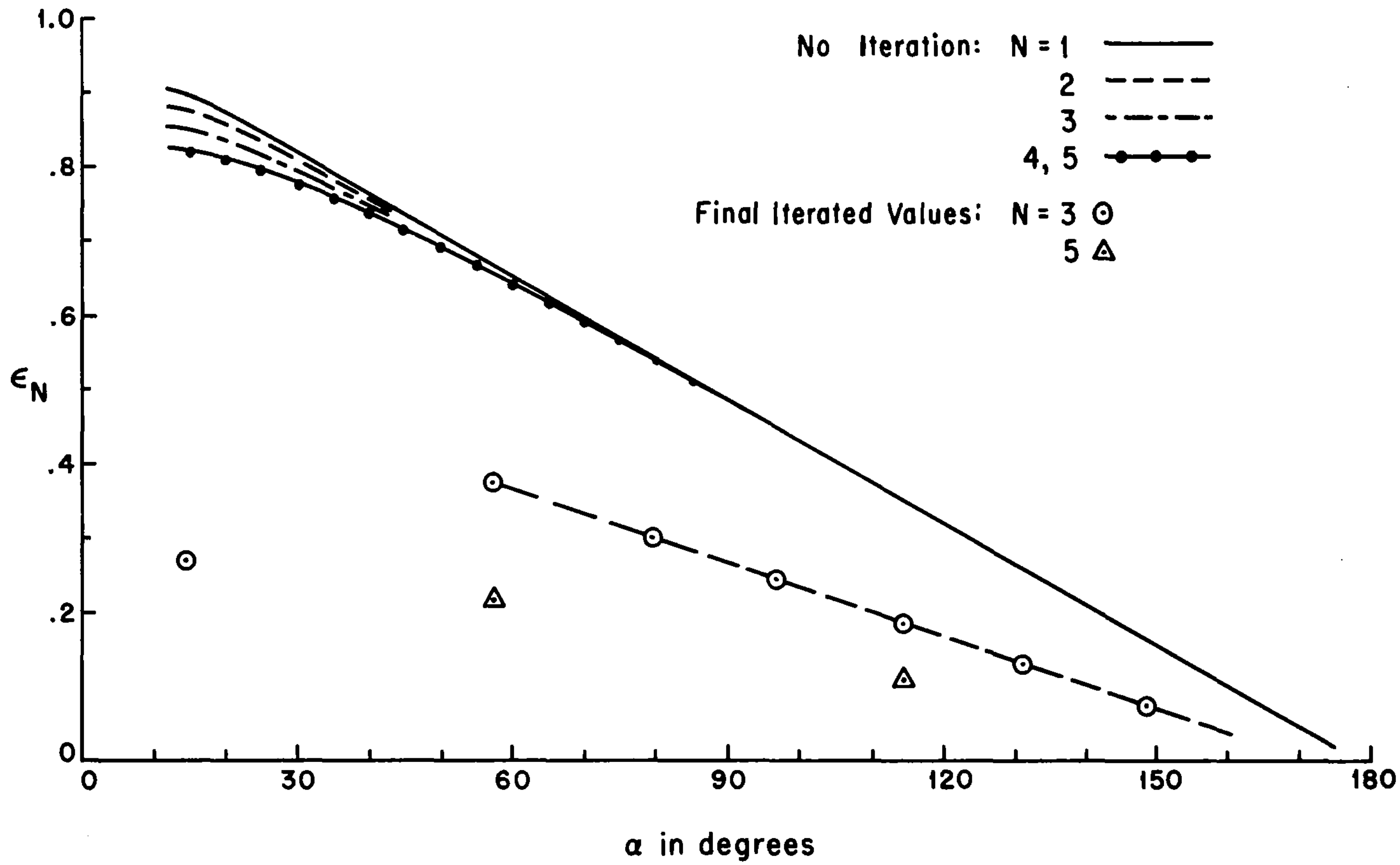


FIG. 2-4: MEAN SQUARE ERROR VS SLOT WIDTH, ka = 15

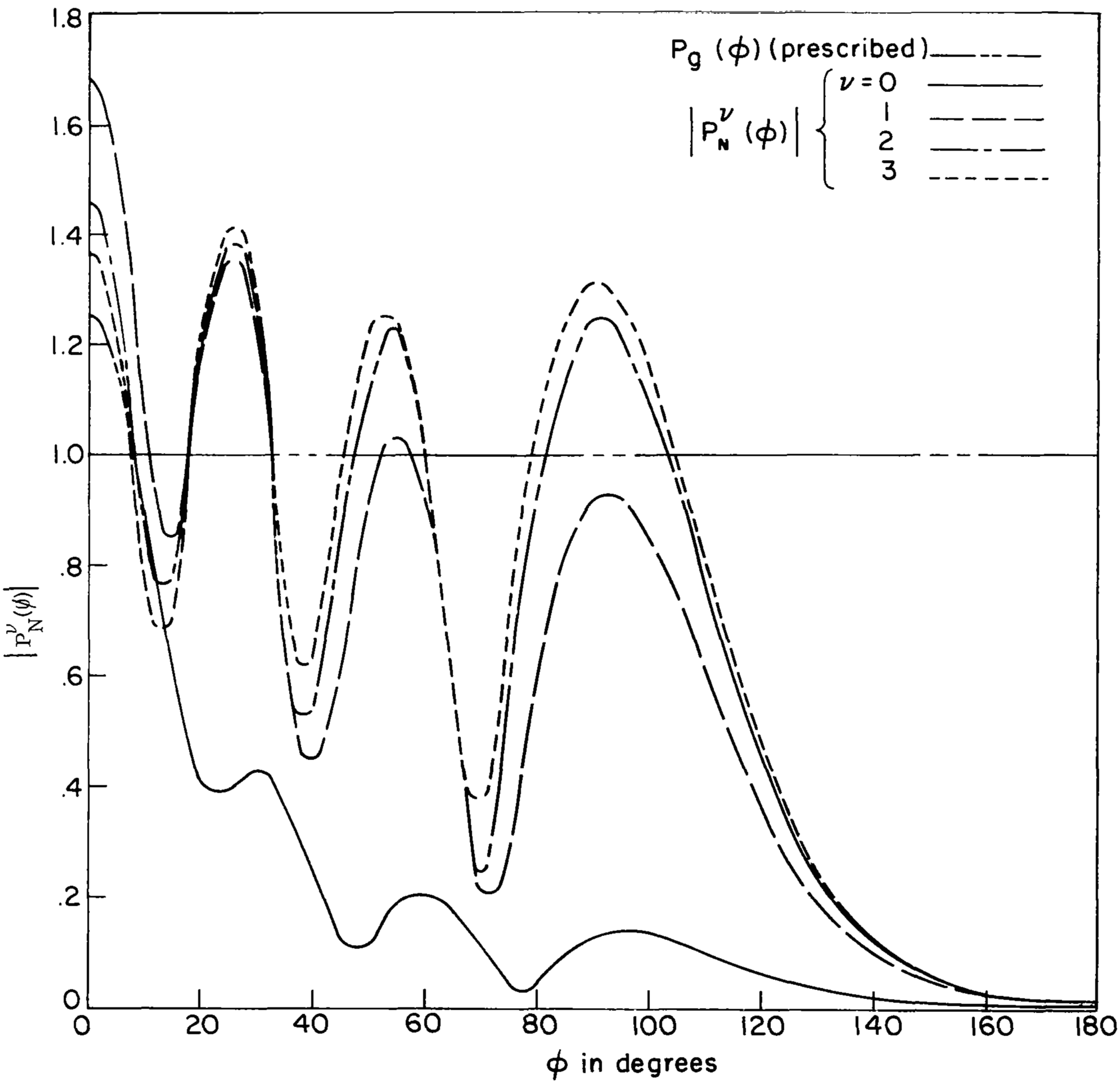


FIG. 2-5: FAR-FIELD AMPLITUDE VS ANGLE, ka = 21, α = 14.3°, N = 3

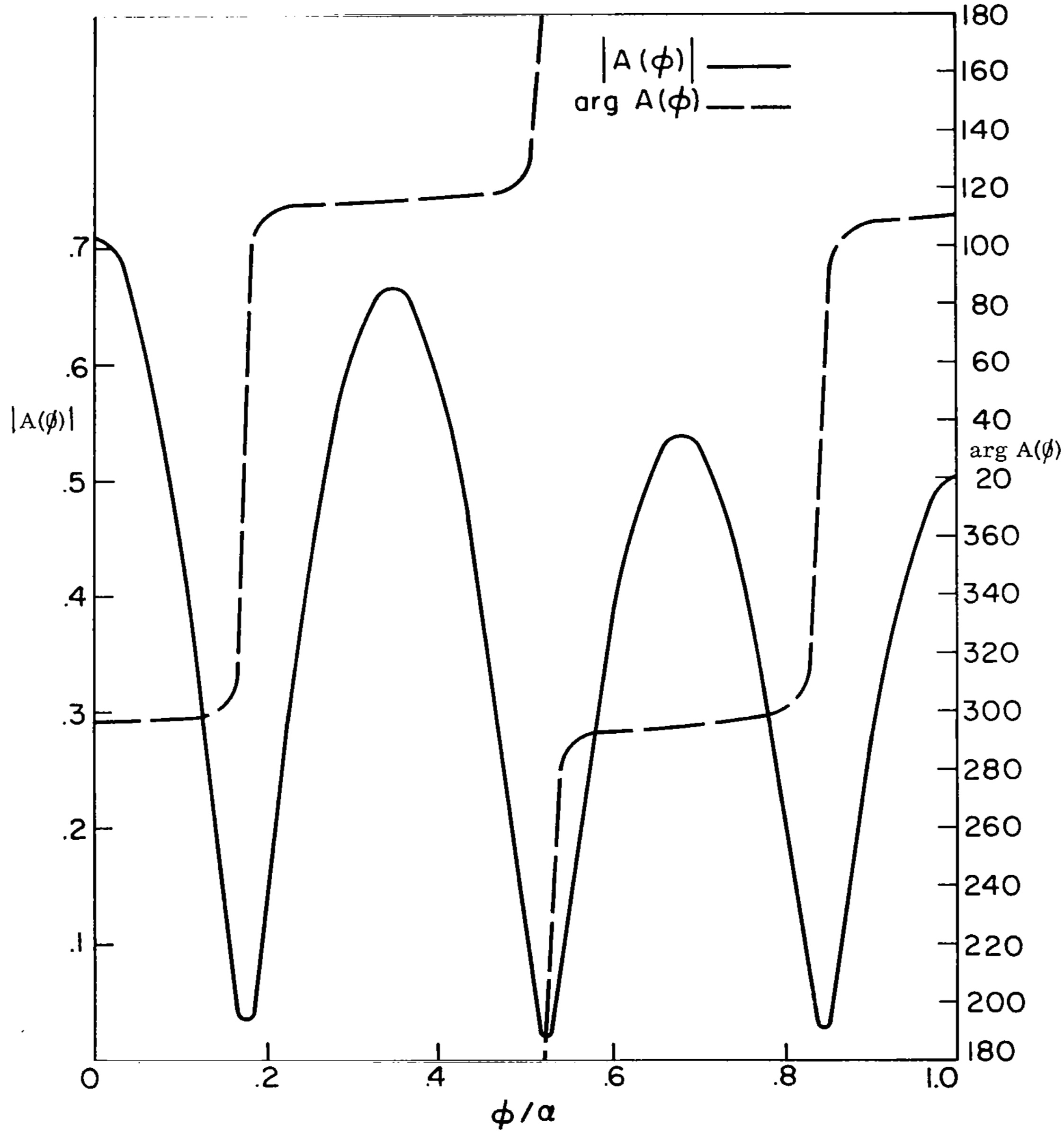


FIG. 2-6: LIMITING APERTURE FIELD DISTRIBUTION; ka=21, α=14. 3°, $\beta=\pi$, N=3.

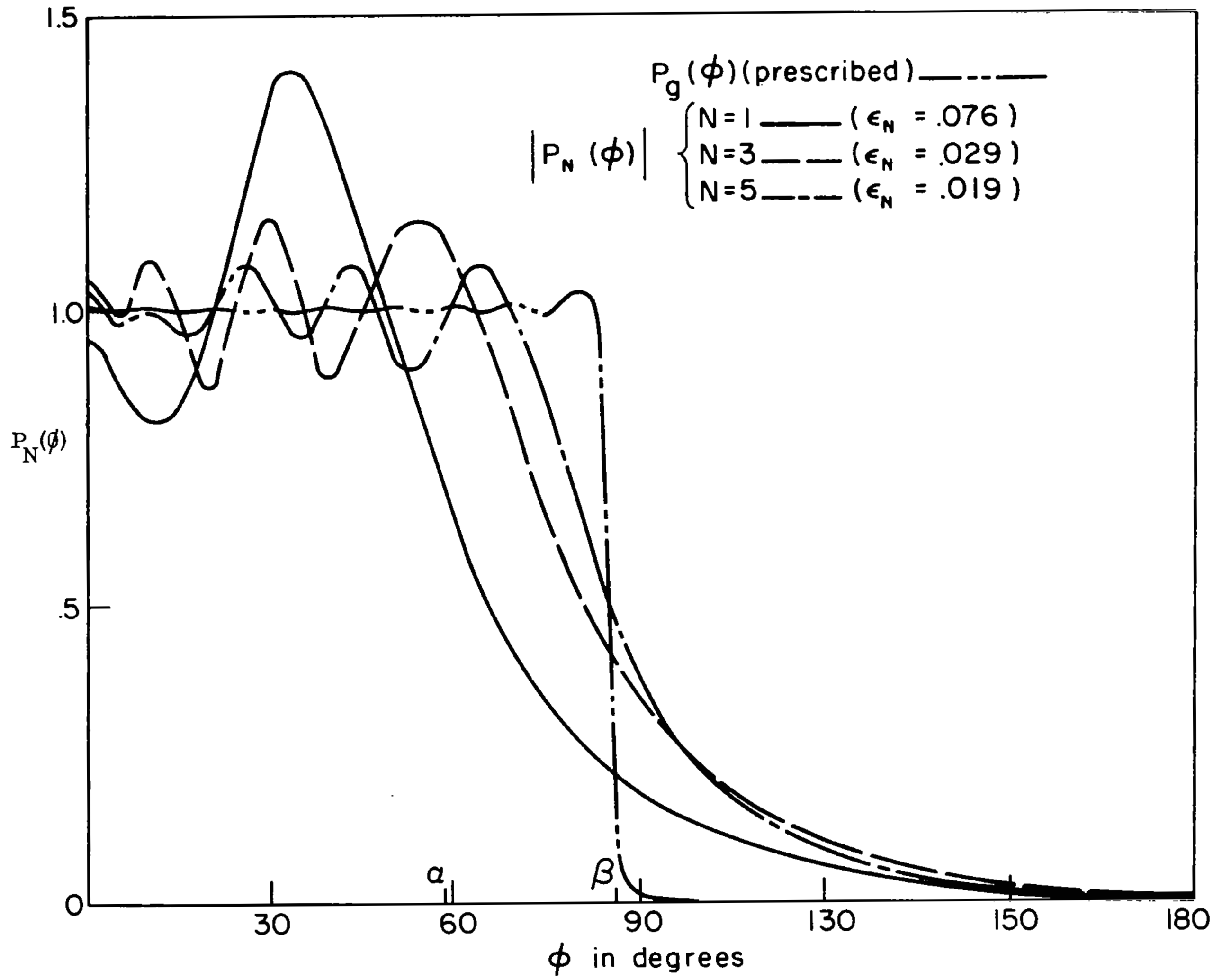


FIG. 2-7: LIMITING FAR-FIELD AMPLITUDE VS ANGLE; ka=21, $\alpha = 57.3^{o}$, $\beta = 85.9^{o}$.

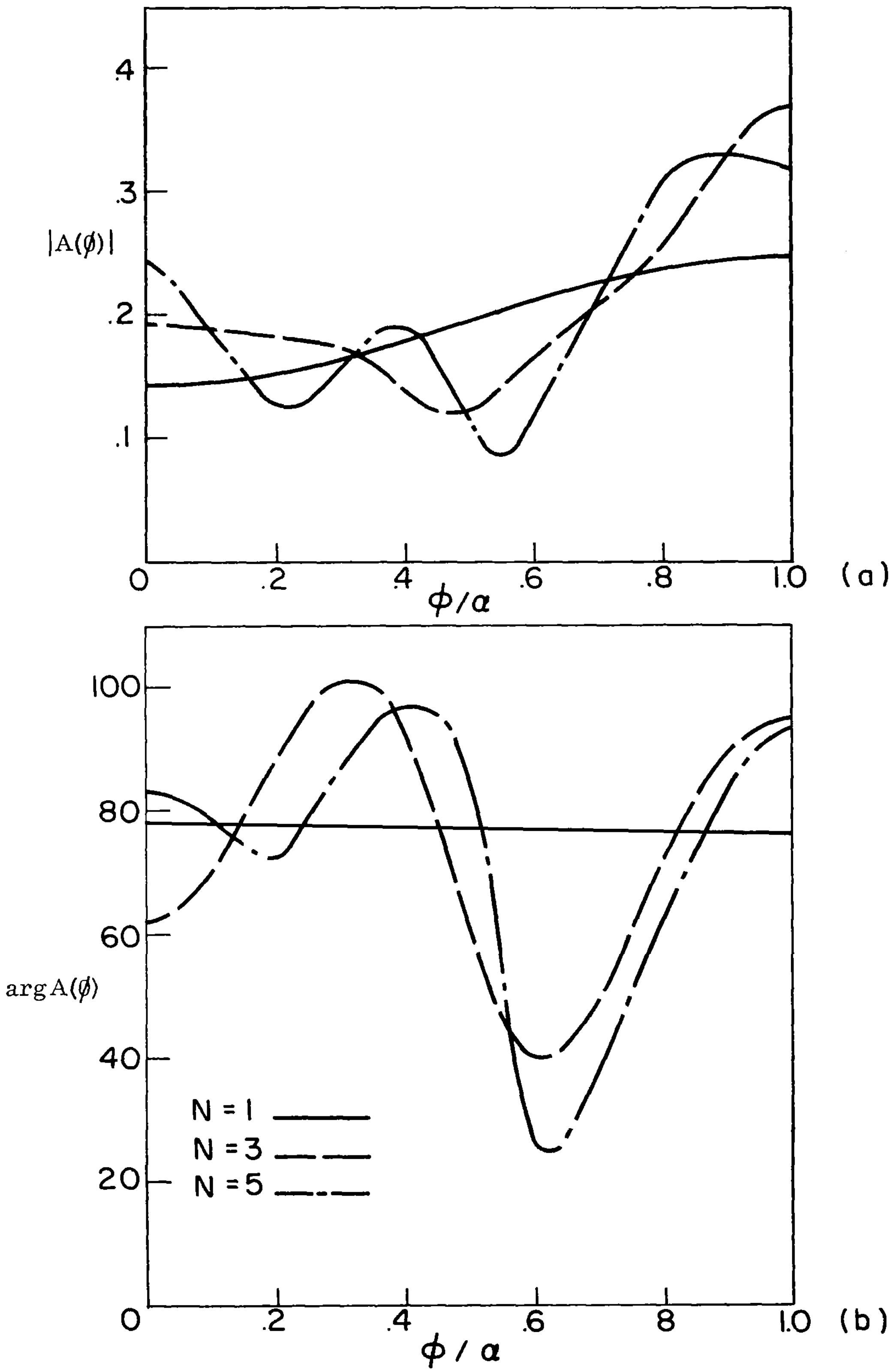


FIG. 2-8: LIMITING APERTURE FIELD DISTRIBUTION; ka = 21, $\alpha = 57.3^{o}$, $\beta = 85.9^{o}$.

The pattern functions and the corresponding aperture fields given in Figs. 2-9 through 2-16 are computed according to the formulation of the synthesis problem in Section 2.5.

The value of ka used in the computation was 15 and α (1/2 angular width of slot) was 1 or 2 radians. The reason for the choice of those relatively large values of α is that the series in Eq. (2.45), as mentioned before, only converges as $1/n^2$ and a simple method for calculating the limit is easiest to obtain for large values of α. The maximum number of terms (N) in the expansion of $A(\phi)$ in Eq. (2.35) was 10, which in most cases gave a satisfactory approximation to the limit $N \longrightarrow \infty$. The number of terms in the series of Eqs. (2.39) and (2.46) was restricted to 39, which due to the factors $H_n^{(1)}(ka)$ and $H_n^{(2)}(ka)$, respectively, in the denominator of the individual terms was quite sufficient. In accordance with this, the prescribed sectorial pattern with 1/2 angular width β=2 radians was defined as given by the first 39 terms in the Fourier expansion, normalized in such a way that the mean square value was equal to one. The iteration procedure was continued until the last value of Δ_{min}^N differed by less than 1 percent from the preceeding one. This required from three to twelve iterations.

The different values of the Lagrange multiplier, μ used in the calculations are listed in Table II-4 together with the obtained values of the minimized quantity Δ and the mean-square difference ϵ between the actual and prescribed patterns. Values for N equal to 9 and 10 are given; the difference between Δ^9 and Δ^{10} is a measure of how close the result is to the limit $N \longrightarrow \infty$. Also listed are the quality factors defined by Eqs. (2.54) and (2.55).

TABLE II-4

ka = 15

		μ	Δ^9	Δ^{10}	ϵ^9	ϵ^{10}	Q^{10}	Γ^{10}
$\beta = \pi$	$\alpha = 1$	0.1	0.379	0.379	0.282	0.282	1.85	0.176
		0.01	0.213	0.213	0.157	0.157	7.74	0.437
	$\alpha = 2$	0.1	0.133	0.127	0.075	0.064	0.78	0.140
		0.01	0.051	0.041	0.036	0.025	1.76	0.174
$\beta = 2$	$\alpha = 1$	0.1	0.163	0.163	0.081	0.081	1.09	0.138
		0.01	0.055	0.055	0.035	0.035	2.22	0.178
	$\alpha = 2$	1	0.098	0.098	0.040	0.040	0.07	0.077
		0.1	0.031	0.030	0.019	0.017	0.14	0.083

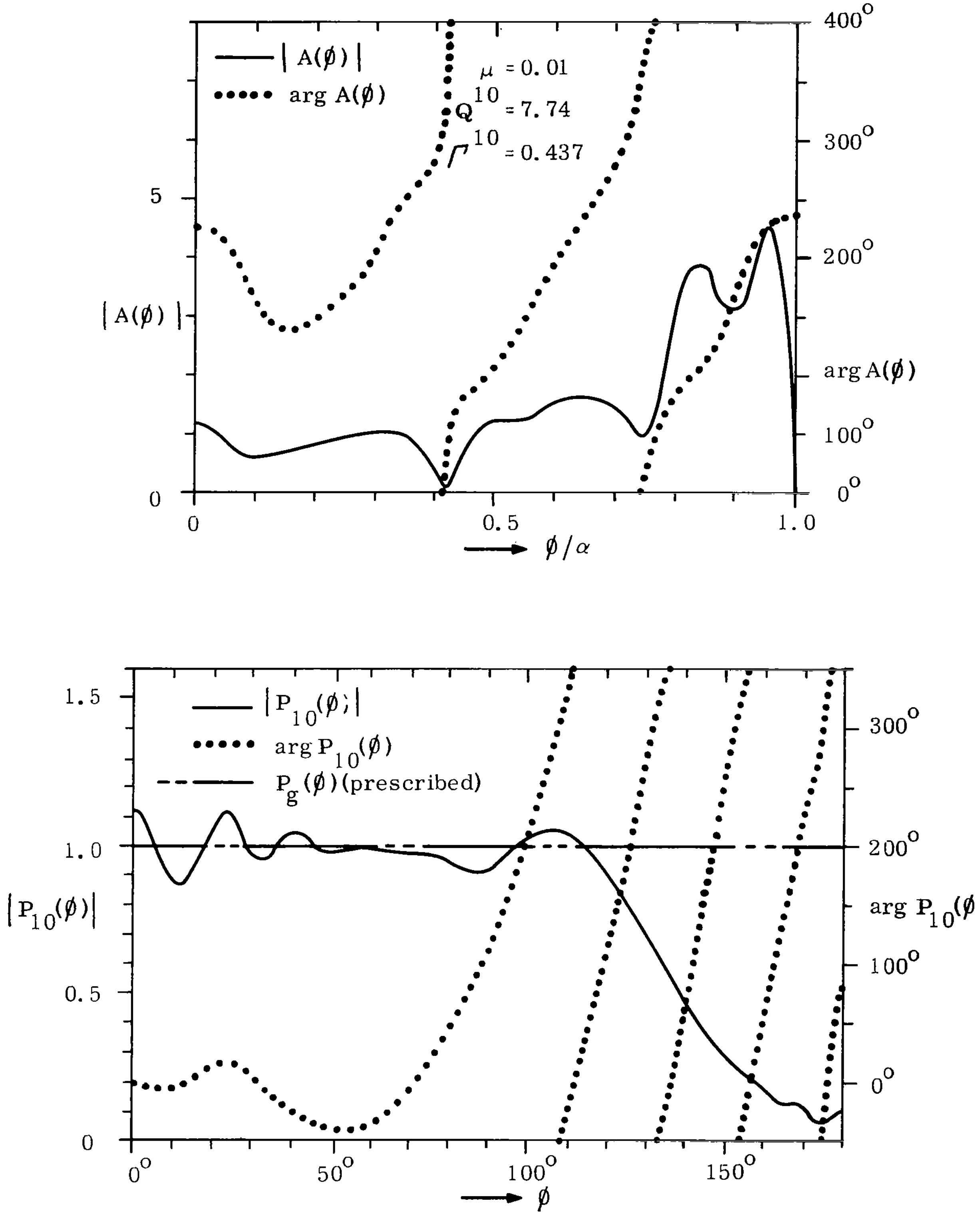


FIG. 2-9: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, $ka = 15$, $\alpha = 57.3^{\circ}$, $\beta = 180^{\circ}$, $N = 10$.

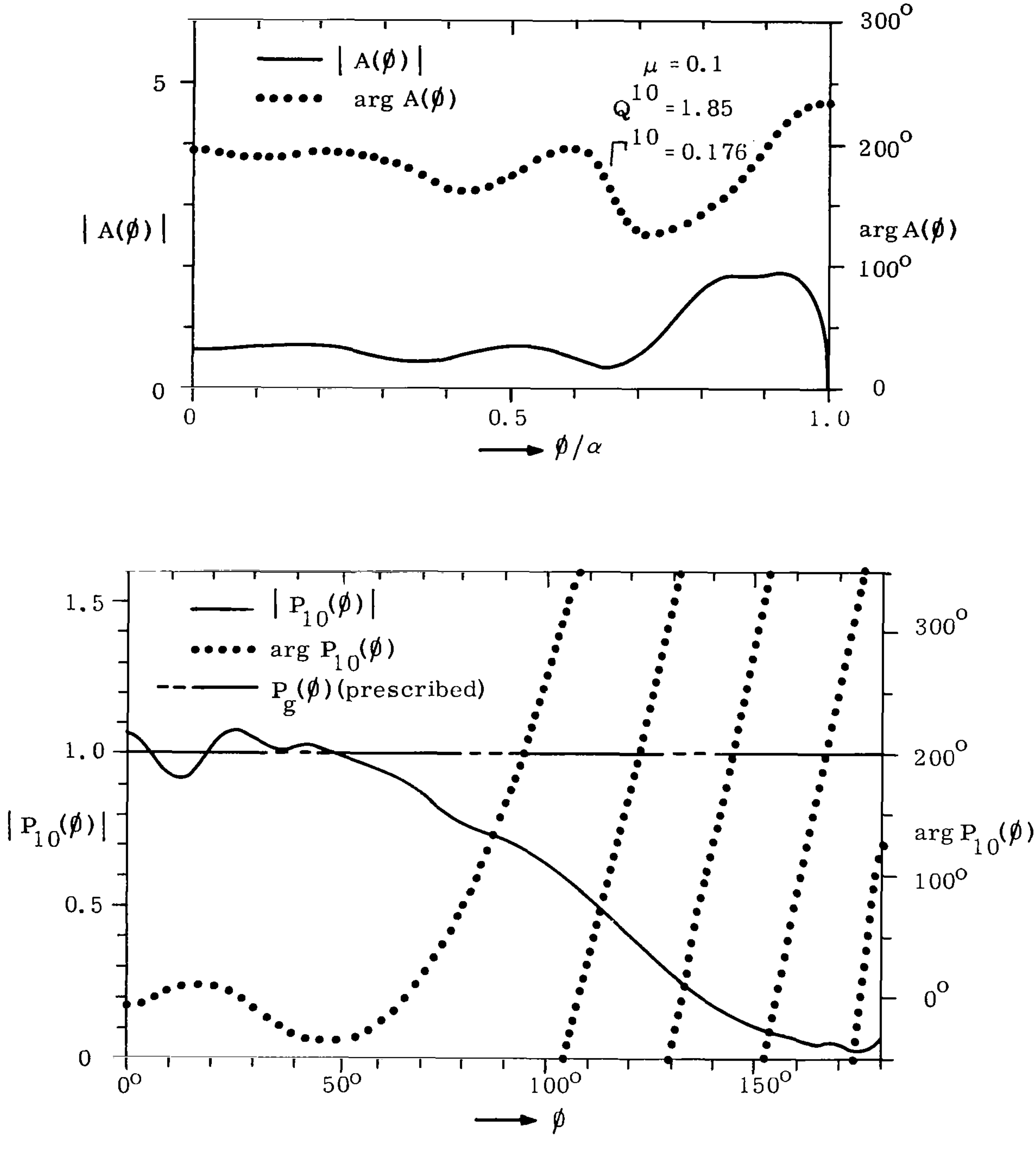


FIG. 2-10: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, $ka = 15$, $\alpha = 57.3^\circ$, $\beta = 180^\circ$, $N = 10$.

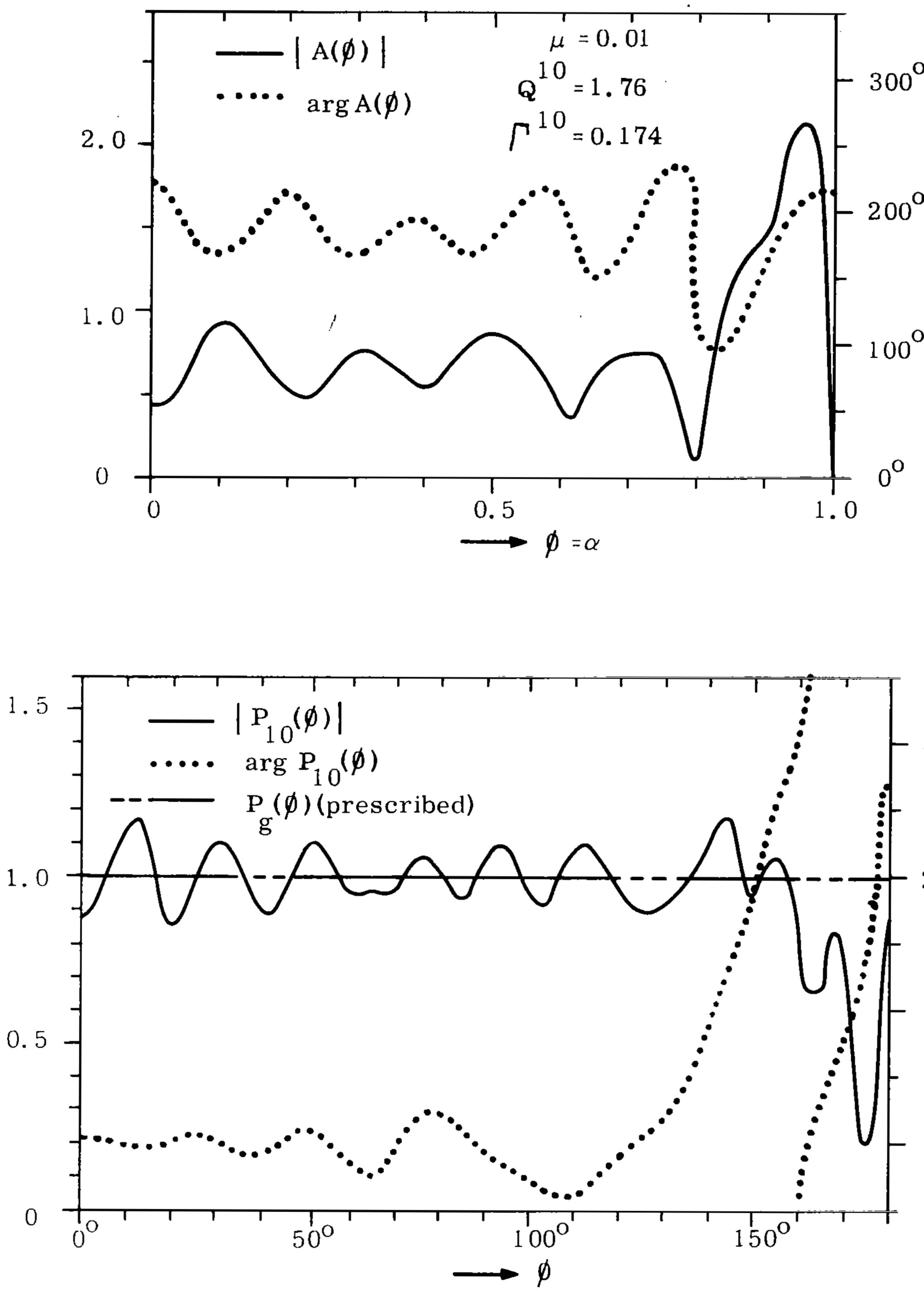


FIG. 2-11: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, ka = 15, $\alpha = 114.6^{\circ}$, $\beta = 180^{\circ}$, N = 10.

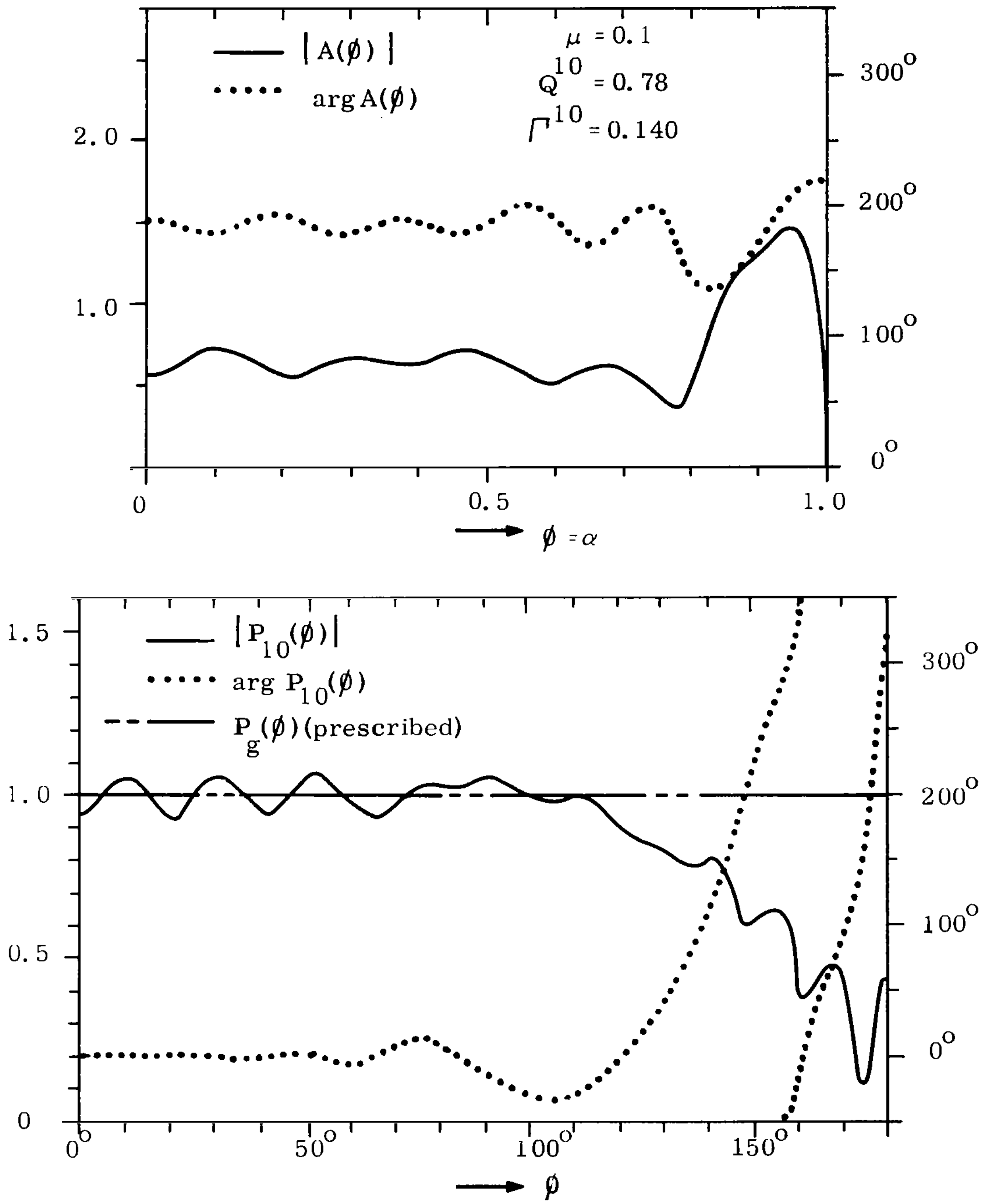
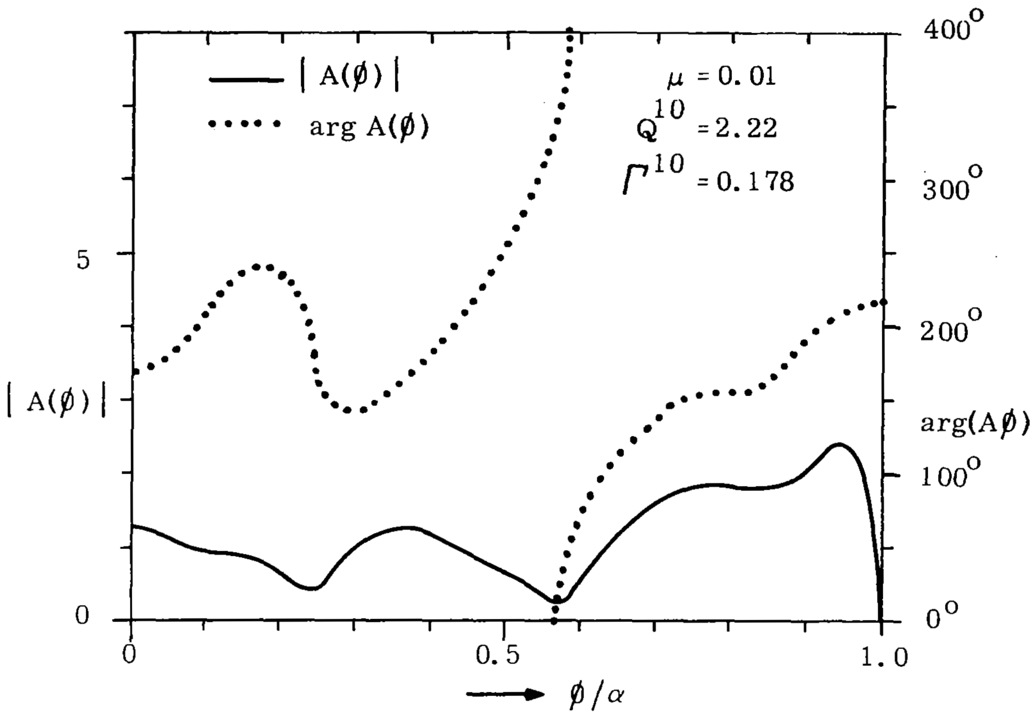


FIG. 2-12: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, $ka = 15$, $\alpha = 114.6^{\circ}$, $\beta = 180^{\circ}$, $N = 10$.



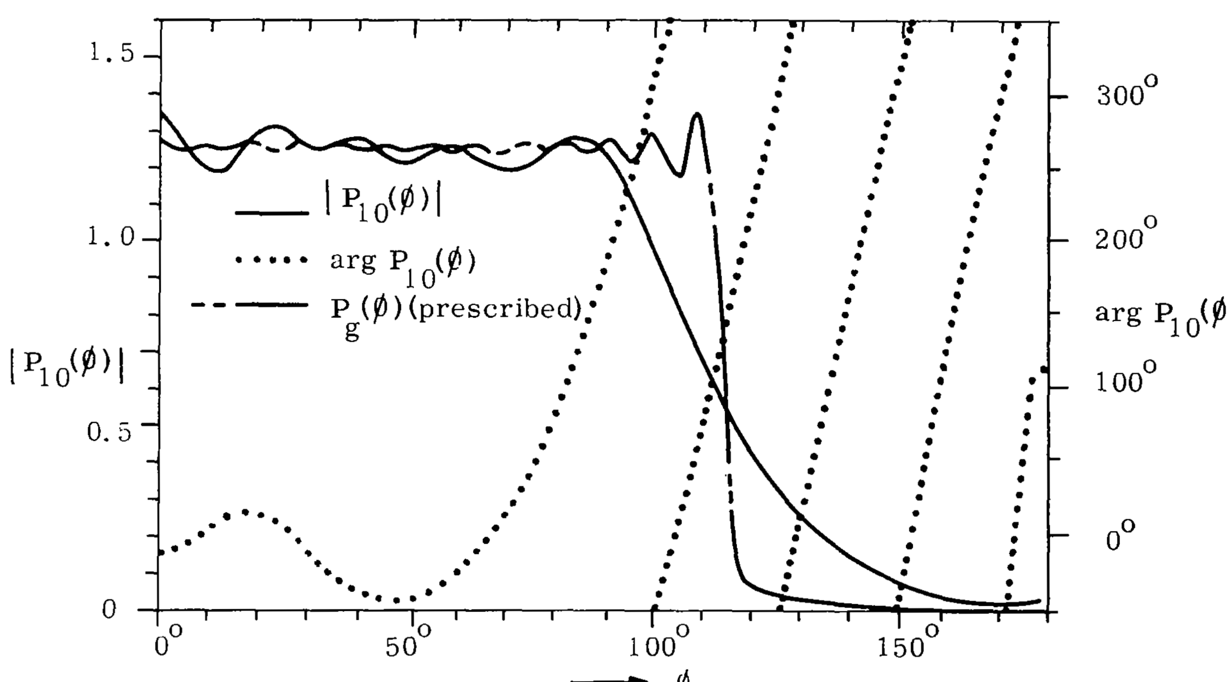


FIG. 2-13: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, $ka = 15$, $\alpha = 57.3^{\circ}$, $\beta = 114.6^{\circ}$, $N = 10$.

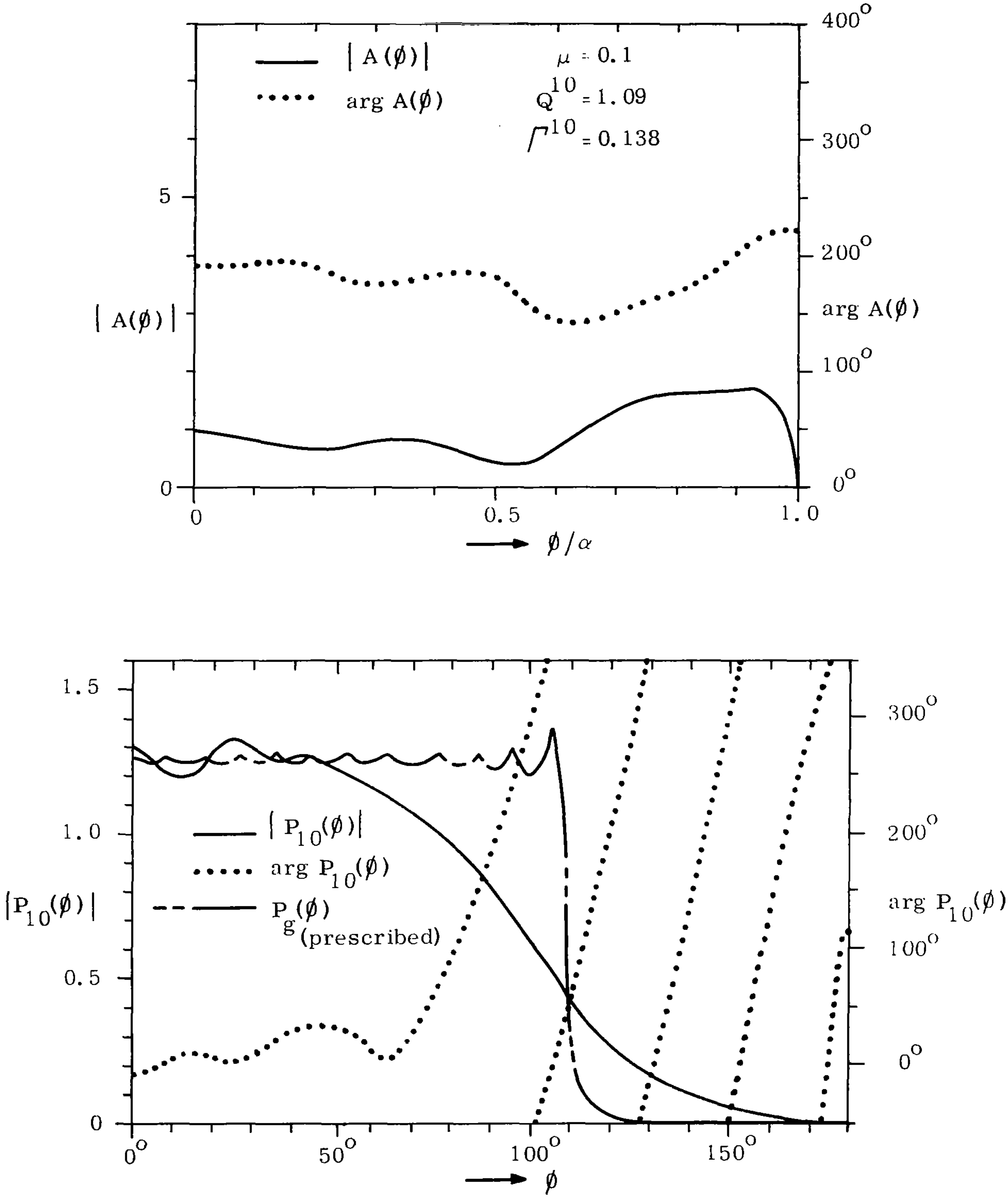


FIG. 2-14: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, ka = 15, $\alpha = 57.3^{\circ}$, $\beta = 114.6^{\circ}$, N = 10.

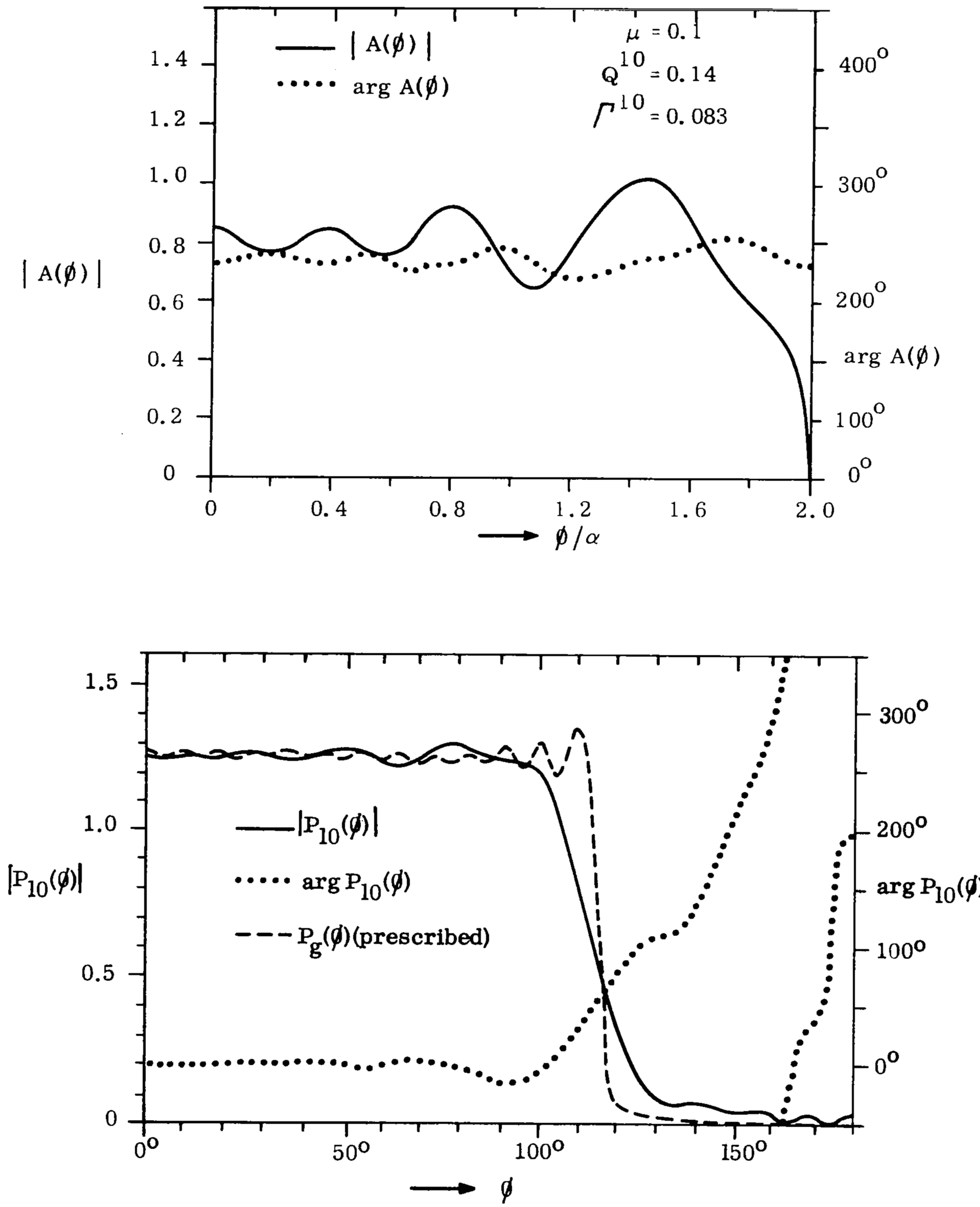
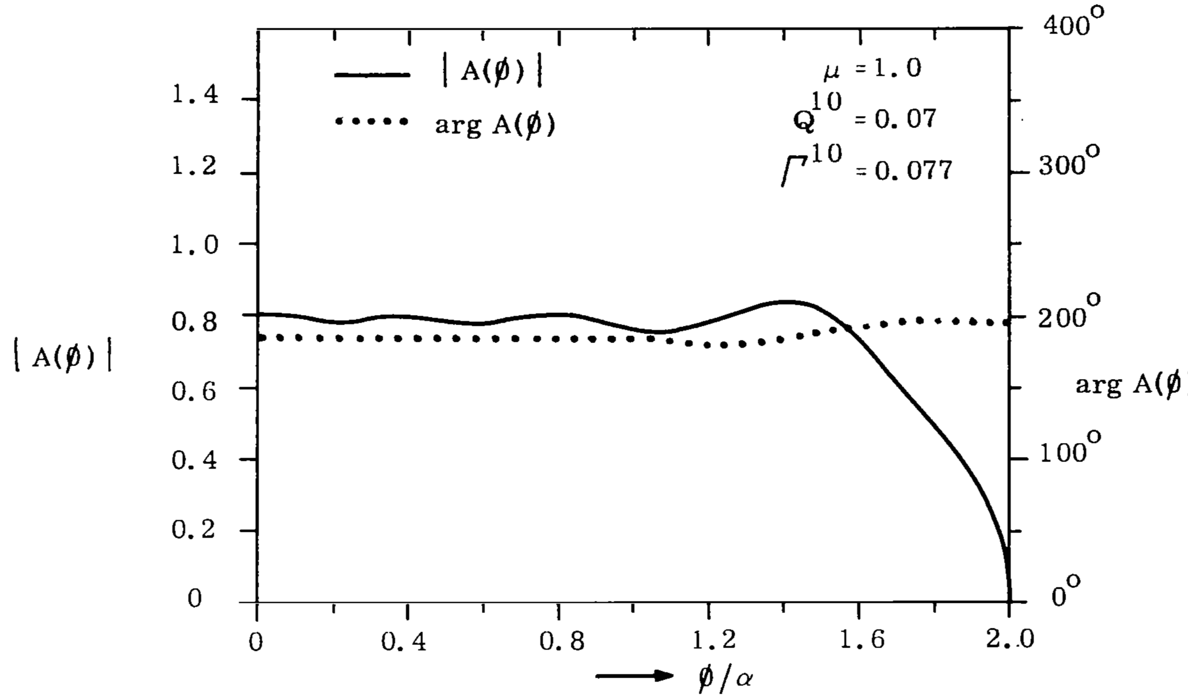


FIG. 2-15: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, ka = 15, $\alpha = 114.6^{\circ}$, $\beta = 114.6^{\circ}$, N = 10.



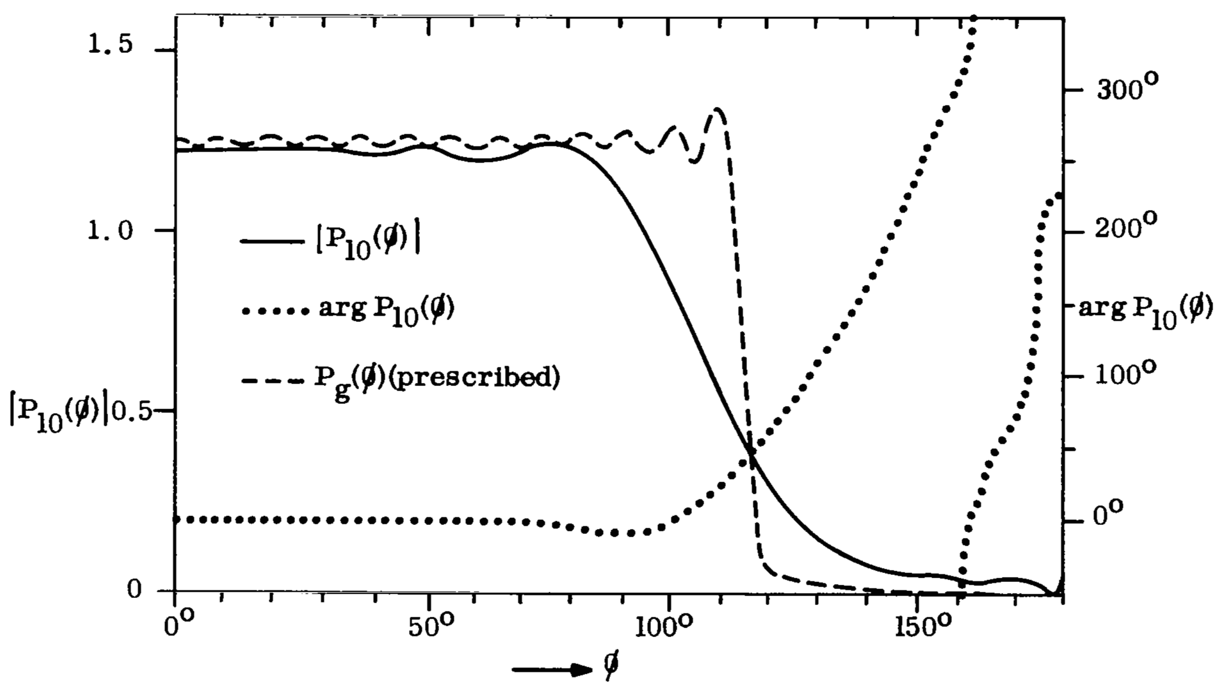


FIG. 2-16: FAR-FIELD AMPLITUDE AND APERTURE FIELD DISTRIBUTION, $ka = 15$, $\alpha = 114.6^{\circ}$, $\beta = 114.6^{\circ}$, $N = 10$.

III

ARRAYS OF AXIAL AND CIRCUMFERENTIAL HALF-WAVELENGTH SLOTS ON AN INFINITE CIRCULAR METAL CYLINDER

3.1 Introduction

In this section, the case of an array of either axial or circumferential slots uniformly spaced around the circumference of an infinite metal cylinder is considered. All slots are assumed to be half a wavelength long and very narrow, so that the voltage distribution along them is sinusoidal. A good fore-and-aft coverage is then achieved in the case of circumferential slots and, to a lesser degree, also for axial slots. The realization of a nearly omnidirectional pattern in a plane perpendicular to the axis of the cylinder (azimuthal plane) is difficult to obtain whenever the cylinder radius is large compared to the wavelength; the considerations which follow are therefore directed to the synthesis of this azimuthal pattern.

If all the feeding voltages across the centers of the slots have the same amplitude and phase, then the best mean-square approximation to an omnidirectional azimuthal field pattern is achieved. In the following sections, formulas are derived which give the minimum mean squared error between the preassigned and the actual patterns, as well as the feeding voltage necessary to produce a far field of prescribed intensity.

Computations were carried out for both the mean squared error and the feeding voltage. The numerical results are tabulated and plotted below for a number of slots, N, varying from 2 to 6, and for values of ka varying from 9.00 to 21.75 ($k = 2\pi/\lambda$ is the free space wave number, and a is the radius of the cylinder). In general, a smaller mean squared error is obtained when the number of slots is increased, for a given value of ka. However, the computed results for the case of axial slots show that this rule is not always valid.

Finally, it is shown how to obtain an omnidirectional equatorial pattern having a preassigned elliptical polarization by alternating axial and circumferential slots around the cylinder and by properly choosing the amplitudes and the relative phase of the two feeding voltages. In particular, a circularly polarized equatorial pattern can be obtained in this way.

3.2 Array of Axial Slots

Let us consider an array of N half-wavelength axial slots equally spaced around the circumference of an infinite circular metal cylinder of radius a surrounded by free space. Let us introduce a system of spherical polar coordinates (r, θ, ϕ) connected to the orthogonal Cartesian coordinates (x, y, z) of Figure 3-1 by the usual relations $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

The slots are symmetrically located with respect to the plane $z = 0$. If we indicate by 2α the angular width of each slot as seen from the cylinder axis, and assume that the first slot is centered at $\phi = 0$, then the electric field produced by the ℓth slot has, at a large distance from the cylinder, only a ϕ-component which in the equatorial plane $\theta = \pi/2$ is given by the well known formula (see, for example Wait, 1955):

$$E_{\phi,\ell} = \frac{e^{ikr}}{r} \frac{V_\ell}{\pi^2 ka} \sum_{n=o}^{\infty} \delta_m \frac{e^{-im\frac{\pi}{2}}}{H_m^{(1)'}(ka)} \beta_m \cos m\left\{\phi - \frac{2\pi}{N}(\ell-1)\right\}, \qquad (\ell=1, 2, \cdots, N), \tag{3.1}$$

where

$$\beta_m = \sin(m\alpha)/(m\alpha), \tag{3.2}$$

$k=2\pi/\lambda$ is the free space wave number, V_ℓ is the voltage across the center of the ℓth slot (that is, the product of the ϕ-component of the electric field at the center of the ℓth slot times the width of the slot), $\delta_o=1$, $\delta_{m>1}=2$, and the prime indicates the derivative of the Hankel function with respect to its argument ka.

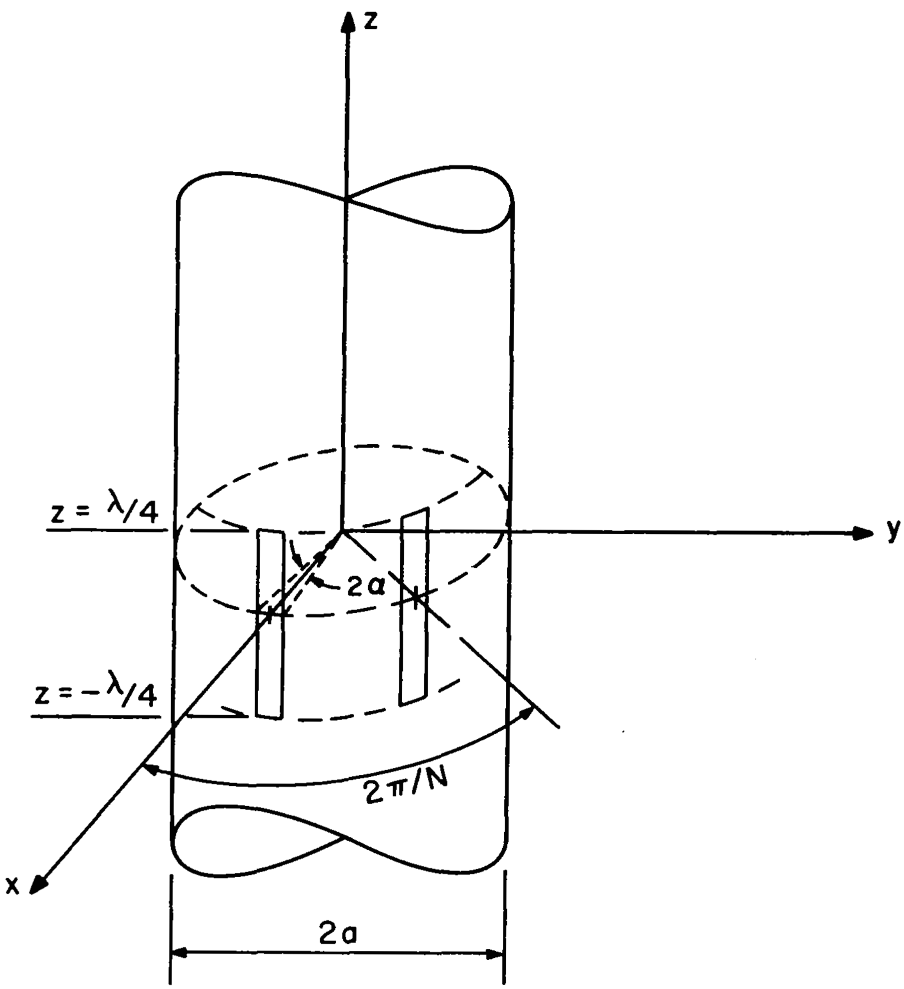


FIG. 3-1: ARRAY OF AXIAL SLOTS

The far field E_ϕ due to the array of slots is obtained by adding together the fields produced by each slot:

$$E_\phi = \sum_{\ell=1}^{N} E_{\phi,\ell} = E_a \frac{e^{ikr}}{kr} P(\phi) , \tag{3.3}$$

where E_a is a normalizing constant with the dimensions of an electric field intensity, and the field pattern $P(\phi)$ is given by

$$P(\phi) = \sum_{m=o}^{\infty} \frac{\delta_m e^{-im\frac{\pi}{2}}}{H_m^{(1)'}(ka)} \beta_m F_m(\phi) , \tag{3.4}$$

with

$$F_m(\phi) = \sum_{\ell=1}^{N} A_\ell \cos m\left\{\phi - \frac{2\pi}{N}(\ell-1)\right\} , \tag{3.5}$$

and

$$A_\ell = \frac{V_\ell}{\pi^2 a E_a} . \tag{3.6}$$

The coefficients A_ℓ are to be chosen in such a way as to approximate the preassigned far field

$$(E_\phi)_{given} = E_a \frac{e^{ikr}}{kr} \tag{3.7}$$

as closely as possible. The mean squared error between preassigned and actual patterns is defined by the relation

$$\epsilon = \frac{1}{2\pi}\int_0^{2\pi} \left|1-P(\phi)\right|^2 d\phi . \tag{3.8}$$

We want to choose A_ℓ so that ϵ be minimum; a simple calculation shows that we must take

$$A_1 = A_2 = \cdots = A_N = \frac{H_o^{(1)'}(ka)}{N(1+B)} \quad , \tag{3.9}$$

where

$$B = 2H_o^{(1)'}(ka)H_o^{(2)'}(ka) \sum_{m=1}^{\infty} \frac{\beta_{mN}^2}{H_{mN}^{(1)'}(ka)H_{mN}^{(2)'}(ka)} \quad . \tag{3.10}$$

If the coefficients A_ℓ are chosen according to (3. 9), then the mean squared error assumes its minimum value:

$$EA = \min \epsilon = B/(1+B) \quad . \tag{3.11}$$

From (3. 6) and (3. 9) it follows that

$$VA = \frac{V_a}{\lambda E_a} = \frac{\pi ka\, H_o^{(1)'}(ka)}{2N(1+B)} \quad , \tag{3.12}$$

where $V_a = V_1 = V_2 = \cdots = V_N$ is the voltage across the center of each slot. Formula (3. 12) gives the feeding voltage as a function of frequency, cylinder radius and number of radiating slots for every preassigned value of the far field intensity.

Computations of both EA and $|VA|$ were performed with the aid of the IBM 7090 computer at The University of Michigan, for the parameter values ka=9. 00(0. 25) 21. 75 and N=2(1)6. In these calculations, the slot was assumed to be of infinitesimal width, that is, the quantity β_{mN} was taken as equal to unity for all values of m and N. Since Wait (1955) has shown that the difference between the radiation pattern of a slot whose width is less than about one-tenth of a wavelength and the

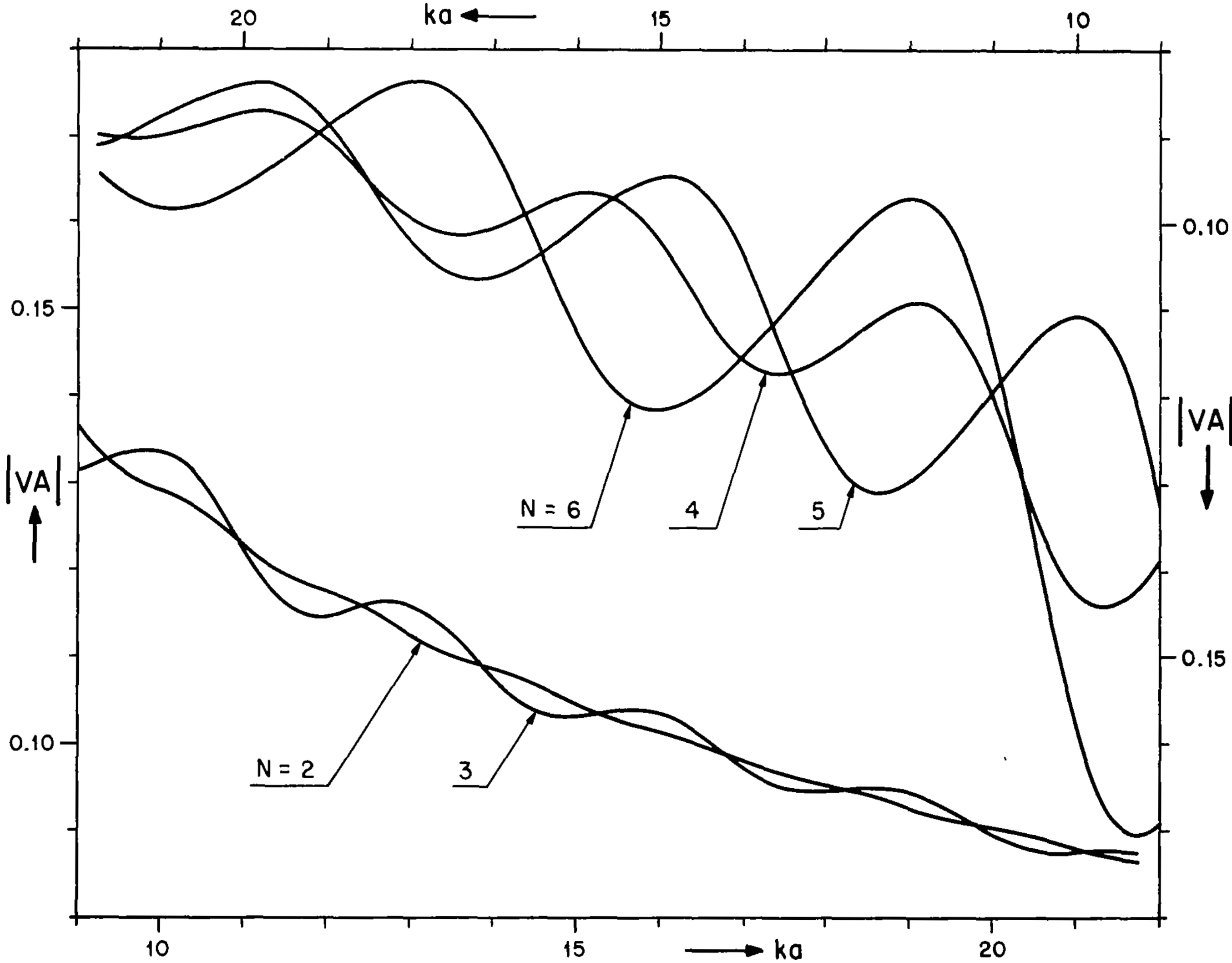


FIG. 3-2: FEEDING VOLTAGE FOR AXIAL SLOTS. Bottom and left coordinate scales are valid for N=2, 3; the right and top scales are valid for N=4, 5 and 6.

radiation pattern of a slot of infinitesimal width is negligible, we may conclude that the numerical results obtained under the hypothesis $\alpha = 0$ remain valid for all $\alpha \lesssim \pi/(10ka)$ radians.

The numerical results are tabulated in Section 3. 4; only three or four figures of the seven that were obtained for each number are given. The same results are plotted in Figs. 3-2 and 3-3. It is seen that for a given N, the dimensionless parameter $|VA|$ tends to decrease as ka increases, whereas EA increases with ka; also, both $|VA|$ and EA present an oscillatory behaviour which becomes more and more pronounced as N increases. For a given ka, the mean squared error EA generally decreases when the number of slots is increased; however, it is easily seen from Fig. 3-3 that this is not always the case: for example, the mean squared error for five slots is less than that for six slots in the range $11.5 < ka < 13$.

The radiation pattern corresponding to the minimum mean squared error is given by

$$\left[P(\phi)\right]_{opt.} = \frac{1+b(\phi)}{1+B} , \tag{3.13}$$

where

$$b(\phi) = 2H_0^{(1)'}(ka) \sum_{m=1}^{\infty} \frac{\beta_{mN} e^{-imN\frac{\pi}{2}}}{H_{mN}^{(1)'}(ka)} \cos(mN\phi) . \tag{3.14}$$

The pattern (3. 13) is symmetrical with respect to $\phi = 0$ and periodic with period $2\pi/N$; it is therefore sufficient to calculate it in the range $0 \leqslant \phi \leqslant \pi/N$.

3. 3 Array of Circumferential Slots

Let us now consider an array of N half-wavelength circumferential slots equally spaced around the circumference of the infinite metal cylinder of Fig. 3-4. If the first slot is centered at $\phi = 0$, then the electric field produced by the ℓth slot has, at a large distance from the cylinder, only a z-component which in the

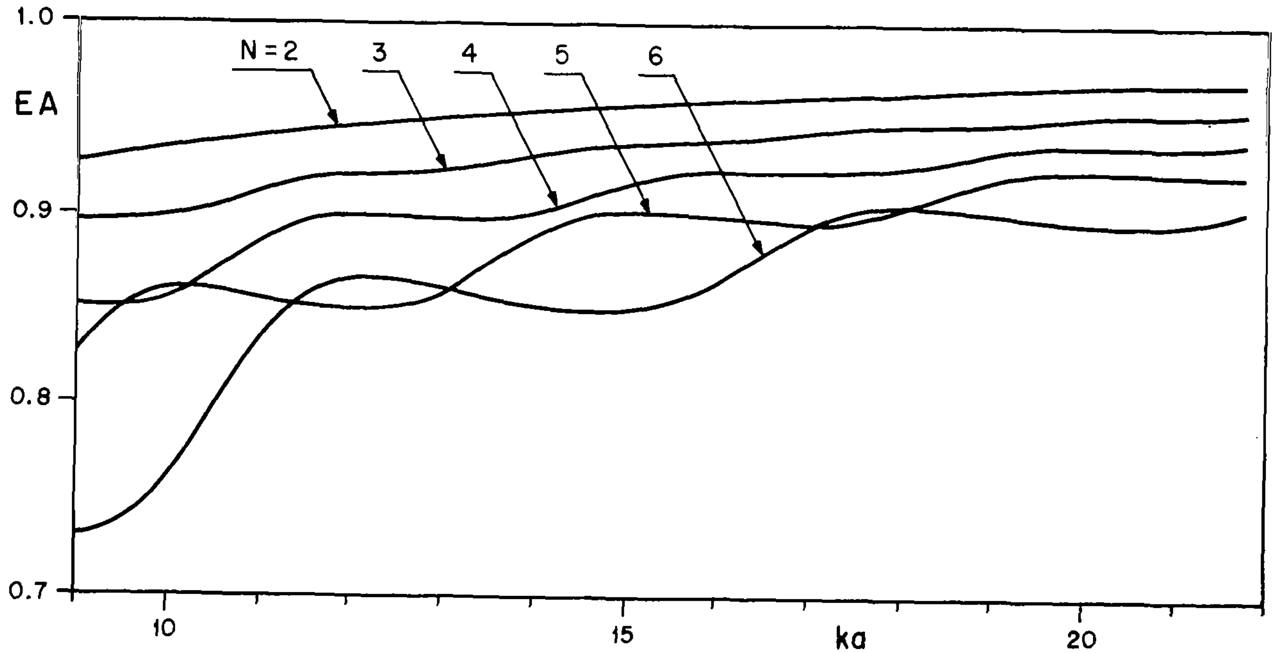


FIG. 3-3: MINIMUM MEAN SQUARED ERROR FOR AXIAL SLOTS

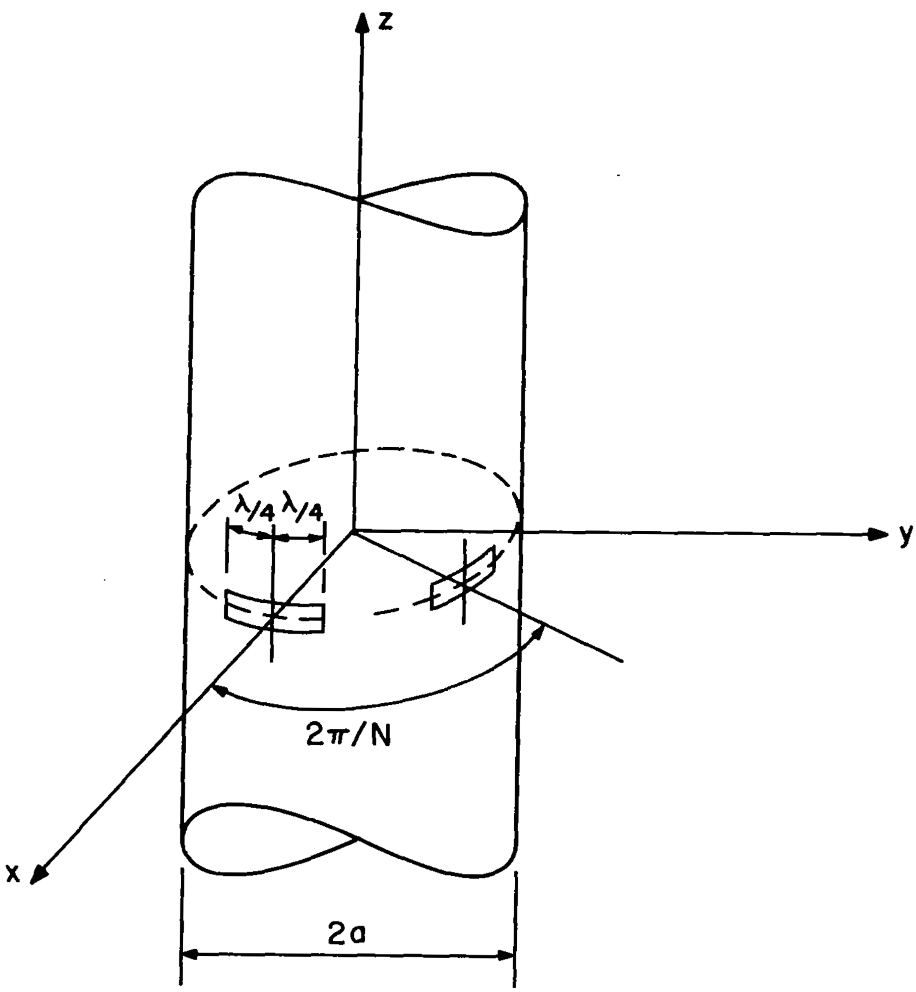


FIG. 3-4: ARRAY OF CIRCUMFERENTIAL SLOTS

equatorial plane $\theta = \pi/2$ is given by the formula (Papas, 1950)

$$E_{z,\ell} = \frac{e^{ikr}}{r} \frac{ka\tilde{V}_\ell}{\pi^2} \sum_{m=0}^{\infty} \delta_m \frac{e^{-i(m+1)\frac{\pi}{2}}}{H_m^{(1)}(ka)} \gamma_m \cos m\left\{\phi - \frac{2\pi}{N}(\ell-1)\right\}, \quad (\ell = 1, 2, \cdots, N), \tag{3.15}$$

where

$$\gamma_m = \frac{\cos\left(\frac{m\pi}{2ka}\right)}{(ka)^2 - m^2}, \tag{3.16}$$

and $\tilde{V}_\ell$ is the voltage across the center of the ℓth slot (that is, the product of the z-component of the electric field at the center of the slot times the width of the slot).

The far field due to the array of N slots is then given by

$$E_z = \sum_{\ell=1}^{N} E_{z,\ell} = E_c \frac{e^{ikr}}{kr} P(\phi), \tag{3.17}$$

where E_c is the normalization constant with the dimensions of an electric field intensity, and

$$P(\phi) = \sum_{m=0}^{\infty} \delta_m \frac{e^{-i(m+1)\frac{\pi}{2}}}{H_m^{(1)}(ka)} \gamma_m \tilde{F}_m(\phi), \tag{3.18}$$

with

$$\tilde{F}_m(\phi) = \sum_{\ell=1}^{N} \tilde{A}_\ell \cos m\left\{\phi - \frac{2\pi}{N}(\ell-1)\right\}, \tag{3.19}$$

and

$$\tilde{A}_\ell = \frac{k^2 a \tilde{V}_\ell}{\pi^2 E_c}. \tag{3.20}$$

The mean squared error between the preassigned far field

$$(E_z)_{given} = E_c \frac{e^{ikr}}{kr}, \tag{3.21}$$

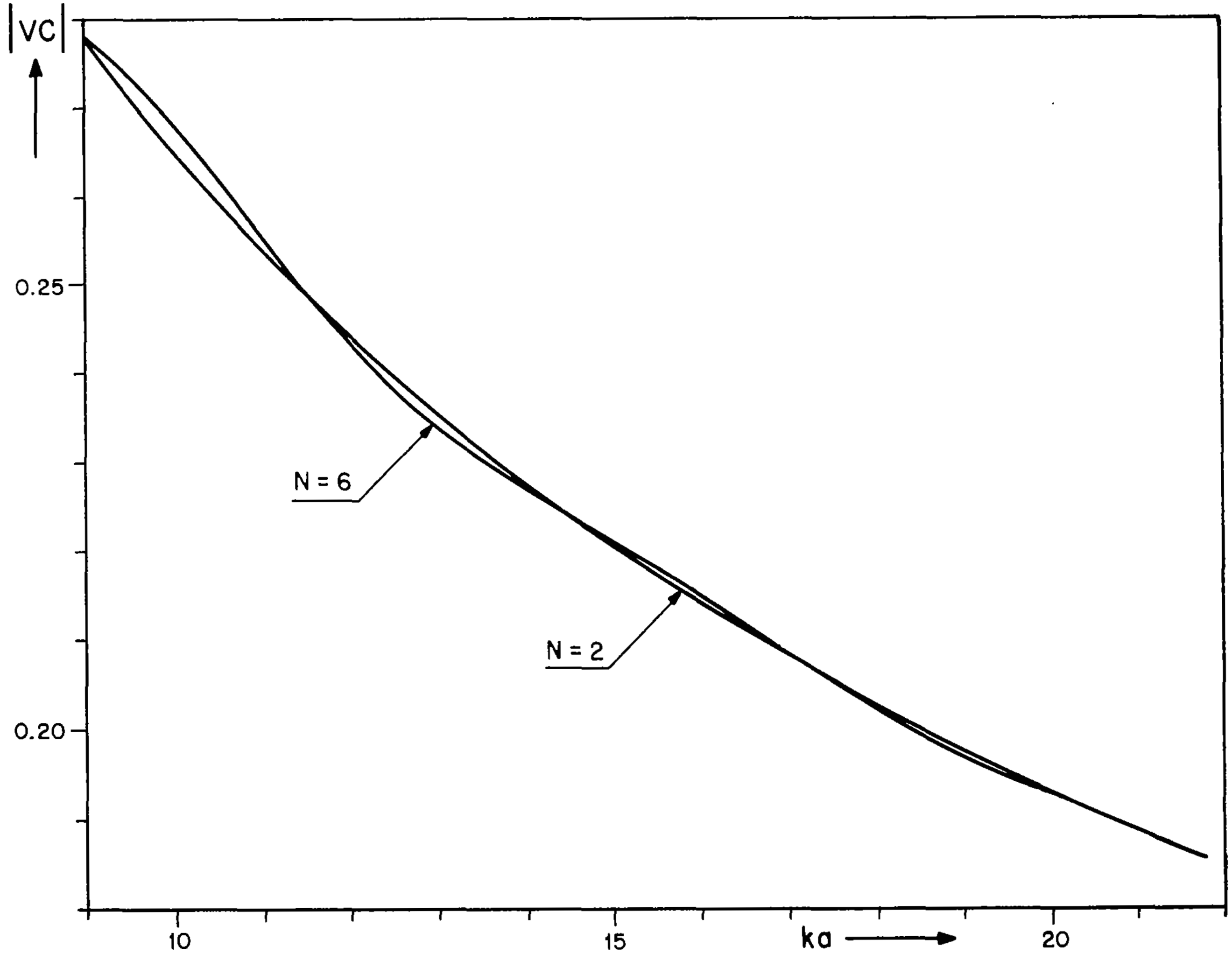


FIG 3-5: FEEDING VOLTAGE FOR CIRCUMFERENTIAL SLOTS

and the actual far field (3.17) is minimized by choosing

$$\tilde{A}_1=\tilde{A}_2=\cdots=\tilde{A}_N=\frac{(ka)^2H_o^{(1)}(ka)e^{i\frac{\pi}{2}}}{N(1+C)}\ , \tag{3.22}$$

where

$$C=2(ka)^4H_o^{(1)}(ka)H_o^{(2)}(ka)\sum_{m=1}^{\infty}\frac{\gamma_{mN}^2}{H_{mN}^{(1)}(ka)H_{mN}^{(2)}(ka)}\ . \tag{3.23}$$

If the coefficients $\tilde{A}_\ell$ are chosen according to (3.22), then it follows that

$$EC = \min\epsilon = C/(1+C)\ , \tag{3.24}$$

and that

$$VC=\frac{V_c}{\lambda E_c}=\frac{\pi kaH_o^{(1)}(ka)e^{i\frac{\pi}{2}}}{2N(1+C)}\ , \tag{3.25}$$

where $V_c=\tilde{V}_1=\tilde{V}_2=\cdots=\tilde{V}_N$ is the voltage across the center of each slot. Formula (3.25) gives the feeding voltage as a function of frequency, cylinder radius and number of radiating slots for every preassigned value of the far field intensity.

Computations of EC and $|VC|$ were carried out using the same values of ka and N that were previously adopted in the case of axial slots. The numerical results are tabulated in Section 3.4 and plotted in Figs. 3-5 and 3-6. It is seen that $|VC|$ decreases rapidly as ka increases, for a given N. If N is not large, then VC is almost independent of N: the curve of $|VC|$ as a function of ka for a given N exhibits small oscillations about the curve N=2, whose amplitudes appear to increase as N becomes larger.

The minimum mean squared error EC increases with ka for a given N, and decreases as N increases for a given ka (Fig. 3-6).

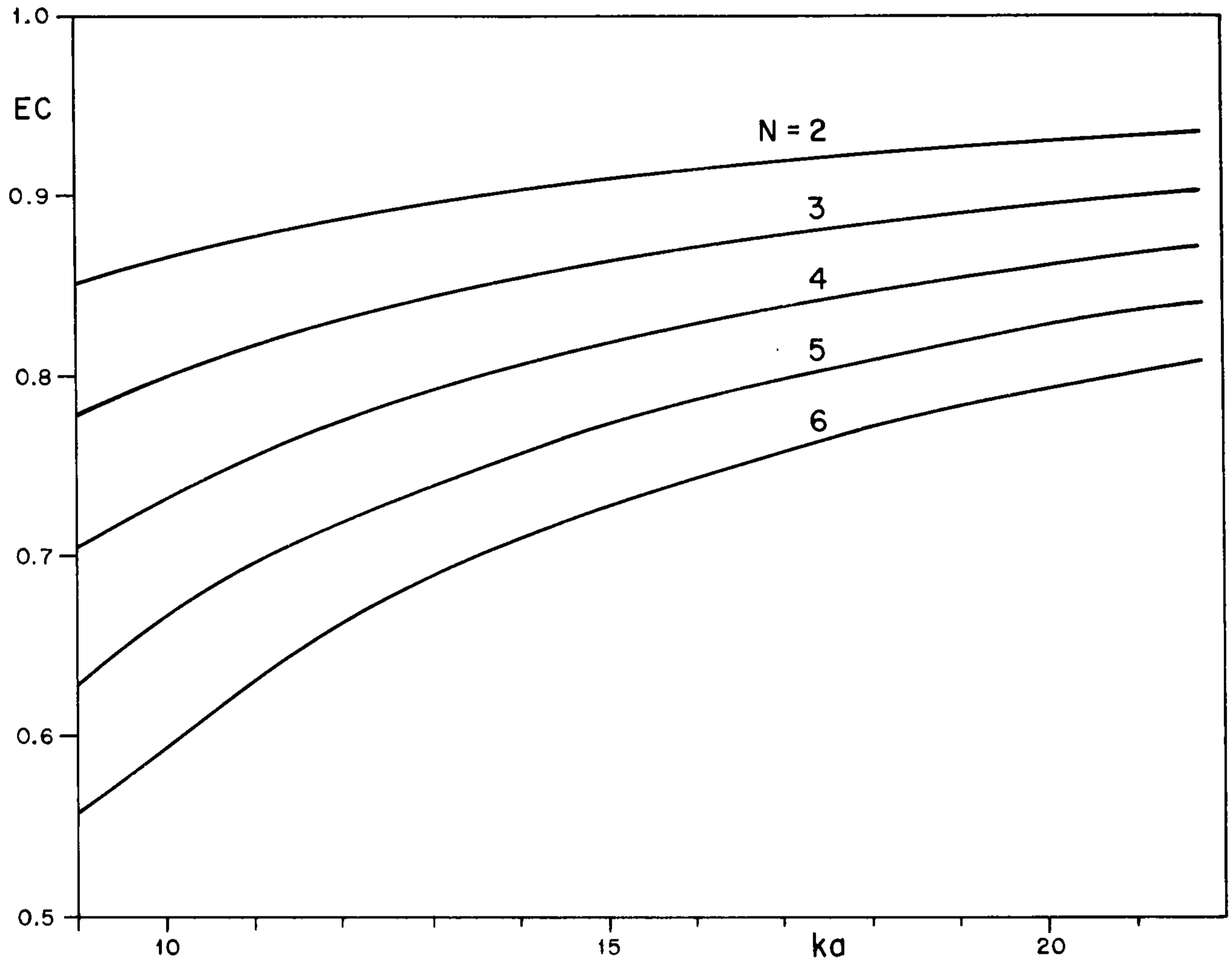


FIG. 3-6: MINIMUM MEAN SQUARED ERROR FOR CIRCUMFERENTIAL SLOTS

The radiation pattern corresponding to the minimum mean squared error is given by

$$\left[P(\phi)\right]_{opt.} = \frac{1+c(\phi)}{1+C} \quad , \tag{3.26}$$

where

$$c(\phi)=2(ka)^2 H_o^{(1)}(ka) \sum_{m=1}^{\infty} \frac{\gamma_{mN} e^{-imN\frac{\pi}{2}}}{H_{mN}^{(1)}(ka)} \cos(mN\phi) \quad . \tag{3.27}$$

As in the case of axial slots, the pattern (3.26) is symmetrical with respect to $\phi=0$ and periodic with period $2\pi/N$, and it is therefore sufficient to calclilate it in the range $0 \leqslant \phi \leqslant \pi/N$.

3.4 Numerical Results

ka	N	VA· 10^4	VC· 10^4	EA· 10^3	EC· 10^3
	2	1366	2774	927	852
	3	1313	2772	895	779
9.00	4	1391	2770	852	705
	5	1315	2788	826	629
	6	1693	2778	731	556
	2	1337	2741	930	856
	3	1321	2740	896	784
9.25	4	1418	2740	851	712
	5	1215	2748	841	639
	6	1703	2757	733	566
	2	1315	2709	932	860
	3	1333	2709	897	789
9.50	4	1436	2711	852	719
	5	1149	2710	852	649
	6	1692	2734	738	575
	2	1300	2678	934	863
	3	1343	2679	897	795
9.75	4	1440	2682	853	726
	5	1115	2673	858	658
	6	1654	2708	747	584
	2	1291	2648	935	866
	3	1343	2650	899	799
10.00	4	1425	2653	856	732
	5	1106	2640	861	666
	6	1586	2681	760	594
	2	1282	2619	936	869
	3	1329	2620	901	804
10.25	4	1388	2624	862	738
	5	1115	2608	861	675
	6	1488	2652	778	603
	2	1269	2592	938	872
	3	1301	2592	904	808
10.50	4	1333	2595	869	744
	5	1135	2579	860	682
	6	1370	2620	798	613
	2	1250	2565	939	875
	3	1262	2565	908	813
10.75	4	1266	2567	877	750
	5	1162	2552	859	689
	6	1250	2588	818	622

ka	N	VA· 10^4	VC· 10^4	EA· 10^3	EC· 10^3
	2	1228	2538	941	878
	3	1220	2538	912	817
11.00	4	1201	2539	885	756
	5	1193	2528	857	696
	6	1143	2555	835	631
	2	1208	2513	943	880
	3	1184	2513	916	821
11.25	4	1148	2512	891	761
	5	1223	2505	855	702
	6	1061	2523	849	640
	2	1193	2489	944	883
	3	1161	2488	918	824
11.50	4	1112	2487	895	766
	5	1253	2484	853	708
	6	1008	2492	858	648
	2	1183	2465	945	885
	3	1149	2464	920	828
11.75	4	1093	2462	898	771
	5	1278	2463	851	713
	6	0980	2462	863	656
	2	1176	2442	946	888
	3	1149	2441	921	831
12.00	4	1094	2439	899	776
	5	1298	2443	851	719
	6	0973	2434	866	664
	2	1167	2419	947	890
	3	1154	2419	921	835
12.25	4	1103	2416	900	780
	5	1309	2423	851	724
	6	0980	2407	866	671
	2	1156	2397	948	892
	3	1160	2397	922	838
12.50	4	1119	2395	899	784
	5	1308	2403	853	729
	6	0997	2382	865	677
	2	1142	2376	949	894
	3	1163	2376	922	841
12.75	4	1137	2375	898	788
	5	1292	2383	856	734
	6	1019	2360	864	684

ka	N	VA·10^4	VC·10^4	EA·10^3	EC·10^3
	2	1126	2355	950	896
	3	1160	2355	923	844
13.00	4	1154	2355	898	791
	5	1259	2362	861	739
	6	1045	2339	861	689
	2	1111	2334	951	898
	3	1147	2335	925	846
13.25	4	1168	2335	898	795
	5	1210	2341	868	743
	6	1073	2320	859	695
	2	1100	2315	952	899
	3	1126	2315	927	849
13.50	4	1174	2316	898	799
	5	1151	2320	875	748
	6	1101	2302	857	700
	2	1092	2295	953	901
	3	1100	2295	929	852
13.75	4	1171	2297	899	802
	5	1089	2299	883	753
	6	1128	2286	855	705
	2	1086	2276	954	902
	3	1074	2277	931	854
14.00	4	1156	2278	901	806
	5	1032	2278	890	757
	6	1153	2270	853	709
	2	1079	2258	954	904
	3	1053	2258	933	857
14.25	4	1130	2260	904	809
	5	0989	2258	896	761
	6	1176	2255	851	714
	2	1069	2240	955	906
	3	1039	2240	935	859
14.50	4	1095	2241	908	812
	5	0961	2238	899	765
	6	1195	2241	850	718
	2	1057	2223	956	908
	3	1033	2223	936	861
14.75	4	1056	2223	912	815
	5	0947	2220	902	769
	6	1208	2227	849	722

ka	N	VA· 10^4	VC· 10^4	EA· 10^3	EC· 10^3
	2	1045	2206	957	909
	3	1033	2206	936	864
15.00	4	1019	2206	916	818
	5	0946	2202	903	773
	6	1215	2212	850	726
	2	1034	2189	958	911
	3	1036	2189	937	866
15.25	4	0990	2189	919	821
	5	0954	2185	903	777
	6	1212	2197	852	731
	2	1026	2173	958	912
	3	1039	2173	937	868
15.50	4	0972	2172	921	824
	5	0967	2169	902	780
	6	1196	2182	855	735
	2	1020	2157	959	913
	3	1039	2157	937	870
15.75	4	0965	2156	922	827
	5	0985	2153	901	783
	6	1168	2167	859	739
	2	1014	2141	960	914
	3	1035	2142	938	872
16.00	4	0966	2140	923	830
	5	1003	2139	900	787
	6	1126	2151	865	743
	2	1007	2126	960	916
	3	1024	2126	939	874
16.25	4	0973	2125	923	832
	5	1022	2125	899	790
	6	1074	2134	873	746
	2	0999	2111	961	917
	3	1008	2111	941	876
16.50	4	0983	2111	923	834
	5	1039	2111	898	793
	6	1016	2118	880	750
	2	0990	2097	961	918
	3	0990	2097	942	877
16.75	4	0995	2097	922	836
	5	1053	2097	897	796
	6	0960	2101	888	754

ka	N	$VA \cdot 10^4$	$VC \cdot 10^4$	$EA \cdot 10^3$	$EC \cdot 10^3$
	2	0980	2082	962	919
	3	0972	2082	944	879
17.00	4	1005	2083	922	839
	5	1062	2084	897	798
	6	0911	2085	894	758
	2	0971	2068	963	921
	3	0958	2068	945	881
17.25	4	1012	2069	922	841
	5	1064	2071	898	801
	6	0875	2068	899	762
	2	0965	2054	963	922
	3	0949	2054	946	882
17.50	4	1014	2055	923	843
	5	1059	2057	899	804
	6	0851	2053	903	765
	2	0959	2041	964	923
	3	0945	2041	946	884
17.75	4	1009	2042	924	845
	5	1044	2044	901	806
	6	0840	2038	905	768
	2	0954	2028	964	924
	3	0945	2028	947	885
18.00	4	0996	2028	925	847
	5	1020	2030	904	809
	6	0839	2023	905	772
	2	0949	2015	965	925
	3	0947	2015	947	887
18.25	4	0976	2015	927	849
	5	0988	2017	908	812
	6	0845	2009	905	775
	2	0942	2002	965	926
	3	0949	2002	947	889
18.50	4	0952	2002	929	851
	5	0952	2003	912	814
	6	0856	1996	905	778
	2	0934	1990	966	927
	3	0947	1990	948	890
18.75	4	0926	1989	932	853
	5	0916	1990	916	817
	6	0871	1984	904	781

ka	N	VA·10^4	VC·10^4	EA·10^3	EC·10^3
	2	0926	1977	966	928
	3	0942	1977	948	891
19.00	4	0904	1977	934	855
	5	0884	1977	919	819
	6	0888	1972	903	783
	2	0919	1965	967	929
	3	0933	1965	949	893
19.25	4	0886	1965	936	857
	5	0861	1964	922	821
	6	0905	1961	901	786
	2	0913	1953	967	929
	3	0921	1953	950	894
19.50	4	0876	1953	937	859
	5	0846	1952	924	824
	6	0923	1950	900	789
	2	0909	1942	967	930
	3	0907	1942	951	895
19.75	4	0872	1941	937	861
	5	0839	1940	925	826
	6	0940	1940	899	791
	2	0904	1930	968	931
	3	0895	1930	952	897
20.00	4	0874	1930	938	863
	5	0840	1929	925	828
	6	0955	1930	898	793
	2	0899	1919	968	932
	3	0885	1919	953	898
20.25	4	0879	1919	938	864
	5	0847	1917	925	830
	6	0968	1920	897	796
	2	0893	1908	969	933
	3	0879	1908	954	899
20.50	4	0887	1908	938	865
	5	0857	1907	925	832
	6	0978	1910	897	798
	2	0886	1897	969	934
	3	0876	1897	954	900
20.75	4	0894	1897	937	867
	5	0869	1896	924	834
	6	0984	1900	897	800

ka	N	$VA \cdot 10^4$	$VC \cdot 10^4$	$EA \cdot 10^3$	$EC \cdot 10^3$
	2	0880	1887	969	934
	3	0876	1887	954	901
21.00	4	0901	1887	937	869
	5	0882	1886	923	836
	6	0984	1890	897	803
	2	0874	1876	970	935
	3	0877	1876	954	903
21.25	4	0904	1877	937	870
	5	0894	1876	923	838
	6	0978	1880	898	805
	2	0869	1866	970	936
	3	0878	1866	955	904
21.50	4	0904	1866	938	872
	5	0904	1866	922	839
	6	0964	1870	900	807
	2	0865	1856	970	936
	3	0876	1856	955	905
21.75	4	0898	1856	939	873
	5	0912	1856	922	841
	6	0943	1860	903	809

3.5 Array of Axial and Circumferential Slots

An elliptically polarized far field pattern in the azimuthal plane may easily be obtained by alternating N uniformly spaced axial slots and N uniformly spaced circumferential slots around the cylinder. It is not necessary to assume that the two angles between the center of a circumferential slot and the centers of the two adjacent axial slots, as seen from the cylinder axis, be equal; we shall only assume that the angle between two adjacent axial slots, or between two adjacent circumferential slots, is $2\pi/N$ radians (Fig. 3-7).

From formulas (3.11),(3.12),(3.24) and (3.25) it follows that

$$\frac{V_a}{V_c} = \frac{1-EA}{1-EC}\,\frac{H_o^{(1)'}(ka)}{H_o^{(1)}(ka)}\,e^{-i\frac{\pi}{2}}\,\frac{E_a}{E_c}\;. \tag{3.28}$$

If the azimuthal patterns (3.13) and (3.26) produced by the axial slots and by the circumferential slots are sensibly omnidirectional, then relation (3.28) gives the ratio of the two feeding voltages as a function of the far field polarization, for prescribed values of N and ka.

For a linearly polarized far field, the quantity E_a/E_c is real. For a circularly polarized far field, one must choose

$$E_a/E_c = e^{\pm i\frac{\pi}{2}}\;,$$

and therefore

$$\left[\frac{V_a}{V_c}\right]_{\text{cir. pol.}} = \pm\,\frac{1-EA}{1-EC}\,\frac{\partial}{\partial(ka)}\ln H_o^{(1)}(ka)\;. \tag{3.29}$$

In the case of a large cylinder, further simplification is achieved by observing

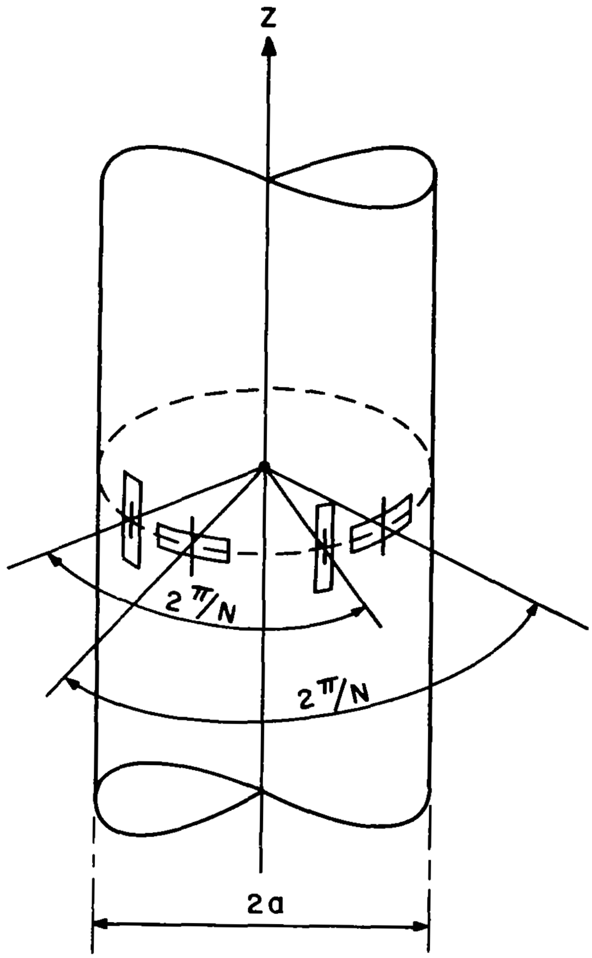


FIG 3-7: ARRAY OF AXIAL AND CIRCUMFERENTIAL SLOTS

that

$$\frac{H_o^{(1)'}(ka)}{H_o^{(1)}(ka)} \; e^{-i\frac{\pi}{2}} \sim 1 \; , \qquad (ka >> 1) \; ;$$

in this case:

$$\frac{V_a}{V_c} \sim \frac{1-EA}{1-EC} \, \frac{E_a}{E_c} \; , \qquad (ka >> 1) \; . \qquad (3.30)$$

3.6 Final Considerations

The main results of Section III may be summarized as follows.

a). If all the feeding voltages have the same amplitude and phase, then the best mean-square approximation to an omnidirectional azimuthal field pattern is achieved; the actual optimum pattern is given by formulas (3.13) and (3.26).

b). The feeding voltage as a function of frequency, cylinder radius, number of slots and far field intensity is given by formulas (3.12) and (3.25), and is tabulated in Section 3.4 and plotted in Figs. 3-2 and 3-5.

c). A far azimuthal field with a prescribed polarization may be obtained through formula (3.28), provided that the actual patterns are sensibly omnidirectional.

The optimization process that was used in this section is based on the minimization of the mean squared error (3.8). In order to have some information on the features of the patterns thus obtained, computations of $P(\phi)$ as given by formula (3.26) were carried out for the case of ka = 10 and five circumferential slots. The results are plotted in Fig. 3-8; it is seen that the amplitude of the field pattern is quite far from unity.

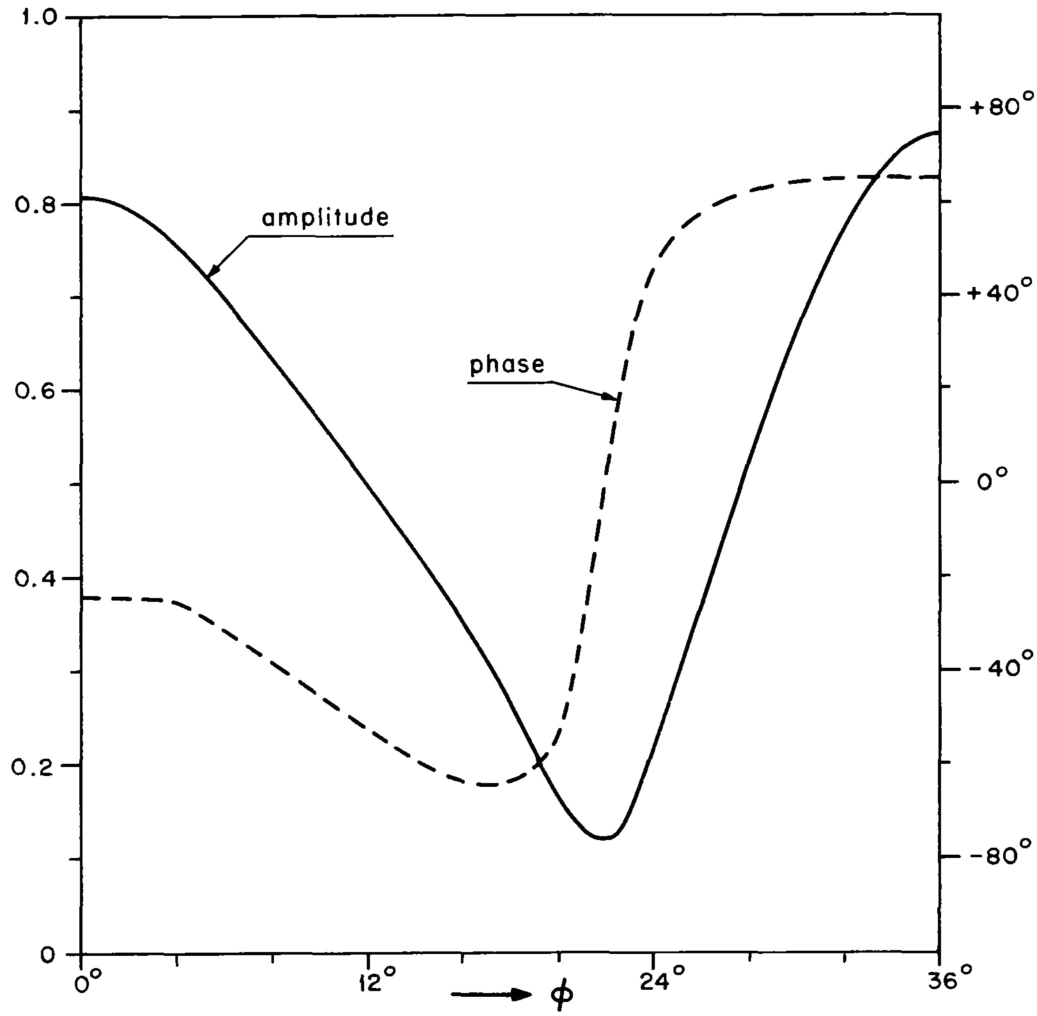


FIG. 3-8: AMPLITUDE AND PHASE OF $\left[P(\phi)\right]_{opt.}$ FOR ka=10 AND FIVE CIRCUMFERENTIAL SLOTS

If one is interested only in approximating the power radiation pattern, then one must conclude that the minimization of the error (3. 8) does not constitute a good criterion of optimization. In this case, one should try to minimize either

$$\epsilon = \frac{1}{2\pi} \int_0^{2\pi} \left| e^{i\Phi(\phi)} - P(\phi) \right|^2 d\phi \tag{3. 31}$$

where $\Phi(\phi)$ is a continuous real function of ϕ to be chosen so that the mean squared error between $|P(\phi)|$ and unity be minimum, or the mean squared error between the actual and the preassigned power patterns:

$$\epsilon_{power} = \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 - |P(\phi)|^2 \right\}^2 d\phi \quad . \tag{3. 32}$$

It appears that the mathematical difficulties encountered in minimizing ϵ_{power} cannot be overcome easily, so that it seems preferable to minimize the right-hand side of (3. 31), e. g. by successive approximations. One may choose:

$$\epsilon_m = \frac{1}{2\pi} \int_0^{2\pi} \left| P_m(\phi) - \frac{P_{m-1}(\phi)}{|P_{m-1}(\phi)|} \right|^2 d\phi, \qquad (m=1, 2, \cdots) \tag{3. 33}$$

with $P_o(\phi) = 1$. and determine the unknown feeding voltages which appear in the expression of $P_m(\phi)$ so as to minimize ϵ_m. The iteration procedure (3. 33) can easily be handled by a computer; however, it remains to be proven that ϵ_m converges to a minimum value of ϵ as m increases. Numerical results based on the approximation procedure (3. 33) have been obtained in Section II for the problem treated there.

APPENDIX

Lemma

For every function $A(\phi)$, square integrable in the interval $-\alpha < \phi \leqslant \alpha$ (notation: $A(\phi) \in L_2^\alpha$) the corresponding function $P(\phi)$ defined by

$$P(\phi) = \int_{-\alpha}^{\alpha} K(\phi - \phi')\, A(\phi')\, d\phi' \;, \qquad -\pi < \phi \leqslant \pi, \quad \alpha \leqslant \pi \tag{A.1}$$

where

$$K(\phi) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\epsilon_n \cos n\phi}{i^n H_n^{(1)}(ka)} \\[2ex] \displaystyle\sum_{n=0}^{\infty} \frac{\epsilon_n \cos n\phi}{i^n H_n^{(1)'}(ka)} \end{cases} \qquad \epsilon_o = 1, \; \epsilon_1 = \epsilon_\eta = \ldots = 2 \tag{A.2}$$

is an analytic function.

We first note that by introduction of the variable $\zeta = e^{i\phi}$, the Fourier series in Eq. (A.2) can be considered as Laurent series which are convergent in every region in the complex ζ-plane defined by $0 < a \leqslant |\zeta| \leqslant b < \infty$, where a, b are positive constants. This is a consequence of the asymptotic behavior of $H_n^{(1)}(ka)$ and $H_n^{(1)'}(ka)$ for ka fixed and $n \to \infty$ through real positive values:

$$\begin{aligned} H_n^{(1)}(ka) &\underset{n\to\infty}{\sim} -i\sqrt{\frac{2}{\pi n}} \left(\frac{2n}{e\,ka}\right)^n \\ H_n^{(1)'}(ka) &\underset{n\to\infty}{\sim} i\sqrt{\frac{2}{\pi n}} \left(\frac{2n}{e\,ka}\right)^{n+1} \end{aligned} \tag{A.3}$$

The Laurent series in ζ represents an analytic function in its region of convergence and, since ζ is an analytic function of ϕ, $K(\phi)$ is a function analytic in the entire complex ϕ-plane (cf. Whittaker and Watson, 1927, p. 160 ff.). From this it follows that $P(\phi)$ is also an analytic function because it has the unique derivative

$$\int_{-\alpha}^{\alpha} \frac{\partial}{\partial \phi} K(\phi - \phi') A(\phi') d\phi' .$$

Theorem

If we take an arbitrary set of functions $\{\psi_n(\phi)\}$ complete in L_2^{α} and construct a new set of functions $\{\Pi_n(\phi)\}$, where $\Pi_n(\phi)$ is the $P(\phi)$ in Eq. (A.1) which corresponds to $A(\phi) = \psi_n(\phi)$, then the set $\{\Pi_n(\phi)\}$ will be closed in L_2^{π}. That is, there is no function belonging to L_2^{π} which is orthogonal to all $\Pi_n(\phi)$.

To prove the theorem, we assume that the contrary is true, i.e. that there is a function $F(\phi) \in L_2^{\pi}$ such that for every n

$$\int_{-\pi}^{\pi} F^*(\phi)\Pi_n(\phi)d\phi = \int_{-\pi}^{\pi} F^*(\phi)d\phi \int_{-\alpha}^{\alpha} K(\phi - \phi')\psi_n(\phi')d\phi' = 0 \qquad \text{(A.4)}$$

where F^* is the complex conjugate of F. It is obvious, by virtue of Fubini's theorem, that we can change the order of integration in Eq. (A.4), and using the fact that according to Eq. (A.2), $K(\phi)$ is an even function we obtain

$$\int_{-\alpha}^{\alpha} \psi_n(\phi')d\phi' \int_{-\pi}^{\pi} F^*(\phi)K(\phi' - \phi)d\phi = \int_{-\alpha}^{\alpha} \psi_n(\phi')G(\phi')d\phi' = 0 \qquad \text{(A.5)}$$

where $G(\phi)$ is thus defined as

$$G(\phi) = \int_{-\pi}^{\pi} K(\phi - \phi')F^{*}(\phi')d\phi' \quad , \qquad -\pi < \phi \leqslant \pi \; . \tag{A.6}$$

As $\{\psi_n(\phi)\}$ is complete in L_2^{α}, Eq. (A.5) can hold only if $G(\phi) \equiv 0$ almost everywhere in the interval $-\alpha < \phi \leqslant \alpha$. But due to Eq. (A.6), $G(\phi)$ satisfies the conditions in the Lemma and is consequently analytic in the interval $-\pi < \phi \leqslant \pi$ and thus $G(\phi) \equiv 0$ in this whole interval. Eq. (A.6) also expresses $G(\phi)$ as the convolution of the functions $K(\phi)$ and $F^{*}(\phi)$. Thus $G(\phi) \equiv 0$ implies that $F^{*}(\phi)$ vanishes almost everywhere, which proves the theorem.

In L_2 completeness and closure are equivalent and we have the following corollary.

<u>Corollary</u>

To every pair of set of functions $\{\psi_n(\phi)\}$ and $\{\Pi_n(\phi)\}$ as defined in the theorem there is a finite set of functions $\{\alpha_n\}_0^N$ such that

$$\int_{-\pi}^{\pi} \left| F(\phi) - \sum_{n=0}^{N} \alpha_n \Pi_n(\phi) \right|^2 d\phi < \epsilon \tag{A.7}$$

for any given arbitrary function $F(\phi) \in L_2^{\pi}$ and $\epsilon > 0$.

This means that if we consider Eq. (A.1) as an integral equation with $P(\phi) = F(\phi)$ there is no solution $A(\phi) \in L_2^{\alpha}$ except when $F(\phi)$ belongs to a certain class of analytic functions. However, we can always find an

$$A(\phi) = \sum_{n=0}^{N} \alpha_n \psi_n(\phi) \in L_2^{\alpha}$$

such that the corresponding

$$P(\phi) = \sum_{n=0}^{N} \alpha_n \Pi_n(\phi)$$

approximates $F(\phi)$ arbitrary close in the mean square sense.

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