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CONTROLLABILITY OF NONLINEAR SYSTEMS

by George W. Haynes

Prepared under Contract No. NAS 2-2351 *by*
MARTIN MARIETTA CORPORATION
Denver, Colo.
for Ames Research Center

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Controllability of Nonlinear Systems*

Summary

This report outlines the method of Hermann for the determination of the controllability of linear and nonlinear control systems. Hermann's method yields an algebraic criterion for the controllability of linear systems with time varying coefficients which supercedes Kalman's integral form, since it does not require knowledge of the fundamental solution of the homogeneous system.

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1. Complete Controllability of Linear Systems

In this section we shall review some of the mathematical consequences of the concept of complete controllability. We take the approach of Hermes[1] by considering a linear system of the form

$$\dot{x}_\alpha = H_{\alpha r}(t)u_r \quad (\alpha = 1 \dots n;\ r = 1 \dots m) \qquad 1.1$$

which, by virtue of a linear transformation to be described later, encompasses all linear systems. Let the n x n matrix $M(t_o, t_1)$ be defined by

$$M_{\alpha\beta}(t_o, t_1) = \int_{t_o}^{t_1} H_{\alpha r}(t) H_{\beta r}(t)\,dt \qquad 1.2$$

then we have the following theorem due to Hermes which in turn is a special case of the theorem due to Kalman[2].

Theorem 1.1

A necessary and sufficient condition for the linear system (1.1) to be completely controllable at t_o, is that there exists at $t_1 > t_o$ such that $M(t_1, t_o)$ is nonsingular.

Proof.

The following proof is essentially that given by Hermes[1]. The sufficiency of the theorem follows from the fact that if $x(t_1)$ is any point in E^n (Enclidean n space) attainable from $x(t_o)$ in time t_1, then if we select

$$u_r(t) = H_{\beta r}(t)\,\xi_\beta \qquad 1.3$$

with $\xi \in E^n$, we obtain

$$x_\alpha(t_1) - x_\alpha(t_o) = \int_{t_o}^{t_1} H_{\alpha r}(t)\, H_{\beta r}(t)dt$$

$$= M_{\alpha\beta}(t_o,t_1)\,\xi_\beta \qquad 1.4$$

Since the matrix $M(t_o,t_1)$ is nonsingular we can solve equation (1.4) for the constant vector ξ and hence by equation (1.3) determine the control that achieves the desired transfer of the state vector x. The necessity of the theorem follows from the fact that if $M(t_o,t_1)$ is singular, then the linear system (1.1) is not completely controllable. If $M(t_o,t_1)$ is singular, then there exists a nontrivial vector $C(t_o,t_1)$ such that

$$C_\alpha(t_o,t_1)\int_{t_o}^{t_1} H_{\alpha r}(t)\, H_{\beta r}(t)dt = 0 \qquad 1.5$$

Multiplying each equation (1.5) by $C_\beta(t_o,t_1)$ and summing yields

$$C_\beta(t_o,t_1)C_\alpha(t_o,t_1)\int_{t_o}^{t_1} H_{\beta r}(t)H_{\beta r}(t)dt$$

$$=\int_{t_o}^{t_1} \left[C_\alpha(t_o,t_1)H_{\alpha r}(t)\right]^2 dt = 0$$

so that

$$C_\alpha(t_o,t_1)H_{\alpha r}(t) \overset{t}{\equiv} 0 \qquad 1.6$$

The dependence of the vector $C(t_o,t_1)$ on t_o and t_1 is immaterial since the identity given by equation (1.6) holds in t for any t_1 and t_o $(t_1 \neq t_o)$. Therefore, we can conclude that if the matrix $M(t_o,t_1)$ is singular, then there exists a constant vector C such that

$$C_\alpha H_{\alpha r}(t) \overset{t}{\equiv} 0 \qquad 1.7$$

If the matrix $M(t_o,t_1)$ is singular and the linear system (1.1) is completely controllable, then we can assume that there exists a control u(t) which transfers the state vector from $x(t_o)$ to $x(t_1) = C$, so that

$$C_\alpha = x_\alpha(t_o) + \int_{t_o}^{t_1} H_{\alpha r}(t)u_r(t)dt$$

Since the initial state $x(t_o)$ is arbitrary, let it be chosen so that $C \cdot x(t_o) = C_\alpha x_\alpha(t_o) = 0$. Therefore we have

$$C^2 = C_\alpha^2 = C_\alpha x_\alpha(t_o) + \int_{t_o}^{t_1} C_\alpha H_{\alpha r}(t)u_r(t)dt = 0$$

which contradicts the fact that C is a nontrivial vector, and completes the proof.

For more general linear systems we have the following theorem due to Kalman[2].

Theorem 1.2

The linear system

$$\dot{x}_\alpha(t) = A_{\alpha\beta}(t)x_\beta(t) + B_{\alpha r}(t)u_r(t) \qquad 1.8$$

is completely controllable at t_o, if and only if the $n \times n$ matrix

$$\int_{t_o}^{t_1} \Phi_{\alpha\gamma}(t_o,t)B_{\gamma r}(t)\Phi_{\beta\tau}(t_o,t)B_{\tau r}(t)dt$$

is nonsingular for some $t_1 > t_o$.

Here $\Phi(t, \tau)$ denotes a fundamental solution of the homogenous system

$$\dot{x}_\alpha(t) = A_{\alpha\beta}(t)x_\beta(t).$$

Proof.

If we transform the linear system (1.8) by

$$x_\alpha(t) = \Phi_{\alpha\beta}(t,t_o)y_\beta(t) \qquad 1.9$$

and note that

$$\Phi_{\alpha\beta}(t,t_o)\,\Phi_{\beta\gamma}(t_o,t) = \delta_{\alpha\gamma}$$

where δ is the Kronecker delta, then we obtain

$$\dot{y}_\alpha(t) = \Phi_{\alpha\beta}(t_o,t)B_{\beta r}(t)u_r(t) \qquad 1.10$$

which is equivalent to the system (1.1). By virtue of the transformation (1.9) being nonsingular, it follows that the linear system (1.8) is completely controllable, if and only if, the linear system (1.10) is completely controllable. Hence applying the results of Theorem 1.1 to the linear system 1.10 completes the proof.

If the matrices A(t) and B(t) associated with the linear system (1.8) are constant matrices, then the test for controllability reduces to one requiring that the rank of the $n \times mn$ matrix, expressed in matrix notation as $\left[B, AB, A^2B, .., A^{n-1}B\right]$, be n.

This requirement was first used by Pontryagin[3] as an assumption in the study of minimal time control systems, and by LaSalle[4] to describe "proper" control systems.

When the matrices A and B are time dependent the test for controllability as given by Theorem 1.2 is not a useful test since it depends on the knowledge of the fundamental solution. However, there does exist an algebraic test for the controllability of the linear system (1.8) which does not require knowledge of the fundamental solution. This method which is due to Hermann[5] applies immediately to nonlinear systems. Before we elucidate this method, we shall review some of the elementary properties of homogeneous and nonhomogeneous systems of partial differential equations pertinent to Hermann's approach.

2. Systems of Nonhomogeneous Linear Partial Differential Equations

Consider the system of partial differential equations

$$\frac{\partial x_\alpha}{\partial y_i} = \psi_{\alpha i}(x_1, \dots, x_n;\ y_1, \dots, y_r)$$

$$= \psi_{\alpha i}(x;y) \qquad (\alpha = 1 \dots n;\ i = 1 \dots r) \tag{2.1}$$

where the ψ's are analytic functions of x and y in some domain $D \subset E^n \times E^r$.

These equations (2.1) are equivalent to the system of pfaffian equations

$$dx_\alpha = \Psi_{\alpha i}(x;y)dy_i \tag{2.2}$$

The conditions of integrability for the system of partial differential equations (2.1) are

$$\frac{\partial \Psi_{\alpha i}}{\partial y_j} + \frac{\partial \Psi_{\alpha i}}{\partial x_\gamma} \Psi_{\gamma j} = \frac{\partial \Psi_{\alpha j}}{\partial y_i} + \frac{\partial \Psi_{\alpha j}}{\partial x_\gamma} \Psi_{\gamma i} \tag{2.3}$$

If these equations are satisfied identically, the system (2.1) is termed completely integrable, and the solution can be constructed in terms of a power series about the point $(\bar{x},\bar{y})$; namely,

$$x_\alpha = \bar{x}_\alpha + \left(\frac{\partial x_\alpha}{\partial y_i}\right)_o (y_i - \bar{y}_i) + \frac{1}{2}\left(\frac{\partial^2 x_\alpha}{\partial y_i \partial y_j}\right)_o (y_i - \bar{y}_i)(y_i - \bar{y}_j) \ldots$$

where

$$\left(\frac{\partial x_\alpha}{\partial y_i}\right)_o = \Psi_i(\bar{x};\bar{y})$$

and the higher derivatives are constructed accordingly. We shall denote this solution by

$$x_\alpha = \phi_\alpha(\bar{x};y - \bar{y}) \tag{2.4}$$

These equations may be regarded as defining a group of transformations of the vector $x \in E^n$ with the components of the $(y - \bar{y})$ vector regarded as r parameters. This interpretation, as will be developed later, has a useful application in the representation of completely controllable systems.

If the system of equations (2.3) is not satisfied, then the partial differential equations (2.1) are not completely integrable. In this case the equations (2.3) would define certain relationships between the x's and the y's, which have to be satisfied together with the system of partial differential equations (2.1). Such systems are called mixed systems, which we shall not pursue any further since our primary interest is concerned with completely integrable systems. Since the functions ϕ defined by (2.4) are such that

$$\bar{x}_\alpha = \phi_\alpha(\bar{x};0)$$

it follows that the Jacobian of the transformation is different from zero in a neighborhood of $\bar{x}$ so that the equations may be solved for $\bar{x}$ to yield

$$\bar{x}_\alpha = f_\alpha(x;y-\bar{y}) \qquad 2.5$$

We note that each component of f is an integral of the system (2.1) and consequently any scalar function of f is also an integral. Since the substitution for x by ϕ in f yields the identity in $\bar{x}$ and $y-\bar{y}$, it follows that the components of f are solutions of the system of r linear homogeneous differential equations

$$\frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial x_\alpha}\Psi_{\alpha i}(x;y) = 0 \qquad 2.6$$

This sytem is often referred to as the system associated with (2.1). A standard result[6] concerning these systems, which we shall state without proof, is:

Theorem 2.1

Given a completely integrable system (2.1) and the associated system (2.6), then if n independent solutions of the associated system (2.6) are equated to arbitrary constants, they define implicitly a solution of (2.1), and conversely a solution of (2.1) determines n independent solutions of the associated system (2.6).

Another important concept regarding the integrals (2.5) is whether the parameters are essential. The parameters y_p explicit in f(x;y) are defined to be essential if it is not possible to determine (r-1) scalar functions of y, denoted by A(y), such that

$$f_{\alpha}(x_1,\ldots,x_n;y_1,\ldots,y_r) \overset{x,y}{\equiv} F_{\alpha}(x_1,\ldots,x_n;A_1(y)\ \ldots\ A_{r-1}(y)) \qquad 2.7$$

If the parameters y_p are not essential, then the transformation (2.5) can be represented with no loss of generality in terms of (r-1) parameters by (2.7), and identical arguments apply to the new parameters for them to be essential. One test[6] for the determination of essential parameters is the following.

Theorem 2.2

A necessary and sufficient condition that the r parameters be essential is that the functions $f_{\alpha}(x;y)$ do not satisfy an equation of the form

$$\lambda_p(y) \frac{\partial f}{\partial y_p} (x:y) \equiv 0 \qquad (p = 1 \ldots r) \qquad 2.8$$

One consequence of this result for $r \leq n$ is that the rank of the Jacobian $\left(\frac{\partial f}{\partial y}\right)$ be r.

3. Systems of Linear Homogeneous Partial Differential Equations

Consider the set of linear operators X on f describing a system of partial differential equations

$$X_a f \equiv A_{ai}(x) \frac{\partial f}{\partial x_i}(x) = 0 \ (a=1,\ldots,p;\ i = 1,\ldots,N;\ p \leq N) \qquad 3.1$$

From the definition of the operator X we have

$$X_a X_b f \equiv A_{ai}(x) \frac{\partial}{\partial x_i} \left(A_{bj}(x) \frac{\partial f}{\partial x_j}(x)\right)$$

$$= A_{ai}(x) \frac{\partial A_{bj}}{\partial x_i}(x) \frac{\partial f}{\partial x}(x) + A_{ai}(x) A_{bj}(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

and hence it follows that

$$(X_a X_b - X_b X_a) f \equiv \left(A_{aj}(x) \frac{\partial A_{bi}}{\partial x_j}(x) - A_{bj}(x) \frac{\partial A_{ai}(x)}{\partial x_j}\right) \frac{\partial f(x)}{\partial x_i} \qquad 3.2$$

The operator defined by

$$(X_a, X_b) f \equiv (X_a X_b - X_b X_a) f$$

is called the Poisson operator, or the commutator of the operators $X_a f$ and $X_b f$. Some other properties of the Poisson operator are:

$$(X_a,X_b)f = -(X_b,X_a)f$$

and the Jacobi identity

$$((X_a,X_b),X_c)f + ((X_b,X_c),X_a)f + ((X_c,X_a),X_b)f = 0$$

Consider the system of homogeneous linear partial differential equations defined by (3.1) for which the rank of the matrix A is p for all $x \in D$; that is, the equations (3.1) are independent. It is immediately evident, by virtue of the independence of the equations (3.1), that if $p = N$ then the only solution possible is a trivial one, namely $f \equiv$ constant. It is possible for a nontrivial solution to exist if $p < N$. From equation (3.2) it follows that any solution of equation (3.1) also satisfies

$$(X_a,X_b)f = 0 \qquad (a,b = 1,\ldots,p) \tag{3.3}$$

If there exist functions γ_{abc} such that

$$(X_a,X_b)f \equiv \gamma_{abc} X_c f \quad (a,b,c = 1,\ldots,p) \tag{3.4}$$

then the Poisson operator does not yield any new partial differential equations, so that the system (3.1) is called a complete system of order p. On the other hand, if there are some commutators that are not expressible in the form (3.4), then these commutators equated to zero represent additional independent partial differential equations which must be satisfied, and accordingly are adjoined to the system (3.1). This process is continued until we obtain either a set N of independent equations, in which case only a trivial solution is possible, or we obtain finally a set $s(s < N)$ of independent equations. In this case the system (3.1) is a complete system of order s, which we shall denote by

$$X_a f = A_{ai}(x) \frac{\partial f}{\partial x_i}(x) = 0 \quad (a = 1 \dots s;\ i = 1 \dots N;\ s < N) \qquad 3.5$$

Since the rank of the matrix A is s, we may express the system (3.5) in Jacobian form

$$\frac{\partial f}{\partial x_a} + \frac{\partial f}{\partial x_\beta} \Psi_{\beta a}(x) = 0 \quad (a = 1, \dots, s;\ \beta = s + 1, \dots N) \qquad 3.6$$

Applying the Poisson operator to these equations we obtain

$$\left(\Psi_{\gamma a} \frac{\partial \Psi_{\beta b}}{\partial x_\gamma} + \frac{\partial \Psi_{\beta b}}{\partial x_a} - \Psi_{\gamma b} \frac{\partial \Psi_{\beta a}}{\partial x_\gamma} - \frac{\partial \Psi_{\beta a}}{\partial x_b} \right) \frac{\partial f}{\partial x_\beta} = 0$$

$$(a, b = 1 \dots s;\ \beta, \gamma = s+1 \dots N) \qquad 3.7$$

In these equations, the derivatives $\frac{\partial f}{\partial x_a}$ $(a = 1 \dots s)$ do not appear, so that the system (3.7) represents a new set of independent equations which contradicts the assumption that the system (3.5) is a complete system of order s. Therefore, in accordance with the completeness assumption we must have

$$\Psi_{\gamma a} \frac{\partial \Psi_{\beta b}}{\partial x_\gamma} + \frac{\partial \Psi_{\beta b}}{\partial x_a} = \Psi_{\gamma b} \frac{\partial \Psi_{\beta a}}{\partial x_\gamma} + \frac{\partial \Psi_{\beta a}}{\partial x_b}$$

Comparing these equations with (2.3), we observe that the system of nonhomogeneous partial differential equations defined by

$$\frac{\partial x_\beta}{\partial x_a} = \Psi_{\beta a}(x) \quad (a = 1 \ldots s; \beta = s+1 \ldots N) \qquad 3.8$$

are completely integrable. In fact the system (3.6) is the system associated with the system (3.8). From Theorem 2.1 we have immediately:

Corollary 3.1

A complete system of s homogeneous linear partial differential equations of the first order in $N(>s)$ variables admits exactly $(N-s)$ independent solutions.

4. Integrability Conditions for a Single Pfaffian

The purpose of this section is to review a special result for a single pfaffian, which has a useful application for those control systems that give rise to a single pfaffian; or the form of the integrating factors can be anticipated when dealing with a pfaffian system. This result[7] is contained in the following:

Theorem 4.1

A necessary and sufficient condition for the pfaffian $A_\alpha(x)dx_\alpha = 0$ $(\alpha = 1\ldots n)$ to be integrable is that

$$\underline{X} \cdot \text{curl}\, \underline{X} \overset{x}{\equiv} 0$$

holds for every three vector $\underline{X}$ whose components are A_α, A_β and A_γ, and the curl is evaluated for the corresponding coordinates x_α, x_β and x_γ.

Proof.

The pfaffian $A_\alpha(x)dx_\alpha = 0$ is integrable if and only if there exists a nontrivial integrating factor $\mu(x)$ and a scalar function V(x) such that

$$\frac{\partial V(x)}{\partial x_\alpha} \overset{x}{\equiv} \mu(x)\, A_\alpha(x) \qquad 4.1$$

Forming the cross derivative of V and under the assumption of certain continuity properties, we obtain

$$\frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} = \frac{\partial \mu(x)}{\partial x_\beta} A_\alpha(x) + \mu(x) \frac{\partial A_\alpha(x)}{\partial x_\beta} = \frac{\partial \mu(x)}{\partial x_\alpha} A_\beta(x) + \mu(x) \frac{\partial A_\beta}{\partial x_\alpha}(x)$$

Rearranging terms yields

$$\frac{\partial \mu(x)}{\partial x_\alpha} A_\beta(x) - \frac{\partial \mu(x)}{\partial x_\beta} A_\alpha(x) = \mu(x)\left[\frac{\partial A_\alpha}{\partial x_\beta}(x) - \frac{\partial A_\beta}{\partial x_\alpha}(x)\right] \qquad 4.2$$

Performing this process for the combinations A_β, A_γ and A_γ, A_α yields

$$\frac{\partial \mu(x)}{\partial x_\beta} A_\gamma(x) - \frac{\partial \mu(x)}{\partial x_\gamma} A_\beta(x) = \mu(x)\left[\frac{\partial A_\beta(x)}{\partial x_\gamma} - \frac{\partial A_\gamma(x)}{\partial x_\beta}\right] \qquad 4.3$$

and

$$\frac{\partial \mu(x)}{\partial x_\gamma} A_\alpha(x) - \frac{\partial \mu(x)}{\partial x_\alpha} A_\gamma(x) = \mu(x)\left[\frac{\partial A_\gamma(x)}{\partial x_\alpha} - \frac{\partial A_\alpha(x)}{\partial x_\gamma}\right] \qquad 4.4$$

Multiplying equation (4.2) by A_γ, equation (4.3) by A_α, and equation (4.4) by A_β and summing yields

$$\left[A_\alpha\left(\frac{\partial A_\beta}{\partial x_\gamma} - \frac{\partial A_\gamma}{\partial x_\beta}\right) + A_\beta\left(\frac{\partial A_\gamma}{\partial x_\alpha} - \frac{\partial A_\alpha}{\partial x_\gamma}\right) + A_\gamma\left(\frac{\partial A_\alpha}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\alpha}\right)\right] \overset{x}{\equiv} 0 \qquad 4.5$$

Since the integrating factor is nontrivial, it may be neglected. This expression must hold for each distinct combination A_α, A_β and A_γ for the pfaffian to be integrable. If we define the three vector $\underline{X}$ with components A_α, A_β, and A_γ, then the integrability condition (4.5) may be expressed in the succinct form

$$\underline{X} \cdot \text{curl } \underline{X} \overset{x}{\equiv} 0$$

which completes the proof.

Let Ψ_r (r = 1 ... (n-1)) be a maximal set of vectors orthogonal to A, that is

$$\Psi_{r\alpha}(x)A_\alpha(x) \overset{x}{\equiv} 0 \qquad 4.6$$

Then by virtue of equation (4.1) we can associate with the pfaffian the system of linear honogeneous equations

$$\Psi_{r\alpha}(x) \frac{\partial V(x)}{\partial x_\alpha} = 0 \qquad 4.7$$

Geometrically speaking, if the pfaffian is integrable, the vector A determines a normal direction and the vectors Ψ_r determine tangent directions at each point of the hypersurface V(x) = constant.

The integrability conditions for the pfaffian can now be couched in terms of completeness of the system of partial differential equations (4.7).

Theorem 4.2

If the pfaffian $A_\alpha(x)dx_\alpha = 0$ is integrable, then the system of (n - 1) partial differential equations

$$\Psi_{r\alpha}(x)\frac{\partial V}{\partial x_\alpha} = 0$$

is a complete system of order (n - 1).

Proof.

Applying the Poisson operator to the p^th and the r^th equations yields

$$\left[\Psi_{p\gamma}\frac{\partial\Psi_{r\alpha}}{\partial x_\gamma} - \Psi_{r\gamma}\frac{\partial\Psi_{p\alpha}}{\partial x_\gamma}\right]\frac{\partial V}{\partial x_\alpha} = 0 \qquad 4.8$$

Since V must satisfy this equation, then it must be some linear combination of the partial differential equations (4.7), otherwise V would not exist and hence by (4.1) the pfaffian would not be integrable. Therefore, it follows that there exist functions $\emptyset$ such that

$$\Psi_{p\gamma}\frac{\partial\Psi_{r\alpha}}{\partial x_\gamma} - \Psi_{r\gamma}\frac{\partial\Psi_{p\alpha}}{\partial x_\gamma} \overset{x}{\equiv} \emptyset_\ell\Psi_{\ell\alpha}$$

By virtue of equations (4.6) these conditions can be expressed as

$$\left(\Psi_{p\gamma}\frac{\partial\Psi_{r\alpha}}{\partial x_\gamma} - \Psi_{r\gamma}\frac{\partial\Psi_{p\alpha}}{\partial x_\gamma}\right)A_\alpha \overset{x}{\equiv} 0 \qquad 4.9$$

$$(p, r = 1 \ldots (n-1) : \alpha, \gamma = 1 \ldots n)$$

Since equation (4.6) is an identity in x, then differentiating it with respect to x_γ, and using this result to simplify (4.9), gives

$$\left(\psi_{p\gamma}\psi_{r\alpha}-\psi_{r\gamma}\psi_{r\alpha}\right)\frac{\partial A_\alpha}{\partial x_\gamma}=0$$

or

$$\psi_{p_\gamma}\psi_{r_\alpha}\left(\frac{\partial A_\alpha}{\partial x_\gamma}-\frac{\partial A_\gamma}{\partial x_\alpha}\right)=0$$

Defining the skew-symmetric matrix w by

$$w_{\alpha\gamma}=\frac{\partial A_\alpha}{\partial x_\gamma}-\frac{\partial A_\gamma}{\partial x_\alpha} \qquad 4.10$$

then the above result assumes the simple form

$$\psi_{p\gamma}\psi_{r\alpha}w_{\alpha\beta}=0 \qquad 4.11$$

This result proved for p and r distinct, also holds when $p=r$ by virtue of the skew-symmetry of the matrix w. Therefore, since the $(n-1)$ independent vectors $\psi_{p\gamma}$ are orthogonal to the vector $\psi_{r\alpha}w_{\alpha\gamma}$ by (4.11), it follows that

$$\psi_{r\alpha}w_{\alpha\gamma}=f_r\,A_\gamma \qquad 4.12$$

where f is an $(n-1)$ component vector of functions which have to be determined. Multiplying equation (4.12) by A_γ and summing determines the components of the vector f as

$$f_r=\frac{\psi_{r\alpha}w_{\alpha\gamma}A_\gamma}{A^2}$$

where A^2 denotes the scalar (or inner) product of the vector A. Substituting for f in equation (4.12) yields

$$\psi_{r\alpha}\left[w_{\alpha\gamma}-\frac{w_{\alpha\tau}A_\tau}{A^2}A_\gamma\right]=0 \qquad 4.13$$

Once again we apply an orthogonality argument to this result and deduce that

$$w_{\alpha\gamma} - \frac{w_{\alpha\tau} A_\tau}{A^2} A_\gamma = Q_\gamma A_\alpha$$

where Q is an n component vector function which has to be determined. However, since w is skew-symmetric, it is easy to verify that

$$Q_\gamma = - \frac{w_{\gamma\tau} A_\tau}{A^2}$$

and hence

$$w_{\alpha\gamma} = \frac{w_{\alpha\tau} A_\tau}{A^2} A_\gamma - \frac{w_{\gamma\tau} A_\tau}{A^2} A_\alpha$$

For three distinct indices α, β, and γ we have

$$A_\alpha w_{\beta\gamma} = \frac{w_{\beta\tau} A_\tau}{A^2} A_\alpha A_\gamma - \frac{w_{\gamma\tau} A_\tau}{A^2} A_\alpha A_\beta$$

$$A_\beta w_{\gamma\alpha} = \frac{w_{\gamma\tau} A_\tau}{A^2} A_\beta A_\alpha - \frac{w_{\alpha\tau} A_\tau}{A^2} A_\beta A_\gamma$$

$$A_\gamma w_{\alpha\beta} = \frac{w_{\alpha\tau} A_\tau}{A^2} A_\gamma A_\beta - \frac{w_{\beta\tau} A_\tau}{A^2} A_\gamma A_\alpha$$

Summing these three equations gives

$$A_\alpha w_{\beta\gamma} + A_\beta w_{\gamma\alpha} + A_\gamma w_{\alpha\beta} = 0$$

which, on recalling the definition of w (4.9), is the integrability condition for the pfaffian. Therefore, if the system of (n - 1) linear partial differential equations (4.7) is a complete system, then the pfaffian (4.1) is integrable and conversely, which completes the proof.

5. Controllability Criterion for Control Systems with Linear Control Vectors

Consider the nonlinear control system with the control vector appearing linearly, defined by

$$\dot{x}_\alpha = A_\alpha(t;x) + B_{\alpha r}(t:x)\ u_r \qquad 5.1$$

$$(\alpha = 1 \ldots n;\ r = 1 \ldots m \leq n)$$

We shall assume that the rank of the matrix B is m, so that there will exist a maximal set of (n - m) vectors Ψ_R orthogonal to B, namely

$$\Psi_{R\alpha}(t;x)\ B_{\alpha r}(t;x) \overset{t,x}{\equiv} 0 \qquad 5.2$$

$$(\alpha = 1 \ldots n;\ r = 1 \ldots m;\ R = 1 \ldots (n-m))$$

By virtue of the vectors Ψ_R we can associate with the system (5.1) the pfaffian system

$$\Psi_{R\alpha}(t;x)dx_\alpha - \Psi_{R\beta}(t;x)\ A_\beta(t;x)dt = 0 \qquad 5.3$$

With regard to the pfaffian system (5.3), Hermes[1] adopted the following definition of controllability.

Definition 1

The system (5.1) is completely controllable for all $(t,x) \in D$ if the associated pfaffian system (5.3) is not integrable for all $(t,x) \in D$.

The usefulness of this definition of controllability is diminished by the fact that there are some inherent difficulties in determining whether or not the pfaffian system (5.3) is integrable. Only in special cases such as when (5.3) defines a single pfaffian or the form of the integrating factors can be anticipated are the results of Theorem 4.1 applicable.

To demonstrate that the pfaffian system (5.3) is integrable, we have to determine the existence of (n - m) integrating factors $\mu_R(t:x)$ and a scalar function V(t;x) such that

$$\mu_R(t;x)\Psi_{R\alpha}(t;x) \overset{t,x}{\equiv} \frac{\partial V(t;x)}{\partial x_\alpha}$$

and

$$\mu_R(t;x)\Psi_{R\beta}(t;x)\; A_\beta(t;x) \overset{t,x}{\equiv} -\frac{\partial V(t:x)}{\partial t} \qquad 5.4$$

Conversely, the demonstration of the nonexistence of either the integrating factors μ_R or the scalar function V, determines the nonintegrability of pfaffian system (5.3). The linear partial differential equations associated with the pfaffian system (5.3), that V has to satisfy, are by virtue of (5.2)

$$\frac{\partial V(t;x)}{\partial t} + \frac{\partial V}{\partial x_\alpha}(t;x)\; A_\alpha(t;x) = 0$$

$$\frac{\partial V}{\partial x_\alpha}(t;x)\; B_{\alpha r}(t;x) = 0 \qquad (5.5)$$

$$(\alpha = 1 \ldots n;\; r = 1 \ldots m)$$

Because of the equivalence between the integrability conditions for a pfaffian system and the completeness of the associated partial differential equations, Definition 1 can be rephrased as follows. The control system (5.1) is completely controllable for all $(t,x) \in D$, if the only possible solution to the system of linear partial differential equations (5.5) is a trivial one. This is essentially the approach adopted by Hermann[5] in his development of the algebraic controllability criterion for linear systems with time varying coefficients. The utility of the method is immediately obvious, since firstly, it circumvents the construction of the orthogonal vectors $\psi_R(t,x)$; and secondly, demonstrating the non-existence of V is a straightforward ritual of applying the Poisson operator to the system (5.5) until $(n+1)$ independent equations have been resurrected. The rephrased version of Definition 1 can be expressed in the following palatable form which appeals to the physical intuition of controllability.

Definition 2

If for a given control system we can determine a scalar function $V(t;x)$ such that $V(t;x) = \text{constant}$ is an integral of the control system independent of the choice of the controls, then the control system is not controllable. Conversely, if no such $V(t;x)$ exists, apart from a trivial solution, then the system is controllable.

It is easy to verify that this definition immediately generates the system of partial differential equations (5.5). It should be noted that this definition of controllability for nonlinear control systems does not guarantee the existence of a control which steers any initial state to any final desired state in finite time. What it does guarantee is that the dimension of the reachable set at any given time is equal to the dimension of the state space. To clarify this point consider the following example.

Example 5.1

Let the nonlinear control system be defined by

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_3^2 \\ \dot{x}_3 &= u \end{aligned} \qquad 5.6$$

Then the system of partial differential equations (5.5) associated with the control system (5.6) is

$$\frac{\partial V}{\partial t} + x_3 \frac{\partial V}{\partial x_1} + x_3^2 \frac{\partial V}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_3} = 0$$

Applying the Poisson operator successively to these equations yields

$$\frac{\partial V}{\partial x_1} + 2x_3 \frac{\partial V}{\partial x_2} = 0$$

and

$$\frac{\partial V}{\partial x_2} = 0$$

thus denying the existence of a nontrivial V. Therefore, in accordance with Definition 2 the system (5.6) would be termed controllable, but it is obvious that there does not exist a control which transfers the state from the origin to any other state possessing negative values of x_2. However, the dimension of the reachable set at any time for the control system (5.6) is 3.

Example 5.2

Consider the control system

$$\dot{x}_1 = x_1 + x_2 u$$
$$\dot{x}_2 = x_2 + x_1 u \qquad 5.7$$

then the partial differential equations (5.5) are

$$\frac{\partial V}{\partial t} + x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2} = 0$$

$$x_2 \frac{\partial V}{\partial x_1} + x_1 \frac{\partial V}{\partial x_2} = 0$$

Applying the Poisson operator to these equations does not yield any new equations, so the system is complete of order 2. Therefore, a nontrivial V exists, and is determined to be

$$V(t,x) \equiv (x_1^2 - x_2^2)\, e^{-2t}.$$

Therefore, all solutions of the control system (5.7) are confined to the surface

$$(x_1^2 - x_2^2)e^{-2t} = \text{constant}$$

irrespective of the choice of the controls, and hence the system (5.7) is not controllable.

6. Hermann's Criterion

We shall review the algebraic controllability criterion developed by Hermann[5] for linear systems with time varying coefficients. Let the control system be defined by

$$\dot{x}_\alpha = A_{\alpha\beta}(t)x_\beta + B_{\alpha r}(t)u_r \qquad 6.1$$

$$(\alpha = 1 \ldots n;\ r = 1 \ldots m)$$

The system of partial differential equations (5.5) associated with the control system (6.1) is

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_\alpha} A_{\alpha\beta}(t)x_\beta = 0 \qquad 6.2$$

$$\frac{\partial V}{\partial x_\alpha} B_{\alpha r}(t) = 0 \qquad 6.3$$

Applying the Poisson operator to this system yields

$$\frac{\partial V}{\partial x_\alpha}\left(A_{\alpha\gamma}(t)B_{\gamma r}(t) - \frac{d}{dt}B_{\alpha r}(t)\right) = 0$$

If we define the operator

$$\Gamma_{\alpha\gamma} \equiv A_{\alpha\gamma}(t) - \delta_{\alpha\gamma}\frac{d}{dt}$$

then the additional equations generated by the Poisson operator can be expressed by

$$\frac{\partial V}{\partial x_\alpha}\Gamma_{\alpha\gamma}B_{\gamma r} = 0 \qquad 6.4$$

Applying the Poisson operator to the systems (6.3) and (6.4) does not generate any new partial differential equations, since the coefficients $\Gamma_{\alpha\gamma} B_{\gamma r}$ are functions of time only. Therefore, using the systems (6.2) with system (6.4) yields

$$\frac{\partial V}{\partial x_\alpha} \Gamma_{\alpha\beta} \Gamma_{\beta\gamma} B_{\gamma r} = 0 \qquad 6.5$$

Hence, we have in general on applying the Poisson operator between system (6.2) and each new system of partial differential equations generated

$$\frac{\partial V}{\partial x_\alpha} \Gamma_{\alpha\beta} \Gamma_{\beta\cdot} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \Gamma_{\cdot\gamma} B_{\gamma r} = 0 \qquad 6.6$$

If from the sets of equations (6.3), (6.4), (6.5) and (6.6) we can select n independent equations, then

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = \cdot \cdot \cdot \cdot \cdot \cdot \cdot = \frac{\partial V}{\partial x_n} = 0$$

and consequently $\frac{\partial V}{\partial t} = 0$ by (6.2), and hence a nontrivial V(t;x) would not exist. The condition that assures this result, when expressed in matrix notation, is rank $\left[B, \Gamma B, \ldots, \Gamma^{n-1} B \right]$ is n for all t. This is the criterion developed by Hermann and has a greater utility than Kalman's integral criterion, since it does not depend on the knowledge of the fundamental solution of the homogeneous system. It is readily apparent that Hermann's criterion generalizes the result for constant coefficient linear systems, because in this case $\Gamma = A$.

7. Equivalence Between Kalman's and Hermann's Criterion

For simplicity, and with no loss of generality, we shall treat the system (1.1)

$$\dot{x}_\alpha = H_{\alpha r}(t)u_r(t) \quad (\alpha = 1 \ldots n;\ r = 1 \ldots m) \tag{7.1}$$

since this form encompasses all linear systems. For this system Hermann's criterion for controllability demands that the rank of the $n \times nm$ matrix

$$\left[H_{\alpha r}(t),\ \frac{d}{dt} H_{\alpha r}(t),\ \ldots\ldots,\ \frac{d^{n-1}}{dt^{n-1}} H_{\alpha r}(t)\right]$$

be n for all t; whereas Kalman's criterion requires that the determinant of the $n \times n$ matrix

$$\int_{t_o}^{t_1} H_{\alpha r}(t)\, H_{\beta r}(t)dt$$

be nonsingular for some $t_1 > t_o$. To demonstrate the equivalence between these criteria we shall first assume that Hermann's criterion is not satisfied, but Kalman's criterion is satisfied for the controllability of the system (7.1). Therefore, if the rank of the matrix

$$\left[H_{\alpha r}(t),\ \frac{dH_{\alpha r}(t)}{dt},\ \frac{d^2H_{\alpha r}(t)}{dt^2},\ \ldots\ldots,\ \frac{d^{n-1}H_{\alpha r}(t)}{dt^{n-1}}\right]$$

is less than n, then there exists an n vector $\emptyset$ such that

$$\emptyset_\alpha(t)\left[H_{\alpha r}(t),\ \frac{dH_{\alpha r}(t)}{dt},\ \frac{d^2H_{\alpha r}(t)}{dt^2},\ \ldots\ldots,\ \frac{d^{n-1}H_{\alpha r}(t)}{dt^{n-1}}\right] \equiv 0$$

Expressing this result in component form yields

$$\begin{aligned}
&\phi_\alpha(t)\, H_{\alpha r}(t) \equiv 0 \\
&\phi_\alpha(t)\, \frac{d}{dt} H_{\alpha r}(t) \equiv 0 \\
&\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&\phi_\alpha(t)\, \frac{d^{n-1} H_{\alpha r}(t)}{dt^{n-1}} \equiv 0
\end{aligned} \tag{7.2}$$

By straightforward differentiation, equations (7.2) imply

$$\begin{aligned}
&\phi_\alpha(t)\, H_{\alpha r}(t) = 0 \\
&\frac{d\, \phi_\alpha(t)}{dt}\, H_{\alpha r}(t) = 0 \\
&\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&\frac{d^{n-1}\, \phi_\alpha(t)}{dt^{n-1}}\, H_{\alpha r}(t) = 0\,.
\end{aligned} \tag{7.3}$$

Since no row of the matrix H(t) can be zero, otherwise the corresponding state component would not be controllable, then it follows that the Wronskian of the vector $\phi(t)$ must vanish. Therefore, it follows that the components of the vector ϕ satisfy at most an (n - 1) order linear differential equation. From the theory of ordinary differential equations[8] any solution of an (n - 1) order linear differential equation can be expressed as a linear combination of (n - 1) independent functions f(t) with constant coefficients C. Therefore, each component of $\emptyset$ can be expressed in the form

$$\emptyset_\alpha(t) = f_1(t)C_{1\alpha} + f_2(t)C_{2\alpha}, \ldots\ldots, + f_{(n-1)}(t)C_{(n-1)\alpha}$$

so that

$$\emptyset_\alpha(t)H_{\alpha r}(t) = f_1(t)\left[C_{1\alpha}H_{\alpha r}(t)\right] + f_2(t)\left[C_{2\alpha}H_{\alpha r}(t)\right] + \ldots + f_{(n-1)}(t)\left[C_{(n-1)\alpha}H_{\alpha r}(t)\right] = 0$$

$$\frac{d\emptyset_\alpha(t)}{dt}H_{\alpha r}(t) = \frac{df_1(t)}{dt}\left[C_{1\alpha}H_{\alpha r}(t)\right] + \frac{df_2}{dt}(t)\left[C_{2\alpha}H_{\alpha r}(t)\right] + \ldots + \frac{df_{(n-1)}(t)}{dt}\left[C_{(n-1)\alpha}H_{\alpha r}(t)\right] = 0$$

$$\ldots\ldots\ldots\ldots$$

$$\frac{d^{n-2}\emptyset_\alpha(t)}{dt^{n-2}} = \frac{d^{n-2}f_1(t)}{dt^{n-2}}\left[C_{1\alpha}H_{\alpha r}(t)\right] + \frac{d^{n-2}f_2(t)}{dt^{n-2}}\left[C_{2\alpha}H_{\alpha r}(t)\right] + \ldots + \frac{d^{n-2}f_{(n-1)}(t)}{dt^{n-2}}\left[C_{(n-1)\alpha}H_{\alpha r}(t)\right] = 0 \qquad 7.4$$

Since the functions f(t) are assumed to be (n-1) independent solutions of an (n - 1) order linear differential equation, then it follows that the Wronskian is different from zero, and hence equations (7.4) can only be satisfied if the components of the (n - 1) vector

$$\left[C_{1\alpha} H_{\alpha r}(t)\right], \left[C_{2\alpha} H_{\alpha r}(t)\right], \ldots . \left[C_{(n-1)\alpha} H_{\alpha r}(t)\right]$$

are zero. Therefore, if the rank of

$$\left[H_{\alpha r}(t), \frac{d}{dt} H_{\alpha r}(t), \frac{d^2 H_{\alpha r}(t)}{dt^2}, \ldots .., \frac{d^{n-1} H_{\alpha r}(t)}{dt^{n-1}}\right]$$

is less than n, then there exists a constant vector C such that

$$C_\alpha H_{\alpha r}(t) \equiv 0 . \qquad 7.5$$

The existence of such a vector, by virtue of (1.7), contradicts the assumption that the determinant of the n x n matrix

$$\int_{t_o}^{t_1} H_{\alpha r}(t) H_{\beta r}(t) \, dt$$

is nonsingular. If on the otherhand we assume that Hermann's criterion is satisfied whereas Kalman's criterion is not, then from the results of Section 1 it follows that there exists a constant vector C such that

$$C_\alpha H_{\alpha r}(t) \overset{t}{\equiv} 0$$

If in (7.2) we let $\phi_\alpha(t) \equiv C_\alpha$, then we contradict the assumption that

$$\left[H_{\alpha r}(t), \frac{d H_{\alpha r}(t)}{dt}, \ldots \ldots, \frac{d^{n-1}}{dt^{n-1}} H_{\alpha r}(t)\right]$$

has rank n; this completes the equivalence between the two criteria. This equivalence was first demonstrated by Hermes[1].

As noted previously, Hermann's criterion is also equivalent to the nonintegrability of the pfaffian system associated with the control system. We shall demonstrate the nonintegrability of the pfaffian system associated with the control system (7.1), because in this case the form of the integrating factors can be anticipated and the results of Theorem 4.1 directly applied. We assume that no row of the matrix H(t) is zero, otherwise the lack of controllability is immediately obvious. In view of this fact the components of any vector orthogonal to the matrix H(t) can be expressed as functions of time alone. If there is a specific dependence of the orthogonal vector on the state vector x, then at most this can only occur as a multiplicative factor, the knowledge of which is inconsequential to the application of Theorem 4.1. Denoting by $\Psi(t)$ the vector orthogonal to H(t), then the pfaffian associated with the control system (7.1) is

$$\Psi_\alpha(t)dx_\alpha = 0 \qquad 7.6$$

Applying Theorem 4.1 we find that a necessary and sufficient condition that the pfaffian (7.6) be integrable is that

$$\frac{d}{dt}\log\Psi_1(t) = \frac{d}{dt}\log\Psi_2(t) = \ldots = \frac{d}{dt}\log\Psi_n(t)$$

Integrating this result yields

$$\frac{\Psi_1(t)}{C_1} = \frac{\Psi_2(t)}{C_2} = \ldots = \frac{\Psi_n(t)}{C_n}$$

where the C's are constants and can be regarded as the components of a nontrivial constant vector C. Since the vector $\Psi(t)$ is proportional to a constant vector C, then this implies that the pfaffian (7.6) is integrable if and only if there exists a constant vector C such that

$$C_\alpha H_{\alpha r}(t) \overset{t}{\equiv} 0$$

The previous arguments concerning this statement now apply, thus illustrating the dual approach to controllability criterion via the integrability conditions for the pfaffian system.

8. Continous Groups of Transformations

In the application of continuous groups of transformations to the determination of controllability, we are mainly concerned with the possibility of representing any control action on the dynamical system in terms of continuous groups of transformations containing a finite number of essential parameters. To illustrate this procedure consider example 5.2 where the system equations are

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 u \\ \dot{x}_2 &= x_2 + x_1 u \end{aligned} \qquad 8.1$$

If we represent the control as $u(t) = \frac{d\,y(t)}{dt}$, then the system (8.1) can be expressed in pfaffian form corresponding to (2.2), by

$$dx_1 = x_1 dt + x_2 dy$$
$$dx_2 = x_2 dt + x_1 dy \qquad 8.2$$

The system of nonhomogeneous partial differential equations (2.1) corresponding to the pfaffian system (8.2) is

$$\frac{\partial x_1}{\partial t} = x_1 ; \qquad \frac{\partial x_1}{\partial y} = x_2$$

$$\frac{\partial x_2}{\partial t} = x_2 ; \qquad \frac{\partial x_2}{\partial y} = x_1 ,$$

furthermore, these equations are completely integrable. Therefore, the solution of the system (8.1) may be expressed in terms of a two parameter transformation by

$$x_1(t,y) = e^t \left[x_1(0,0) \cosh y + x_2(0,0) \sinh y \right]$$
$$x_2(t,y) = e^t \left[x_1(0,0) \sinh y + x_2(0,0) \cosh y \right] \qquad 8.3$$

The significance of the solution (8.3) is that it gives an algebraic representation for all the possible control actions on the dynamic system (8.1). This follows from the fact that the real time solutions of the system (8.1) are generated by the one parameter subgroup obtained by substituting y(t) for y,

$$x_1(t) \equiv x_1(t,y(t)) = e^t \left[x_1(0,0)\cosh y(t) + x_2(0,0)\sinh y(t)\right]$$

$$x_2(t) \equiv x_2(t,y(t)) = e^t \left[x_1(0,0)\sinh y(t) + x_2(0,0)\cosh y(t)\right]$$

Since the solution (8.3) represents all the possible control actions on the dynamic system, then on eliminating the parameter y from (8.3), it can be concluded that all solutions of the system (8.1) are constrained to the surface

$$\left[x_1^2(t) - x_2^2(t)\right] e^{-2t} = x_1^2(0) - x_2^2(0).$$

Since all solutions of the system (8.1) are constrained to this surface independent of the choice of the control u(t), then the system is not controllable.

A different kind of representation was encountered previously in connection with the controllability of linear systems, where the solution of

$$\dot{x}_\alpha(t) = H_{\alpha r}(t)\, u_r(t)$$

was represented by

$$x_\alpha(t_1) = x_\alpha(t_o) + \int_{t_o}^{t_1} H_{\alpha r}(t) H_{\beta r}(t) dt\, \xi_\beta \qquad 8.4$$

for a particular form of the control vector. This representation was particularly fruitful in proving the sufficiency of Theorem 1.1 where the representation (8.4) was considered as a transformation between the vectors

$x(t_1) - x(t_o)$ and ξ. On the otherhand the representation (8.4) can be interpreted as a transformation between $x(t_1)$ and $x(t_o)$, with t_1 and the n components of ξ regarded as $(n+1)$ parameters. The condition that the linear system be controllable corresponds to the n parameters ξ_α being essential. If the n parameters ξ_α are not essential, then it follows from Theorem 1.1 that the linear system is not controllable; however, the procedure of eliminating the parameters, as illustrated in the previous example, to generate the constraining hypersurfaces, does not apply in this case. This is because the one parameter subgroup of transformations obtained by substituting $\xi(t_1)$ for ξ, does not yield all the possible control actions on the linear dynamical system. The combination of these two ideas leads to the following sequential method for the determination of the controllability of control systems. For simplicity of exposition we shall treat the linear system

$$\overset{\circ}{x}_\alpha = H_\alpha(t)\, u \qquad (\alpha = 1 \ldots n) \tag{8.5}$$

where u is a single component control. Hermann's controllability criterion for this system requires that rank

$$\left[H_\alpha(t), \; \frac{dH_\alpha(t)}{dt}, \; \ldots\ldots, \; \frac{d^{n-1}}{dt^{n-1}} H_\alpha(t) \right]$$

be n for all t. The sequential method proceeds as follows, and simply mimics the procedure adopted for example 5.2. We try to determine the existence of a scalar function of time $\lambda(t)$ such that if the control

$u(t)$ is given the representation

$$u(t) = \lambda(t) \frac{dy(t)}{dt}$$

then the nonhomogeneous partial differential equations associated with the linear system (8.5) are completely integrable. For this representation of the control, the nonhomogeneous partial differential equations are

$$\frac{\partial x_\alpha}{\partial t} = 0; \qquad \frac{\partial x_\alpha}{\partial y} = H_\alpha(t)\lambda(t) \tag{8.6}$$

The equations (8.6) are completely integrable if and only if

$$\frac{dH_\alpha(t)}{dt} \lambda(t) + H_\alpha(t) \frac{d\lambda(t)}{dt} = 0 \tag{8.7}$$

It is to be observed that if there exists a $\lambda(t)$ such that (8.7) is satisfied, then Hermann's controllability criterion is not satisfied. In fact, for this situation we can give an algebraic representation for all the possible control actions on the linear system in terms of two parameters by

$$x_\alpha(t,y) = x_\alpha(0 \cdot 0) + H_\alpha(t)\lambda(t)y \tag{8.8}$$

Eliminating the parameter y from these expressions yields (n-1) constraining hypersurfaces, so that the dimension of the reachable set at any time would be one.

If there does not exist a $\lambda(t)$ such that the integrability conditions (8.7) are satisfied, then we attempt to determine if all the possible control actions on the linear system can be represented in terms of three parameters. To do this we define a transformation from x to x^1 by

$$x_\alpha = x_\alpha^1 + H_\alpha(t)\, u^1 \qquad 8.9$$

where the new control u^1 is given by

$$\frac{du^1}{dt} = u$$

The transformed system becomes

$$\frac{dx_\alpha^1}{dt} = -\frac{dH_\alpha(t)}{dt}\, u^1 \qquad 8.10$$

which is of the same form as the system (8.5) and hence we repeat the same process. If there does exist a $\lambda(t)$ such that the representation

$$u^1(t) = \lambda(t)\,\frac{dy(t)}{dt}$$

yields a completely integrable set of nonhomogeneous partial differential associated with the system (8.10), then the vector x^1 can be represented in terms of a two parameter group of transformations by

$$x_\alpha^1(t,y) = x_\alpha^1(0\cdot 0) - \frac{dH_\alpha(t)}{dt}\,\lambda(t)y$$

Therefore, the vector x can be expressed in terms of a three parameter group of transformations by virtue of (8.9) as

$$x_\alpha(t,u^1,y) = x_\alpha(0\cdot 0\cdot 0) + H_\alpha(t)u^1 - \frac{dH_\alpha(t)}{dt}\,\lambda(t)y$$

Obviously, to demonstrate the controllability of the system (8.5) we simply continue this process until we have resurrected n parameters not including the parameter t. The condition for these n parameters to be essential then yields Hermann's criterion. This method has a distinct advantage when the control system is not controllable, since it determines the dimension of that portion of the control system that is controllable.

9. Controllable Systems

In this section we shall catalogue those control systems that are known to be controllable.

1) The linear system with constant coefficients

$$\frac{d^n x}{dt^n} + a_1 \frac{dx^{n-1}}{dt} + \cdots\cdots + a_{n-1} \frac{dx}{dt} + a_n x = u$$

is controllable.

2) The linear system with time varying coefficients

$$\frac{d^n x}{dt^n} + a_1(t) \frac{dx^{n-1}}{dt} + \cdots\cdots + a_{n-1}(t) \frac{dx}{dt} + a_n(t) x = u$$

is controllable.

3) The quasi-linear system

$$\frac{d^n x}{dt^n} + A\left(t; x; \frac{dx}{dt}; \cdots\cdots; \frac{d^{n-1}x}{dt^{n-1}} \right) = u$$

is controllable. Using Hermann's method the proof of each of these statements is trivial.

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