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COMPUTATION OF GENERAL PLANETARY PERTURBATIONS FOR RESONANCE CASES

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INTRODUCTION

Many investigators have studied resonant motions in celestial mechanics. These problems are of theoretical interest from the point of view of stability and for understanding the characteristics of the motion. The practical importance of these problems comes from the existence of so many cases of resonant motions in the solar system. Hagihara's work (Reference 3) contains an excellent summary of studies of these problems. The work of Schubart (Reference 5) has added significantly to the understanding of motions of resonant asteroids. By a numerical averaging process on a computer he was able to isolate the secular and critical terms of the disturbing function and then investigate the stability and characteristics of the motion for a wide range of values of the parameters.

The cases of resonance require special techniques when general perturbations are to be computed. For this purpose it is necessary to represent the motion in a form compatible with the usual trigonometric series so that accurate results may be obtained including higher order effects from all the disturbing planets. Such a form is obtained in the present study by using an argument in the trigonometric terms such that the resonance of the mean motions is taken into account. In this way the critical terms are removed and the effects appear in the secular terms. The resulting series have the same form as in ordinary nonresonance cases. The mean anomalies of the two bodies have been replaced by a single related "anomaly" in the arguments of the trigonometric terms.

THE REFERENCE ORBITS

The perturbations of the coordinates are computed to give the deviation of the motion of a planet from the motion in a fixed Keplerian ellipse. There is some latitude in the choice of the elements of the reference ellipse, but to keep the perturbations small, the reference motion should be a close approximation to the actual motion. When the mean motions, n and n', of two planets are in the ratio of two integers,

$$\frac{n}{n'} = \frac{p}{q}, \ p, \ q \text{ are integers,}$$

the motions are said to be commensurable and this is a case of resonance which requires some special consideration in the computation of general perturbations.

An extensive expansion of the disturbing forces from the mutual attractions of two planets is given in Reference 2. In the usual nonresonance case the perturbations are obtained as trigonometric series in terms of the two mean anomalies, g and g', with various powers of the time as factors. This form results from the fact that the position vectors $\vec{r}_0$ and $\vec{r}_0'$ in the two reference orbits are periodic functions of the mean anomalies g and g' respectively. In the expansions of the components of the disturbing force, the terms in the series have the form

$$\begin{matrix}\cos \\ \sin\end{matrix}(ig + jg')$$

where i and j are integers. When these terms are integrated with respect to the time, the coeffecients are multiplied by the factor

$$\frac{n}{in + jn'}\,.$$

In cases of resonance where

$$qn = pn'$$

as before the terms with the argument

$$qg - pg'$$

or any multiple of this become critical since

$$qn - pn' = 0.$$

There are several possibilities for treating such cases. The purpose here is to give a method wherein the form of the series is similar to that of the usual nonresonance case. The mean anomalies, g and g', are linear functions of the time, t :

$$g = nt + g_0$$
$$g' = n't + g_0'.$$

Putting

$$n^* = \frac{n}{p} = \frac{n'}{q} \text{ and } g^* = n^* t$$

there follows

$$g = pn^* t + g_0 = pg^* + g_0$$

$$g' = qn^* t + g_0' = qg^* + g_0'$$

Now if g^* is increased by 2π then g and g' are increased by $2p\pi$ and $2q\pi$ respectively. Since $\vec{r}_0$ and $\vec{r}_0'$ are periodic functions of g and g' respectively, both $\vec{r}_0$ and $\vec{r}_0'$ are periodic functions of g^*. The former development in terms of g and g' can be replaced, in the resonance case, by a development in terms of the single angle g^*, and in this form there are no critical terms. In the results the affects of the critical terms will appear in secular form. The development given on pages 5 through 11 of Reference 2 can be used in the computation except that the double harmonic analysis is now replaced by simple harmonic analysis in the variable g^*.

In the application of this method, the mean motions of the two planets in the reference orbits must be exactly commensurable. The deviations from the reference orbits are properly taken into account by adjusting the constants of integration in the series.

FORMULAS FOR THE PERTURBATIONS

The form presented in the previous section for representing the motion in cases of resonance using trigonometric series can be applied using nearly any decomposition of the perturbations. Musen's (Reference 4) method of perturbations of the rectangular coordinates is used in the present case.

The position vector, $\vec{r}$, of a planet is given by

$$\vec{r} = (1 + \alpha)\,\vec{r}_0 + \beta\vec{w} + \gamma a \vec{R}$$

where α, β, γ are the components of the perturbations, $\vec{r}_0$ is the position vector in the fixed reference ellipse, $\underline{a}$ is the semi-major axis of the reference ellipse, $\vec{R}$ is the unit vector in the direction of the angular momentum of the motion in the reference ellipse and

$$\vec{w} = \frac{1}{n}\frac{d\vec{r}_0}{dt},$$

n being the mean motion in the reference ellipse and t is the time.

The perturbations are computed from the formulas

$$\alpha = \int (M_1 D + M_2 E)\, d(nt)$$

$$\beta = \int (M_4 - 2\alpha)\, d(nt)$$

$$\gamma = \int M_3 D\, d(nt)$$

to which the contributions from the constants of integration are to be added. The integrands are obtained from the formulas:

$$M_1 = \vec{S}_1 \cdot (a^2\vec{F})$$

$$M_2 = \vec{S}_2 \cdot (a^2\vec{F})$$

$$M_3 = \frac{r_0}{a}\vec{R} \cdot (a^2\vec{F})$$

$$M_4 = \int M_2\, d(nt)$$

$$\vec{S}_1 = \vec{P} \cos \epsilon + \vec{Q} \frac{\sin \epsilon}{\sqrt{1 - e^2}}$$

$$\vec{S}_2 = - \vec{P} \sin \epsilon + \vec{Q} \frac{\cos \epsilon - e}{\sqrt{1 - e^2}}$$

$$D = \frac{a}{r_0} [\sin (\eta - \epsilon) - e \sin \eta + e \sin \epsilon]$$

and

$$E = 2 \frac{a}{r_0} [1 - \cos (\eta - \epsilon)]$$

where e is the eccentricity of the reference ellipse, ϵ is the eccentric anomaly in the reference ellipse, η is the eccentric anomaly held constant during the integration and replaced by ϵ afterward, and $\vec{P}, \vec{Q}$ are the standard unit vectors associated with $\vec{R}$. The factor $a^2 \vec{F}$ is the disturbing force which can be expressed as

$$a^2 \vec{F} = a^2 \vec{f}_I + a^2 \vec{f}_1 t + a^2 \vec{f}_2 t^2 + \cdots \cdots .$$

The various $\vec{f}$'s which are factors of the powers of the time are given on pages 6-10 of Reference 2.

The $\vec{f}$'s depend on the reference position vectors of $\vec{r}_0$ and $\vec{r}_0'$ which are periodic functions of g^*. Similarly $\vec{S}_1$, $\vec{S}_2$, and $\vec{r}_0$ are periodic functions of g^*. Thus M_1, M_2, and M_3 can be written in the form

$$M_i = \mathfrak{m}_{i,I} + \mathfrak{m}_{i,1} t + \mathfrak{m}_{i,2} t^2 + \cdots$$

where

$$\mathfrak{m}_{1,j} = \vec{S}_1 \cdot a^2 \vec{f}_j$$

$$\mathfrak{M}_{2,j} = \vec{S}_2 \cdot a^2 \vec{f}_j$$

$$\mathfrak{M}_{3,j} = \frac{r_0}{a} \vec{R} \cdot a^2 \vec{f}_j$$

for j = I, 1, 2, 3, ... Therefore, by simple harmonic analysis each $\mathfrak{M}_{i,j}$ can be obtained as a trigonometric series in g^*.

The expressions for D and E also can be represented as trigonometric series in g^* and η. These series along with those for the M_i's form the integrands in the expressions for α, β, and γ. Terms under the integral have the general form

$$[C \cos(ax + b) + S \sin(ax + b)]\, x^p.$$

Upon integration this takes the form

$$\left[C_p \cos(ax + b) + S_p \sin(ax + b)\right] x^p + \left[C_{p-1} \cos(ax + b) + S_{p-1} \sin(ax + b)\right] x^{p-1}$$
$$+ \left[C_{p-2} \cos(ax + b) + S_{p-2} \sin(ax + b)\right] x^{p-2} + \cdots$$
$$+ \left[C_0 \cos(ax + b) + S_0 \sin(ax + b)\right]$$

where

$$C_p = -\frac{S}{a}, \qquad S_p = \frac{C}{a},$$

$$C_{p-1} = \frac{p}{a} S_p, \qquad S_{p-1} = -\frac{p}{a} C_p,$$

$$C_{p-2} = \frac{p-1}{a} S_{p-1}, \qquad S_{p-2} = -\frac{p-1}{a} C_{p-1},$$

$$C_{p-3} = \frac{p-2}{a} S_{p-2}, \qquad S_{p-3} = -\frac{p-2}{a} C_{p-2}, \quad \cdots\cdots$$

and finally

$$C_0 = \frac{1}{a} S_1 , \qquad S_0 = -\frac{1}{a} C_1 .$$

The special case where $a = 0$ must be treated separately. Here the term under the integral has the general form $k t^p$ where k is a constant. Integration yields terms of the form $(k/p+1)t^{p+1}$ which introduces factors with one higher power of the time and since β involved double integration factors are generated for two higher powers of the time. In this manner, secular terms and terms with higher powers of the time enter the expansions.

The final expressions for α, β, and γ have the general form

$$\sum_{p=0}^{s} t^p \sum_i \left[C_{p,i} \cos (i g^*) + S_{p,i} \sin (i g^*)\right]$$

where s is the highest power of the time carried in the computation and $g^* = n^* t$.

APPLICATION

The case of the mutual attractions of Pluto and Neptune demonstrates the capabilities of the method. The values used for the reference orbit elements are given in Table I. These are approximate mean elements with the mean motion and semi-major axis of Pluto adjusted to satisfy the resonance criterion, $3n_9 = 2n_8$ where n_9 is the mean motion of Pluto and n_8 is the mean motion of Neptune.

Table I
Reference Elements of the Planets Pluto and Neptune

Planet	Semi-Major Axis in a.u. a	Eccentricity e	Mean Motion in Degrees/Day n	Longitude of Ascending Node Ω	Argument of Perihelion ω	Inclination i	Mean Anomaly at Epoch (JD 2415200.5) g_0
Neptune	30.005162	.008956	$^\circ$.0059818842	131°.2332	275°.9147	1°.7745	39°.1226
Pluto	39.382760	.248895	$^\circ$.0039879228	109°.6750	113°.9034	17°.1434	230°.0159

The coefficients of $T^k \cos(ig^*)$ and $T^k \sin(ig^*)$ in the series for the perturbations are given in Tables II and III. For comparison purposes, the mutual attractions of Pluto and Neptune were computed in Chebyshev series, as in Reference 1. The series in Table II and the Chebyshev series were evaluated at intervals of 800 days to give residuals in α, β and γ for two centuries on either side of the epoch. Figure 1 shows the trend of the residuals over the interval. The unit of time for the secular terms in the trigonometric series is one century (36525 days) and as expected the residuals increase in size beyond one unit from the epoch. The largest deviation of the four centuries is $.7 \times 10^{-8}$ or less than $0''.002$ of arc in β.

Table II

Coefficients in the Series for the Perturbations of Pluto Due to Neptune

			$\alpha \cdot 10^6$		$\beta \cdot 10^6$		$\gamma \cdot 10^6$	
i	j	k	cos	sin	cos	sin	cos	sin
0	0	0	7.7108	0.	0.	0.	3.0499	0.
1	0	0	116.4417	76.7695	237.3396	-250.1348	10.2250	-4.3397
2	0	0	20.4367	-6.8767	-0.0000	0.0000	-0.0000	-0.0000
3	0	0	13.8694	-10.2346	-5.1872	-12.4812	1.1377	-6.1786
4	0	0	7.8812	-3.0165	-7.5918	-11.2540	-2.2129	2.6157
5	0	0	4.6002	0.0170	-1.3825	-6.2142	-0.6797	0.6606
6	0	0	1.4731	-0.3964	-0.6860	-2.3503	-0.2082	0.2638
7	0	0	0.3957	-0.2325	-0.4682	-0.7887	-0.1218	0.3940
8	0	0	0.4284	-0.2927	-0.4400	-0.4676	-0.0971	0.2568
9	0	0	0.2797	-0.2631	-0.3533	-0.3140	-0.0672	0.1060
10	0	0	0.1484	-0.1084	-0.1940	-0.1977	-0.0233	0.0645
11	0	0	0.0969	-0.0598	-0.1064	-0.1157	0.0002	0.0557
12	0	0	0.0501	-0.0554	-0.0755	-0.0573	-0.0027	0.0398
13	0	0	0.0262	-0.0368	-0.0556	-0.0305	-0.0054	0.0243
14	0	0	0.0191	-0.0241	-0.0388	-0.0209	-0.0012	0.0141
15	0	0	0.0116	-0.0180	-0.0257	-0.0134	0.0016	0.0093
16	0	0	0.0062	-0.0121	-0.0167	-0.0071	0.0014	0.0072
17	0	0	0.0041	-0.0080	-0.0116	-0.0036	0.0007	0.0051
18	0	0	0.0024	-0.0057	-0.0083	-0.0020	0.0006	0.0031
19	0	0	0.0012	-0.0039	-0.0057	-0.0011	0.0007	0.0019
20	0	0	0.0006	-0.0027	-0.0038	-0.0006	0.0007	0.0014
0	0	1	92.0432	0.	0.	0.	1.5160	0.
1	0	1	-0.0945	-0.0592	-0.8880	0.7988	-0.0007	0.0034
2	0	1	-14.0529	-81.7795	-163.6035	28.0584	10.0917	3.2832
3	0	1	-0.0485	-0.0008	0.0381	0.0756	-0.0024	-0.0021
4	0	1	8.7911	5.1016	5.1041	-8.7645	-1.1050	0.6854
5	0	1	0.0046	0.0016	-0.0032	0.0037	0.0013	-0.0016
6	0	1	-1.7591	0.6397	0.4274	1.1732	0.0345	-0.2376
7	0	1	-0.0040	0.0014	0.0014	0.0011	0.0002	-0.0001
8	0	1	0.1385	-0.3828	-0.1910	-0.0684	0.0345	0.0387
9	0	1	0.0002	-0.0009	0.0002	0.0007	0.0000	-0.0002
10	0	1	0.0486	0.0846	0.0339	-0.0192	-0.0124	0.0003
11	0	1	-0.0001	0.0005	0.0001	0.0002	0.0001	-0.0001
12	0	1	-0.0246	-0.0042	-0.0013	0.0063	0.0020	-0.0025
13	0	1	-0.0002	-0.0000	0.0001	0.0001	0.0000	-0.0000
14	0	1	0.0050	-0.0042	-0.0012	-0.0014	0.0002	0.0008
16	0	1	-0.0000	0.0018	0.0005	0.0000	-0.0002	-0.0001
18	0	1	-0.0004	-0.0003	-0.0001	0.0001	0.0001	-0.0000
20	0	1	0.0001	-0.0000	0.0000	-0.0000	-0.0000	0.0000

Table II (Continued)
Coefficients in the Series for the Perturbations of Pluto Due to Neptune

			$\alpha.10^6$		$\beta.10^6$		$\gamma.10^6$	
i	j	k	cos	sin	cos	sin	cos	sin
0	0	2	0.0191	0.	-188.5243	0.	0.0010	0.
1	0	2	0.1408	-0.1433	-0.3141	-0.4173	0.0059	-0.0025
2	0	2	0.0390	0.0329	0.0700	-0.0767	0.0008	-0.0025
3	0	2	0.0017	0.0206	-0.0003	0.0093	-0.0005	-0.0018
4	0	2	-0.0144	0.0004	-0.0032	0.0083	0.0012	0.0005
5	0	2	-0.0093	-0.0046	-0.0021	0.0032	0.0006	0.0003
6	0	2	0.0008	-0.0048	-0.0014	0.0013	-0.0001	0.0000
7	0	2	0.0020	-0.0018	-0.0008	0.0007	0.0001	-0.0000
8	0	2	0.0004	0.0011	-0.0004	0.0005	0.0002	-0.0001
9	0	2	-0.0005	0.0008	-0.0001	0.0003	0.0001	-0.0000
10	0	2	-0.0007	-0.0002	-0.0001	0.0002	0.0001	0.0000
11	0	2	-0.0003	-0.0003	-0.0001	0.0002	0.0000	0.0000
12	0	2	0.0001	-0.0002	-0.0001	0.0001	0.0000	-0.0000
13	0	2	0.0001	-0.0000	-0.0000	0.0001	0.0000	-0.0000
14	0	2	-0.0000	0.0001	-0.0000	0.0001	0.0000	-0.0000
15	0	2	-0.0001	0.0000	-0.0000	0.0000	0.0000	-0.0000
0	0	3	-0.0122	0.	-0.0345	0.	-0.0005	0.
1	0	3	0.0001	0.0002	0.0008	-0.0007	0.0000	-0.0000
2	0	3	-0.0298	0.0179	0.0172	0.0055	-0.0014	0.0029
3	0	3	-0.0000	-0.0001	0.0000	-0.0001	0.0000	0.0000
4	0	3	0.0041	-0.0123	-0.0001	0.0011	-0.0003	-0.0006
6	0	3	0.0024	0.0044	-0.0002	-0.0001	0.0002	0.0000
8	0	3	-0.0018	-0.0004	0.0000	-0.0000	-0.0000	0.0000
10	0	3	0.0005	-0.0004	0.0000	0.0000	0.0000	-0.0000
12	0	3	-0.0000	0.0002	-0.0000	-0.0000	0.0000	0.0000
14	0	3	-0.0001	-0.0001	0.0000	-0.0000	-0.0000	0.0000
0	0	4	-0.0196	0.	0.0227	0.	-0.0000	0.
1	0	4	-0.0001	-0.0001	0.0001	0.0002	-0.0000	0.0000
2	0	4	0.0092	-0.0110	-0.0000	0.0000	-0.0000	0.0000
3	0	4	0.0001	0.0000	0.0000	-0.0000	0.0000	0.0000
4	0	4	0.0009	0.0051	0.0000	-0.0000	-0.0000	0.0000
6	0	4	-0.0016	-0.0009	0.0000	0.0000	0.0000	-0.0000
8	0	4	0.0006	-0.0002	0.0000	-0.0000	-0.0000	0.0000
10	0	4	-0.0001	0.0002	0.0000	-0.0000	-0.0000	-0.0000
12	0	4	-0.0000	-0.0001	0.0000	0.0000	0.0000	-0.0000

Table III

Coefficients in the Series for the Perturbations of Neptune Due to Pluto

i	j	k	$\alpha \cdot 10^6$ cos	$\alpha \cdot 10^6$ sin	$\beta \cdot 10^6$ cos	$\beta \cdot 10^6$ sin	$\gamma \cdot 10^6$ cos	$\gamma \cdot 10^6$ sin
0	0	0	0.0361	0.	0.	0.	-0.1957	0.
1	0	0	-2.8317	-1.9734	-10.7879	13.4496	-0.4121	0.1763
2	0	0	7.1479	0.2280	0.2795	-19.3522	-0.5690	-0.0708
3	0	0	-0.7132	0.1012	-0.0000	-0.0000	-0.0000	0.0000
4	0	0	-1.8191	0.9415	1.6611	2.9897	0.2614	-0.2765
5	0	0	-0.6387	0.3230	0.4981	1.0080	0.0808	-0.0527
6	0	0	-0.3069	0.1337	0.1731	0.4327	0.0608	-0.0472
7	0	0	-0.1487	0.0794	0.0885	0.1765	0.0296	-0.0324
8	0	0	-0.0709	0.0467	0.0580	0.0799	0.0146	-0.0103
9	0	0	-0.0391	0.0288	0.0397	0.0462	0.0076	-0.0066
10	0	0	-0.0231	0.0195	0.0233	0.0279	0.0029	-0.0067
11	0	0	-0.0132	0.0128	0.0132	0.0155	0.0021	-0.0046
12	0	0	-0.0078	0.0081	0.0087	0.0081	0.0020	-0.0027
13	0	0	-0.0047	0.0054	0.0063	0.0046	0.0010	-0.0016
14	0	0	-0.0027	0.0037	0.0043	0.0029	0.0003	-0.0010
15	0	0	-0.0016	0.0025	0.0027	0.0018	0.0002	-0.0008
16	0	0	-0.0010	0.0017	0.0018	0.0010	0.0001	-0.0006
17	0	0	-0.0006	0.0012	0.0013	0.0005	0.0001	-0.0004
18	0	0	-0.0003	0.0008	0.0009	0.0003	0.0000	-0.0002
19	0	0	-0.0002	0.0006	0.0006	0.0002	-0.0000	-0.0002
20	0	0	-0.0001	0.0004	0.0004	0.0001	-0.0000	-0.0001
0	0	1	-4.0541	0.	0.	0.	-0.0027	0.
1	0	1	0.0065	0.0060	0.0621	-0.0524	0.0003	-0.0001
2	0	1	-0.0005	-0.0002	0.0034	0.0046	0.0004	0.0002
3	0	1	1.0487	3.4695	6.9393	-2.0959	-0.3872	-0.8137
4	0	1	0.0024	-0.0007	-0.0012	-0.0042	-0.0006	0.0003
5	0	1	0.0006	-0.0002	-0.0004	-0.0008	-0.0002	0.0001
6	0	1	0.0137	0.0089	0.0089	-0.0138	-0.0037	-0.0017
7	0	1	0.0002	-0.0001	-0.0001	-0.0002	-0.0000	0.0000
8	0	1	0.0001	-0.0000	-0.0001	-0.0001	-0.0000	0.0000
9	0	1	0.0002	-0.0000	-0.0000	-0.0001	-0.0000	0.0000
0	0	2	-0.0007	0.	11.5777	0.	-0.0001	0.
1	0	2	-0.0049	0.0030	0.0154	0.0235	-0.0001	-0.0000
2	0	2	-0.0027	-0.0016	-0.0047	0.0071	0.0001	-0.0001
3	0	2	-0.0003	-0.0019	-0.0044	0.0010	0.0001	0.0001
4	0	2	0.0004	0.0005	0.0008	-0.0006	-0.0001	-0.0002
5	0	2	0.0003	0.0001	0.0001	-0.0004	-0.0000	-0.0001
6	0	2	0.0002	0.0001	0.0001	-0.0002	-0.0000	0.0000
7	0	2	0.0001	0.0001	0.0001	-0.0001	-0.0000	0.0000
8	0	2	0.0001	0.0000	0.0001	-0.0001	-0.0000	-0.0000
0	0	3	0.0010	0.	0.0013	0.	0.0000	0.
1	0	3	-0.0000	-0.0000	-0.0001	0.0000	-0.0000	0.0000
3	0	3	0.0005	-0.0001	-0.0003	-0.0011	-0.0000	0.0001
0	0	4	-0.0001	0.	-0.0014	0.	0.0000	0.

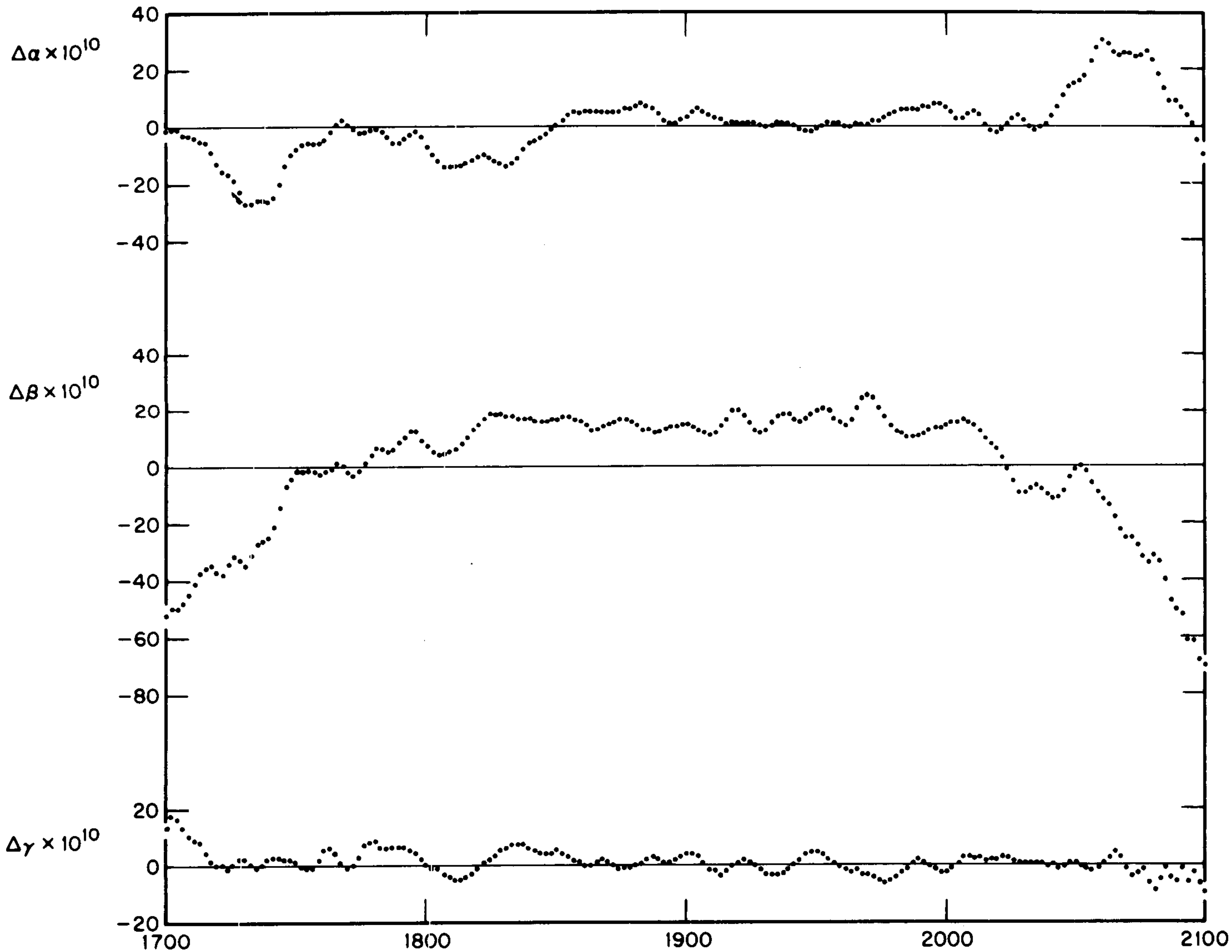

Figure 1–The Residuals in α, β, γ from a Comparison of the Trigonometric Series with a Corresponding Chebyshev Series

Iteration to convergence for the mutual attraction of Pluto and Neptune requires four iterations and six minutes of machine time on an IBM 7094 "Moonlight" system.

Another case handled by this method of trigonometric expansion is the perturbations of the minor planet Hilda due to Jupiter. Again there exists a three to two resonance. Table IV gives the elements for the reference orbits of the planets.

Table IV
Reference Elements of the Planets Hilda and Jupiter

Planet	Semi-Major Axis in a.u. a	Eccentricity e	Mean Motion in Degrees/Day n	Longitude of Ascending Node Ω	Argument of Perihelion ω	Inclination i	Mean Anomaly at Epoch (JD 2433200.5) g_0
Hilda	3.969423	.153760	$^\circ$.12462789	228°.3400	49°.2610	7°.8460	245°.9780
Jupiter	5.203063	.048410	$^\circ$.08308526	99°.9479	274°.0669	1°.3071	295°.5194

Table V gives the coefficients of the trigonometric series for α, β, and γ. The unit of time for this series is the period of Hilda divided by 2π. The reduced time unit is necessary because of the large perturbations of Jupiter on the minor planet. With a one century time unit, the series coefficients are large and convergence is slow.

Table V
Coefficients in the Series for the Perturbations of Hilda Due to Jupiter

			$\alpha . 10^6$		$\beta . 10^6$		$\gamma . 10^6$	
i	j	k	cos	sin	cos	sin	cos	sin
0	0	0	-120.9943	0.	0.	0.	-0.5198	0.
1	0	0	-306.6840	-841.0596	-3435.7379	875.6191	51.0465	40.1660
2	0	0	1521.6919	-2997.8475	-8238.6464	-4140.0171	-46.4376	1.9010
3	0	0	-86.2587	-430.1434	-0.0000	-0.0000	0.0000	-0.0000
4	0	0	727.0608	718.3318	1209.8296	-1193.0804	4.4585	-62.4435
5	0	0	-211.9126	41.6055	4.7176	346.9106	4.1683	19.2708
6	0	0	228.5893	-25.3987	-34.1563	-256.9403	-7.1305	-7.8846
7	0	0	-146.2929	96.7089	104.5142	156.8829	10.1052	4.1389
8	0	0	16.6290	-70.0961	-75.4031	-21.4159	-5.1518	0.7114
9	0	0	0.0076	56.6001	50.9787	-3.4319	2.4647	-2.0367
10	0	0	-18.2119	-33.8711	-29.2623	17.5199	-1.1573	2.2910
11	0	0	19.7151	8.1139	7.0920	-18.0331	-0.1560	-1.5206
12	0	0	-15.2318	-0.5679	0.9553	12.5697	0.6249	0.8126
13	0	0	9.0409	-4.8708	-4.6395	-7.0993	-0.6756	-0.3321
14	0	0	-2.8680	6.0194	5.1815	2.1245	0.5027	-0.0581
15	0	0	0.0546	-4.6413	-3.7536	0.4394	-0.2825	0.2190
16	0	0	1.6686	2.7666	2.1013	-1.5645	0.1043	-0.2365
17	0	0	-2.0506	-0.9682	-0.6626	1.7236	0.0262	0.1831
18	0	0	1.5886	-0.0717	-0.2072	-1.2785	-0.0846	-0.1046
19	0	0	-0.9470	0.6482	0.5896	0.7116	0.0915	0.0358
20	0	0	0.3344	-0.7647	-0.6330	-0.2205	-0.0713	0.0123
21	0	0	0.0619	0.5945	0.4745	-0.0952	0.0408	-0.0347

Table V (Continued)
Coefficients in the Series for the Perturbations of Hilda to Jupiter

i	j	k	$\alpha.10^6$ cos	$\alpha.10^6$ sin	$\beta.10^6$ cos	$\beta.10^6$ sin	$\gamma.10^6$ cos	$\gamma.10^6$ sin
22	0	0	-0.2670	-0.3515	-0.2628	0.2357	-0.0132	0.0374
23	0	0	0.3036	0.1205	0.0780	-0.2481	-0.0058	-0.0291
24	0	0	-0.2362	0.0360	0.0435	0.1864	0.0148	0.0166
25	0	0	0.1379	-0.1136	-0.0978	-0.1026	-0.0158	-0.0051
26	0	0	-0.0452	0.1257	0.1015	0.0289	0.0123	-0.0027
27	0	0	-0.0185	-0.0976	-0.0761	0.0199	-0.0069	0.0065
28	0	0	0.0494	0.0562	0.0416	-0.0417	0.0021	-0.0069
29	0	0	-0.0536	-0.0176	-0.0110	0.0428	0.0013	0.0053
30	0	0	0.0414	-0.0091	-0.0092	-0.0320	-0.0029	-0.0030
31	0	0	-0.0236	0.0218	0.0182	0.0173	0.0030	0.0008
32	0	0	0.0070	-0.0233	-0.0185	-0.0043	-0.0023	0.0006
33	0	0	0.0044	0.0179	0.0137	-0.0043	0.0013	-0.0013
34	0	0	-0.0098	-0.0101	-0.0074	0.0081	-0.0004	0.0014
35	0	0	0.0103	0.0028	0.0017	-0.0081	-0.0003	-0.0010
36	0	0	-0.0079	0.0021	0.0020	0.0060	0.0006	0.0006
37	0	0	0.0044	-0.0044	-0.0036	-0.0032	-0.0006	-0.0001
38	0	0	-0.0012	0.0046	0.0036	0.0007	0.0005	-0.0001
39	0	0	-0.0010	-0.0035	-0.0027	0.0010	-0.0003	0.0003
40	0	0	0.0020	0.0019	0.0014	-0.0017	0.0001	-0.0003
41	0	0	-0.0021	-0.0005	-0.0003	0.0016	0.0001	0.0002
42	0	0	0.0016	-0.0005	-0.0005	-0.0012	-0.0001	-0.0001
43	0	0	-0.0009	0.0009	0.0008	0.0006	0.0001	0.0000
44	0	0	0.0002	-0.0010	-0.0007	-0.0001	-0.0001	0.0000
45	0	0	0.0002	0.0007	0.0005	-0.0002	0.0001	-0.0001
46	0	0	-0.0004	-0.0004	-0.0003	0.0004	-0.0000	0.0001
47	0	0	0.0004	0.0001	0.0000	-0.0003	-0.0000	-0.0000
48	0	0	-0.0003	0.0001	0.0001	0.0002	0.0000	0.0000
49	0	0	0.0002	-0.0002	-0.0002	-0.0001	-0.0000	-0.0000
50	0	0	-0.0000	0.0002	0.0002	0.0000	0.0000	-0.0000
51	0	0	-0.0001	-0.0002	-0.0001	0.0001	-0.0000	0.0000
52	0	0	0.0001	0.0001	0.0001	-0.0001	0.0000	-0.0000
53	0	0	-0.0001	-0.0000	-0.0000	0.0001	0.0000	0.0000
54	0	0	0.0001	-0.0000	-0.0000	-0.0001	-0.0000	-0.0000
0	0	1	95.9995	0.	0.	0.	-4.8773	0.
1	0	1	5.7703	5.8104	10.9310	-17.2214	-0.0497	0.3242
2	0	1	-1.0392	-0.3414	-8.7731	7.1091	-0.1154	-0.0403
3	0	1	-564.2549	92.2646	185.1807	1127.3962	20.0066	31.7850
4	0	1	1.1026	1.8002	-0.2593	-3.3625	0.0734	0.1157
5	0	1	-2.1400	3.3088	-0.5565	0.3601	-0.1196	0.0147
6	0	1	11.2625	-42.9095	-42.9219	-11.5484	-2.8424	0.4044
7	0	1	-0.6873	-0.1982	0.6858	-0.3163	-0.0040	-0.0392
8	0	1	-0.7996	-1.0232	-0.0838	0.1660	0.0110	-0.0084
9	0	1	4.0918	3.1953	2.1231	-2.8368	0.0842	-0.3227
10	0	1	0.1513	-0.1104	0.0840	0.1904	0.0145	0.0055
11	0	1	0.3233	-0.1223	-0.0931	-0.0432	-0.0045	0.0037
12	0	1	-0.6156	0.3736	0.2387	0.3072	0.0385	0.0254
13	0	1	-0.0153	0.0396	-0.0604	0.0286	-0.0028	0.0049
14	0	1	0.0167	0.0825	0.0194	-0.0393	-0.0010	-0.0032
15	0	1	-0.0332	-0.1003	-0.0402	0.0349	-0.0045	0.0050
16	0	1	-0.0010	-0.0196	-0.0123	-0.0219	-0.0019	-0.0011
17	0	1	-0.0192	0.0067	0.0166	0.0080	0.0016	-0.0003
18	0	1	0.0131	-0.0079	-0.0118	-0.0041	-0.0010	-0.0003
19	0	1	0.0102	0.0051	0.0088	-0.0058	0.0004	-0.0009
20	0	1	-0.0050	-0.0047	-0.0033	0.0074	0.0001	0.0008
21	0	1	0.0049	0.0007	-0.0002	-0.0055	-0.0002	-0.0004
22	0	1	-0.0037	0.0045	0.0028	0.0038	0.0004	0.0002
23	0	1	0.0015	-0.0032	-0.0034	-0.0014	-0.0004	0.0001
24	0	1	0.0004	0.0029	0.0027	-0.0004	0.0002	-0.0002
25	0	1	-0.0020	-0.0020	-0.0017	0.0014	-0.0001	0.0002
26	0	1	0.0018	0.0006	0.0006	-0.0016	-0.0000	-0.0002
27	0	1	-0.0015	0.0003	0.0002	0.0013	0.0001	0.0001
28	0	1	0.0010	-0.0009	-0.0007	-0.0008	-0.0001	-0.0000

Table V (Continued)

Coefficients in the Series for the Perturbations of Hilda Due to Jupiter

i	j	k	$\alpha.10^6$ cos	$\alpha.10^6$ sin	$\beta.10^6$ cos	$\beta.10^6$ sin	$\gamma.10^6$ cos	$\gamma.10^6$ sin
29	0	1	-0.0003	0.0009	0.0008	0.0002	0.0001	-0.0000
30	0	1	-0.0002	-0.0008	-0.0006	0.0001	-0.0001	0.0000
31	0	1	0.0004	0.0005	0.0004	-0.0003	0.0000	-0.0001
32	0	1	-0.0005	-0.0001	-0.0001	0.0004	0.0000	0.0000
33	0	1	0.0004	-0.0001	-0.0001	-0.0003	-0.0000	-0.0000
34	0	1	-0.0002	0.0002	0.0002	0.0002	0.0000	0.0000
35	0	1	0.0001	-0.0002	-0.0002	-0.0000	-0.0000	0.0000
36	0	1	0.0000	0.0002	0.0001	-0.0000	0.0000	-0.0000
37	0	1	-0.0001	-0.0001	-0.0001	0.0001	-0.0000	0.0000
38	0	1	0.0001	0.0000	0.0000	-0.0001	-0.0000	-0.0000
39	0	1	-0.0001	0.0000	0.0000	0.0001	0.0000	0.0000
40	0	1	0.0001	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000
41	0	1	-0.0000	0.0001	0.0000	0.0000	0.0000	-0.0000
0	0	2	-0.5686	0.	-53.7978	0.	-0.0252	0.
1	0	2	-0.0831	0.0701	0.6767	-0.5963	0.0101	-0.0142
2	0	2	-0.1962	-0.1008	0.4002	-0.4359	0.0017	0.0143
3	0	2	-0.1275	0.1611	0.6963	0.4037	-0.0232	0.0017
4	0	2	-0.0434	-0.0090	0.1174	0.1888	0.0022	0.0007
5	0	2	0.1065	-0.0841	-0.0652	-0.0136	-0.0018	0.0006
6	0	2	0.1753	-0.0209	0.0438	0.0957	0.0078	0.0058
7	0	2	-0.0016	0.0016	-0.0329	0.0159	-0.0003	0.0011
8	0	2	0.0124	0.0327	0.0064	-0.0181	-0.0003	-0.0008
9	0	2	-0.0219	0.0678	-0.0129	0.0090	-0.0018	0.0012
10	0	2	-0.0004	-0.0048	-0.0041	-0.0087	-0.0005	-0.0002
11	0	2	-0.0080	0.0019	0.0059	0.0025	0.0004	-0.0001
12	0	2	-0.0164	-0.0148	-0.0032	-0.0015	-0.0002	-0.0003
13	0	2	0.0029	0.0010	0.0028	-0.0015	0.0001	-0.0002
14	0	2	-0.0009	-0.0018	-0.0010	0.0022	0.0000	0.0002
15	0	2	0.0059	-0.0028	0.0001	-0.0014	-0.0000	-0.0001
16	0	2	-0.0009	0.0013	0.0007	0.0011	0.0001	0.0000
17	0	2	0.0005	-0.0006	-0.0009	-0.0004	-0.0001	0.0000
18	0	2	0.0003	0.0020	0.0007	-0.0000	0.0000	-0.0000
19	0	2	-0.0005	-0.0005	-0.0004	0.0003	-0.0000	0.0000
20	0	2	0.0004	0.0002	0.0002	-0.0004	-0.0000	-0.0000
21	0	2	-0.0006	-0.0000	0.0000	0.0003	0.0000	0.0000
22	0	2	0.0003	-0.0002	-0.0001	-0.0002	-0.0000	-0.0000
23	0	2	-0.0001	0.0002	0.0002	0.0001	0.0000	-0.0000
24	0	2	0.0000	-0.0002	-0.0001	0.0000	-0.0000	0.0000
25	0	2	0.0001	0.0001	0.0001	-0.0001	0.0000	-0.0000
26	0	2	-0.0001	-0.0000	-0.0000	0.0001	0.0000	0.0000
27	0	2	0.0001	0.0000	-0.0000	-0.0001	-0.0000	-0.0000
28	0	2	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0	0	3	0.0014	0.	0.2027	0.	0.0002	0.
1	0	3	0.0003	-0.0006	0.0046	0.0075	-0.0000	0.0001
2	0	3	0.0005	0.0002	-0.0025	0.0006	0.0000	-0.0001
3	0	3	0.0045	0.0201	-0.0597	0.0003	-0.0011	0.0006
4	0	3	-0.0001	0.0000	-0.0000	-0.0009	-0.0000	-0.0000
5	0	3	-0.0008	-0.0001	0.0003	-0.0002	0.0000	-0.0000
6	0	3	-0.0109	-0.0037	0.0012	-0.0033	-0.0000	-0.0002
7	0	3	0.0001	-0.0002	0.0002	0.0001	0.0000	-0.0000
8	0	3	0.0002	-0.0002	-0.0000	0.0001	0.0000	0.0000
9	0	3	0.0023	-0.0024	0.0003	0.0003	0.0000	0.0000
10	0	3	0.0000	0.0001	-0.0001	0.0001	0.0000	0.0000
11	0	3	0.0000	0.0001	-0.0000	-0.0000	-0.0000	-0.0000
12	0	3	0.0003	0.0008	-0.0000	0.0000	-0.0000	0.0000
15	0	3	-0.0002	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0	0	4	-0.0015	0.	0.0036	0.	-0.0000	0.
1	0	4	0.0000	-0.0000	-0.0003	0.0004	-0.0000	0.0000
3	0	4	0.0003	-0.0007	0.0002	0.0000	0.0000	-0.0000
6	0	4	0.0001	0.0002	-0.0000	0.0000	-0.0000	0.0000
9	0	4	-0.0001	0.0000	-0.0000	-0.0000	-0.0000	-0.0000

REFERENCES

1. Carpenter, L.,"Planetary Perturbations in Chebyshev Series", NASA Technical Note D-3168, 1966.

2. Carpenter, L., "Computation of General Planetary Perturbations, Part III – An Expansion of the Disturbing Force", NASA Technical Note D-3078, 1966.

3. Hagihara, Yusuke, "Gaps in the Distribution of Asteroids", Smithsonian Contributions to Astrophysics, Vol. 5, No. 6, pp. 59-67.

4. Musen, P. and Carpenter, L., "On the General Planetary Perturbations in Rectangular Coordinates", J. Geophysical Research 68(9): 2727-2734, May 1, 1963.

5. Schubart, J., "Long Period Effects in Nearly Commensurable Cases of the Restricted Three Body Problem", Cambridge, Massachusetts: Smithsonian Astrophysical Observatory, SAO Special Report 149, April 15, 1964.