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CROSS SECTION FOR ENERGY TRANSFER BETWEEN TWO MOVING PARTICLES

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Cross Section for Energy Transfer Between Two Moving Particles

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Abstract

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The classical cross section $\sigma_{\Delta E}$, for producing a specified energy transfer ΔE in the collision of two particles 1,2 having arbitrary masses and velocities $\mathbf{v}_1, \mathbf{v}_2$ in the laboratory system, is derived. The effective average (for fixed speeds v_1, v_2) of $\sigma_{\Delta E}$ over all directions of the particle velocities $\mathbf{v}_1$ and/or $\mathbf{v}_2$ then is computed. These results are required in the classical calculations of atomic collision cross sections via the procedures recently proposed by Gryzinski. The method will yield the average of any $F(v,V, \cos\bar{\theta})$ over all directions of the particle velocities, where $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$; $\mathbf{V}$ is the velocity of the center of mass; and $\bar{\theta}$ is the angle between $\mathbf{v}$ and $\mathbf{V}$. Author

I. Introduction

Recently Gryzinski has published three long papers[1-3] detailing his procedures for performing classical (non-quantum) calculations of atomic collision cross sections. The utility of these procedures in electron-atom and electron-molecule collisions has been examined by Bauer and Bartky.[4] For such collisions one requires the cross section $\sigma_{\Delta E}(\mathbf{v}_1, \mathbf{v}_2)$ for producing an energy transfer ΔE in the collision of two electrons moving with arbitrary velocities $\mathbf{v}_1, \mathbf{v}_2$ in the laboratory system. There also is required $\sigma_{\Delta E}^{eff}(v_1,v_2)$ the effective average of $\sigma_{\Delta E}(\mathbf{v}_1, \mathbf{v}_2)$ over all orientations of $\mathbf{v}_1$ and/or $\mathbf{v}_2$ for fixed speeds v_1, v_2. Gryzinski has derived expressions for these quantities, but use of these formulas is complicated by an extremely awkward notation; moreover Gryzinski's expressions involve some subsidiary approximations. For these reasons, Stabler[5] has rederived--and obtained in much simpler form--the exact expressions for $\langle\sigma_{\Delta E}\rangle_{eff}$ and $\sigma_{\Delta E}$ in electron-electron collisions. Similar expressions have been obtained by Ochkur and Petrun'kin.[6] However, these authors[5,6] have rederived $\sigma_{\Delta E}$ only for electron-electron collisions, i.e., for colliding particles of equal mass, whereas for calculations of, e.g., ion-atom collisions by Gryzinski's procedures one requires $\sigma_{\Delta E}$ and $\sigma_{\Delta E}^{eff}$ for collisions of unequally massive charged particles.

This paper derives the required exact formulas for $\sigma_{\Delta E}$ and $\sigma_{\Delta E}^{eff}$ in the unequal mass case. Application of these formulas to examination of the utility of Gryzinski's procedures in charge transfer reactions is under way (in cooperation with Hsiang Tai and Jean Welker). This paper obtains the final formula for $\sigma_{\Delta E}^{eff}(v_1,v_2)$ in only one case, namely Coulomb collisions; it will be clear, however, that the method of performing the average over all orientations in applicable to arbitrary interactions, as well as to the averages of quantities other than $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$.

II. Calculation of $\sigma_{\Delta E}$

I consider a collision between particles 1 and 2, whose initial velocities in the laboratory system are $\underset{\sim}{v}_1 = v_1\underset{\sim}{n}_1$ and $\underset{\sim}{v}_2 = v_2\underset{\sim}{n}_2$ respectively. Their laboratory velocities after the collision will be $\underset{\sim}{v}_1' = v_1'\underset{\sim}{n}_1'$ and $\underset{\sim}{v}_2' = v_2'\underset{\sim}{n}_2'$. Correspondingly, the velocities of these particles measured by an observer moving with the center of mass are $\bar{\underset{\sim}{v}}_1 = \bar{v}_1\bar{\underset{\sim}{n}}_1$, $\bar{\underset{\sim}{v}}_2 = \bar{v}_2\bar{\underset{\sim}{n}}_2$ (initial) and $\bar{\underset{\sim}{v}}_1' = \bar{v}_1'\bar{\underset{\sim}{n}}_1'$, $\bar{\underset{\sim}{v}}_2' = \bar{v}_2'\bar{\underset{\sim}{n}}_2'$ (final). It is presumed that the coordinate axes of the laboratory and center of mass observers are parallel, so that the components of the vectors defined above are consistent with

$$\underset{\sim}{v}_1 = \underset{\sim}{V} + \bar{\underset{\sim}{v}}_1, \text{ etc.} \tag{1}$$

where $\underset{\sim}{V}$ is the center of mass velocity measured by the laboratory observer.

$$\begin{aligned} \underset{\sim}{V} &= V\underset{\sim}{n}_V = M^{-1}(m_1\underset{\sim}{v}_1 + m_2\underset{\sim}{v}_2) \\ M &= m_1 + m_2 \end{aligned} \tag{2}$$

Also

$$\begin{aligned} \bar{\underset{\sim}{v}}_1 &= m_2M^{-1}\underset{\sim}{v} \\ \bar{\underset{\sim}{v}}_2 &= -m_1M^{-1}\underset{\sim}{v} \end{aligned} \tag{3a}$$

$$\begin{aligned} \bar{\underset{\sim}{v}}_1' &= m_2M^{-1}\underset{\sim}{v}' \\ \bar{\underset{\sim}{v}}_2' &= -m_1M^{-1}\underset{\sim}{v}' \end{aligned} \tag{3b}$$

where

$$\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{v}_1 - \underset{\sim}{v}_2 = v\bar{\underset{\sim}{n}} \\ \underset{\sim}{v}' &= \underset{\sim}{v}_1' - \underset{\sim}{v}_2' = v\bar{\underset{\sim}{n}}' \end{aligned} \tag{4}$$

are the relative velocities before and after the collision.

For given $\mathbf{v}_1, \mathbf{v}_2$ the vectors $\mathbf{V}, \mathbf{v}$ are determined, so that for given $\mathbf{v}_1, \mathbf{v}_2$ the polar axis of a fixed system of spherical coordinates can be chosen along $\mathbf{V}$; in this system the polar and azimuth angles of $\bar{\mathbf{n}}$ and $\bar{\mathbf{n}}'$ are $\bar{\theta}, \bar{\phi}$ and $\bar{\theta}', \bar{\phi}'$ respectively. Now suppose 1 is regarded as the "incident" particle. Then the energy gain ΔE by particle 2 (as seen in the laboratory system) is[7]

$$\Delta E = \frac{1}{2} m_2 v_2'^2 - \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_1 v_1'^2$$
$$= m_2 \mathbf{V} \cdot (\bar{\mathbf{v}}_2' - \bar{\mathbf{v}}_2) = \mu v V (\cos \bar{\theta} - \cos \bar{\theta}'), \quad (5)$$

where $\mu = m_1 m_2 M^{-1}$ is the reduced mass. Eq. (5) shows that for given $\mathbf{v}_1, \mathbf{v}_2$ the quantity ΔE is a function only of $\bar{\theta}'$. In fact

$$d(\Delta E) = \mu v V \sin \bar{\theta}' \, d\bar{\theta}' \quad (6)$$

Let $\sigma(\mathbf{v}_1, \mathbf{v}_2)$ be the total cross section for given $\mathbf{v}_1, \mathbf{v}_2$. Then the quantity $\sigma_{\Delta E}(\mathbf{v}_1, \mathbf{v}_2)$ is defined by

$$\sigma(\mathbf{v}_1, \mathbf{v}_2) = \int d(\Delta E) \sigma_{\Delta E}(\mathbf{v}_1, \mathbf{v}_2) \quad (7)$$

But if $\bar{\sigma}(v; \bar{\mathbf{n}} \rightarrow \bar{\mathbf{n}}')$ is the corresponding differential cross section for scattering in the center of mass system (wherein the collision can change only the direction but not the magnitude of the relative velocity), it also is true that

$$\sigma(\mathbf{v}_1, \mathbf{v}_2) = \int d\bar{\mathbf{n}}' \bar{\sigma}(v; \bar{\mathbf{n}} \rightarrow \bar{\mathbf{n}}') \quad (8a)$$

$$= \int d\bar{\phi}' d\bar{\theta}' \sin \bar{\theta}' \; \bar{\sigma}(v; \bar{\mathbf{n}} \rightarrow \bar{\mathbf{n}}') \quad (8b)$$

$$= \frac{1}{\mu v V} \int d(\Delta E) d\bar{\phi}' \bar{\sigma}(v; \bar{\mathbf{n}} \rightarrow \bar{\mathbf{n}}'), \quad (8c)$$

using (6).

Eqs. (7) and (8c) imply

$$\sigma_{\Delta E}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = \frac{1}{\mu v V} \int d\bar{\phi}' \bar{\sigma}(v; \underset{\sim}{\bar{n}} \to \underset{\sim}{\bar{n}}') \tag{9}$$

For fixed $\underset{\sim}{v}_1, \underset{\sim}{v}_2$, i.e., for fixed $\bar{\theta}, \bar{\phi}$, the right side of (9) is a function of $\bar{\theta}'$ and therefore, by (6), of ΔE. For every value of $\bar{\theta}'$ the integral in Eq. (9) runs over all values of $\bar{\phi}'$ from 0 to 2π, because (for any initial $\bar{\theta}, \bar{\phi}$) the final relative velocity $\underset{\sim}{v}'$ can have any direction in space. The cross section $\bar{\sigma}(v; \underset{\sim}{\bar{n}} \to \underset{\sim}{\bar{n}}')$, though dependent only on the angle between $\underset{\sim}{\bar{n}}$ and $\underset{\sim}{\bar{n}}'$, can be a function of $\bar{\phi}'$.

The results so far hold for any $\bar{\sigma}$. For definiteness, I now specialize to the Coulomb case,

$$\bar{\sigma}(v; \underset{\sim}{\bar{n}} \to \underset{\sim}{\bar{n}}') = \left(\frac{Z_1 Z_2 e^2}{2\mu v^2}\right)^2 \csc^4 \frac{1}{2}\chi \tag{10}$$

where the center of mass system scattering angle χ is the angle between $\underset{\sim}{\bar{n}}$ and $\underset{\sim}{\bar{n}}'$; and $Z_1 e$, $Z_2 e$ are the charges carried by particles 1,2. Substituting Eq. (10) in Eq. (9), and employing

$$\sin^4 \frac{1}{2}\chi = \frac{1}{4}(1 - \cos\chi)^2 \tag{11a}$$

$$\cos\chi = \cos\bar{\theta}\cos\bar{\theta}' + \sin\bar{\theta}\sin\bar{\theta}'\cos(\bar{\phi} - \bar{\phi}') \tag{11b}$$

one finds

$$\sigma_{\Delta E}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = \frac{1}{\mu v V}\left(\frac{Z_1 Z_2 e^2}{\mu v^2}\right)^2 \int_0^{2\pi} d\phi \, \frac{1}{(a - b\cos\phi)^2} \tag{12}$$

where

$$a = 1 - \cos\bar{\theta}\cos\bar{\theta}'$$
$$b = \sin\bar{\theta}\sin\bar{\theta}' \tag{13}$$

When[8] $a^2 \geqslant b^2$, as is the case for a,b of (13)

$$\int_0^{2\pi} d\phi \frac{1}{(a-b\cos\phi)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}} \tag{14}$$

Thus

$$\sigma_{\Delta E}(\underset{\sim}{v}_1,\underset{\sim}{v}_2) = \frac{2\pi}{\mu v V}\left(\frac{Z_1 Z_2 e^2}{\mu v^2}\right)^2 \frac{(1 - \cos\bar{\theta}\cos\bar{\theta}')}{|\cos\bar{\theta} - \cos\bar{\theta}'|^3} \tag{15}$$

Or, using (5),

$$\sigma_{\Delta E}(\underset{\sim}{v}_1,\underset{\sim}{v}_2) = \frac{2\pi(Z_1 Z_2 e^2)^2 V^2}{v^2|\Delta E|^3} \left(1 - \cos^2\bar{\theta} + \frac{\Delta E}{\mu v V}\cos\bar{\theta}\right) \tag{16a}$$

with the restriction, also from (5), that

$$-1 \leq \cos\bar{\theta} - \frac{\Delta E}{\mu v V} \leq 1 \tag{16b}$$

which guarantees $\sigma_{\Delta E} \geqslant 0$. For given $\underset{\sim}{v}_1, \underset{\sim}{v}_2$, if (16b) is not satisfied, then

$$\sigma_{\Delta E}(\underset{\sim}{v}_1,\underset{\sim}{v}_2) = 0, \tag{16c}$$

i.e., values of ΔE for which (16b) fails cannot occur.

Eqs. (16) are the desired result for $\sigma_{\Delta E}$, in what proves to be a convenient form for calculating $\langle\sigma_{\Delta E}\rangle$. In terms of $\underset{\sim}{v}_1, \underset{\sim}{v}_2$, the quantities v, V, $\cos\bar{\theta}$ are, using Eqs. (2) and (4),

$$v = (v_1^2 + v_2^2 - 2v_1 v_2 \underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2)^{1/2} \tag{17a}$$

$$V = M^{-1}(m_1^2 v_1^2 + m_2^2 v_2^2 + 2m_1 m_2 v_1 v_2 \underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2)^{1/2} \tag{17b}$$

$$\cos\bar{\theta} = (vV)^{-1}\underset{\sim}{v}\cdot\underset{\sim}{V} = (MvV)^{-1}[m_1 v_1^2 - m_2 v_2^2 + (m_2 - m_1)v_1 v_2 \underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2] \tag{17c}$$

It can be shown that in the special case $m_1 = m_2 = m$, Eqs. (16) reduce to the seemingly very different expression for $\sigma_{\Delta E}$ given by Stabler,[5] namely[9] his Eq. (8).

III. Calculation of $\sigma_{\Delta E}^{eff}$

Suppose the target particle 2 has an isotropic velocity distribution in the laboratory system. Then for any actual $\underset{\sim}{v}_1$ the effective $\sigma_{\Delta E}$ is defined by

$$v_1\sigma_{\Delta E}^{eff} = \frac{1}{4\pi}\int d\underset{\sim}{n}_2|\underset{\sim}{v}_1 - v_2\underset{\sim}{n}_2|\sigma_{\Delta E}(\underset{\sim}{v}_1,\underset{\sim}{v}_2) \tag{18}$$

This definition of the effective $\sigma_{\Delta E}$ is appropriate when, e.g., the particles 2 are bound electrons in stationary atoms being ionized by a beam of protons (particles 1). If the atoms have velocity $\underset{\sim}{v}_a \neq 0$ in the laboratory system, e.g., if the atoms form a beam, the velocity distribution of 2, though isotropic in a coordinate system moving with the atoms, is not isotropic in the laboratory system. In this event, realizing that the total reaction rate (e.g., the total rate of ionization) is independent of the observer's velocity, the simplest procedure is to compute the total reaction rate in the system where the velocity of 1 now is $\underset{\sim}{v}_1 + \underset{\sim}{v}_a$.

Once, as in (18), the distribution of $\underset{\sim}{v}_2$ is accepted as isotropic, the value of $\sigma_{\Delta E}^{eff}$ obviously cannot depend on the direction of $\underset{\sim}{n}_1$. In other words $\sigma_{\Delta E}^{eff}$ now depends only on the magnitudes of $\underset{\sim}{v}_1,\underset{\sim}{v}_2$, and so can be averaged over $\underset{\sim}{n}_1$ as well as $\underset{\sim}{n}_2$. For the Coulomb case, therefore, using (16)

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{(Z_1Z_2e^2)^2}{8\pi|\Delta E|^3 v_1}\int d\underset{\sim}{n}_1 d\underset{\sim}{n}_2 \frac{V^2}{v}\left(1 - \cos^2\bar{\theta} + \frac{\Delta E}{\mu v V}\cos\bar{\theta}\right) \tag{19}$$

where $v,V,\cos\bar{\theta}$ are given by Eqs. (17), and the allowed ranges of $\underset{\sim}{n}_1,\underset{\sim}{n}_2$ must be consistent with (16b), i.e., in (19) appear only those $\underset{\sim}{n}_1,\underset{\sim}{n}_2$ for which $\sigma_{\Delta E}(\underset{\sim}{v}_1,\underset{\sim}{v}_2) \neq 0$. Specificially, for given $v_1,v_2,\Delta E$ the integral (19) runs only over those directions $\underset{\sim}{n}_1,\underset{\sim}{n}_2$ for which

$$-1 + \frac{\Delta E}{\mu v V} \leq \cos\bar{\theta} \leq 1 \qquad \Delta E \geq 0 \tag{20a}$$

$$-1 \leq \cos\bar{\theta} \leq 1 + \frac{\Delta E}{\mu v V} \qquad \Delta E \leq 0 \tag{20b}$$

Despite its apparent complexity, the integral (19) can be evaluated in closed form. For any integrand $F(\underset{\sim}{n}_1, \underset{\sim}{n}_2, v_1, v_2)$

$$\int F d\underset{\sim}{n}_1 d\underset{\sim}{n}_2 = \frac{1}{v_1^2 v_2^2}\int v_1^2 d\underset{\sim}{n}_1 v_2^2 d\underset{\sim}{n}_2 F(\underset{\sim}{n}_1, \underset{\sim}{n}_2, v_1, v_2)$$

$$= \frac{1}{v_1^2 v_2^2}\int \hat{v}_1^2 d\underset{\sim}{n}_1 d\hat{v}_1 \hat{v}_2^2 d\underset{\sim}{n}_2 d\hat{v}_2 \delta(\hat{v}_1 - v_1)\delta(\hat{v}_2 - v_2) F(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \hat{v}_1, \hat{v}_2)$$

But $\hat{v}_1^2 d\underset{\sim}{n}_1 d\hat{v}_1$ is the volume element $d\hat{\underset{\sim}{v}}_1$ in the space formed by the components of the vector $\hat{\underset{\sim}{v}}_1 = \hat{v}_1\underset{\sim}{n}_1$. Thus Eq. (19) can be replaced by

$$\sigma_{\Delta E}^{eff} = \frac{(Z_1 Z_2 e^2)^2}{8\pi|\Delta E|^3 v_1^3 v_2^2}\int d\hat{\underset{\sim}{v}}_1 d\hat{\underset{\sim}{v}}_2 \delta(\hat{v}_1 - v_1)\delta(\hat{v}_2 - v_2)\frac{V^2}{v}(1 - \cos^2\bar{\theta} + \frac{\Delta E}{\mu v V}\cos\bar{\theta}) \tag{21}$$

with the understanding that under the integral sign $\hat{v}_1, \hat{v}_2$ now replace v_1, v_2 in Eqs. (17) for v, V, cos $\bar{\theta}$. Consequently, recalling (2) and (4), the equations relating $\hat{\underset{\sim}{v}}_1, \hat{\underset{\sim}{v}}_2$ to $\underset{\sim}{v}$, $\underset{\sim}{V}$ in (21) must be

$$\begin{aligned} \hat{\underset{\sim}{v}}_1 &= \hat{v}_1\underset{\sim}{n}_1 = \underset{\sim}{V} + m_2 M^{-1}\underset{\sim}{v} \\ \hat{\underset{\sim}{v}}_2 &= \hat{v}_2\underset{\sim}{n}_2 = \underset{\sim}{V} - m_1 M^{-1}\underset{\sim}{v} \end{aligned} \tag{22}$$

With (22), the Jacobian of the transformation from $d\hat{\underset{\sim}{v}}_1 d\hat{\underset{\sim}{v}}_2$ to $d\underset{\sim}{v}d\underset{\sim}{V}$ is unity. Hence,

$$\sigma_{\Delta E}^{eff} = \frac{(Z_1 Z_2 e^2)^2}{8\pi|\Delta E|^3 v_1^3 v_2^2}\int d\underset{\sim}{v}d\underset{\sim}{V}\delta(\hat{v}_1 - v_1)\delta(\hat{v}_2 - v_2)\frac{V^2}{v}(1 - \cos^2\bar{\theta} + \frac{\Delta E}{\mu v V}\cos\bar{\theta}) \tag{23}$$

wherein, recalling $\bar{\theta} = \cos^{-1}(\bar{n}\cdot n_v)$,

$$\hat{v}_1 = (V^2 + m_2^2M^{-2}v^2 + 2m_2M^{-1}vV\cos\bar{\theta})^{1/2} \tag{24a}$$

$$\hat{v}_2 = (V^2 + m_1^2M^{-2}v^2 - 2m_1M^{-1}vV\cos\bar{\theta})^{1/2} \tag{24b}$$

Since (20) and the integrand in (23) do not involve n_v or the azimuth angle $\bar{\Phi}$, Eq. (23) simplifies to

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^3v_2^2}\int_0^\infty dv\int_0^\infty dV\int d\bar{\theta}\,\sin\bar{\theta}$$

$$\times\, vV^4\delta(\hat{v}_1-v_1)\delta(\hat{v}_2-v_2)(1-\cos^2\bar{\theta}+\frac{\Delta E}{\mu vV}\cos\bar{\theta}) \tag{25}$$

where the limits of integration over $\bar{\theta}$ are determined by (20).

Integrate (25) over the allowed range of $\cos\bar{\theta}$, recalling that

$$\int dx f(x)\delta[g(x)] = \sum_i\left\{\left|\left(\frac{dg}{dx}\right)^{-1}\right|f(x)\right\}_{x=x_i} \tag{26}$$

where x_i are the roots of $g(x) = 0$ in the integration interval. Because of (24a), the quantity $\hat{v}_1-v_1$ as a function of $\cos\bar{\theta}$ vanishes only at

$$\cos\bar{\theta}_i = (2m_2M^{-1}vV)^{-1}(v_1^2 - V^2 - m_2^2M^{-2}v^2) \tag{27}$$

Thus

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^3v_2^2}\int dv\int dV vV^4\left(\frac{m_2M^{-1}vV}{v_1}\right)^{-1}(1-\cos^2\bar{\theta}_i+\frac{\Delta E}{\mu vV}\cos\bar{\theta}_i)$$

$$\times\,\delta\left[\left(\frac{MV^2}{m_2}+\frac{m_1v^2}{M}-\frac{m_1v_1^2}{m_2}\right)^{1/2}-v_2\right] \tag{28}$$

integrated over that portion of the first quadrant of the v,V plane for which cos $\bar{\theta}_i$ from (27) lies within the limits on cos $\bar{\theta}$ specified by (20). These restrictions on v,V implied by substituting (27) in (20) take the form, for positive or negative ΔE,

$$(V - m_2M^{-1}v)^2 \leq v_{1s}^2 \tag{29a}$$

$$v_{1g}^2 \leq (V + m_2M^{-1}v)^2 \tag{29b}$$

where, recalling Eq. (5),

$$v_{1s} = \text{smaller of } v_1, v_1'$$
$$v_{1g} = \text{greater of } v_1, v_1' \tag{30a}$$

Of course

$$v_1' = [v_1^2 - (2/m_1)(\Delta E)]^{1/2}$$
$$v_2' = [v_2^2 + (2/m_2)(\Delta E)]^{1/2} \tag{30b}$$

Eqs. (29) imply that (28) is integrated over the portion of the first quadrant of the v,V plane lying below the line (termed line (a))

$$V - m_2M^{-1}v = v_{1s}; \tag{31a}$$

lying above the line (termed line (b))

$$m_2M^{-1}v - V = v_{1s}; \tag{31b}$$

and lying above the line (termed line (c))

$$V + m_2M^{-1}v = v_{1g}. \tag{31c}$$

The shaded region in Fig. 1 is this allowed portion of the v,V plane.

The δ-function in (28) vanishes unless

$$\frac{1}{2} MV^2 + \frac{1}{2} \mu v^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = E = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{32}$$

where E is the total energy in the laboratory system. In other words, the quantities v,V in (28) indeed must have values consistent with conservation of energy. Eq. (32) is an ellipse in the v,V plane. Then, integrating (28) over V, and again using (26),

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{\pi(Z_1 Z_2 e^2)^2}{|\Delta E|^3 v_1^2 v_2} \int_{v_\ell}^{v_u} dv V_i^2 (1 - \cos^2\bar{\theta}_i + \frac{\Delta E}{\mu v V_i} \cos \bar{\theta}_i) \tag{33}$$

integrated in the range $v_\ell \le v \le v_u$ for which points v,V on the ellipse (32) lie in the shaded region of Fig. 1. Here, for given v_1, v_2

$$V_i(v) = [M^{-1}(2E - \mu v^2)]^{1/2} = [M^{-1}(m_1 v_1^2 + m_2 v_2^2 - \mu v^2)]^{1/2} \tag{34a}$$

and, in (33), V_i replaces V in the definition of (27), i.e., now

$$\cos \bar{\theta}_i = (2vV_i)^{-1}[v_1^2 - v_2^2 + M^{-1}(m_1 - m_2)v^2] \tag{34b}$$

as one expects from Eqs. (17a) and (17c).

Eqs. (34) reduce (33) to a simple integral over v, yielding finally

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{\pi(Z_1 Z_2 e^2)^2}{4|\Delta E|^3 v_1^2 v_2} \{(v_1^2 - v_2^2)(v_2'^2 - v_1'^2)(v_\ell^{-1} - v_u^{-1})$$

$$+ (v_1^2 + v_2^2 + v_1'^2 + v_2'^2)(v_u - v_\ell) - \frac{1}{3}(v_u^3 - v_\ell^3)\} \tag{35}$$

where v_1', v_2' are given by (30b). The integration limits v_ℓ, v_u in (33) and (35) remain to be determined. Otherwise, (35) is the desired result for $\sigma_{\Delta E}^{eff}$.

IV. Determination of v_ℓ, v_u

Evidently v_ℓ, v_u are the values of v at which the ellipse (32) intersects the boundaries of the shaded region in Fig. 1. From Eqs. (31c) and (32) one sees that the ellipse always has two real intersections with line (c), of which both, or only one, or neither may lie on the boundary of the shaded region, depending on the values of v_1, v_2. These intersections occur at $v = v_\gamma$ and $v = v_\delta$, given by

$$\left.\begin{aligned} v_\gamma &= v_1 - v_2 \\ v_\delta &= v_1 + v_2 \end{aligned}\right\} \quad \Delta E \geqslant 0, \text{ i.e., } v_{1g} = v_1 \tag{36a}$$

$$\left.\begin{aligned} v_\gamma &= v_1' - v_2' \\ v_\delta &= v_1' + v_2' \end{aligned}\right\} \quad \Delta E \leqslant 0, \text{ i.e., } v_{1g} = v_1' \tag{36b}$$

where $v_\gamma \leq v_\delta$. Similarly, in the first quadrant of the v,V plane lines (a) and (b) each have at most one intersection with the ellipse, at $v = v_\alpha$ and $v = v_\beta$ respectively, given by

$$\left.\begin{aligned} v_\alpha &= v_2' - v_1' \\ v_\beta &= v_2' + v_1' \end{aligned}\right\} \quad \Delta E \geqslant 0, \text{ i.e., } v_{1s} = v_1' \tag{37a}$$

$$\left.\begin{aligned} v_\alpha &= v_2 - v_1 \\ v_\beta &= v_2 + v_1 \end{aligned}\right\} \quad \Delta E \leqslant 0, \text{ i.e., } v_{1s} = v_1 \tag{37b}$$

Because the ellipse (32) is everywhere concave downward in the first quadrant, it must intersect the boundary of the shaded region no more than twice; it may not intersect the boundary of the shaded region at all. Thus, referring to Fig. 1, it is clear that the only possible limits of integration in (33) are:

$$\begin{aligned} &\text{(i)} \quad v_\ell = v_\alpha,\ v_u = v_\beta \\ &\text{(ii)} \quad v_\ell = v_\alpha,\ v_u = v_\delta \\ &\text{(iii)} \quad v_\ell = v_\gamma,\ v_u = v_\delta \\ &\text{(iv)} \quad v_\ell = v_\gamma,\ v_u = v_\beta \\ &\text{(v)} \quad \text{no intersections, } \sigma_{\Delta E}^{eff}(v_1,v_2) = 0 \end{aligned} \tag{38}$$

The conditions for the above cases to occur are[10] (again referring to Fig. 1):

(i) $v_\gamma \leq v_{ac}$, $v_{bc} \leq v_\delta$; or equivalently, $v_{ac} \leq v_\alpha$, $v_{bc} \leq v_\beta$

(ii) $v_\gamma \leq v_{ac} \leq v_\delta \leq v_{bc}$; or equivalently, $v_{ac} \leq v_\alpha$, $v_\beta \leq v_{bc}$

(iii) $v_{ac} \leq v_\gamma$, $v_\delta \leq v_{bc}$

(iv) $v_{ac} \leq v_\gamma \leq v_{bc} \leq v_\delta$, or equivalently, $v_\alpha \leq v_{ac}$, $v_{bc} \leq v_\beta$

(v) either $v_\delta \leq v_{ac}$ or $v_{bc} \leq v_\gamma$

when v_{ac} is the value of v at the intersection of lines (a) and (c); v_{bc} is the value of v at the intersection of lines (b) and (c). These values are, for positive or negative ΔE,

$$v_{ac} = (2m_2)^{-1}M(v_{1g} - v_{1s}) = (2m_2)^{-1}M|v_1 - v_1'| \tag{40a}$$

$$v_{bc} = (2m_2)^{-1}M(v_{1g} + v_{1s}) = (2m_2)^{-1}M(v_1 + v_1') \tag{40b}$$

Eqs. (36) and (40) imply[10] that for $\Delta E \geqslant 0$ cases (i) - (v) of (38) correspond to the following limits in Eq. (35), and occur under the following circumstances:

(i) $$v_\ell = v_2' - v_1' \ , \ v_u = v_2' + v_1' \tag{41a}$$

when

$$(2m_2)^{-1}[Mv_1' + |m_1 - m_2|v_1] \leq v_2 \tag{41b}$$

$$\text{or } (2m_2)^{-1}[Mv_1 + |m_1 - m_2|v_1'] \leq v_2' \tag{41c}$$

or

$$\Delta E \geq \frac{4m_1m_2}{M^2}[E_1 - E_2 + |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|] \tag{41d}$$

$$\text{and } 2m_2v_2 \geq |m_1 - m_2|v_1 \tag{41e}$$

(ii) $$v_\ell = v_2' - v_1' \ , \ v_u = v_1 + v_2 \tag{42a}$$

when $m_1 > m_2$ and

$$(2m_2)^{-1}|Mv_1' - (m_1-m_2)v_1| \leq v_2 \leq (2m_2)^{-1}[Mv_1' + (m_1-m_2)v_1] \tag{42b}$$

$$\text{or } (2m_2)^{-1}|Mv_1 - (m_1 - m_2)v_1'| \leq v_2' \leq (2m_2)^{-1}[Mv_1 + (m_1 - m_2)v_1'] \tag{42c}$$

(iii) $$v_\ell = v_1 - v_2 \ , \ v_u = v_1 + v_2 \tag{43a}$$

$$0 \leq v_2 \leq (2m_2)^{-1}[Mv_1' - |m_1 - m_2|v_1] \tag{43b}$$

$$\text{or } 0 \leq \Delta E \leq \frac{4m_1m_2}{M^2}[E_1 - E_2 - |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|] \tag{43c}$$

(iv) $$v_\ell = v_1 - v_2 \ , \ v_u = v_2' + v_1' \tag{44a}$$

when $m_1 < m_2$ and

$$(2m_2)^{-1}|Mv_1' - (m_2-m_1)v_1| \leq v_2 \leq (2m_2)^{-1}[Mv_1' + (m_2-m_1)v_1] \tag{44b}$$

$$\text{or } (2m_2)^{-1}|Mv_1 - (m_2-m_1)v_1'| \leq v_2' \leq (2m_2)^{-1}[Mv_1 + (m_2-m_1)v_1'] \tag{44c}$$

(v) $\sigma_{\Delta E}^{eff}(v_1,v_2) = 0$ (45a)

when $0 \lesssim v_2 \lesssim (2m_2)^{-1}[|m_1 - m_2|v_1 - Mv_1']$ (45b)

or
$$\left\{ \begin{array}{l} \Delta E \gtrsim \frac{4m_1m_2}{M^2}[E_1 - E_2 + |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|] \quad (45c) \\ \text{and } 2m_2v_2 \lesssim |m_1 - m_2|v_1 \quad (45d) \end{array} \right\}$$

Eq. (41b) is obtained from the first set of conditions for case (i) in (39), namely from $v_\gamma \lesssim v_{ac}$, $v_{bc} \lesssim v_\delta$; Eq. (41c) is obtained from the equivalent set $v_{ac} \lesssim v_\alpha$, $v_{bc} \lesssim v_\beta$. Thus Eq. (41b) and (41c) must be equivalent statements of the same restriction on the values of v_1, v_2, ΔE, i.e., if either of (41b), (41c) holds for given v_1, v_2, ΔE then both of them must hold. Indeed, the equivalence of (42b) and (42c) can be demonstrated[10] directly, without reference to their common genesis in (39). The pair of equations (41d), (41e) is inferred from (41b); the pair (41d), (41e) (but not (41d) alone) furnishes another equivalent statement of the restriction imposed by (41b) and (41c). Similarly,[10] (42b) is equivalent to (42c); (43b) is equivalent to (43c); (44b) is equivalent to (44c); (45b) is equivalent to the pair (45c), (45d).

Eqs. (30a) and (36) - (40) show that for $\Delta E \lesssim 0$ the limits for cases (i) - (v) are obtained from those for $\Delta E \gtrsim 0$ simply by interchanging primed and unprimed quantities, i.e., by interchange of v_1, v_1' and of v_2, v_2'. With these interchanges, Eqs. (41a) - (41c) immediately yield[10] the limits and equivalent conditions for case (i) when $\Delta E \lesssim 0$; in fact the pair (41b), (41c) are invariant under these interchanges, i.e., (41b) and (41c) are equally valid for $\Delta E \gtrsim 0$ and $\Delta E \lesssim 0$. An almost equally trivial argument[10] shows (41d) retains the same form when $\Delta E \lesssim 0$. For $\Delta E \lesssim 0$, however, the subsidiary condition (41e) is implied[10] by (41d). In other words, for $\Delta E \lesssim 0$ Eqs. (41b), (41c) and (41d) (now without (41e)) remain equivalent conditions for case (i), although the limits v_ℓ, v_u in case (i) are not the same for $\Delta E \geq 0$ and $\Delta E \leq 0$. Similarly,[10] although the limits are trivially changed by the interchange of v_1, v_1' and of v_2, v_2' in cases (ii) - (iv), Eqs. (42b), (42c), (44b) and (44c) are equally valid for positive and negative ΔE. The cited interchange does not leave Eqs. (43b) and (45b) invariant, but does yield the correct conditions for cases (iii) and (v) respectively when $\Delta E \lesssim 0$. But the ΔE conditions corresponding to (43c) and (45c) for $\Delta E \lesssim 0$ have rather different forms[10] for positive and negative ΔE, which forms are best obtained from the $\Delta E \lesssim 0$ analogues of Eqs. (48) - (52) below. Of course, because v_1', v_2' in (30b) must be real, Eq. (45a) holds unless

$$-\frac{1}{2} m_2 v_2^2 \leq \Delta E \leq \frac{1}{2} m_1 v_1^2 \qquad (46)$$

which expresses the fact that the particle losing energy in the collision cannot lose more than its initial kinetic energy.

Eqs. (35) and (41) - (46) complete the specification of $\sigma_{\Delta E}^{eff}(v_1, v_2)$. As they stand, however, Eqs. (41) - (45) are somewhat inconvenient when one wishes to study the dependence of $\sigma_{\Delta E}^{eff}$ on v_1 for fixed v_2, as, e.g., when studying the ionization of bound electrons 2 by incident protons 1. For this purpose it proves more convenient to eliminate first the $\delta(\hat{v}_2 - v_2)$ in (25), rather than--as previously--the $\delta(\hat{v}_1 - v_1)$. In this event Eq. (35) still holds, and there are again five different sets of limits (38), but now[10]

$$\left.\begin{aligned} v_\alpha &= v_1 - v_2 \\ v_\beta &= v_1 + v_2 \\ v_\gamma &= v_2' - v_1' \\ v_\delta &= v_2' + v_1' \end{aligned}\right\} \quad \Delta E \geq 0 \qquad (47a)$$

$$v_\alpha = v_1' - v_2'$$

$$v_\beta = v_1' + v_2' \qquad\qquad \Delta E \lessgtr 0 \qquad\qquad (47b)$$

$$v_\gamma = v_2 - v_1$$

$$v_\delta = v_2 + v_1$$

Comparing Eqs. (36) and (37) with Eqs. (47), it appears that an alternative set of limits and conditions for cases (i) - (v) of (38) when $\Delta E \gtrless 0$ can be obtained from Eqs. (41) - (45) by interchange first of primed and unprimed particle velocities, and then by interchange of the subscripts 1 and 2, i.e., by interchange of v_1, v_2', of v_2, v_1', and of m_1, m_2. In fact, it can be seen[10] that elimination first of $\delta(\hat{v}_2 - v_2)$ in (25) leads to the following set of limits in (35), to be inserted under the following circumstances (for $\Delta E \gtrless 0$):

(i) $$v_\ell = v_1 - v_2, \; v_u = v_1 + v_2 \qquad (48a)$$

when
$$(2m_1)^{-1}[Mv_2 + |m_1 - m_2| v_2'] \leqslant v_1' \qquad (48b)$$

or
$$(2m_1)^{-1}[Mv_2' + |m_1 - m_2| v_2] \leqslant v_1 \qquad (48c)$$

or
$$\Delta E \leqslant \frac{4m_1 m_2}{M^2} [E_1 - E_2 - |E_2 \frac{v_1}{v_2} - E_1 \frac{v_2}{v_1}|] \qquad (48d)$$

(ii) $$v_\ell = v_1 - v_2, \; v_u = v_2' + v_1' \qquad (49a)$$

when $m_2 > m_1$ and

$$(2m_1)^{-1}|Mv_2 - (m_2 - m_1)v_2'| \leqslant v_1' \leqslant (2m_1)^{-1}[Mv_2 + (m_2 - m_1)v_2'] \qquad (49b)$$

or
$$(2m_1)^{-1}|Mv_2' - (m_2 - m_1)v_2| \leqslant v_1 \leqslant (2m_1)^{-1}[Mv_2' + (m_2 - m_1)v_2] \qquad (49c)$$

(iii) $$v_\ell = v_2' - v_1',\ v_u = v_2' + v_1' \tag{50a}$$

$$0 \leqslant v_1' \leqslant (2m_1)^{-1}[Mv_2 - |m_1 - m_2|v_2'] \tag{50b}$$

or
$$\left\{\begin{array}{l} \Delta E \geqslant 4m_1m_2[E_1 - E_2 + |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|] \quad (50c) \\ \text{and } 2m_2v_2 \geqslant |m_1 - m_2|v_1 \quad (50d) \end{array}\right\}$$

(iv) $$v_\ell = v_2' - v_1',\ v_u = v_1 + v_2 \tag{51a}$$

when $m_2 < m_1$ and

$$(2m_1)^{-1}|Mv_2 - (m_1-m_2)v_2'| \leqslant v_1' \leqslant (2m_1)^{-1}[Mv_2 + (m_1-m_2)v_2'] \tag{51b}$$

or
$$(2m_1)^{-1}|Mv_2' - (m_1-m_2)v_2| \leqslant v_1 \leqslant (2m_1)^{-1}[Mv_2' + (m_1-m_2)v_2] \tag{51c}$$

(v) $$\sigma_{\Delta E}^{eff}(v_1,v_2) = 0 \tag{52a}$$

when

$$0 \leqslant v_1' \leqslant (2m_1)^{-1}[|m_1-m_2|v_2' - Mv_2] \tag{52b}$$

or
$$\left\{\begin{array}{l} \Delta E \geqslant \frac{4m_1m_2}{M^2}[E_1 - E_2 + |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|] \\ \text{and } 2m_2v_2 \leqslant |m_1 - m_2|v_1 \end{array}\right\} \tag{52c}$$

Comparison of Eqs. (41) - (45) and (48) - (52) shows that the limits (41a) and (50a) are identical. In other words, the present case (iii), obtained by eliminating first the $\delta(\hat{v}_2-v_2)$ in (25), must be identical with the previous case (i), obtained by eliminating first the $\delta(\hat{v}_1-v_1)$. Correspondingly, the equivalent conditions (41b), (41c) and the pair (41d), (41e), must each be equivalent to (50b) and to the pair (50c), (50d). These equivalences can be proved.[10] Similarly it can be proved[10] that Eqs. (42) are equivalent to Eqs. (51); Eqs. (43) are equivalent to (48); Eqs. (44) are equivalent to (49); Eqs. (45) to (52).

Eqs. (48) - (52) pertain only when $\Delta E \geq 0$. As previously, however, limits v_u, v_ℓ and conditions when $\Delta E \leq 0$ are obtainable from Eqs. (48) - (52) by interchange of v_1, v_1' and of v_2, v_2'. In summary,[10] for $\Delta E \leq 0$:

(i) $v_\ell = v_2 - v_2, v_u = v_2 + v_1$

when Eq. (41b); or (41c); or (41d);

or $0 \leq v_1 \leq (2m_1)^{-1}[Mv_2' - |m_1 - m_2|v_2]$

or $\frac{4m_1m_2}{M^2}\,[E_1 - E_2 + |E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2}|\,] \leq \Delta E$

(ii) $v_\ell = v_2 - v_1,\ v_u = v_1' + v_2'$

when $m_1 > m_2$ and

Eq. (42b); or (42c); or (51b); or (51c).

(iii) $v_\ell = v_1' - v_2',\ v_u = v_1' + v_2'$

when $0 \leq v_2' \leq (2m_2)^{-1}[Mv_1 - |m_1 - m_2|v_1']$

or Eq. (48b); or (48c);

or $\left\{\begin{array}{l}\text{Eq. (48d)}\\ \text{and } 2m_1v_1 \geq |m_1 - m_2|v_2\end{array}\right\}$

(iv) $v_\ell = v_1' - v_2',\ v_u = v_2 + v_1$

when $m_1 < m_2$ and

Eq. (44b); or (44c); or (49b); or (49c).

(v) $\sigma_{\Delta E}^{eff}(v_1, v_2) = 0$

when $\quad 0 \leq v_2' \leq (2m_2)^{-1}[|m_1 - m_2|v_1' - Mv_1]$

or $\quad 0 \leq v_1 \leq (2m_1)^{-1}[|m_1 - m_2|v_2 - Mv_2']$

or $\quad \left\{ \begin{array}{l} \text{Eq. (48d)} \\ \text{and } 2m_1 v_1 \leq |m_1 - m_2|v_2 \end{array} \right\}$

Comparing Eqs. (19) and (33), it is clear that the calculation of $\sigma_{\Delta E}^{eff}(v_1,v_2)$ for any (central) interaction would not be essentially different from the calculation performed here of $\sigma_{\Delta E}^{eff}$ for Coulomb collisions. Whenever, as in (10), the angular variation of $\bar{\sigma}'$ depends solely on the angle χ between $\bar{n}$ and $\bar{n}'$ $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ defined by (9) will depend only on $\cos\bar{\theta}$ and $\cos\bar{\theta}'$ But $\cos\bar{\theta}'$ then can be eliminated in favor of ΔE via (5), so that $\sigma_{\Delta E}^{eff}(v_1,v_2)$ defined by (18) will be an average over all $\mathbf{n}_1,\mathbf{n}_2$ of a $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ depending only on V,v and $\cos\bar{\theta}$, where Eqs. (17) and (20) continue to hold. Thus one will be led to a single integral involving $\sigma_{\Delta E}$ of form (33), between upper and lower limits v_ℓ, v_u given by precisely the formulas developed in this section. Similar remarks pertain to an average over all $\mathbf{n}_1,\mathbf{n}_2$ of any function of v,V, $\cos\bar{\theta}$, where these quantities obey Eqs. (17). Of course only in special cases, such as the Coulomb case, will the aforementioned integral from v_ℓ to v_u be doable in closed form.

The fact that there are four sets of limits v_ℓ, v_u, plus the case (v) $\sigma_{\Delta E}^{eff} = 0$, can be interpreted, as can restrictions like the pair (41d), (41e), but I shall not do so here. I mention that the result (35) does reduce to Stabler's[5,9] Eq. (15) when $m_1 = m_2$.

V. Additional Details and Amplifications

To keep the argument from bogging down in details, many of the assertions made in the previous section were not thoroughly justified or sufficiently discussed. This section amplifies those assertions, and provides added details of their justifications.

Eqs. (39)

The ellipse (32) has semi-axes

$$A = \left(\frac{M}{m_2} v_1^2 + \frac{M}{m_1} v_2^2\right)^{1/2} = \left(\frac{M}{m_2} v_1'^2 + \frac{M}{m_1} v_2'^2\right)^{1/2} \tag{53a}$$

$$B = \left(\frac{m_1}{M} v_1^2 + \frac{m_2}{M} v_2^2\right)^{1/2} = \left(\frac{m_1}{M} v_1'^2 + \frac{m_2}{M} v_2'^2\right)^{1/2} \tag{53b}$$

along v and V respectively. In other words, A is the intersection of the ellipse (d) with the v-axis in Fig. 1; B is the intersection of the ellipse with the V-axis.

Suppose for definiteness $\Delta E \geqslant 0$, so that $v_1' \leqslant v_1$, and $v_{1s} = v_1'$, $v_{1g} = v_1$ in Fig. 1. Then for any given ΔE it is possible that

$$B \gg v_{1g} \tag{54a}$$

or, it is possible that

$$B \ll v_{1s} \tag{54b}$$

A sufficient condition for (54a) is

$$v_2 \gg v_1 \tag{55a}$$

For (54b) it suffices that

$$m_2 v_2^2 << m_1 v_1^2$$

and $$m_1 << m_2 \qquad (55b)$$

and $$\Delta E << \frac{1}{2} m_1 v_1^2$$

The conditions (55a) and (55b) can occur. In other words, it is possible that the ellipse intersects the V-axis well above the intercept of line (c), or well below the intercept of line (a). On the other hand, as $\Delta E \rightarrow \frac{1}{2} mv_1^2$ for fixed v_1, v_2, the lines (a) and (b) in Fig. 1 coalesce. Thus it is conceivable that v_γ and v_δ both may be less than v_{ac}; or that one or both may be between v_{ac} and v_{bc}; or that both may exceed v_{bc}. There always are two real intersections v_γ, v_δ with the ellipse, however, which at worst may coincide. From (36a), this occurs only when $v_2 = 0$, and corresponds to the line (c) being tangent to the ellipse.

Now because the ellipse is everywhere concave downward, it lies above the line (c) for $v_\gamma \leq v \leq v_\delta$, and lies below line (c) for $v \lesssim v_\gamma$ and $v_\delta \leqslant v$. Thus if $v_\gamma \leqslant v_{ac}$ and $v_{bc} \leq v_\delta$, then surely $v_{ac} \lesssim v_\alpha$ and $v_{bc} \lesssim v_\beta$. Conversely if $v_{ac} \lesssim v_\alpha$ and $v_{bc} \lesssim v_\beta$, the ellipse is lying above the line (c) for $v_\alpha \lesssim v \lesssim v_\beta$, because the lines (a) and (b) slope up and (c) slopes down. Hence the ellipse must come down to (c) to the right of v_{bc}, i.e., $v_{bc} \lesssim v_\delta$; similarly, the ellipse must come up to (c) at a point to the left of v_{ac}, i.e., $v_\gamma \lesssim v_{ac}$, because the ellipse already lies higher than does the line (c) at $v = v_{ac}$.

The above argument shows the two alternative statements of the condition (i), Eq. (39), really are equivalent. Similarly, if $v_\delta \lesssim v_{bc}$, then $v_\beta \lesssim v_{bc}$, and vice versa, because the ellipse lies below (c) for $v \gtrsim v_\delta$. If $v_\gamma \lesssim v_{ac} \lesssim v_\delta$ the ellipse must intersect line (a) at a point between v_γ and v_δ lying above line (c), i.e., at a point to the right of v_{ac}; if $v_{ac} \lesssim v_\alpha$, then surely $v_\gamma \leqslant v_{ac} \leqslant v_\delta$. In this fashion the conditions for (ii), Eq. (39), are explained and shown to be equivalent.

Now consider (iii), Eq. (39), and refer to (iii), Eq. (38). If $v_\ell = v_\gamma$ and $v_u = v_\delta$, the ellipse is lying above the shaded area of Fig. 1 only in the range $v_\gamma \leqslant v \leqslant v_\delta$. So surely $v_{ac} \leqslant v_\gamma$ and $v_\delta \leqslant v_{bc}$. If $v_{ac} \leqslant v_\gamma$, then $v_\alpha \leqslant v_{ac}$; if $v_\delta \leqslant v_{bc}$, then $v_\beta \leqslant v_{bc}$. On the other hand, $v_\alpha \leqslant v_{ac}$, $v_\beta \leqslant v_{bc}$ does not guarantee case (iii), because these conditions are compatable with both v_γ and $v_\delta \leqslant v_{ac}$, or with both v_γ and $v_\delta \leqslant v_{bc}$. In other words, without some additional prescription on v_γ or v_δ, the conditions $v_\alpha \leqslant v_{ac}$, $v_\beta \leqslant v_{bc}$ are not equivalent to $v_{ac} \leqslant v_\gamma$, $v_\delta \leqslant v_{bc}$. This is the reason there is but one condition for case (iii), Eq. (39), instead of two equivalent conditions as in cases (i) and (ii). The conditions for cases (iv) and (v) are understood similarly.

It is worth while to show that case (v) really can occur. In fact, I shall show $v_{bc} \leqslant v_\gamma$ can occur and can be consistent with $v_\alpha \leqslant v_{ac}$, $v_\beta \leqslant v_{bc}$, thus illustrating the assertion that these conditions on v_α, v_β are not sufficient to guarantee case (iii) holds. Using Eq. (31c) and (40b), at $v = v_{bc}$ (and $\Delta E \gtrless 0$), the value of V is

$$V_{bc} = v_1 - \frac{m_2}{M}\frac{M}{2m_2}(v_1 + v_1') = \frac{1}{2}(v_1 - v_1') \qquad (56)$$

If the V-intercept B of (53b) is less than this V_{bc}, then the ellipse surely lies below line (c) for $v \leqslant v_{bc}$, and so v_{bc} surely is less than the first intersection v_γ of the ellipse with line (c). So a sufficient condition for $v_{bc} < v_\gamma$ is

$$\frac{m_1}{M} v_1^2 + \frac{m_2}{M} v_2^2 < \frac{1}{4}\left(v_1 - v_1'\right)^2 \tag{57}$$

Eq. (57) is satisfied when $v_2 = 0$, $v_1' = 0$ (its smallest value), and

$$\frac{m_1}{M} < \frac{1}{4}, \quad \text{i.e.,} \quad 3m_1 < m_2 \tag{58}$$

Hence $v_{bc} < v_\gamma$, corresponding to case (v) of Eqs. (38) and (39) occurs.

The conditions $v_\alpha \leq v_{ac}$, $v_\beta \leq v_{bc}$ are

$$v_2' - v_1' \leq \frac{M}{2m_2}\left(v_1 - v_1'\right) \tag{59a}$$

$$v_2' + v_1' \leq \frac{M}{2m_2}\left(v_1 + v_1'\right) \tag{59b}$$

When $v_2 = 0$, $v_1' = 0$,

$$v_2' = \left(\frac{2\Delta E}{m_2}\right)^{1/2} = \left(\frac{2}{m_2}\,\frac{1}{2}\,m_1 v_1^2\right)^{1/2} = \left(\frac{m_1}{m_2}\right)^{1/2} v_1 \tag{60}$$

Using (60), Eqs. (59) will be satisfied if

$$\left(\frac{m_1}{m_2}\right)^{1/2} \leq \frac{M}{2m_2} \tag{61a}$$

i.e., if

$$4m_1m_2 \leq \left(m_1 + m_2\right)^2 \tag{61b}$$

which also holds because

$$0 \leq \left(m_1 - m_2\right)^2 \tag{61c}$$

So Eqs. (59) can be satisfied, and yet v_{bc} can be v_γ.

<u>Eqs. (41)</u>

For $\Delta E > 0$, the conditions $v_\gamma \leq v_{ac}$, $v_{bc} \leq v_\delta$ of (i), Eq. (39), are:

$$v_1 - v_2 \leq \left(2m_2\right)^{-1} M\left(v_1 - v_1'\right) \tag{62a}$$

and

$$\left(2m_2\right)^{-1} M\left(v_1 + v_1'\right) \leq v_1 + v_2 \tag{62b}$$

or

$$\left(m_2 - m_1\right)v_1 + Mv_1' \leq 2m_2v_2 \tag{63a}$$

and

$$\left(m_1 - m_2\right)v_1 + Mv_1' \leq 2m_2v_2 \tag{63b}$$

Eqs. (63a) and (63b) are both encompassed in the single condition (41b).

The other set of conditions in (i), Eq. (39), namely $v_{ac} \leq v_\alpha$, $v_{bc} \leq v_\beta$, are:

$$\left(2m_2\right)^{-1} M\left(v_1 - v_1'\right) \leqslant v_2' - v_1' \qquad (64a)$$

and

$$\left(2m_2\right)^{-1} M\left(v_1 + v_1'\right) \leqslant v_2' + v_1' \qquad (64b)$$

or

$$Mv_1 + \left(m_2 - m_1\right) v_1' \leqslant 2m_2 v_2' \qquad (65a)$$

and

$$Mv_1 + \left(m_1 - m_2\right) v_1' \leqslant 2m_2 v_2' \qquad (65b)$$

Eqs. (65) are both encompassed in the single condition (41c).

Squaring both sides of (41b) yields

$$M^2 v_1'^2 + \left(m_1 - m_2\right)^2 v_1^2 + 2M\left|m_1 - m_2\right| v_1 v_1' \leqslant 4m_2^2 v_2^2 \qquad (66a)$$

Similarly, Eq. (41c) yields

$$M^2 v_1^2 + \left(m_1 - m_2\right)^2 v_1'^2 + 2M\left|m_1 - m_2\right| v_1 v_1' \leqslant 4m_2^2 v_2'^2 \qquad (66b)$$

Thus Eqs. (66a) and (66b) are identical if

$$M^2 v_1'^2 + \left(m_1 - m_2\right)^2 v_1^2 - 4m_2^2 v_2^2 = M^2 v_1^2 + \left(m_1 - m_2\right)^2 v_1'^2 - 4m_2^2 v_2'^2 \qquad (67a)$$

Using (30b), Eq. (67a) requires

$$M^2\left(v_1^2 - \frac{2\Delta E}{m_1}\right) + \left(m_1 - m_2\right)^2 v_1^2 - 4m_2^2 v_2^2 = M^2 v_1^2 + \left(m_1 - m_2\right)^2\left(v_1^2 - \frac{2\Delta E}{m_1}\right) - 4m_2^2\left(v_2^2 + \frac{2\Delta E}{m_2}\right) \qquad (67b)$$

i.e.,

$$\frac{-M^2}{m_1} = -\frac{(m_1-m_2)^2}{m_1} - 4m_2 \tag{68}$$

which is true. Conversely, starting with the obviously true (68) one can derive (67b) and (67a), and so one can conclude that if (66a) holds then (66b) holds, and vice versa. But taking the square root of both sides, Eq. (66a) implies

$$\left| Mv_1' + |m_1 - m_2| v_1 \right| \leq 2m_2 v_2$$

which is identical with (41b) because both Mv_1' and $|m_1 - m_2| v_1$ are intrinsically positive. This argument shows Eq. (66a) implies Eq. (41b). Similarly, Eq. (66b) implies Eq. (41c). Therefore it has been proved that Eqs. (41b) and (41c) are equivalent.

Next write (41b) in the form

$$Mv_1' \leq 2m_2 v_2 - |m_1-m_2| v_1 \tag{69}$$

$$M^2 v_1'^2 = M^2\left(v_1^2 - \frac{2\Delta E}{m_1}\right) \leq 4m_2^2 v_2^2 + (m_1-m_2)^2 v_1^2 - 4m_2 |m_1-m_2| v_1 v_2 \tag{70a}$$

or

$$4m_1 m_2 v_1^2 - 4m_2^2 v_2^2 + 4m_2 |m_1 - m_2| v_1 v_2 \leq M^2 \frac{2\Delta E}{m_1} \tag{70b}$$

Eq. (70b) becomes

$$\Delta E \geq \frac{4m_1m_2}{M^2}\left[\frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2 + \left|\frac{1}{2}m_1v_1v_2 - \frac{1}{2}m_2v_1v_2\right|\right] \tag{71}$$

Hence, since $E_1 = m_1v_1^2/2$, $E_2 = m_2v_2^2/2$, it has been shown that Eq. (41b) implies the condition on ΔE in (41d). Conversely, starting with (71) one can infer (70b) and (70a). Thus, taking the square root of both sides of (70a), Eq. (71) implies

$$Mv_1' \leq \left|2m_2v_2 - |m_1-m_2|v_1\right| \tag{72}$$

Eq. (72) is not immediately equivalent to (69), however. To obtain Eq. (69), it is necessary to impose the additional condition that

$$2m_2v_2 \geq |m_1-m_2|\ v_1 \tag{73}$$

Therefore it has been shown that Eq. (41d) will not of itself imply Eq. (41b), but that the pair of conditions (41d), (41e) does imply (41b). On the other hand, starting with (41b) one obtains (69) from which one knows (73) must hold, because Mv_1' is intrinsically positive. Consequently, because it already has been proved that (41b) implies (41d), Eq. (41b) both implies and is implied by the pair of conditions (41d), (41e), i. e., Eq. (41b) and the pair (41d), (41e) are equivalent. Note that (41e) is not encompassed in (41d); when $v_2 = 0$, (41e) fails, but (41d) will be satisfied if ΔE is as large as E, its maximum allowed value.

Eqs. (42)

The conditions $v_\gamma \leq v_{ac} \leq v_\delta \leq v_{bc}$ of (ii), Eq. (39), are ($\Delta E \geq 0$):

$$v_1 - v_2 \leq (2m_2)^{-1} M(v_1 - v_1') \leq v_1 + v_2 \leq (2m_2)^{-1} M(v_1 + v_1') \tag{74}$$

These three inequalities yield, in turn

$$Mv_1' - (m_1 - m_2)v_1 \leq 2m_2 v_2 \tag{75a}$$

$$(m_1 - m_2)v_1 - Mv_1' \leq 2m_2 v_2 \tag{75b}$$

$$2m_2 v_2 \leq Mv_1' + (m_1 - m_2)v_1 \tag{75c}$$

Each of (75a), (75b), (75c) must be satisfied. Eqs. (75a) and (75b) can be combined into

$$\left| Mv_1' - (m_1 - m_2)v_1 \right| \leq 2m_2 v_2 \tag{75d}$$

which implies both (75a) and (75b). Thus the first set of conditions for case (ii), Eq. (39), reduce to (42b). So far there has been no condition that $m_1 > m_2$. But if $m_1 < m_2$, the left side of the inequality (42b) would exceed the right side, i.e., Eq. (42b) could not possibly be satisfied. Therefore the limits $v_\ell = v_\alpha$, $v_u = v_\delta$ of case (ii), Eq. (38), occur only when $m_1 > m_2$ and when v_2 obeys the inequality (42b).

The conditions $v_{ac} \leqslant v_\alpha$, $v_\beta \leqslant v_{bc}$ of (ii), Eq. (39), are

$$(2m_2)^{-1} M(v_1 - v_1') \leqslant v_2' - v_1'$$

and (76)

$$v_2' + v_1' \leqslant (2m_2)^{-1} M(v_1 + v_1')$$

which immediately reduce to (42c), Note that when $\Delta E \geqslant 0$

$$Mv_1 \geqslant (m_1 - m_2)v_1' \tag{77}$$

because $v_1 \geqslant v_1'$ and $M \geqslant (m_1 - m_2)$. Hence the absolute value sign can be removed in (42c). However, retaining the absolute value sign keeps the forms of (42b) and (42c) as alike as possible, which proves convenient when the situation $\Delta E \leqslant 0$ is considered (see below).

To show the equivalence of Eqs. (42b) and (42c), proceed as follows.

From the right inequalities in (42b) and (42c),

$$4m_2^2 v_2^2 \leqslant M^2 v_1'^2 + (m_1 - m_2)^2 v_1^2 + 2M(m_1 - m_2)v_1 v_1' \tag{78a}$$

$$4m_2^2 v_2'^2 \leqslant M^2 v_1^2 + (m_1 - m_2)^2 v_1'^2 + 2M(m_1 - m_2)v_1 v_1' \tag{78b}$$

Eqs. (78a) and (78b) are identical if (67a) holds, after which one follows on to the obviously true (68), just as before. Conversely, because (67a) holds, either of (78a) or (78b) implies the other. But (78a) and (78b) imply, respectively,

$$2m_2 v_2 \leqslant \left| Mv_1' + (m_1 - m_2)v_1 \right|$$

$$2m_2 v_2' \leqslant \left| Mv_1 + (m_1 - m_2)v_1' \right|$$

from which the absolute value signs can be removed because all terms under the absolute value signs are positive (with $m_1 > m_2$). Thus the equivalence of the right inequalities in (42b) and (42c) is proved. Similarly, starting from (67a), one infers that each of

$$M^2 v_1'^2 + (m_1 - m_2)^2 v_1^2 - 2M(m_1 - m_2) v_1 v_1' \leq 4m_2^2 v_2^2 \tag{79a}$$

$$M^2 v_1^2 + (m_1 - m_2)^2 v_1'^2 - 2M(m_1 - m_2) v_1 v_1' \leq 4m_2^2 v_2'^2 \tag{79b}$$

imply the other. But (79a) and (79b) imply, respectively,

$$\left| Mv_1' - (m_1 - m_2) v_1 \right| \leq 2m_2 v_2 \tag{80a}$$

$$\left| Mv_1 - (m_1 - m_2) v_1' \right| \leq 2m_2 v_2' \tag{80b}$$

Eqs. (80a) and (80b) are the left inequalities of Eqs. (42b) and (42c) respectively.

This completes the proof that (42b) and (42c) are equivalent. A condition on ΔE like (41d) can be derived in this case (ii), but because of the fact that v_2 is bounded both from above and from below in this present case (ii), the ΔE condition now is rather more awkward than was (41d).

Eqs. (43)

The conditions $v_{ac} \leq v_\gamma$, $v_\delta \leq v_{bc}$ of (iii), Eq. (39), are ($\Delta E \geq 0$):

$$(2m_2)^{-1} M(v_1 - v_1') \leq v_1 - v_2 \tag{81a}$$

and

$$v_1 + v_2 \leq (2m_2)^{-1} M(v_1 + v_1') \tag{81b}$$

which become, respectively,

$$2m_2 v_2 \leq Mv_1' + (m_2 - m_1)v_1 \tag{82a}$$

and

$$2m_2 v_2 \leq Mv_1' + (m_1 - m_2)v_1 \tag{82b}$$

The smaller of the right sides of (82a) and (82b) is

$$Mv_1' - |m_1 - m_2| \, v_1$$

Thus if (43b) holds, both (82a) and (82b) hold. In other words, (43b) is the condition for case (iii), Eqs. (39).

Writing (43b) in the form

$$2m_2 v_2 + |m_1 - m_2| \, v_1 \leq Mv_1' \tag{83a}$$

and squaring both sides leads to (43c), in the same fashion as (41b) led to (71). Conversely, working back from (43c) one can infer

$$\left| 2m_2 v_2 + |m_1 - m_2| \, |v_1| \right| \leq Mv_1' \tag{83b}$$

But (83b) is equivalent to (83a), because the terms under the absolute value sign in (83b) are intrinsically positive. Thus (43c) alone is equivalent to (43b); a subsidiary condition like

(41e) is not needed in this case.

Eqs. (44) and (45)

Eqs. (44b) and (44c) are deduced, and their equivalence established, in essentially the same manner as in the above amplification of Eqs. (42).

The condition for case (v), Eq. (39) is either $v_\delta < v_{ac}$ or $v_{bc} < v_\gamma$. These reduce, respectively, to:

$$2m_2v_2 < (m_1-m_2)v_1 - Mv_1' \tag{84a}$$

or

$$2m_2v_2 < (m_2-m_1)v_1 - Mv_1' \tag{84b}$$

The larger of the right sides of (84a) and (84b) is

$$|m_1-m_2|v_1 - Mv_1'$$

Thus if (45b) holds, either (84a) or (84b) will hold. Thus (45b) is the condition for case (v), Eq. (38), i. e., is the condition for (45a) to hold.

Write Eq. (45b) in the form

$$Mv_1' < |m_1-m_2|v_1 - 2m_2v_2 \tag{85}$$

and compare with Eq. (69). Then it is clear squaring both sides of (85) will lead to (71), i.e., to the ΔE condition of (41d) which is identical with the ΔE condition of (45c). Conversely, from this ΔE condition one can again infer (72), which with the new additional condition

$$|m_1-m_2|v_1 \geq 2m_2v_2 \tag{86}$$

becomes (85). Therewith the equivalence of (45b) and the pair of conditions (45c), (45d) is demonstrated. Also, letting now $E_1 = 0$, ones sees (45d) is not encompassed in (45c).

Eqs. (45b) and the pair (45c), (45d) are general conditions for $\sigma_{\Delta E}^{eff}(v_1,v_2)$ to vanish. Of course, therefore, Eqs. (45b) - (45d) are satisfied (as can be verified) when $v_2 = 0$, $v_1' = 0$, $3m_1 < m_2$, the particular illustrative circumstances under which $\sigma_{\Delta E}^{eff} = 0$ was previously established, by showing that in these particular circumstances Eq. (57) holds.

Relations Corresponding to Eqs. (41) - (45) when $\Delta E \leq 0$

The limits and the conditions for cases (i) - (v), Eqs. (38) - (39), depend only on the values of v_α, v_β, v_γ, v_δ, v_{ac}, v_{bc}. Evidently (36b) and (37b) are obtained from (36a) and (37a) respectively by writing v_1 for v_1', v_1' for v_1, v_2 for v_2', v_2' for v_2. Eq. (40b) is invariant under this interchange. Eq. (40a) reads

$$v_{ac} = (2m_2)^{-1} M(v_1-v_1') \qquad \Delta E \geq 0$$

and

$$v_{ac} = (2m_2)^{-1} M(v_1'-v_1) \qquad \Delta E \leq 0$$

recalling (30b). So v_{ac} for $\Delta E \leq 0$ also is obtained from v_{ac} for $\Delta E \geq 0$ by interchange of v_1, v_1' and of v_2, v_2'. Thus for $\Delta E \leq 0$ the limits and inequalities following directly from Eqs. (38) and (39) are immediately obtained from making this interchange in those of

Eqs. (41) - (45) which for $\Delta E \geq 0$ followed directly from (38) - (39).

Specifically, then, the limits in cases (i) - (iv) are

$$
\begin{array}{ll}
\text{(i)} & v_\ell = v_2 - v_1,\ v_u = v_2 + v_1 \\
\text{(ii)} & v_\ell = v_2 - v_1,\ v_u = v_1' + v_2' \\
\text{(iii)} & v_\ell = v_1' - v_2',\ v_u = v_2' + v_1' \\
\text{(iv)} & v_\ell = v_1' - v_2',\ v_u = v_2 + v_1
\end{array}
\qquad (87)
$$

and of course (45a) continues to state (without replacing v_1 by v_1', or v_2 by v_2') the value of $\sigma_{\Delta E}^{eff}$ when case (v) occurs with $\Delta E \leq 0$.

Interchange of v_1, v_1' and of v_2, v_2' converts (41b) to (41c) and vice versa. Examining Eqs. (66a)-(68) it is seen that they hold for $\Delta E \leq 0$ just as well as for $\Delta E \geq 0$. So (41b) and (41c), the presumably equivalent conditions inferred from (i) of Eq. (39) for $\Delta E \leq 0$, are proved equivalent precisely as in the circumstance $\Delta E \geq 0$. Similarly, Eqs. (69)-(73) do not depend on the sign of ΔE, so that the pair (41d), (41e) is equivalent to Eq. (41b) for $\Delta E \leq 0$ as well as for $\Delta E \geq 0$. However, when $\Delta E \leq 0$, the condition (41e) is encompassed in (41d), i.e., for $\Delta E \leq 0$ it is not necessary to supplement (41d) with (41e). The demonstration of this assertion goes as follows:

If $\Delta E \leq 0$ satisfies (41d), then surely

$$E_2 \geq E_1 + \left| E_1 \frac{v_2}{v_1} - E_2 \frac{v_1}{v_2} \right| \qquad (88a)$$

In terms of the velocities, Eq. (88a) is

$$m_2 v_2^2 - m_1 v_1^2 - \left| m_1 - m_2 \right| v_1 v_2 \geq 0 \qquad (88b)$$

i.e.,

$$m_2v_2^2 - m_1v_1^2 - (m_1 - m_2)v_1v_2 \geqslant 0 \qquad m_1 > m_2$$

$$m_2v_2^2 - m_1v_1^2 - (m_2 - m_1)v_1v_2 \geqslant 0 \qquad m_1 < m_2 \qquad (88c)$$

These equations factor into

$$(m_2v_2 - m_1v_1)(v_2 + v_1) \geqslant 0 \qquad m_1 > m_2$$

$$(m_2v_2 + m_1v_1)(v_2 - v_1) \geqslant 0 \qquad m_1 < m_2$$

So one can conclude

$$m_2v_2 \geqslant m_1v_1 \qquad m_1 > m_2 \qquad (88d)$$

$$v_2 \geqslant v_1 \qquad m_1 < m_2 \qquad (88e)$$

Now consider (41e), which is

$$2m_2v_2 \geqslant (m_1 - m_2)v_1 \quad \text{when} \quad m_1 > m_2 \qquad (88f)$$

$$2m_2v_2 \geqslant (m_2 - m_1)v_1 \quad \text{when} \quad m_1 < m_2 \qquad (88g)$$

But (88f) obviously follows from (88d), (88g) follows from (88e), remembering $m_2 > m_1$.

It has been proved, therefore, that when $\Delta E < 0$ Eqs. (41b), (41c), and (41d) alone--without (41e)--are equivalent conditions for case (i). Of course, with $\Delta E < 0$ one must take the limits for case (i) from (87), not from (41a).

One sees similarly that the pair of Eqs. (42b) and (42c) are invariant under interchange of v_1, v_1' and of v_2, v_2' (the reason the absolute value sign in (42c) was retained. Correspondingly, the argument that Eqs. (42b) and (42c) are equivalent remains valid for $\Delta E \leq 0$.

When $\Delta E \leq 0$, the condition for case (iii) is

$$\text{(iii)} \qquad 0 \leq v_2' \leq (2m_2)^{-1} \left[Mv_1 - |m_1 - m_2| v_1' \right] \tag{89a}$$

Because (43b) is not one of a pair of inequalities, as was the situation for (41b), (41c) and for (42b), (42c), one sees that the condition for case (iii), $\Delta E \leq 0$ is not the same as for case (iii), $\Delta E \geq 0$. Moreover it is not a matter of my having overlooked the second inequality forming a pair with (43b); Eq. (89a) for $\Delta E \geq 0$ really is not equivalent to (43b). For instance, with $\Delta E \geq 0$, consider the circumstance $v_2 = 0$, $v_1' = 0$. Then, recalling (60), Eq. (89a) is satisfied if

$$\left(\frac{m_1}{m_2}\right)^{\frac{1}{2}} v_1 \leq \frac{M}{2m_2} v_1$$

which is identical with Eq. (61a), i.e., is always satisfied, as was seen from (61c). So (89a) is satisfied when $v_2 = 0$, $v_1' = 0$. On the other hand, (43b) is not, because with $v_1' = 0$ the right side of (43b) is negative.

The analogue of (43c) when $\Delta E \leq 0$ is (48d) (in other words (43c) still, except of course that now $\Delta E \leq 0$) provided also

$$2m_1 v_1 \geq |m_1 - m_2| v_2 \qquad (89b)$$

The derivation of (89b) is explained below, under the heading, "Relations Corresponding to Eqs. (48) - (52) when $\Delta E \leq 0$."

Eqs. (44b) and (44c), like (42b) and (42c), form an equivalent pair invariant under interchange of v_1, v_1' and of v_2, v_2', and therefore equally valid for positive and negative ΔE. The condition for case (v), $\sigma_{\Delta E}^{eff}(v_1, v_2) = 0$, when $\Delta E \leq 0$ is after interchanging in (45b)

$$\text{(v)} \qquad 0 \leq v_2' \leq (2m_2)^{-1}[|m_1 - m_2| v_1' - Mv_1] \qquad (89c)$$

The analogue of (45c), like the analogue of (43c) is derived below. It turns out that for $\Delta E \leq 0$, the condition (89c) is equivalent to the assertion that (48d) again holds but that now (89b) fails, i.e., in case (v) instead of (89b)

$$2m_1 v_1 \leq |m_1 - m_2| v_2 \qquad (89b)$$

Eq. (89d) is derived below under the same heading as (89b).

$$2m_1 v_1 > |m_1 - m_2| v_2 \qquad (89d)$$

Eqs. (47)

To eliminate first $\delta(\hat{v}_2 - v_2)$ in (25), use (24b) rather than (as previously) (24a) to express $\cos\bar{\theta}$ in terms of v,V. Then Eq. (27) is replaced by

$$\cos\bar{\theta}_i = (2m_1M^{-1}vV)^{-1}\,(V^2 + m_1M^{-2}v^2 - v_2^2) \tag{90a}$$

and in (26)

$$\frac{dg}{dx} \equiv \left.\frac{d\hat{v}_2}{d\cos\bar{\theta}}\right)_{\bar{\theta}=\bar{\theta}_i} = \frac{-m_1M^{-1}vV}{v_2} \tag{90b}$$

Thus (28) now is replaced by

$$\sigma_{\Delta E}^{eff}(v_1,v_2) = \frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^3v_2^2}\int dv\int dV vV^4\left(\frac{m_1M^{-1}vV}{v_2}\right)^{-1}(1-\cos^2\bar{\theta}_i + \frac{\Delta E}{\mu vV}\cos\bar{\theta}_i)$$

$$\times\delta[\{V^2 + m_2^2M^{-2}v^2 + \frac{2m_2M^{-1}vV}{2m_1M^{-1}vV}(V^2 + m_1^2M^{-2}v^2 - v_2^2)\}^{1/2} - v_1] \tag{91a}$$

The δ function in (91a) is $\delta(\hat{v}_1 - v_1)$, reexpressed in terms of v,V using (24a) and (90a). This δ function becomes

$$\delta\left[\left(\frac{MV^2}{m_1} + \frac{m_2v^2}{M} - \frac{m_2}{m_1}v_2^2\right)^{1/2} - v_1\right] \tag{91b}$$

The restrictions in v,V implied by substituting (90a) in (20) take the form

$$-1 + \frac{\Delta E}{\mu v V} \leq \frac{V^2 + m_1^2 M^{-2} v^2 - v_2^2}{2 m_1 M^{-1} v V} \leq 1 \qquad \Delta E \geq 0 \qquad (92a)$$

$$-1 \leq \frac{V^2 + m_1^2 M^{-2} v^2 - v_2^2}{2 m_1 M^{-1} v V} \leq 1 + \frac{\Delta E}{\mu v V} \qquad \Delta E \leq 0 \qquad (92b)$$

In other words

$$\left.\begin{aligned} (V - m_1 M^{-1} v)^2 &\leq v_2^2 \\ v_2^2 + \frac{2\Delta E}{m_2} = v_2'^2 &\leq (V + m_1 M^{-1} v)^2 \end{aligned}\right\} \qquad \Delta E \geq 0 \qquad (93a)$$

$$\left.\begin{aligned} v_2^2 &\leq (V + m_1 M^{-1} v)^2 \\ (V - m_1 M^{-1} v)^2 &\leq v_2^2 + \frac{2\Delta E}{m_2} = v_2'^2 \end{aligned}\right\} \qquad \Delta E \leq 0 \qquad (93b)$$

Eqs. (93a) and (93b), for $\Delta E \geq 0$ and $\Delta E \leq 0$, are summarized by

$$(V - m_1 M^{-1} v)^2 \leq v_{2s}^2 \qquad (94a)$$

$$v_{2g}^2 \leq (V + m_1 M^{-1} v)^2 \qquad (94b)$$

where

$$\begin{aligned} v_{2s} &= \text{smaller of } v_2, v_2' \\ v_{2g} &= \text{larger of } v_2, v_2' \end{aligned} \qquad (95)$$

Eqs. (94) imply that (91a) is integrated over the portion of the first quadrent of the v,V plan lying below the line [again termed (a)]

$$V - m_1M^{-1}v = v_{2s} \qquad (96a)$$

lying above the line [again termed (b)]

$$m_1M^{-1}v - V = v_{2s} \qquad (96b)$$

and lying above the line [again termed (c)]

$$V + m_1M^{-1}v = v_{2g} \qquad (96c)$$

The δ function (91b) vanishes unless (32) holds. Integrating over V, the δ function contribution from (91b) now involves

$$\frac{dg}{dx} = \frac{d}{dV}\left(\left(\frac{MV^2}{m_1} + \frac{m_2v^2}{M} - \frac{m_2}{m_1}v_2^2\right)^{\frac{1}{2}}\right)_{V=V_i} = \frac{MV_i}{m_1v_1} \qquad (97)$$

with $V_i(v)$ still given by (34a). Therefore, in Eqs. (91), after integrating over V, the factors under the integral sign multiplying the terms in $\cos\bar{\theta}_i$ become

$$v\bar{V}_i^4 \; \frac{v_2}{m_1M^{-1}vV_i} \; \frac{m_1v_1}{MV_i} = v_1v_2V_i^2 \qquad (98)$$

Moreover, substituting (34a) in (90a), one sees that Eq. (34b) continues to give $\cos\bar{\theta}_i$ in terms of v_1,v_2,v and V_i, again as one expects.

It follows that Eqs. (33) and (35) still hold when one eliminates first the $\delta(\hat{v}_2 - v_2)$ in (25), but that the limits

v_ℓ, v_u now are determined by the intersections of lines (96a), (96b) and (96c) with each other and with the ellipse (32). Moreover, let v_α, v_β, v_γ, v_δ, v_{ac}, v_{bc} be defined as previously (e.g., v_{ac} is value of v at the intersection of lines (a) and (c), v_γ is the smaller v at the two intersections made by line (c) with the ellipse), so that Eqs. (38) and (39) continue to specify the limits v_ℓ, v_u to be inserted in (35). Then one readily sees the values of v_α, v_β, v_γ, v_δ are as quoted in Eqs. (47).

In particular, suppose $\Delta E \geqslant 0$, so that

$$v_{2s} = v_2, \quad v_{2g} = v_2' \tag{99}$$

Hence, substituting (96a) in (32),

$$\frac{1}{2}M(v_0 + m_1 M^{-1} v)^2 + \frac{1}{2}\mu v^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$(M - m_2)v_2^2 + \left(\frac{m_1^2}{M} + \frac{m_1 m_2}{M}\right)v^2 + 2m_1 v_2 v = m_1 v_1^2$$

$$v_2^2 + v^2 + 2v_2 v = v_1^2$$

$$(v + v_2)^2 = v_1^2 \tag{100}$$

Eq. (100) has the two roots

$$v + v_2 = v_1$$

$$v + v_2 = -v_1$$

of which, by definition, v_α is the root which possibly (though not necessarily) can lie in the first quadrant. Evidently, therefore

$$v_\alpha = v_1 - v_2$$

as in (47a).

The other results in (47a) and (47b) are derived similarly. A more simple argument is to note that when $\Delta E \geq 0$, Eq. (31a) converts to Eq. (96a) if first the prime is removed from v_1', and then the subscripts 1, 2 are interchanged in v_1, v_2, and in v_1', v_2'; when $\Delta E \leq 0$, Eq. (31a) converts to Eq. (96a) if first the prime is added to v_1, and then the subscripts 1, 2 are interchanged. The same operations convert Eq. (31b) to Eq. (96b), for $\Delta E \geq 0$ and $\Delta E \leq 0$ Eq. (31c) is converted to (96c) by interchange of 1, 2 after adding the prime to $v_1 (\Delta E \geq 0)$, or dropping the prime from $v_1' (\Delta E \leq 0)$. In other words, Eqs. (31a), (31b), (31c) are converted to the corresponding Eqs. (96a), (96b), (96c) by first interchanging v_1, v_2 with their corresponding primed quantities v_1', v_2', and then interchanging all subscripts 1, 2. But, because of the last equality in Eq. (32), these interchanges leave Eq. (32) unaltered. So Eqs. (47) must be obtainable from Eqs. (36) and (37) by first interchanging primed and unprimed particle velocities, and then by interchange of subscripts 1, 2. Correspondingly, from Eqs. (40) the intersections v_{ac}, v_{bc} now are

$$v_{ac} = (2m_1)^{-1} \; M|v_2' - v_2| \tag{101a}$$

$$v_{bc} = (2m_1)^{-1} \; M(v_2' + v_2) \tag{101b}$$

as can be verified directly from Eqs. (96).

<u>Eqs. (48)</u>

With $\Delta E \geq 0$, using Eq. (47a), case (i) of Eq. (38) now corresponds to the limits

$$v_\ell = v_\alpha = v_1 - v_2$$

$$v_u = v_\beta = v_1 + v_2$$

as in Eqs. (48a), and consistent with the interchange rules which have been cited. Similarly, Eqs. (41b), (41c) yield the corresponding criteria (48b), (48c) for the occurrence of the limits (48a). Eq. (48d) is obtained from (48c) by exactly the same argument, Eqs. (69)-(71), as was used to obtain (41d) from (41b). Because (48c) differs from (41b) by the interchange only of subscripts 1, 2 (primed and unprimed quantities are not interchanged in going from (41b) to (48c)!), the corresponding interchange in (41d) should yield the ΔE condition equivalent to (48c). There is the proviso, however, that in getting (41d) from (41b), ΔE enters through

$$v_1'^2 = v_1^2 - \frac{2\Delta E}{m_1}$$

whereas, in proceeding from (48c), ΔE enters through

$$v'^{2}_{2} = v^{2}_{2} + \frac{2\Delta E}{m_2}$$

So the presently desired analogue of (41d) is obtained not merely by interchanging subscripts 1, 2, but also by changing the sign of ΔE. Performing these operations on (41d) yields

$$-\Delta E \geq \frac{4m_1m_2}{M^2}\left[E_2 - E_1 + \left|E_2\frac{v_1}{v_2} - E_1\frac{v_2}{v_1}\right|\right] \tag{102}$$

which is the ΔE condition (48d).

To check the correctness of this argument, I will obtain (102) directly from (48c). Write (48c) in the form

$$Mv_2' \leq 2m_1v_1 - |m_1 - m_2|v_2 \tag{103}$$

Squaring both sides of (103) yields

$$M^2v_2'^2 = M^2\left(v_2^2 + \frac{2\Delta E}{m_2}\right) \leq 4m_1{}^2v_1 + (m_1 - m_2)^2v_2^2 - 4m_1|m_1 - m_2|v_1v_2$$

$$M^2\,\frac{2\Delta E}{m_2} \leq 4m_1{}^2v_1^2 - 4m_1m_2v_2^2 - 4m_1|m_1 - m_2|v_1v_2$$

$$\Delta E \leq \frac{4m_1m_2}{M^2}\left[\frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2 - \frac{1}{2}\left|m_1v_1v_2 - m_2v_1v_2\right|\right] \tag{104}$$

Eq. (104) is the ΔE condition (48d). Conversely, working back from (104), one gets

$$Mv_2' \leqslant \left| 2m_1v_1 - \left|m_1 - m_2\right|v_2 \right| \tag{105}$$

which is the analogue of (72). So to make (104) equivalent to (103) requires the extra condition

$$2m_1v_1 \geqslant \left|m_1 - m_2\right|v_2 \tag{106}$$

which is the analogue of (41e).

The condition (106) is not needed as a supplement to (48d), however, because the ΔE⩾0 Eq. (106) is implied by (48d), just as (41e) was implied by (41d) when ΔE⩽0. The argument follows the lines of Eqs. (88a) - (88g). If (48d) holds and ΔE⩾0, then

$$E_1 \geqslant E_2 + \left| E_2\frac{v_1}{v_2} - E_1\frac{v_2}{v_1} \right| \tag{107a}$$

In terms of the velocities

$$m_1v_1^2 - m_2v_2^2 - \left|m_2 - m_1\right|v_1v_2 \geqslant 0 \tag{107b}$$

i.e.

$$\begin{aligned} m_1v_1^2 - m_2v_2^2 - (m_2 - m_1)v_1v_2 &\geqslant 0 \qquad m_2 > m_1 \\ m_1v_1^2 - m_2v_2^2 - (m_1 - m_2)v_1v_2 &\geqslant 0 \qquad m_1 > m_2 \end{aligned} \tag{107c}$$

These factor into

$$(m_1v_1 - m_2v_2)\ (v_1 + v_2)\ \geqslant\ 0 \qquad m_2 > m_1$$

$$(m_1v_1 + m_2\ v_2)\ (v_1 - v_2)\ \geqslant\ 0 \qquad m_1 > m_2$$

yielding

$$m_1v_1\ \geqslant\ m_2v_2 \qquad m_2 > m_1 \qquad (107d)$$

$$v_1\ \geqslant\ v_2 \qquad m_1 > m_2 \qquad (107e)$$

Eq. (106) is

$$2m_1v_1 \geqslant (m_2 - m_1)v_2 \quad \text{when} \quad m_2 > m_1 \qquad (107f)$$

$$2m_1v_1 \geqslant (m_1 - m_2)v_2 \quad \text{when} \quad m_1 > m_2 \qquad (107g)$$

Eq. (107d) implies (107f); Eq. (107e) implies (107g).

It has been proved, therefore, that when $\Delta E \geqslant 0$, Eqs. (48b), (48c), and (48d) are equivalent conditions for use of the limits (48a). But these limits are the same as (43a). Thus the conditions for (43a) must be the same as for (48a). Indeed, (48d) is identical with (43c). So, when $\Delta E \geqslant 0$, the conditions (43b), (43c), (48b) and (48c) each all are equivalent, remembering (43b), (43c), (48b) and (48c) each already have been proved equivalent to (48d).

Eqs. (49)

The interchange of v_1, v_2', of v_2, v_1' and of m_1, m_2 in Eqs. (42) yields Eqs. (49). But now the limits in Eq. (49a), presently case (ii), are identical with the limits in (44a), previously termed case (iv). Eqs. (44) and (49) both apply only when $m_1 < m_2$, moreover. So it must be possible to show the condition (44b), (44c), (49b), and (49c) all are equivalent.

To show the equivalence of these conditions, note first that Eqs. (79)-(79) have proved the left inequalities in Eqs. (42b), (42c) are equivalent, as are the right inequalities in (42b), (42c). So the left (and right) inequalities in (44b) and (44c) must be equivalent; similarly the left (and right) inequalities in (49b) and (49c) are equivalent. So it is sufficient to show: (a) one of the right inequalities in (44b) and (44c) is equivalent to one of the left inequalities in (49b), (49c); (b) one of the left inequalities in (44b), (44c) is equivalent to one of the right inequalities in (49b), (49c).

I shall compare the right inequality in (44b) with the left inequality in (49c). These take the respective forms

$$2m_2v_2 - (m_2 - m_1)v_1 \leqslant Mv_1' \tag{108a}$$

$$Mv_2' \leqslant 2m_1v_1 + (m_2 - m_1)v_2 \tag{108b}$$

because the absolute value sign can be removed from (49c) when $v_2' \geqslant v_2$, i.e., when $\Delta E \geqslant 0$.

Squaring (108a)

$$4m_2^2v_2^2 + (m_2 - m_1)^2v_1^2 - 4m_2(m_2 - m_1)v_1v_2 \leqslant M^2v_1'^2 = M^2\left(v_1^2 - \frac{2\Delta E}{m_1}\right)$$

$$M^2\,\frac{2\Delta E}{m_1} \leqslant 4m_1m_2v_1^2 - 4m_2^2v_2^2 + 4m_2(m_2 - m_1)v_1v_2$$

$$2\Delta E \leqslant \frac{4m_1m_2}{M^2}\,[m_1v_1^2 - m_2v_2^2 + (m_2 - m_1)v_1v_2] \qquad (109)$$

Squaring (108b)

$$M^2v_2'^2 = M^2\left(v_2^2 + \frac{2\Delta E}{m_2}\right) \leqslant 4m_1^2v_1^2 + (m_2 - m_1)^2v_2^2 + 4m_1(m_2 - m_1)v_1v_2$$

$$M^2\,\frac{2\Delta E}{m_2} \leqslant 4m_1^2v_1^2 - 4m_1m_2v_2^2 + 4m_1(m_2 - m_1)v_1v_2$$

which becomes identical with (109).

Now starting from (109) one works back to (108b), without subsidiary conditions, i.e., conditions (109) and (108b) are equivalent. Starting from (109) one also works back to

$$\left|2m_2v_2 - (m_2 - m_1)v_1\right| \leqslant Mv_1' \qquad (110a)$$

which is equivalent to (108a) only if

$$2m_2v_2 \geqslant (m_2 - m_1)v_1 \qquad (110b)$$

So either right inequality in (44) is equivalent to (109) supplemented by (110b); either left inequality in (49) is equivalent to (109) above.

Next compare the left inequality in (44c) with the right inequality in (49b). These take the respective forms

$$Mv_1 \leq 2m_2v_2' + (m_2 - m_1)v_1' \qquad (111a)$$

$$2m_1v_1' - (m_2 - m_1)v_2' \leq Mv_2 \qquad (111b)$$

because the absolute value sign can be removed from (44c) when $\Delta E \geq 0$. Eq. (111a) yields

$$M^2v_1^2 = M^2\left(v_1'^2 + \frac{2\Delta E}{m_1}\right) \leq 4m_2^2v_2'^2 + (m_2 - m_1)v_1'^2 + 4m_2(m_2 - m_1)v_1'v_2'$$

$$M^2\frac{2\Delta E}{m_1} \leq 4m_2^2v_2'^2 - 4m_1m_2v_1'^2 + 4m_2(m_2 - m_1)v_1'v_2'$$

$$2\Delta E \leq \frac{4m_1m_2}{M^2}[m_2v_2'^2 - m_1v_1'^2 + (m_2 - m_1)v_1'v_2'] \qquad (112a)$$

Eq. (111b) yields

$$4m_1^2v_1'^2 + (m_2 - m_1)^2v_2'^2 - 4m_1(m_2 - m_1)v_1'v_2' \leq M^2v_2^2 = M^2\left(v_2'^2 - \frac{2\Delta E}{m_2}\right)$$

$$M^2\frac{2\Delta E}{m_2} \leq 4m_1m_2v_2'^2 - 4m_1^2v_1'^2 + 4m_1(m_2 - m_1)v_1'v_2'$$

which becomes identical with (112).

Starting from (112), one works back to (111a); starting from (112) one also works back to

$$\left|2m_1v_1' - (m_2 - m_1)v_2'\right| \leqslant Mv_2 \tag{113a}$$

which is equivalent to (111b) only if

$$2m_1v_1' - (m_2 - m_1)v_2' \geqslant 0 \tag{113b}$$

So either left inequality in (44) is equivalent to (112) alone; either right inequality in (49) is equivalent to (112) supplemented by (113b).

The above results, together with the results stated following Eq. (110b), may be summarized as follows. The conditions (109) plus (112) plus (110b) are equivalent to the entire set of inequalities (44); the conditions (109) plus (112) plus (113b) are equivalent to the entire set of inequalities (49). In addition, (109) alone is equivalent to the left inequality in (49). I next show that (109) plus (112) plus (110b) also imply the right inequality in (49).

Eqs. (109), (112) and (110b) imply the inequalities (44). The left inequality (44c) is

$$Mv_1 - (m_2 - m_1)v_1' \leqslant 2m_2v_2'$$

i.e.,

$$Mv_1 - 2m_2v_2' \leqslant (m_2 - m_1)v_1'$$

$$M(m_2 - m_1)v_1 - 2m_2(m_2 - m_1)v_2' \leqslant (m_2 - m_1)^2v_1'$$

$$M(m_2 - m_1)v_1 - 2m_2(m_2 - m_1)v_2' \leqslant M^2v_1' - 4m_1m_2v_1'$$

$$4m_1m_2v_1' - 2m_2(m_2 - m_1)v_2' \leqslant M^2v_1' - M(m_2 - m_1)v_1$$

$$M^{-1} [2m_1v_1' - (m_2 - m_1)v_2'] \leqslant (2m_2)^{-1} [Mv_1' - (m_2 - m_1)v_1] \tag{114a}$$

But from the left inequality (44b), recalling Eqs. (75),

$$Mv_1' - (m_2 - m_1)v_1 \leqslant 2m_2v_2 \tag{114b}$$

Using (114b) in (114a),

$$M^{-1} [2m_1v_1' - (m_2 - m_1)v_2'] \leqslant v_2$$

$$2m_1v_1' \leqslant Mv_2 + (m_2 - m_1)v_2' \tag{114c}$$

Eq. (114c) is the right inequality in (49b), which already is known to be equivalent to the right inequality (49c).

This proves (109), (112), and (110b), imply the entire set of inequalities (49), which in turn imply the set (109), (112) and (113b). On the other hand, I now show (109), (112) and (113b) imply the set (109), (112) and (110b). Eqs. (109), (112) and (113b) are known to imply the entire set (49) plus the left inequality (44). From (49c)

$$Mv_2' - (m_2 - m_1)v_2 \leqslant 2m_1v_1$$

$$Mv_2' - 2m_1v_1 \leqslant (m_2 - m_1)v_2$$

$$M(m_2 - m_1)v_2' - 2m_1(m_2 - m_1)v_1 \leqslant (m_2 - m_1)^2v_2 = M^2v_2 - 4m_1m_2v_2$$

$$4m_1m_2v_2 - 2m_1(m_2 - m_1)v_1 \leqslant M^2v_2 - M(m_2 - m_1)v_2'$$

$$M^{-1}[2m_2v_2 - (m_1 - m_2)v_1] \leqslant (2m_1)^{-1}[Mv_2 - (m_2 - m_1)v_2']$$

So, using the left inequality (49b)

$$M^{-1}[2m_2v_2 - (m_1 - m_2)v_1] \leqslant v_1'$$

$$2m_2v_2 \leqslant Mv_1' + (m_1 - m_2)v_1$$

which is the right inequality (44b).

Therefore (109), (112) and (113b) imply the entire set (44) which in turn imply (109), (112) and (110b). This proves (109). (112) and (110b) are equivalent to each other, and so the entire set (44) are equivalent to the entire set (49).

Eqs. (50)

The limits (50a) coincide with (41a). Therefore (50b) must be equivalent to (41b) or (41c). The direct proof of the equivalence is as follows. Write (50b) and (41c) in the forms, respectively

$$2m_1v_1' + |m_1 - m_2|v_2' \leqslant Mv_2 \qquad (115a)$$

$$Mv_1 \leqslant 2m_2v_2' - |m_1 - m_2|v_1' \qquad (115b)$$

Squaring (115a)

$$4m_1^2 v_1'^2 + (m_1 - m_2)^2 v_2'^2 + 4m_1 |m_1 - m_2| v_1'v_2' \leqslant M^2 v_2^2$$

$$4m_1 |m_1 - m_2| v_1'v_2' \leqslant M^2 v_2^2 - 4m_1^2 v_1'^2 - (m_2 - m_1)^2 v_2'^2 \qquad (115c)$$

Squaring (115b)

$$M^2 v_1^2 \leqslant 4m_2^2 v_2'^2 + (m_1 - m_2)^2 v_1'^2 - 4m_2 |m_1 - m_2| v_1'v_2'$$

$$4m_2 |m_1 - m_2| v_1'v_2' \leqslant 4m_2^2 v_2'^2 + (m_1 - m_2)^2 v_1'^2 - M^2 v_1^2 \qquad (115d)$$

So (115c) is identical with (115d) if

$$m_2[M^2 v_2^2 - 4m_1^2 v_1'^2 - (m_1 - m_2)^2 v_2'^2] = m_1[4m_2^2 v_2'^2 + (m_1 - m_2)^2 v_1'^2 - M^2 v_1^2] \qquad (115e)$$

i.e. if,

$$m_2[4m_1m_2v_1'^2 - M^2 \frac{2\Delta E}{m_2} - 4m_1^2 v_1^2 + 4m_1^2 \frac{2\Delta E}{m_1}] =$$

$$m_1[4m_2^2 v_2'^2 - 4m_1m_2v_1^2 - (m_1 - m_2)^2 \frac{2\Delta E}{m_1}]$$

which holds because

$$-M^2 + 4m_1m_2 = -(m_1 - m_2)^2$$

Conversely, working back from the identity (115e), one infers that (115d) and (115c) are equivalent. Eq. (115c) implies (115a); Eq. (115d) implies

$$Mv_1 \leqslant \left| 2m_2 v_2' - |m_1 - m_2| v_1' \right| \qquad (116a)$$

Thus (41e) is equivalent to (50b) plus

$$2m_2v_2' \geq |m_1 - m_2|v_1' \tag{116b}$$

However, (116b) is implied by (50b), as follows:

$$2m_1v_1' \leq Mv_2 - |m_1 - m_2|v_2'$$

$$2m_1|m_1 - m_2|v_1' \leq M|m_1 - m_2|v_2 - (m_1 - m_2)^2v_2'$$

$$2m_1|m_1 - m_2|v_1' \leq M|m_1 - m_2|v_2 - M^2v_2' + 4m_1m_2v_2'$$

$$M^2v_2' - M|m_1 - m_2|v_2 \leq 4m_1m_2v_2' - 2m_1|m_1 - m_2|v_1'$$

$$M[Mv_2' - |m_1 - m_2|v_2] \leq 2m_1\,[2m_2v_2' - |m_1 - m_2|v_1'] \tag{116c}$$

But the left side of (116c) necessarily is positive for $\Delta E \geq 0$. So the right side of (116c) is positive, i.e., (116b) holds.

This completes the proof that (50b) is equivalent to (41b) or (41c). The pair of conditions (50c), (50d) now is simply a rewrite of the pair (41d), (41e); there is no simple way to deduce this ΔE condition directly from (50b).

Eqs. (51) - (52)

Eqs. (51) bear the same relation to Eqs. (42) as (49) did to (44), there is no need to discuss them further. Eq. (52b) must be equivalent to (45b). The proof of this equivalence resembles the proof of the equivalence of (49) and (41).

Eq. (52b) takes the form

$$2m_1 v_1' + Mv_2 \leq |m_1 - m_2| v_2' \tag{117a}$$

Eq. (45b) is

$$2m_2 v_2 + Mv_1' \leq |m_1 - m_2| v_1 \tag{117b}$$

Squaring (117a)

$$4m_1^2 v_1'^2 + M^2 v_2^2 + 4m_1 M v_1' v_2 \leq (m_1 - m_2)^2 v_2'^2$$

$$4m_1 M v_1' v_2 \leq (m_1 - m_2)^2 v_2'^2 - M^2 v_2^2 - 4m_1^2 v_1'^2 \tag{117c}$$

Squaring (117b)

$$4m_2^2 v_2^2 + M^2 v_1'^2 + 4m_2 M v_1' v_2 \leq (m_1 - m_2)^2 v_1^2$$

$$4m_2 M v_1' v_2 \leq (m_1 - m_2)^2 v_1^2 - M^2 v_1'^2 - 4m_2^2 v_2^2 \tag{117d}$$

So (117c) and (117d) are equivalent if

$$m_2[(m_1 - m_2)^2 v_2'^2 - M^2 v_2^2 - 4m_1^2 v_1'^2] = m_1[(m_1 - m_2)^2 v_1^2 - M^2 v_1'^2 - 4m_2^2 v_2^2] \tag{117e}$$

i.e., if

$$m_2[-4m_1 m_2 v_2^2 + (m_1 - m_2)^2 \frac{2\Delta E}{m_2} - 4m_1^2 v_1'^2] =$$

$$m_1[-4m_1 m_2 v_1'^2 + (m_1 - m_2)^2 \frac{2\Delta E}{m_1} - 4m_2^2 v_2^2]$$

which is an identity. So working back, the equivalence of (117c) and (117d) implies the equivalence of (117a) and (117b) without subsidiary conditions.

Relations Corresponding to Eqs. (48) - (52) when $\Delta E \leq 0$

The pair of Eqs. (48b), (48c) are invariant under the interchange of v_1, v_1' and of v_2, v_2'. Thus (48d) continues to hold, but working back one gets only (105), i.e., (106) is required. But with $\Delta E \leq 0$ the argument (Eqs. (107)) showing (106) followed from (48d) no longer is valid. So (48d) must be supplemented by (106).

Eqs. (49b), (49c) form another invariant pair.

Eq. (50b) becomes

$$0 \leq v_1 \leq (2m_1)^{-1}[Mv_2' - |m_1 - m_2|v_2] \tag{118a}$$

from which follows

$$2m_1v_1 + |m_1 - m_2|v_2 \leq Mv_2' \tag{118b}$$

So, in the usual way

$$4m_1^2v_1^2 + (m_1 - m_2)^2v_2^2 + 4m_1|m_1 - m_2|v_1v_2 \leq M^2\left(v_2^2 + \frac{2\Delta E}{m_2}\right)$$

$$4m_1^2v_1^2 - 4m_1m_2v_2^2 + 4m_1|m_1 - m_2|v_1v_2 \leq M^2\,\frac{2\Delta E}{m_2}$$

$$\frac{4m_1m_2}{M^2}\left[E_1 - E_2 + \left|E_1\frac{v_2}{v_1} - E_2\frac{v_1}{v_2}\right|\right] \leq \Delta E \tag{118c}$$

Eq. (118c) is equivalent to (118a), without any auxiliary conditions.

Eq. (52b) becomes

$$0 \leqslant v_1 \leqslant (2m_1)^{-1} [|m_1 - m_2|v_2 - Mv_2'] \tag{119a}$$

implying

$$Mv_2' \leqslant |m_1 - m_2|v_2 - 2m_1v_1$$

$$M^2v_2'^2 = M^2\left(v_2^2 + \frac{2\Delta E}{m_2}\right) \leqslant (m_1 - m_2)^2v_2^2 + 4m_1^2v_1^2 - 4m_1|m_1 - m_2|v_1v_2$$

$$M^2\,\frac{2\Delta E}{m_2} \leqslant -4m_1m_2v_2^2 + 4m_1^2v_1^2 - 4m_1|m_1 - m_2|v_1v_2$$

$$\Delta E \leqslant \frac{4m_1m_2}{M^2}\left[E_1 - E_2 - \left|E_1\frac{v_2}{v_1} - E_2\frac{v_1}{v_2}\right|\right] \tag{119b}$$

which must be supplemented by

$$2m_1v_1 \leqslant |m_1 - m_2|v_2 \tag{119c}$$

These results, and the results of Eqs. (87) - (89), are summarized at the end of section IV.

References and Footnotes

1. M. Gryzinski, Phys. Rev. 138, A305 (1965).
2. M. Gryzinski, Phys. Rec. 138, A322 (1965).
3. M. Gryzinski, Phys. Rev. 138, A336 (1965).
4. E. Bauer and C. D. Bartky, J. Chem. Phys. 43, 2466 (1965).
5. Robert C. Stabler, Phys. Rev. 133, A1268 (1964).
6. V. I. Ochkur and A. M. Petrun'kin, Opt. Spectry. (U.S.S.R.) 14, 245 (1963) [Opt. i Spectroskopiya 14, 457 (1963)].
7. This result, and some other equations obtained in this paper, can be found in Gryzinski's papers (references 1 - 3), or in the earlier work of Chandrasekhar. S. Chandrasekhar, Astrophys. J. 93, 285 and 323 (1941); R. E. Williamson and S. Chandrasekhar, Astrophys. J. 93, 305 (1941).
8. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis", Cambridge 1940), p. 113.
9. Note that although Stabler regards 1 as the "target" electron and 2 as the "incident" electron, his ΔE, defined by his Eq. (1), is identical with my ΔE of Eq. (5).
10. See section V.

Figure 1 Caption

Integration region (shaded) in the v,V plane for Eq. (28). Lines (a), (b), (c) are plots of Eqs. (31a), (31b), (31c) respectively. Lines (a), (c) intersect at $v = v_{ac}$; lines (b), (c) at $v = v_{bc}$. The ellipse (d) is a plot of Eq. (32), for the case that its intersections with the boundaries of the shaded region occur on lines (a), (b), at $v = v_{\alpha}$, v_{β} respectively. In this case, the limits of integration in (33) are $v_{\ell} = v_{\alpha}$ and $v_u = v_{\beta}$.

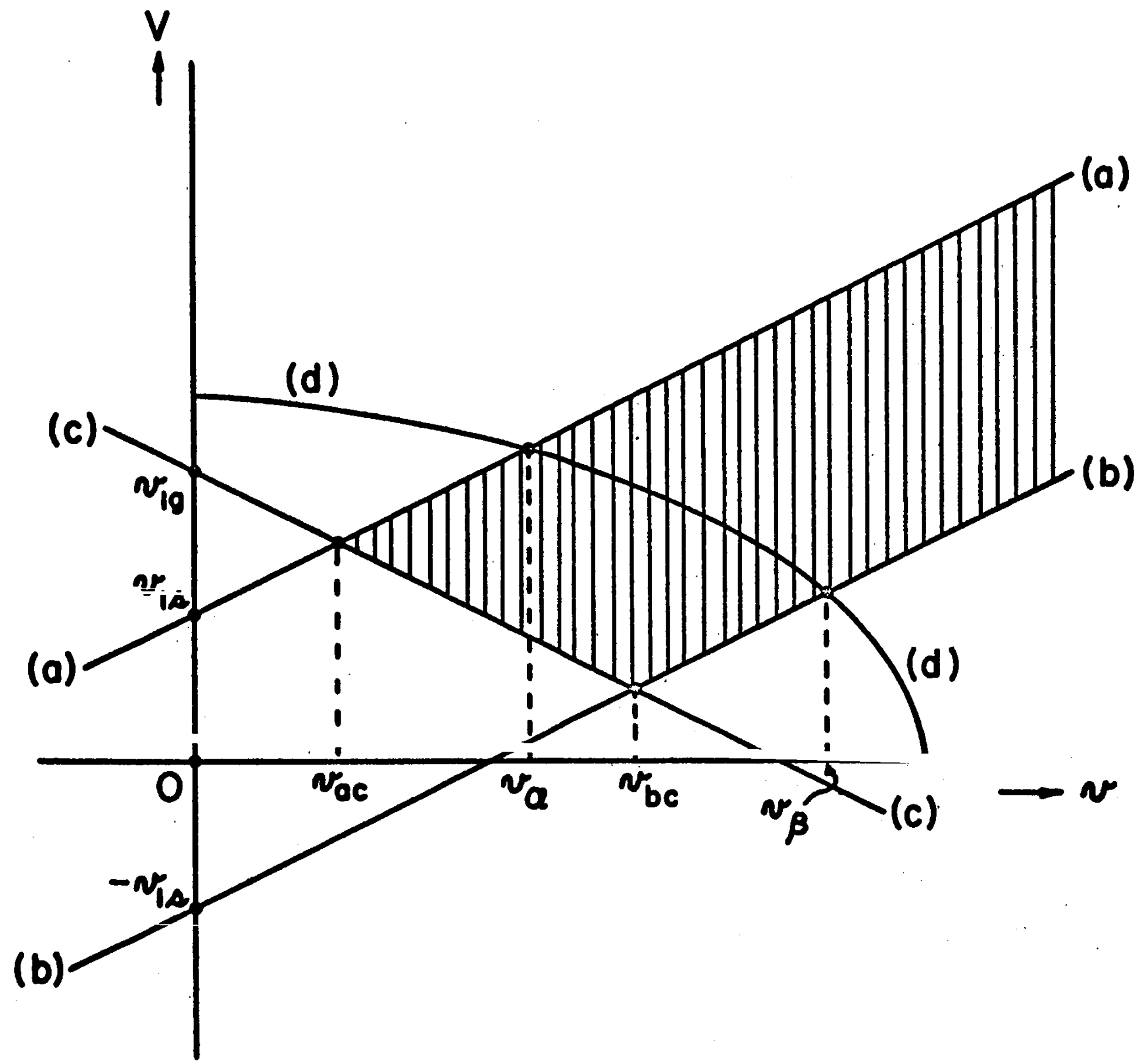

Fig. 1