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GUIDANCE SYSTEM COMPONENT CORRELATION TESTING AND STUDIES

Research Note Prepared by

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INTRODUCTION

The first six months of research on NASA Grant NGR-22-009-078 investigated the application of statistical techniques to the experimental and analytical study of inertial guidance components.

The completed research involved both instrumentation and data collection. Instrumentation required for support of the inertial component was obtained, modified or built, after which drift data of an inertial gyroscope was obtained. These data consist of torque-to-balance measurements taken at hourly intervals for over 450 hours. A second test consisted of measurements taken over approximately the same number of 15 minute intervals.

Concurrent with these activities an examination of the applicability of Markov process theory was conducted. Parallel efforts on the part of a graduate student, Lt. John R. Cooper, H. M. S. Royal Navy, were conducted using already available gyro drift test data and adapting previously established modelling techniques. This resulted in a Master's thesis entitled, "A Statistical Analysis of Gyro Drift Test Data."

The appendices explain definitions and the mathematical manipulations involved in various types of Markov chains. The notation is mainly that used in "Finite Markov Chains" by John G. Kemeny and J. Laurie Snell published by Van Nostrand Company, Inc., Princeton, New Jersey. Other sources for some of the applicable set theory concepts are "Markov Chains with Stationary Transition Probabilities" by Kai Lai Chung published by Springer-Verlag, Berlin, Germany, and "Introduction to Probability Theory and Its Application" by W. Feller published by J. Wiley and Sons, Inc., New York, New York.

All proofs regarding results stated in these appendices are contained in these references.

1. INSTRUMENTATION AND DATA COLLECTION

The collected data were concerned with small variations in inertial gyroscope performance over a period of seven hundred hours. Consequently, all supporting electronics and test instrumentation had to be capable of long term stability. The inertial component investigated was a GG-159 gyroscope provided on a loan basis by Aeronautical Division of Honeywell, Inc., Minneapolis, Minnesota. A gyro mounting fixture and electronic circuit designs used previously for testing similar gyros was provided by the M.I.T. Instrumentation Laboratory's "Skipper B" group under the direction of Mr. Richard E. Marshall. The circuitry was re-designed and adapted to the purposes of this investigation.

Fig. 1 is a block diagram of the manner in which the gyroscope was tested. For any constant rate input, a compensation current was applied to the torque generator thereby cancelling the rate input. Deviations from this constant rate input (e.g., gyro drift) were compensated by a closed loop around the gyro. The compensation current in this control loop is proportional to the gyro drift.

During long-term operation of the gyroscope, parameters such as gyro drift, wheel voltage, wheel power and gyro temperature were recorded. Of these, only gyro drift could be adequately recorded since fluctuations in the other parameters were so small as to be unmeasurable without an order to magnitude improvement in instrumentation sensitivity. Such improvement would require either a major effort in circuit design or an equipment procurement exceeding the available funding. These means were unjustified in view of the fact that the research goal was basically to model gyro drift.

Collected data consisted of approximately four consecutive weeks of continuous test with useable data for 6, 7, 9 and 4 consecutive day intervals. Interruptions in useable data occurred because: On two occasions air conditioner mis-operation caused the room temperature to fall from 70°F to 55°F; once the data recorder ran out of paper in the early morning hours; and once the recording

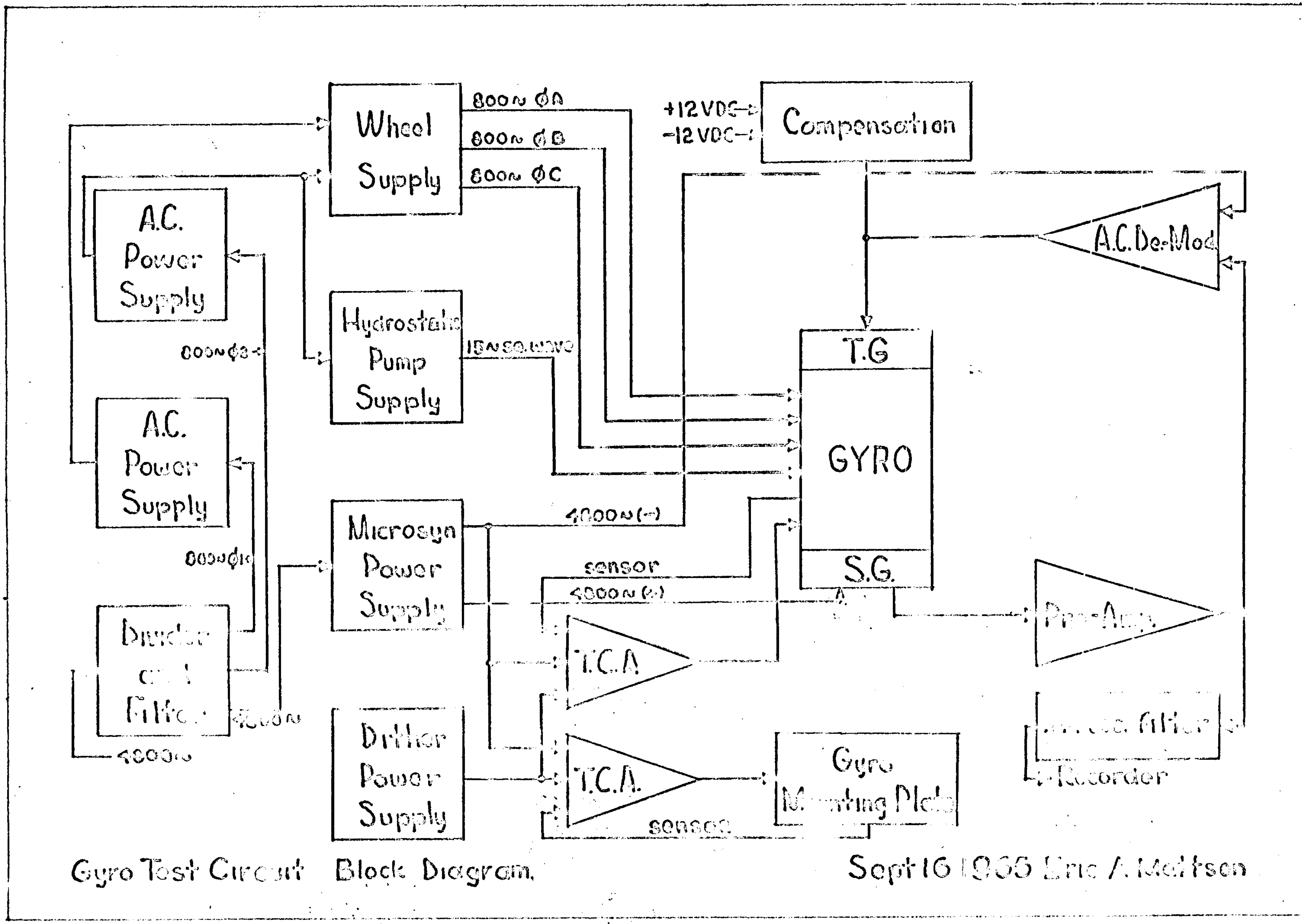
Wheel
Supply
800~ ØA
800~ ØB
800~ ØC
+12VDC
-12VDC
Compensation
A.C.
Power
Supply
A.C.
Power
Supply
800~Ø3
Hydrostatic
Pump
Supply
15~sq.wave
T.G
GYRO
S.G.
A.C. De-Mod
Microsyn
Power
Supply
4000~(-)
sensor
4000~(+)
T.C.A.
T.C.A.
Dither
Power
Supply
Gyro
Mounting Plate
sensor
Recorder
Gyro Test Circuit Block Diagram.
Sept 16 1965

Fig. 1

interval was changed from hourly to quarter-hourly (during the last four days). In all of these cases the gyroscope was operating continuously and only the data recording process was interrupted. This resulted in approximately 450 hourly gyro drift measurements and approximately the same number of quarter-hour interval measurements. These data are at present being analyzed and will be reported later.

Future Effort

Further work will be directed towards the analysis of data already acquired. NASA-ERC has suggested that the primary research goal of this grant shift to computational support for general theoretical navigation and guidance studies conducted at EAL. It is anticipated that by the end of the present period of the grant such a shift will be accomplished.

2. GYRO DRIFT RATE AS A MARKOV CHAIN

The application of Markov chain theory to gyro drift rate, or to any measurement which has a random drift, is based on the assumption that incremental drift rate is a stationary random process. That is, given an initial drift rate, the change in drift rate over the next interval is independent of all the previous intervals except the immediately preceding interval. This is the description of a first order Markov process. If the drift rates are defined as the states, or events, the transitions to any other states (the incremental drift rates) are the bases for the transition matrix. If the statistics of the incremental drift rate can be determined, the transition matrix can be obtained. Each element of the transition matrix is the probability of occurrence of a change in the magnitude of drift rate (incremental drift rate) equal to the difference between the two drift rate states, the present drift rate and the next ensuing drift rate. Once the transition matrix is known then the entire statistical process is known given the initial conditions.

The primary problem in the case of gyro drift is establishing whether incremental drift rate is a stationary random process. Previous work done by Weinstock [1] and Cooper [2] establish that this assumption is not valid except for certain conditions which were dependent on gyro orientation and data time intervals. In Cooper's case there was cause to suspect the data since it was not considered a true random sample. The data were obtained from 50 gyroscopes which had all passed a specific test of their drift rate within a given tolerance. In this way, the data were not a true sample since data from a gyroscope with poor performance are eliminated. It is therefore necessary to obtain data according to some true statistical plan. The best method is to place a single gyro in a fixed orientation and collect drift data until a sample is obtained which is large enough to yield a good approximation for the transition probabilities.

With data taken specifically for this purpose, the doubts encountered using other data will be removed.

The application of the Markov Chain theory may be extended to include gyro drift situations not described by the simple case. These would include:

1. The investigation of using shorter intervals for data collection.

2. The application of higher order Markov processes, that is, the effect on gyro drift may be examined as a function of what has occurred two or more intervals previously.

3. The effect of making the length of the data invervals change according to some statistically distributed function.

4. The analysis of data from gyroscopes which have passed a given standard test as a Markov chain with absorbing states.

5. The analysis of Markov chains with different transition probability distributions.

APPENDIX A

SIMPLE MARKOV CHAIN THEORY

The Markov process or chain is defined as a process in which the state at a future time has a conditional probability dependent only on the state it is in at the present time. This definition permits both continuous and discrete chains. The latter type will be of more interest in the following discussion. If the magnitudes of the changes in state are restricted to unit changes, the special case Markov chain is then sometimes referred to as the random walk.

The sequence described by a given Markov chain is a function of the probabilities for transition from a given state to another. Let E_{jn} be the event that a certain state exists at some time (or trial) n, then defining p_{ij} as the conditional probability that given E_i has occurred that E_j will occur on the next time interval (or trial) then

$$E_{jn} = p_{ij} E_{i(n-1)}$$

since the p_{ij} (called transition probability) is assumed stationary or time invariant. This notation permits the probability for a given sequence E_{j0}, E_{j1}, E_{j2} . . . E_{jn} to be written:

$$P[E_{j0} E_{j1} E_{j2} \cdots E_{jn}] = a_{j0}\, p_{j0\,j1}\, p_{j1\,j2} \cdots p_{jm-2\,jm-1}\, p_{jm-1\,jm}$$

where a_{j0} is the probability of being in the given initial state.

Given a number of states and their related transition probabilities the transition from a given state into any other state including itself can be written

$$p_i = p_{i1} + p_{i2} + p_{i3} + \cdots = \sum_{j=1}^{n} p_{ij} = 1 .$$

Since the transition must take place, the sum of the probabilities is one. This result can be expressed in the form of a matrix

$$[P] = \begin{bmatrix} p_{00} & p_{01} & & p_{0n} \\ p_{10} & p_{11} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n0} & p_{ni} & \cdots & p_{nn} \end{bmatrix}$$

called the transition matrix which expresses the transition probabilities from any given state to any other state in compact form. The row sums equal one because transition from a given state must occur as expressed previously. The transition matrix and the initial distribution completely define a Markov chain.

Given an initial distribution $\{E_{i0}\}$ the distribution after one trial is the matrix product

$$\{E_{i1}\} = [P] \{E_{i0}\}$$

where $\{\ \}$ indicates a column matrix. After two trials

$$\{E_{i2}\} = [P] \{E_{i1}\} = [P][P] \{E_{i0}\} = [P]^2 \{E_{i0}\}$$

This process may continued at will, resulting in

$$\{E_{in}\} = [P]^n \{E_{i0}\}$$

This shows that the higher transition probabilities–the probability of a change from a given state to another given state in a particular number of trials– may be obtained by raising the transition matrix to the power equal to the desired number of trials.

At this point, the concepts of closed sets, transient and persistent states and absorbing states are of interest.

A closed set is defined as set of states from which a state externnal to this set can not be reach. This set may be accessible from the outside but once entered there is no exit. If a closed set contains one state, it is called an absorbing state. The set of all states in a system may contain more than one closed set. Within a closed set, transitions are possible between every pair of states but not necessarily in one step. These states in a closed set are further categorized by the probability whether starting from a particular state the system returns to that state. If this is a certainty, the state is called persistent, or, in other words, the state continually recurs in the sequence. The probability for persistent state recurrence is therefore equal to one. States whose recurrence probabilities are less than one are called transient because there is a finite probability that these states may never recur.

The following transition matrix will indicate the relationships to the previous definitions.

	E_0	E_1	E_2	E_3	E_4	E_5
E_0	p_{00}	p_{01}	p_{02}	p_{03}	p_{04}	p_{05}
E_1	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}
E_2	p_{20}	p_{21}	p_{22}	p_{23}	p_{24}	p_{25}
E_3	p_{30}	p_{31}	p_{32}	p_{33}	p_{34}	p_{35}
E_4	p_{40}	p_{41}	p_{42}	p_{43}	p_{44}	p_{45}
E_5	p_{50}	p_{51}	p_{52}	p_{53}	p_{54}	p_{55}

Given the set of 6 states, let E_0 and E_1 represent a closed set C_1 of persistent states, E_2 and E_3 represent one closed set C_2 of transient states and E_4 and E_5 represent another closed set C_3 of transient states may always be rearranged and re-numbered to conform to this model.

For C_1, a closed set, the transition probabilities to the other states are all zero

$$p_{on} = p_{in} = 0, \quad n > 1$$

For C_2, transitions to C_3 are forbidden

$$p_{2n} = p_{3n} = 0, \quad n > 3$$

Similarly for C_3, transitions are possible only to the preceding sets. The resulting transition matrix is

	E_0	E_1	E_2	E_3	E_4	E_5
E_0	p_{00}	p_{01}	0	0	0	0
E_1	p_{10}	p_{11}	0	0	0	0
E_2	p_{20}	p_{21}	p_{22}	p_{23}	0	0
E_3	p_{30}	p_{31}	p_{32}	p_{33}	0	0
E_4	p_{40}	p_{41}	p_{42}	p_{43}	p_{44}	p_{45}
E_5	p_{50}	p_{51}	p_{42}	p_{53}	p_{54}	p_{55}

If C_2 were another set of persistent states, the sub-matrix of transition probabilities to C_1 would be zero.

By reducing the transition matrix to the indicated partitioned form

C_1	0	0
R_2	C_2	0
R_3		C_3

the effect of taking powers of the matrix can be seen. The "C" matrices individually will be raised to the n power and the new "R" would then represent the higher transition probabilities from set to set.

By the use of transition matrices, various types of chains may be studied. Of particular interest are regular Markov chains and absorbing Markov Chains. The former having no transient states and the latter containing absorbing states.

The types of information which can be determined are:

Starting in a given state what is the probability of reaching another given state in n steps?

What is the mean number of steps required to pass between two given states?

What is the variance of the mean number of steps required to pass between the two given states?

These questions may be related to gyro drift as follows:

Given an initial state of gyro drift what is the probability of exceeding a given limit in n steps?

What is the mean number of steps required to pass from a given gyro drift to a given limit?

What is the variance of the mean number of steps required to pass from a given gyro drift to a given limit?

APPENDIX B

MARKOV CHAIN WITH ABSORBING STATES

In the previous section the concepts of closed sets and persistent and transient states were discussed. In this section the discussion will cover a particular type of Markov chain. This chain will have as upper and lower limits, two closed set composed of one state each i.e., absorbing states. The transition matrix is:

	E_0	E_1	E_2	E_3	E_4
E_0	1	0	0	0	0
E_1	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}
E_2	p_{20}	p_{21}	p_{22}	p_{23}	p_{24}
E_3	p_{30}	p_{31}	p_{32}	p_{33}	p_{34}
E_4	0	0	0	0	1

By re-ordering and re-numbering the matrix may be arranged in canonic form to:

	E_0	E_1	E_2	E_3	E_4
E_0	1	0	0	0	0
E_1	0	1	0	0	0
E_2	p_{20}	p_{21}	p_{22}	p_{23}	p_{24}
E_3	p_{30}	p_{31}	p_{32}	p_{33}	p_{34}
E_4	p_{40}	p_{41}	p_{42}	p_{43}	p_{44}

or, in different notation, the partitioned matrix becomes

$$\overline{P} = \begin{array}{c|c} I & O \\ \hline R & Q \end{array}$$

where I is the identity matrix and O is a matrix of all zero's.

<u>The Fundamental Matrix for Absorbing Markov Chains</u>

Since Q^n tends to zero for an absorbing chain, then $(\overline{I}-\overline{Q})^{-1}$, then the mean number of steps t (including the first) required to reach an absorbing state starting from a transient state is

$$\{M_i[t]\} = \tau = N\xi$$

where ξ is a column vector of 1's

The variance of t is:

$$\{Var_i[t]\} = \tau_2 = (2N-I)\ \tau - \tau_{sq}$$

where $\tau = N.\xi$

τ_{sq} = the matrix t with each of its elements squared (not equal to $[t]^2$ unless t is a diagonal matrix).

For the probability that starting in a transient state the system ends up in an absorbing state:

$$B = NR$$

The relationship between Markov chains with absorbing states and gyro drift is that this chain may model data obtained from gyroscopes which pass some gyrodrift test with a certain standard of performance. The data derived from those gyroscopes which have not passed the test, having been removed, may have the effect similar to a Markov chain with a absorbing states. The gyroscopes having been rejected are absorbed, so to speak, and are no longer considered in the sample.

APPENDIX C

REGULAR MARKOV CHAIN

A regular Markov chain is a chain which has no transient states and has a single closed set. The regular chain has no absorbing states and after n steps the system may be in any of the states; there are no zeros in the transition matrix.

In this case, the transition matrix p^n approaches a limit A as n increases. The matrix a has identical rows with all positive components. A has the property AP = PA = A and the A row vector is that probability vector which for large n gives the probability of being in a certain state.

The Fundamental Matrix for Regular Chains

As in the case of Markov chains with absorbing states, regular Markov chains have fundamental matrices associated with them.

The fundamental matrix Z for the Markov chain defined by the transition matrix P is defined as

$$Z = [I - (p - A)]^{-1}$$

where I is the identity matrix and A is the limiting matrix of P.

The fundamental matrix may then be used to obtain the mean first passage time matrix M. The mean first passage time matrix is the matrix of the average time of (average number of steps) the first passing into a given state starting from a given state (f_j)

$$m_{ij} = M_i [f_j]$$

$$M = (I - Z + E Z_{dg}) D$$

where I is the identify matrix E is a square matrix aff of whose components are 1's

Z_{dg} is the diagonal matrix with components Z_{ii} of the fundamental matrix Z and D is the diagonal matrix with components $d_{ii} = \frac{1}{a_i}$ where a_i is the i^{th} component of A. The variance of M is found by

$$W = \{ M_i [f_i^2] \} = M(2 Z_{dg} D-I) + 2 [ZM-E(ZM)_{dg}]$$

This type of Markov chain may be applicable to the general case of gyroscope drift data. The general case would contain data which would be bounded but with all levels of drift a possibility, and all levels likely to recur.

REFERENCES

1. Weinstock, Herbert, "Statistical Analyses of Five Inertial-Reference Servo Runs Preliminary Results," Report #E-1694, M.I.T. Instrumentation Laboratory, Cambridge, Massachusetts, 02139, November 1964.

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