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# ON THE FORMULATION OF EQUATIONS OF MOTION OF AN ECCENTRICALLY STIFFENED SHALLOW CIRCULAR CYLINDRICAL SHELL.

by

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## SUMMARY

A shallow circular cylindrical shell with closely spaced stringer- and ring-stiffeners is considered. Equilibrium equations are derived on the basis of assumed states of stress in the monocoque cylinder and stiffeners through superposition and smearing-out of the stiffening effects. Inertia terms of such an equivalent shell are incorporated and coupled equations of motion are stated in linear and nonlinear form. Partial decoupling of the radial displacement equation is shown to be possible for the orthotropic shell (zero eccentricity). On neglecting tangential inertia effects the equations of motion are formulated through the use of a stress function which results in a system of two nonlinear partial differential equations in the radial displacements and the stress function. Since some of these equations have not appeared in the open literature an effort has been made to check them by reduction to well-known expressions for the orthotropic and isotropic shell.

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## 1. INTRODUCTION

The dynamic stability of monocoque and stiffened cylindrical shells is the topic presently under investigation at the Applied Mechanics Laboratory of Syracuse University. This interim report covers the initial activity of the writer in formulating the pertinent equations. It is therefore not the purpose to arrive at a preset goal, but rather to report an initial stage of development leading to the dynamic response.

## 2. SCOPE

Differential equations of motion in linear and nonlinear form are derived for eccentrically stiffened shells. The stiffeners are assumed to be closely spaced and consist of stringers and rings equally spaced in the longitudinal and circumferential direction, respectively.

## 3. ACCOMPLISHMENTS

A simpified linearly elastic stress-strain distribution is assumed in the monocoque shell and the stiffened part from which the stress resultants, bending- and twisting moments are derived in terms of strains and curvature changes. The additive stiffness provided by the stringers and rings is assumed to be "smeared-out" over the shell which is justifiable for stiffener spacings that are small with respect to the half-wavelength of the buckling pattern. The stress-, moment-and twist resultants are given in matrix form in terms of strains and curvature changes. It is shown that when linear strain-displacement relations are used these equations become identical with those of reference (1). On using nonlinear strain-displacement relations the equations of reference (2) are obtained. In these two references the principle of stationary total potential is employed in which the stress-, moment and twist resultants are inherently defined and the equations of equilibrium and boundary conditions follow from Euler's variational equations. The equilibrium equations used in this paper accomodate components of the membrane forces in the radial direction and also include inertia forces. The equations of motion follow through the introduction of the displacements and their space and time derivatives. When these equations are linearized and reduced to the isotropic shell, Flügge's shallow shell equations of motion of reference (3) result.

An attempt is made for the linear case to obtain an equation of motion in the radial displacement w alone, while accounting for tangential plane inertia terms. This partial decoupling is achieved only for the orthotropic case since the eccentricity of the stiffeners upsets the symmetry requirement of the decoupling procedure. A reduction to the isotropic case checks with the equation obtained in reference (5).

Nonlinear equations of motion are derived on the basis of a stress function f(x,y) when tangential inertias are neglected. This method reduces the problem to the solution of two coupled fourth order nonlinear partial differential equations in f and w which is analogous to the von Kármán large deflection equations for the flat plate under static conditions.

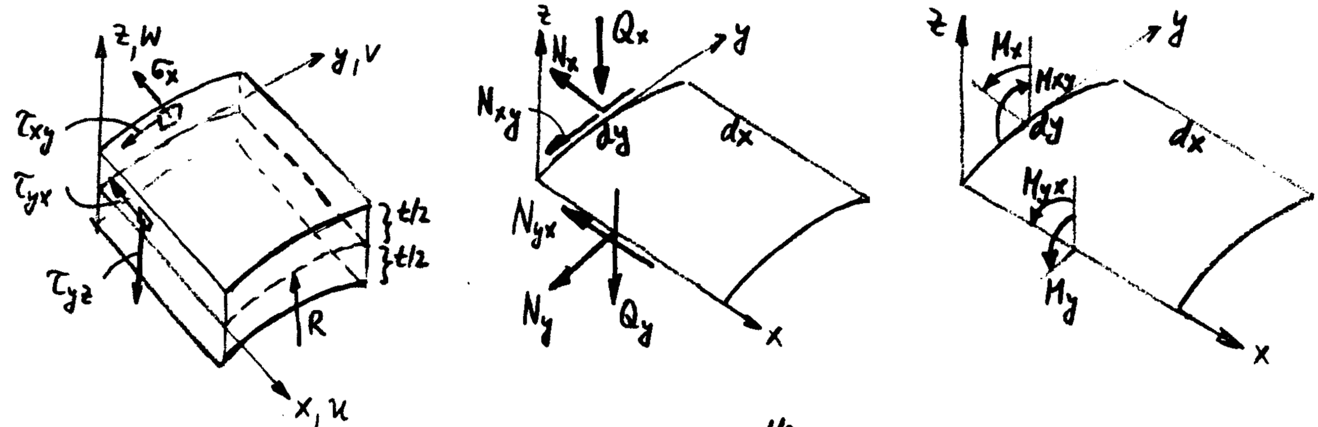
## 4. THE STRESS RESULTANTS, MOMENTS AND TWISTS FOR THE SHALLOW MONOCOQUE CYLINDRICAL SHELL

The plane stress-strain relations of the engineering theory of elasticity are assumed to be valid, e.g.

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\left(\varepsilon_{xT} + \nu\,\varepsilon_{yT}\right) \\ \sigma_y &= \frac{E}{1-\nu^2}\left(\varepsilon_{yT} + \nu\,\varepsilon_{xT}\right) \\ \tau_{xy} &= G\,\gamma_{xyT} \end{aligned}\right\} \quad (1)$$

where the subscript T on the strains relates to the total strain at any height in the thickness direction.

A thin shallow cylindrical shell of thickness t and middle surface radius R is assumed $(t/R \ll 1)$.
With the notation shown below the stress resultants, shear forces, moments and twists are defined as:




$$
\left.
\begin{aligned}
N_x^{(m)} &= \int_{-t/2}^{t/2} \sigma_x \, dz & N_y^{(m)} &= \int_{-t/2}^{t/2} \sigma_y \, dz \\
N_{xy}^{(m)} &= \int_{-t/2}^{t/2} \tau_{xy} \, dz & N_{yx}^{(m)} &= \int_{-t/2}^{t/2} \tau_{yx} \, dz \\
Q_x^{(m)} &= \int_{-t/2}^{t/2} \tau_{xz} \, dz & Q_y^{(m)} &= \int_{-t/2}^{t/2} \tau_{yz} \, dz \\
M_x^{(m)} &= \int_{-t/2}^{t/2} \sigma_x \, z \, dz & M_y^{(m)} &= \int_{-t/2}^{t/2} \sigma_y \, z \, dz \\
M_{xy}^{(m)} &= -\int_{-t/2}^{t/2} \tau_{xy} \, z \, dz & M_{yx}^{(m)} &= \int_{-t/2}^{t/2} \tau_{yx} \, z \, dz
\end{aligned}
\right\} \quad (2)
$$

where the superscript (m) stands for momocoque.

The assumption, that cross-sections normal to the middle surface remain normal to it after deformation, leads to the following relation between the total- and middle surface strains:

$$\left.\begin{aligned}
\varepsilon_{xT} &= \varepsilon_x - z\varkappa_x \\
\varepsilon_{yT} &= \varepsilon_y - z\varkappa_y \\
\gamma_{xyT} &= \gamma_{xy} - 2z\varkappa_{xy}
\end{aligned}\right\} \quad (3)$$

where $\varkappa_x, \varkappa_y$ and $\varkappa_{xy}$ are the changes in curvature and twist of the middle surface.
Introducing (3) into(1) and then into (2) and integrating over the thickness leads to:

$$\left.\begin{aligned}
N_x^{(m)} &= \frac{Et}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \\
N_y^{(m)} &= \frac{Et}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \\
N_{xy}^{(m)} &= N_{yx}^{(m)} = G\gamma_{xy} \\
M_x^{(m)} &= -\frac{Et^3}{12(1-\nu^2)}\left(\varkappa_x + \nu\varkappa_y\right) \\
M_y^{(m)} &= -\frac{Et^3}{12(1-\nu^2)}\left(\varkappa_y + \nu\varkappa_x\right) \\
M_{xy}^{(m)} &= -M_{yx}^{(m)} = \frac{Et^3}{12(1+\nu)}\varkappa_{xy}
\end{aligned}\right\} \quad (4)$$

In the sequel the following abbreviations will be used:
( A summary of all abbreviated parameters is given in the Appendix)

$$\left.\begin{aligned}
K &= \frac{Et}{1-\nu^2} \\
K_\nu &= \frac{\nu Et}{1-\nu^2} \\
K_G &= \frac{Et}{2(1+\nu)} = Gt \\
K_P &= K_\nu + K_G = \frac{Et}{2(1-\nu)} \\
D &= \frac{Et^3}{12(1-\nu^2)} \\
D_\nu &= \frac{\nu Et^3}{12(1-\nu^2)} \\
D_G &= \frac{Et^3}{12(1+\nu)} = \frac{Gt^3}{6}
\end{aligned}\right\} \tag{5}$$

With (5) equations (4) can be written as:

$$\left.\begin{aligned}
N_x^{(m)} &= K\varepsilon_x + K_\nu \varepsilon_y \\
N_y^{(m)} &= K\varepsilon_y + K_\nu \varepsilon_x \\
N_{xy}^{(m)} &= K_G \gamma_{xy}
\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} M_x^{(m)} &= -(D\varkappa_x + D_\nu \varkappa_y) \\ M_y^{(m)} &= -(D\varkappa_y + D_\nu \varkappa_x) \\ M_{xy}^{(m)} &= M_{yx}^{(m)} = D_G \varkappa_{xy} \end{aligned}\right\} \quad (6)$$

or in matrix form:

$$\begin{Bmatrix} N_x^{(m)} \\ N_y^{(m)} \\ N_{xy}^{(m)} \\ M_x^{(m)} \\ M_y^{(m)} \\ M_{xy}^{(m)} \end{Bmatrix} = \begin{pmatrix} K & K_\nu & 0 & 0 & 0 & 0 \\ K_\nu & K & 0 & 0 & 0 & 0 \\ 0 & 0 & K_G & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & -D_\nu & 0 \\ 0 & 0 & 0 & -D_\nu & D & 0 \\ 0 & 0 & 0 & 0 & 0 & D_G \end{pmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \varkappa_x \\ \varkappa_y \\ \varkappa_{xy} \end{Bmatrix} \quad (7)$$

## 5. THE STRESS RESULTANTS, MOMENTS AND TWISTS FOR THE SHALLOW STIFFENED SHELL

It is assumed that the stringers and rings can be treated as beams with zero Poisson ratio. The normal stresses are assumed zero for the stringers in the y-direction and for the rings in the x-direction.

$$\left.\begin{aligned} \sigma_x^{(S)} &= E_S \varepsilon_x - E_S z \varkappa_x \\ \sigma_y^{(R)} &= E_R \varepsilon_y - E_R z \varkappa_y \end{aligned}\right\} \quad (8)$$

Where S and R stand for stringer and ring, respectively.

a) <u>The Contribution of Stringers and Rings to the Stress Resultants</u>

Integrating (8) over stringer- and ring cross-section, respectively, (denoted by $A_S$ and $A_R$), we note that the strains and curvature changes can be treated as constants if their variation is small in the neighborhood of the cross-sectional areas.
Summing forces in the x-direction, contributed by the stringer, we can write,

$$F_x = E_s A_s \varepsilon_x - E_s \varkappa_x \int_{A_s} z \, dA_s = E_s A_s \varepsilon_x - E_s A_s \bar{z}_s \varkappa_x$$

where $\bar{z}_S$ is the centroidal distance of $A_S$ from the middle surface. If this contribution is smeared over the stringer spacing d, we obtain a stress resultant $N_x^{(S)}$ per unit circumference contributed by the stringer, e.g.

$$N_x^{(s)} = \frac{E_s A_s}{d} \varepsilon_x - \frac{E_s A_s \bar{z}_s}{d} \qquad (9)$$

Identical considerations apply to the ring cross-section with the result:

$$N_y^{(R)} = \frac{E_R A_R}{\ell} \varepsilon_y - \frac{E_R A_R \bar{z}_R}{\ell} \qquad (10)$$

where $\ell$ is the ring spacing.
The contribution of the ring and stringer cross-sections to the shear stress resultant are assumed to be negligibly small.

b) <u>The Contribution of Stringers and Rings to the Moments and Twists.</u>

Taking moments of the forces due to the stresses of equations (8) about the local coordinate axes of the middle surface results in:

$$M^{(s)} = \int_{A_s} (E_s \varepsilon_x z - E_s \varkappa_x z^2) dA_s = E_s A_s \bar{z}_s \varepsilon_x - E_s I_{so} \varkappa_x$$

where $I_{So} = \int_{A_s} z^2 dA_s$ is the area moment of inertia of the stringer cross-section with respect to the local y-axis of the middle surface. Smearing out this moment contribution over the stringer spacing d, we obtain:

$$M_x^{(s)} = \frac{E_s A_s \bar{z}_s}{d} \varepsilon_x - \frac{E_s I_{so}}{d} \varkappa_x \qquad (11)$$

Similarly, there results for the ring,

$$M_y^{(R)} = \frac{E_R A_R \bar{z}_R}{\ell} \varepsilon_y - \frac{E_R I_{Ro}}{\ell} \varkappa_y \qquad (12)$$

Twisting of the rings and stringers occurs due to the twisting curvature change $\varkappa_{xy}$ of the middle surface. The assumption that plane cross-section remain plane and normal to the middle surface is discarded for the stringers and rings under twisting. If the latter consist of tubular cross-sections they provide considerably to the twisting rigidity in comparison with the monocoque shell. For simplicity it is assumed that the effect of any possible joint of the stiffening elements can be ignored and the latter can warp freely. Since $\varkappa_{xy}$ is the twist per unit length the contribution of the stringer to the twisting torque is,

$$M_t^{(s)} = G_s J_s \varkappa_{xy}$$

where $J_S$ is the torsion constant for the stringer cross-section. Smearing-out the twisting torque over the stringer spacing d we obtain,

$$M_{xy}^{(s)} = \frac{G_s J_s}{d} \varkappa_{xy} \qquad (13)$$

In analogy, there is a contribution from the ring,

$$M_{yx}^{(R)} = -\frac{G_R J_R}{\ell} \varkappa_{xy} \qquad (14)$$

c) The Total Effect of Stiffeners and Monocoque Shell

By superposing corresponding quantities from equations (4),(9),(10), (13) and (14), the following stress resultants, moments and twists can be written:

$$\left.\begin{aligned}
N_x &= N_x^{(m)} + N_x^{(s)} = K\varepsilon_x + K_\nu \varepsilon_y + \frac{E_s A_s}{d}\varepsilon_x - \frac{E_s A_s \bar{z}_s}{d}\varkappa_x \\
N_y &= N_y^{(m)} + N_y^{(R)} = K\varepsilon_y + K_\nu \varepsilon_x + \frac{E_R A_R}{\ell}\varepsilon_y - \frac{E_R A_R}{\ell}\bar{z}_R \varkappa_y \\
N_{xy} &= N_{yx} = N_{xy}^{(m)} = N_{yx}^{(m)} = K_G \gamma_{xy} \\
M_x &= M_x^{(m)} + M_x^{(s)} = -D\varkappa_x - D_\nu \varkappa_y + \frac{E_s A_s \bar{z}_s}{d}\varepsilon_x - \frac{E_s I_{so}}{d}\varkappa_x \\
M_y &= M_y^{(m)} + M_y^{(R)} = -D\varkappa_y - D_\nu \varkappa_x + \frac{E_R A_R \bar{z}_R}{\ell}\varepsilon_y - \frac{E_R I_{Ro}}{\ell}\varkappa_y \\
M_{xy} &= M_{xy}^{(m)} + M_{xy}^{(s)} = D_G \varkappa_{xy} + \frac{G_s J_s}{d}\varkappa_{xy} \\
M_{yx} &= \qquad -D_G \varkappa_{xy} - \frac{G_R J_R}{\ell}\varkappa_{xy}
\end{aligned}\right\} \quad (15)$$

The following additional parameters are defined:
(Included in the list of the appendix)

$$\left.\begin{aligned}
K_s &= \frac{E_s A_s}{d} \\
K_R &= \frac{E_R A_R}{\ell} \\
F_{sb} &= \frac{E_s A_s}{d}\bar{z}_s \\
F_{Rb} &= \frac{E_R A_R}{\ell}\bar{z}_R
\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} D_S &= \frac{E_S I_{SO}}{d} = \frac{E_S}{d}\left[I_{SC} + \bar{z}_S^2 A_S\right] \\ D_R &= \frac{E_R I_{RO}}{\ell} = \frac{E_R}{\ell}\left[I_{RC} + \bar{z}_R^2 A_R\right] \\ D_{GS} &= \frac{G_S J_S}{d} \\ D_{GR} &= \frac{G_R J_R}{\ell} \end{aligned}\right\} \quad (16)$$

where $I_{SC}$ and $I_{RC}$ are the area moments of inertia with respect to parallel axes through the centroids of stringer and ring cross-sections.

$F_{Sb}$ and $F_{Rb}$ are algebraic quantities and they are positive in our notation for external stringers and rings.

When stringer and ring cross-sections are symmetrical with respect to the middle surface the eccentricities $\bar{z}_S$ and $\bar{z}_R$ are zero and therefore, $F_{Sb} = F_{Rb} = 0$. In this case there are only 12 stiffness-rigidity quantities and we speak of an orthotropic shell which corresponds to that of reference (4).

Considering equations (15), certain parameters can be grouped together and it is convenient to introduce certain combined stiffness -rigidity parameters which are defined as follows:
( Also listed in the Appendix)

$$\left.\begin{aligned} K_{HS} &= K + K_S = \frac{Et}{1-\nu^2} + \frac{E_S A_S}{d} \\ K_{HR} &= K + K_R = \frac{Et}{1-\nu^2} + \frac{E_R A_R}{\ell} \\ D_{HS} &= D + D_S = \frac{Et^3}{12(1-\nu^2)} + \frac{E_S}{d}\left[I_{SC} + \bar{z}_S^2 A_S\right] \\ D_{HR} &= D + D_R = \frac{Et^3}{12(1-\nu^2)} + \frac{E_R}{\ell}\left[I_{RC} + \bar{z}_R^2 A_R\right] \\ D_{HGS} &= D_G + D_{GS} = \frac{Gt^3}{6} + \frac{G_S J_S}{d} \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned}
D_{MGR} &= D_G + D_{GR} = \frac{Gt^3}{6} + \frac{G_R J_R}{\ell} \\
D_1 &= D_G + \frac{D_{GS} + D_{GR}}{2} \\
D_2 &= D_\nu + \frac{D_{MGR} + D_{MGS}}{2}
\end{aligned}\right\} \quad (17)$$

With these abbreviations equations (15) become:

$$\left.\begin{aligned}
N_x &= K_{MS}\,\varepsilon_x + K_\nu\,\varepsilon_y - F_{Sb}\,\varkappa_x \\
N_y &= K_\nu\,\varepsilon_x + K_{MR}\,\varepsilon_y - F_{Rb}\,\varkappa_y \\
N_{xy} &= N_{yx} = K_G\,\gamma_{xy} \\
M_x &= K_{Sb}\,\varepsilon_x - D_{MS}\,\varkappa_x - D_\nu\,\varkappa_y \\
M_y &= K_{Rb}\,\varepsilon_y - D_\nu\,\varkappa_x - D_{MR}\,\varkappa_y \\
M_{xy} &= D_{MGS}\,\varkappa_{xy} \\
M_{yx} &= -D_{MGR}\,\varkappa_{xy}
\end{aligned}\right\} \quad (18)$$

It must be noted that the twisting moments $M_{xy}$ and $M_{yx}$ are not of equal magnitude. In order to cast equations (18) into a matrix, let us define:

$$\bar{M}_{xy} = \frac{1}{2}\left(M_{xy} - M_{yx}\right) = \left(D_G + \frac{D_{GS} + D_{GR}}{2}\right)\varkappa_{xy} = D_1\,\varkappa_{xy} \quad (19)$$

$\bar{M}_{xy}$ can be interpreted physically as an average twisting moment by which the differences of torsional stiffness of stringers and rings is averaged out such that the magnitude of $\bar{M}_{xy}$ and $\bar{M}_{yx}$ are equal.

Equations (18) can be written in matrix form as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ \bar{M}_{xy} \end{Bmatrix} = \begin{pmatrix} K_{HS} & K_\nu & 0 & -F_{Sb} & 0 & 0 \\ K_\nu & K_{HR} & 0 & -F_{Rb} & 0 & 0 \\ 0 & 0 & K_G & 0 & 0 & 0 \\ F_{Sb} & 0 & 0 & -D_{HS} & -D_\nu & 0 \\ 0 & F_{Rb} & 0 & -D_\nu & -D_{HR} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 \end{pmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \varkappa_x \\ \varkappa_y \\ \varkappa_{xy} \end{Bmatrix} \qquad (20)$$

## 6. GEOMETRICALLY LINEAR RELATIONS FOR STRESS RESULTANTS, MOMENTS AND TWISTS FOR THE ECCENTRICALLY STIFFENED CIRCULAR CYLINDER

It is assumed that the strains, curvature changes and twists are linearly related to the middle surface displacements and their spatial derivatives in the following manner:

$$\left.\begin{aligned} \varepsilon_x &= u_{,x} \\ \varepsilon_y &= v_{,y} + \frac{w}{R} \\ \gamma_{xy} &= u_{,y} + v_{,x} \\ \varkappa_x &= w_{,xx} \\ \varkappa_y &= w_{,yy} \\ \varkappa_{xy} &= w_{,xy} \end{aligned}\right\} \qquad (21)$$

where a comma followed by the subscripted independent variable denotes partial differentiation with respect to that variable in the usual manner.

Introducing (21) into (18) results in:

$$\left.\begin{aligned}
N_x &= K_{HS}\, u_{,x} + K_\nu \left(v_{,y} + \frac{w}{R}\right) - F_{Sb}\, w_{,xx} \\
N_y &= K_\nu\, u_{,x} + K_{HR} \left(v_{,y} + \frac{w}{R}\right) - F_{Rb}\, w_{,yy} \\
N_{xy} &= N_{yx} = K_G \left(u_{,y} + v_{,x}\right) \\
M_x &= F_{Sb}\, u_{,x} - D_{HS}\, w_{,xx} - D_\nu\, w_{,yy} \\
M_y &= F_{Rb} \left(v_{,y} + \frac{w}{R}\right) - D_\nu\, w_{,xx} - D_{HR}\, w_{,yy} \\
M_{xy} &= D_{HGS}\, w_{,xy} \\
M_{yx} &= -D_{HGR}\, w_{,xy}
\end{aligned}\right\} \qquad (22)$$

It can be shown easily that equations (22) are identical with those of reference (1) if the proper substitutions are made for the stiffness and rigidity quantities.

## 7. GEOMETRICALLY NONLINEAR RELATIONS OF STRESS RESULTANTS, MOMENTS AND TWISTS FOR THE ECCENTRICALLY STIFFENED CIRCULAR CYLINDRICAL SHELL

The influence of a large radial displacement w is taken into account so that instead of (21) we have,

$$\begin{aligned}
\varepsilon_x &= u_{,x} + \tfrac{1}{2} w_{,x}^2 \\
\varepsilon_y &= v_{,y} + \tfrac{1}{2} w_{,y}^2 + \frac{w}{R} \\
\gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}\, w_{,y} \\
\varkappa_x &= w_{,xx} \\
\varkappa_y &= w_{,yy} \\
\varkappa_{xy} &= w_{,xy}
\end{aligned}$$

Introducing (21) into (18) yields:

$$\left.\begin{aligned}
N_x &= K_{HS}\left(u_{,x}+\tfrac{1}{2}w_{,x}^2\right)+K_\nu\left(v_{,y}+\tfrac{1}{2}w_{,y}^2+\tfrac{w}{R}\right)-F_{Sb}\,w_{,xx}\\
N_y &= K_\nu\left(u_{,x}+\tfrac{1}{2}w_{,x}^2\right)+K_{HR}\left(v_{,y}+\tfrac{1}{2}w_{,y}^2+\tfrac{w}{R}\right)-F_{Rb}\,w_{,yy}\\
N_{xy} &= N_{yx}= K_G\left(u_{,y}+v_{,x}+w_{,x}w_{,y}\right)\\
M_x &= F_{Sb}\left(u_{,x}+\tfrac{1}{2}w_{,x}^2\right)-D_{HS}\,w_{,xx}-D_\nu\,w_{,yy}\\
M_y &= F_{Rb}\left(v_{,y}+\tfrac{1}{2}w_{,y}^2+\tfrac{w}{R}\right)-D_\nu\,w_{,xx}-D_{HR}\,w_{,yy}\\
M_{xy} &= D_{HGS}\,w_{,xy}\\
M_{yx} &= -D_{HGR}\,w_{,xy}
\end{aligned}\right\}\quad(24)$$

With the proper substitution for the stiffeness and rigidity parameters and a slight rearrangement, equations (22) can be shown to be identical with those of reference (2).

In both, references (1) and (2), the stress resultants, moments and twists are arrived at through a formulation of the surface integrals of the strain energy density due to stretching, twisting and bending. Stress resultants, moments and twists are then defined as those forces and moments per unit length which yield the same strain energy density if they act through the their respective displacements and angles.

## 8. EQUILIBRIUM EQUATIONS FOR THE ECCENTRICALLY STIFFENED SMEARED-OUT SHELL ELEMENT

The equilibrium equations are formulated on the basis of the notations given in Section 4. Force equilibrium equations are written in the x-, y, and z-directions. Radial components of the membrane forces are taken into account. The following equations result:

$$\left.\begin{aligned}
N_{x,x}+N_{yx,y}+X &= 0\\
N_{y,y}+N_{xy,x}+Y &= 0
\end{aligned}\right\}\quad(25)$$

$$\left.\begin{aligned} &N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \frac{N_y}{R} + (N_{x,x} + N_{yx,y}) w_{,x} \\ &+ (N_{y,y} + N_{xy,x}) w_{,y} + Q_{x,x} + Q_{y,y} + Z = 0 \end{aligned}\right\} \quad (25)$$

where X, Y and Z are forces per unit surface.
Moment equilibrium equations are written about the x- and y-axes while the equilibrium equation about the z-axis is identically satisfied with our assumption of a shallow shell.
Discarding the latter equation, there remain:

$$\left.\begin{aligned} &M_{y,y} - M_{xy,x} - Q_y = 0 \\ &M_{x,x} + M_{yx,y} - Q_x = 0 \end{aligned}\right\} \quad (26)$$

On differentiating the first of (26) with respect to y, the second of (26) with respect to x and introducing the result into (25), the shear forces can be eliminated. If inertia forces are the only body forces, we can define an equivalent mass per unit area of the smeared-out equivalent stiffened shell as:

$$\bar{m} = \rho t + \rho_s \frac{A_s}{d} + \rho_R \frac{A_R}{\ell} \quad (27)$$

where $\rho$ refers to the mass per unit volume and the subscripts are selfexplanatory. Thus the following equations are obtained:

$$\left.\begin{aligned} &N_{x,x} + N_{yx,y} = \bar{m}\ddot{u} \\ &N_{y,y} + N_{xy,x} = \bar{m}\ddot{v} \\ &N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \frac{N_y}{R} + (N_{x,x} + N_{yx,y}) w_{,x} \\ &+ (N_{y,y} + N_{xy,x}) w_{,y} + M_{x,xx} + M_{yx,xy} + M_{y,yy} - M_{xy,xy} = \bar{m}\ddot{w} \end{aligned}\right\} \quad (28)$$

where dots indicate differentiation with respect to time.
It must be noted that rotatory inertia terms have been neglected. They could be of considerable influence if the eccentricities and the masses of the stiffeners are sizable. This does not lend itself to be easily included in the above approach, since besides the stiffener stiffness, the stiffener mass is "smeared-out" too.

## 9. COUPLED EQUATIONS OF MOTION OF THE SMEARED-OUT ECCENTRICALLY STIFFENED CIRCULAR CYLINDRICAL SHELL

The equations of motion are obtained when the appropriate derivatives of either the linear set (22) or the nonlinear set (24) of equations are introduced into equations (28). Since the third of (28) contains products of the stress resultants ($N_x$, $N_y$, $N_{xy}$) and its derivatives with w-derivatives, the third equation of motion becomes nonlinear regardless whether (24) or (28) are used, unless the secondary effect of the membrane forces is neglected in (24).

### a) Linearized Equations of Motion

The effect of the membrane forces is neglected in the third equation of the set (28). This set then reduces to:

$$\left.\begin{aligned} N_{x,x} + N_{yx,y} &= \bar{m}\,\ddot{u} \\ N_{y,y} + N_{xy,x} &= \bar{m}\,\ddot{v} \\ M_{x,xx} + M_{yx,xy} - M_{xy,xy} + M_{y,yy} - \frac{N_y}{R} &= \bar{m}\,\ddot{w} \end{aligned}\right\} \quad (29)$$

Introducing the appropriate derivatives of (22) and employing the parameters $K_P$ and $D_2$ from (5) and (17) and after some regrouping the following set of equations can be written:

$$\left.\begin{aligned} K_{HS}\,u_{,xx} + K_G\,u_{,yy} + K_P\,v_{,xy} + K_\nu \frac{w_{,x}}{R} - F_{Sb}\,w_{,xxx} &= \bar{m}\,\ddot{u} \\ K_P\,u_{,xy} + K_{HR}\,v_{,yy} + K_G\,v_{,xx} + K_{HR}\frac{w_{,y}}{R} - F_{Rb}\,w_{,yyy} &= \bar{m}\,\ddot{v} \\ F_{Sb}\,u_{,xxx} - K_\nu \frac{u_{,x}}{R} + F_{Rb}\,v_{,yyy} - K_{HR}\frac{v_{,y}}{R} - D_{HS}\,w_{,xxxx} & \\ - 2D_2\,w_{,xxyy} - D_{HR}\,w_{,yyyy} + 2F_{Rb}\frac{w_{,yy}}{R} - K_{HR}\frac{w}{R^2} &= \bar{m}\,\ddot{w} \end{aligned}\right\} \quad (30)$$

The isotropic case for the linear equations of motion of the monocoque cylindrical shell follows from the above when the reinforcing parameters are reduced according to the following scheme:

$$\left.\begin{array}{lcl} \bar{m} & \rightarrow & m \\ K_{HS} & \rightarrow & K \\ F_{Sb} & \rightarrow & 0 \\ K_{HR} & \rightarrow & K \\ F_{Rb} & \rightarrow & 0 \\ D_{HS} & \rightarrow & D \\ D_{HR} & \rightarrow & D \\ D_{2} & \rightarrow & D \end{array}\right\} \quad (31)$$

With (31) the equations of motion of the isotropic shell become,

$$\left.\begin{array}{ll} K u_{,xx} + K_G u_{,yy} + K_P v_{,xy} + K_\nu \frac{w_{,x}}{R} & = m\ddot{u} \\ K_P u_{,xy} + K v_{,yy} + K_G v_{,xx} + K \frac{w_{,y}}{R} & = m\ddot{v} \\ -K_\nu \frac{u_{,x}}{R} - K \frac{v_{,y}}{R} - D\nabla^4 w - K \frac{w}{R^2} & = m\ddot{w} \end{array}\right\} \quad (32)$$

It can be readily verified that equations (32) are identical with Flügge's equations of motion of the "complete vibration theory of a circular cylindrical shell"[2] when use is made of the assumption of shallowness, e.g. $\frac{1}{12}$ (t/R) << 1. (Reference (3))

b) <u>Complete Nonlinear Equations of Motion</u>

When the complete set (24) is inserted into equations (28) and use is made of the first two of (28) to simplify the expressions, the following equations can be written after considerable regrouping :

$$\left.\begin{array}{ll} \underline{K_{HS} u_{,xx} + K_G u_{,yy} + K_P v_{,xy} + K_\nu \frac{w_{,x}}{R} - F_{Sb} w_{,xxx}} & \\ + K_{HS} w_{,xx} w_{,x} + K_G w_{,yy} w_{,x} + K_P w_{,xy} w_{,y} & = \bar{m}\ddot{u} \end{array}\right\} \quad (33)$$

$$\left.\begin{array}{l}
K_P\, u_{,xy} + K_{HR}\, v_{,yy} + K_G\, v_{,xx} + K_{HR} \frac{w_{,y}}{R} - F_{Rb}\, w_{,yyy} \\
\text{-------------------------------------------------} \\
+ K_{HR}\, w_{,yy}\, w_{,y} + K_G\, w_{,xx}\, w_{,y} + K_P\, w_{,xy}\, w_{,x} \qquad = \bar{m}\, \ddot{v} \\
\\
K_{Sb}\, u_{,xxx} - K_\nu \frac{u_{,x}}{R} + F_{Rb}\, v_{,yyy} - K_{HR} \frac{v_{,y}}{R} - D_{HS}\, w_{,xxxx} \\
-2 D_2\, w_{,xxyy} - D_{HR}\, w_{,yyyy} + 2 F_{Rb} \frac{w_{,yy}}{R} - K_{HR} \frac{w}{R^2} \\
\text{-------------------------------------------------} \\
+ K_{HS}\, u_{,x}\, w_{,xx} + K_\nu\, u_{,x}\, w_{,yy} + 2 K_G\, u_{,y}\, w_{,xy} \\
+ K_{HR}\, v_{,y}\, w_{,yy} + K_\nu\, v_{,y}\, w_{,xx} + 2 K_G\, v_{,x}\, w_{,xy} \\
+ K_\nu \frac{w\, w_{,xx}}{R} - \frac{K_\nu}{2} \frac{w_{,x}^2}{R} - \frac{K_{HR}}{2} \frac{w_{,y}^2}{R} + K_{HR} \frac{w\, w_{,yy}}{R} \\
+ F_{Sb}\, w_{,xxx}\, w_{,x} + F_{Rb}\, w_{,yyy}\, w_{,y} \\
\text{-------------------------------------------------} \\
+ \frac{K_{HS}}{2} w_{,xx}\, w_{,x}^2 + K_\nu \left( w_{,xx}\, w_{,y}^2 + w_{,yy}\, w_{,x}^2 \right) + \frac{K_{HR}}{2} w_{,yy}\, w_{,y}^2 \\
+ 2 K_G\, w_{,x}\, w_{,y}\, w_{,xy} \qquad = \bar{m}(\ddot{w} - \ddot{u}\, w_{,x} - \ddot{v}\, w_{,y})
\end{array}\right\} \quad (33)$$

The set (33) is the coupled system of nonlinear partial differential equations of motion of the eccentrically stiffened circular cylindrical shell. The dashed lines indicate separation of the terms into groups of the first, second and third degree.
A second degree approximation of (33) can therefore be obtained by deleting the last terms in the third equation of (33).

## 10. <u>ON THE PARTIAL DECOUPLING OF THE w-EQUATION OF MOTION FROM THE COUPLED LINEAR SYSTEM INCLUDING TANGENTIAL INERTIA</u>

The linearized coupled system of equations of motion (30) is repeated here for convenience:

$$K_{HS}\, u_{,xx} + K_G\, u_{,yy} + K_P\, v_{,xy} + K_\nu \frac{w_{,x}}{R} - F_{Sb}\, w_{,xxx} = \bar{m}\, \ddot{u} \qquad (34)$$

$$K_P\, u_{,xy} + K_{MR}\, v_{,yy} + K_G\, v_{,xx} + K_{MR}\frac{w_{,y}}{R} - F_{Rb}\, w_{,yyy} = \bar{m}\,\ddot{v} \qquad (35)$$

$$F_{sb}\, u_{,xxx} - K_\nu \frac{u_{,x}}{R} + F_{Rb}\, v_{,yyy} - K_{MR}\frac{v_{,y}}{R} - D_{MS}\, w_{,xxxx}$$
$$-2\, D_2\, w_{,xxyy} - D_{MR}\, w_{,yyyy} + 2\, F_{Rb}\frac{w_{,yy}}{R} - K_{MR}\frac{w}{R^2} = \bar{m}\,\ddot{w} \qquad (36)$$

The following decoupling procedure is followed:
In order to obtain an equation in u and w alone, we partially differentiate (35) with respect to x and y in sequence and then substitute for the v-terms from (34), which are obtained by separately differentiating (34) twice with respect to x, twice with respect to y and twice with respect to time.
In order to obtain an equation in v and w alone, (34) is partially differentiated with respect to x and y in sequence and the u-terms are then expressed from (35), after the latter is separately differentiated twice with respect to x, y and time.
After considerable manipulation and regrouping the following two equations are obtained:

$$\left.\begin{aligned} &\mathcal{L}^4(u_{,x}) + K_{MR}K_\nu\frac{w_{,xxyy}}{R} + K_{MR}F_{sb}\, w_{,xxxxyy} + K_G K_\nu \frac{w_{,xxxx}}{R} \\ &- F_{sb}K_G\, w_{,xxxxxx} - K_{MR}K_P\frac{w_{,xxyy}}{R} + F_{Rb}K_P\, w_{,xxyyyy} \\ &= \bar{m}\frac{\partial^2}{\partial t^2}\left[(K_{MS}+K_G)u_{,xxx} + (K_{MR}+K_G)u_{,xyy} - \bar{m}\,\ddot{u}_{,x} + K_\nu\frac{w_{,xx}}{R} - F_{sb}\, w_{,xxxx}\right] \end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned} &\mathcal{L}^4(v_{,y}) + K_{MS}K_{MR}\frac{w_{,xxyy}}{R} + K_{MS}F_{Rb}\, w_{,xxyyyy} + K_G K_{MR}\frac{w_{,yyyy}}{R} \\ &- F_{Rb}K_G\, w_{,yyyyyy} - K_\nu K_P\frac{w_{,xxyy}}{R} + F_{sb}K_P\, w_{,xxxxyy} \\ &= \bar{m}\frac{\partial^2}{\partial t^2}\left[(K_{MR}+K_G)v_{,yyy} + (K_{MS}+K_G)v_{,xxy} - \bar{m}\,\ddot{v}_{,y} + K_{MR}\frac{w_{,yy}}{R} - F_{Rb}\, w_{,yyyy}\right] \end{aligned}\right\} \quad (38)$$

where the linear operator $\mathcal{L}^4$ is defined as:

$$\mathcal{L}^4 = K_{MS}K_G\frac{\partial^4}{\partial x^4} + (K_{MR}K_{MS} + K_G^2 - K_P^2)\frac{\partial^4}{\partial x^2\partial y^2} + K_{MR}K_G\frac{\partial^4}{\partial y^4} \qquad (39)$$

Equation (37) contains only terms in u and w, while (38) relates v and w.
Multiplying (37) by $K_\nu$ and (38) by $K_{HR}$ and adding yields the following expression:

$$\begin{rcases}\mathcal{L}^4 (K_\nu u_{,x} + K_{HR} v_{,y}) + \frac{1}{R} [K_G K_\nu^2 w_{,xxxx} + K_{HR} (K_\nu^2 - 2K_P K_\nu + K_{HR} K_{HS}) w_{,xxyy} \\ + K_G K_{HR}^2 w_{,yyyy}] - F_{Sb} K_G K_\nu w_{,xxxxxx} + F_{Sb} K_{HR} (K_\nu + K_P) w_{,xxxxyy} \\ + F_{Rb} (K_P K_\nu + K_{HS} K_{HR}) w_{,xxyyyy} - F_{Rb} K_G K_{HR} w_{,yyyyyy} \\ = \bar{m} \frac{\partial^2}{\partial t^2} \left[ \mathcal{L}^2 (K_\nu u_{,x} + K_{HR} v_{,y}) - \bar{m} \frac{\partial^2}{\partial t^2} (K_\nu u_{,x} + K_{HR} v_{,y}) + K_\nu^2 \frac{w_{,xx}}{R} \right. \\ \left. + K_{HR}^2 \frac{w_{,yy}}{R} - F_{Sb} K_\nu w_{,xxxx} - F_{Rb} K_{HR} w_{,yyyy} \right]\end{rcases} (40)$$

where the linear operator $\mathcal{L}^2$ is defined as:

$$\mathcal{L}^2 = (K_{HS} + K_G) \frac{\partial^2}{\partial x^2} + (K_{HR} + K_G) \frac{\partial^2}{\partial y^2} \qquad (41)$$

Considering (36) with (40), we would like to eliminate the u and v terms of (40) with the help of (36).
It can be seen that this is not possible for the eccentrically stiffened shell. It may however be achieved for the orthotropic shell. In the latter case, $F_{Sb} = F_{Rb} = 0$ in equation (36), and we may solve for:

$$K_\nu u_{,x} + K_{HR} v_{,y} = -R \left[ D_{HS} w_{,xxxx} + 2D_2 w_{,xxyy} + D_{HR} w_{,yyyy} + K_{HR} \frac{w}{R^2} + \bar{m} \ddot{w} \right] \qquad (42)$$

When (42) is introduced into equation (40), with $F_{Sb} = F_{Rb} = 0$, the following partially uncoupled linear partial differential equation in w results for the orthotropically stiffened cylindrical shell:

$$\begin{rcases}\mathcal{L}^4 \left\{ \left( \mathcal{L}_D^4 + \frac{K_{HR}}{R^2} + \bar{m} \frac{\partial^2}{\partial t^2} \right) w \right\} - \frac{1}{R^2} [K_G K_\nu^2 w_{,xxxx} + K_{HR} (K_\nu^2 - 2K_P K_\nu \\ + K_{HR} K_{HS}) w_{,xxyy} + K_G K_{HR}^2 w_{,yyyy}] \\ = \bar{m} \frac{\partial^2}{\partial t^2} \left\{ \mathcal{L}^2 \left[ \left( \mathcal{L}_D^4 + \frac{K_{HR}}{R^2} + \bar{m} \frac{\partial^2}{\partial t^2} \right) w \right] - \bar{m} \frac{\partial^2}{\partial t^2} \left[ \left( \mathcal{L}_D^4 + \frac{K_{HR}}{R^2} + \bar{m} \frac{\partial^2}{\partial t^2} \right) w \right] \right. \\ \left. + K_\nu^2 \frac{w_{,xx}}{R} + K_{HR}^2 \frac{w_{,yy}}{R} \right\}\end{rcases} (43)$$

Where the linear operator $\mathcal{L}_D^4$ is defined as:

$$\mathcal{L}_D^4 = D_{HS}\frac{\partial^4}{\partial x^4} + 2D_2\frac{\partial^4}{\partial x^2 \partial y^2} + D_{HR}\frac{\partial^4}{\partial y^4} \qquad (44)$$

Equation (43) is called partially decoupled since w must still satisfy (37) in u and w, as well as (38) in v and w.
(43) can also be reduced to the case of the isotropic shell. With the appropriate substitutions and after considerable algebra, the following equation can be obtained:

$$\left.\begin{aligned}&\frac{t^2}{12}\nabla^8 w + \frac{1-\nu^2}{R^2}w_{,xxxx} = -\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left\{\left(\frac{1-\nu^2}{E}\rho\frac{\partial^2}{\partial t^2} - \frac{3-\nu}{2}\nabla^2\right)\left(\frac{1-\nu^2}{E}\rho\frac{\partial^2 w}{\partial t^2} + \right.\right.\\&\left.\left. + \frac{w}{R^2} + \frac{t^2}{12}\nabla^4 w\right) + \frac{1-\nu}{2}\nabla^4 w + \frac{\nu^2}{R^2}w_{,xx} + \frac{1}{R^2}w_{,yy}\right\}\end{aligned}\right\} (45)$$

Equation (45) is identical with that derived in reference (5). For the static case, (45) reduces to the well-known linear Donnell equation of reference (6).

## 11. THE NONLINEAR EQUATIONS OF MOTION OF THE ECCENTRICALLY STIFFENED SHELL IN TERMS OF A STRESS FUNCTION AND ZERO TANGENTIAL INERTIA

A stress function f(x,y) is defined such that

$$\left.\begin{aligned}N_x &= f_{,yy}\\ N_y &= f_{,xx}\\ N_{xy} &= -f_{,xy}\end{aligned}\right\} \quad (46)$$

With $\ddot{u}=\ddot{v}=0$, the first two equilibrium equations of the system (28) are identically satisfied. The third of (28) can be written with (46) and the M's substituted from (24) such that,

$$-(D_{HS}\, w_{,xxxx} + 2D_2\, w_{,xxyy} + D_{HR}\, w_{,yyyy}) + f_{,yy}\, w_{,xx} - 2f_{,xy}\, w_{,xy} + f_{,xx}\, w_{,yy} - \frac{f_{,xx}}{R} + F_{Sb}(u_{,xxx} + w_{,xx}^2 + w_{,x}\, w_{,xxx}) + F_{Rb}\left(v_{,yyy} + w_{,yy}^2 + w_{,y}\, w_{,yyy} + \frac{w_{,yy}}{R}\right) = \bar{m}\,\ddot{w}$$

or on using the operator $\mathcal{L}_D^4$ by (44)

$$\left.\begin{aligned} & -\mathcal{L}_D^4 w + f_{,yy}\, w_{,xx} - 2f_{,xy}\, w_{,xy} + f_{,xx}\, w_{,yy} - \frac{f_{,xx}}{R} \\ & + F_{Sb}(u_{,xxx} + w_{,xx}^2 + w_{,x}\, w_{,xxx}) + F_{Rb}(v_{,yyy} + w_{,yy}^2 + w_{,y}\, w_{,yyy} + \frac{w_{,yy}}{R}) = \bar{m}\,\ddot{w} \end{aligned}\right\} \quad (47)$$

The terms in u and v can be eliminated with the help of the first two equations of (24). In these the N's are expressed by the stress function f. After some algebra, the following equations are obtained:

$$\left.\begin{aligned} u_{,xxx} = {} & \frac{1}{K_{HR}K_{HS} - K_\nu^2}\left[K_{HR}\, f_{,xxyy} - K_\nu\, f_{,xxxx} + K_{HR}\, F_{Sb}\, w_{,xxxx} - K_\nu\, F_{Rb}\, w_{,xxyy}\right] \\ & - w_{,xx}^2 - w_{,x}\, w_{,xxx} \end{aligned}\right\} \quad (48)$$

$$\left.\begin{aligned} v_{,yyy} = {} & \frac{1}{K_{HR}K_{HS} - K_\nu^2}\left[K_{HS}\, f_{,xxyy} - K_\nu\, f_{,yyyy} + K_{HS}\, F_{Rb}\, w_{,yyyy} - K_\nu\, F_{Sb}\, w_{,xxyy}\right] \\ & - w_{,yy}^2 - w_{,y}\, w_{,yy} - \frac{w_{,yy}}{R} \end{aligned}\right\} \quad (49)$$

Substituting (48) and (49) into (47) and combining appropriate terms, yields the following equation:

$$\left.\begin{aligned} & -\Big[\Big(D_{HS} - \frac{K_{HR}\, F_{Sb}^2}{K_{HR}K_{HS} - K_\nu^2}\Big) w_{,xxxx} + 2\Big(D_2 - \frac{K_\nu\, F_{Sb}\, F_{Rb}}{K_{HR}K_{HS} - K_\nu^2}\Big) w_{,xxyy} \\ & + \Big(D_{HR} - \frac{K_{HS}\, F_{Rb}^2}{K_{HR}K_{HS} - K_\nu^2}\Big) w_{,yyyy}\Big] - \frac{1}{K_{HR}K_{HS} - K_\nu^2}\Big[K_\nu\, F_{Sb}\, f_{,xxxx} \\ & - (K_{HR}\, F_{Sb} + K_{HS}\, F_{Rb})\, f_{,xxyy} + K_\nu\, F_{Rb}\, f_{,yyyy}\Big] \\ & + f_{,xx}\, w_{,yy} - 2f_{,xy}\, w_{,xy} + f_{,yy}\, w_{,xx} - \frac{f_{,xx}}{R} = \bar{m}\,\ddot{w} \end{aligned}\right\} \quad (50)$$

The nonlinear compatibility equation is taken from reference (7), page 417, stated for the plate, and is modified for the cylindrical shell as follows:

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = w_{,xy}^2 - w_{,xx} w_{,yy} + \frac{w_{,xx}}{R} \tag{51}$$

The strain terms can be expressed from the first three equations of (18). Introducing f into the N's, and w's into the $\varkappa$'s, they become:

$$\left.\begin{aligned} K_{MS}\,\varepsilon_x + K_\nu\,\varepsilon_y &= f_{,yy} + F_{Sb}\,w_{,xx} \\ K_\nu\,\varepsilon_x + K_{MR}\,\varepsilon_y &= f_{,xx} + F_{Rb}\,w_{,yy} \\ \gamma_{xy} &= -\frac{1}{K_G} f_{,xy} \end{aligned}\right\} \tag{52}$$

Solving for the strains and differentiating appropriately, the left side of (51) can be expressed from (52), which leads to the counterpart of (50):

$$\left.\begin{aligned} &\frac{1}{K_{MR}K_{MS}-K_\nu^2}\Big\{\Big[K_{MS} f_{,xxxx} + 2\Big(\frac{K_{MR}K_{MS}-K_\nu^2}{2K_G} - K_\nu\Big) f_{,xxyy} + K_{MR} f_{,yyyy}\Big] \\ &\qquad - \Big[K_\nu F_{Sb}\, w_{,xxxx} - (K_{MR} F_{Sb} + K_{MS} F_{Rb})\, w_{,xxyy} + K_\nu F_{Rb}\, w_{,yyyy}\Big]\Big\} \\ &= w_{,xy}^2 - w_{,xx} w_{,yy} + \frac{w_{,xx}}{R} \end{aligned}\right\} \tag{53}$$

The equations (50) and (53) form a nonlinear system of partial differential equations realating the stress function f and the radial displacement w for the eccentrically stiffened circular cylindrical shell. Their relationship is similar to that of the von Karman equations for the flat plate.

The bulk of these equations can be reduced by defining the following parameters: (Also listed in the Appendix)

$$\left.\begin{aligned} D_{11} &= D_{MS} - \frac{K_{MR} F_{Sb}^2}{K_{MR}K_{MS}-K_\nu^2} \\ D_{12} &= D_2 - \frac{K_\nu F_{Sb} F_{Rb}}{K_{MR}K_{MS}-K_\nu^2} \end{aligned}\right\} \tag{54}$$

$$
\left.
\begin{aligned}
D_{22} &= D_{MR} - \frac{K_{MS} F_{Rb}^{2}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
S_{11} &= \frac{K_{\nu} F_{Sb}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
S_{12} &= \frac{1}{2} \frac{K_{MR} F_{Sb} + K_{MS} F_{Rb}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
S_{22} &= \frac{K_{\nu} F_{Rb}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
A_{11} &= \frac{K_{MS}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
A_{12} &= \frac{1}{2K_{G}} - \frac{K_{\nu}}{K_{MR} K_{MS} - K_{\nu}^{2}} \\
A_{22} &= \frac{K_{MR}}{K_{MR} K_{MS} - K_{\nu}^{2}}
\end{aligned}
\right\} \quad (54)
$$

With these abbreviations the two equations can be written as:

$$
\left.
\begin{aligned}
& D_{11} w_{,xxxx} + 2 D_{12} w_{,xxyy} + D_{22} w_{,yyyy} + S_{11} f_{,xxxx} - 2 S_{12} f_{,xxyy} \\
& + S_{22} f_{,yyyy} - f_{,xx} w_{,yy} + 2 f_{,xy} w_{,xy} - f_{,yy} w_{,xx} + \frac{f_{,xx}}{R} + \bar{m} \ddot{w} = 0
\end{aligned}
\right\} \quad (55)
$$

$$
\left.
\begin{aligned}
& A_{11} f_{,xxxx} + 2 A_{12} f_{,xxyy} + A_{22} f_{,yyyy} - S_{11} w_{,xxxx} + 2 S_{12} w_{,xxyy} \\
& - S_{22} w_{,yyyy} - w_{,xy}^{2} + w_{,xx} w_{,yy} - \frac{w_{,xx}}{R} = 0
\end{aligned}
\right\} \quad (56)
$$

The mathematical formulation of the free vibration of an eccentrically stiffened cylindrical shell is therefore given by the set of equations (55) and (56) when tangential inertia is neglected. With an appropriate set of 8 boundary and 2 initial conditions, a solution should be possible, even though a closed-form solution seems remote due to mathematical difficulties.

So far then, various equations for the dynamics of eccentrically stiffened circular cylindrical shell have been formulated through derivation. Immediate future work will concentrate on methods of solutions for specific problems. In general, solutions are needed for cases where $X=X(t)$, $Y=Y(t)$, $Z=Z(t)$, besides inertia, include externally time-varying surface forces for prescribed boundary conditions, or, on the other hand, prescribed time-varying boundary displacements.

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## APPENDIX: SUMMARY OF ABBREVIATED PARAMETERS

| Stiffness (lb/in) | Flexural rigidity (lb-in) | Mixed parameters |
|---|---|---|
| | | **Force (lb)** |
| $K = \frac{Et}{1-\nu^2}$ | $D = \frac{Et^3}{12(1-\nu^2)}$ | |
| $K_\nu = \frac{\nu Et}{1-\nu^2}$ | $D_\nu = \frac{\nu Et^3}{12(1-\nu^2)}$ | $F_{Sb} = \frac{E_S A_S}{d}\bar{z}_S$ |
| $K_G = \frac{Et}{2(1+\nu)} = Gt$ | $D_G = \frac{Et^3}{12(1+\nu)} = \frac{Gt^3}{6}$ | $F_{Rb} = \frac{E_R A_R}{\ell}\bar{z}_R$ |
| $K_2 = K_\nu + K_G = \frac{Et}{2(1-\nu)}$ | $D_S = \frac{E_S I_{SO}}{d} = \frac{E_S}{d}[I_{SC} + \bar{z}_S^2 A_S]$ | **Displacement (in)** |
| $K_S = \frac{E_S A_S}{d}$ | $D_R = \frac{E_R I_{RO}}{\ell} = \frac{E_R}{\ell}[I_{RC} + \bar{z}_R^2 A_R]$ | $S_{11} = \frac{K_\nu F_{Sb}}{K_{HR}K_{HS} - K_\nu^2}$ |
| $K_R = \frac{E_R A_R}{\ell}$ | $D_{GS} = \frac{G_S J_S}{d}$ | $S_{12} = \frac{1}{2}\frac{K_{HR}F_{Sb} + K_{HR}F_{Rb}}{K_{HR}K_{HS} - K_\nu^2}$ |
| $K_{HS} = K + K_S = \frac{Et}{1-\nu^2} + \frac{E_S A_S}{d}$ | $D_{GR} = \frac{G_R J_R}{\ell}$ | $S_{22} = \frac{K_\nu F_{Rb}}{K_{HR}K_{HS} - K_\nu^2}$ |
| $K_{HR} = K + K_R = \frac{Et}{1-\nu^2} + \frac{E_R A_R}{\ell}$ | $D_{HS} = D + D_S = \frac{Et^3}{12(1-\nu^2)} + \frac{E_S}{d}[I_{SC} + \bar{z}_S^2 A_S]$ | **Flexibility (in/lb)** |
| | $D_{HR} = D + D_R = \frac{Et^3}{12(1-\nu^2)} + \frac{E_R}{\ell}[I_{RC} + \bar{z}_R^2 A_R]$ | $A_{11} = \frac{K_{HS}}{K_{HR}K_{HS} - K_\nu^2}$ |
| | $D_{HGS} = D_G + D_{GS} = \frac{Gt^3}{6} + \frac{G_S J_S}{d}$ | $A_{12} = \frac{1}{2K_G} - \frac{K_\nu}{K_{HR}K_{HS} - K_\nu^2}$ |
| | $D_{HGR} = D_G + D_{GR} = \frac{Gt^3}{6} + \frac{G_R J_R}{\ell}$ | $A_{22} = \frac{K_{HR}}{K_{HR}K_{HS} - K_\nu^2}$ |
| | $D_1 = D_G + \frac{D_{GS} + D_{GR}}{2}$ | |
| | $D_2 = D_\nu + \frac{D_{HGR} + D_{HGS}}{2}$ | |
| | $D_{11} = D_{HS} - \frac{K_{HR}F_{Sb}^2}{K_{HR}K_{HS} - K_\nu^2}$ | |
| | $D_{12} = D_2 - \frac{K_\nu F_{Sb}F_{Rb}}{K_{HR}K_{HS} - K_\nu^2}$ | |
| | $D_{22} = D_{HR} - \frac{K_{HS}F_{Rb}^2}{K_{HR}K_{HS} - K_\nu^2}$ | |
| Stiffness (lb/in) | Flexural Rigidity (lb-in) | Mixed Parameters |