NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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September 1, 1965

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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I. INTRODUCTION

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Because a superconductor is diamagnetic, it experiences a force when immersed in a nonuniform magnetic field. This Report gives the result of the calculation of the force acting on a sphere of zero permeability in an arbitrary magnetic field. The solution is obtained by expressing the applied field in a series of complete spherical harmonic functions, determining the induced field required to satisfy the boundary condition $B_{normal} = 0$, then integrating the Maxwell stress tensor over the sphere's surface. The final result appears as sums of products of the expansion coefficients of the applied field. To make the result more useful, it is also shown how to express the field due to an array of coils having concurrent axes in a series of spherical harmonics and, for the purpose of computing restoring forces, the applied field is expanded in a Taylor series expansion in the displacements. Two simple examples are shown.

II. SPHERICAL HARMONIC FUNCTIONS

The complete set of normalized spherical harmonic functions

$$Y_l^m(\mu, \phi) = (-1)^{\frac{1}{2}(m+|m|)} \left[(2l+1)(l-|m|)!/2(l+|m|)!\right]^{\frac{1}{2}} \frac{P_l^{|m|}(\mu)\, e^{im\phi}}{(2\pi)^{\frac{1}{2}}} \tag{1}$$

where

$$P_l^{|m|}(\mu) = \frac{1}{2^l\, l!}(1-\mu^2)^{|m|/2} \frac{d^{l+|m|}}{d\mu^{l+|m|}}(\mu^2-1)^l$$

is chosen for representing the magnetic field because of its appropriateness to a spherical boundary and because of the following useful properties (Ref. 1):

$$\iint Y_l^{m*}\, Y_p^q\, d\mu d\phi = \begin{cases} 1 \text{ if } l = p \text{ and } m = q \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

$$L^2\, Y_l^m \equiv -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_l^m = l(l+1)\, Y_l^m$$

$$L_z\, Y_l^m \equiv -i\frac{\partial}{\partial\phi} Y_l^m = m\, Y_l^m \tag{3}$$

$$L_+\, Y_l^m \equiv e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_l^m = G_l^m\, Y_l^{m+1} \tag{4}$$

$$L_-\, Y_l^m \equiv e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_l^m = H_l^m\, Y_l^{m-1} \tag{5}$$

$$\cos\theta\, Y_l^m = A_l^m\, Y_{l+1}^m + B_l^m\, Y_{l-1}^m \tag{6}$$

$$\sin\theta\, e^{i\phi}\, Y_l^m = -C_l^m\, Y_{l+1}^{m+1} + D_l^m\, Y_{l-1}^{m+1} \tag{7}$$

$$\sin\theta\, e^{-i\phi}\, Y_l^m = E_l^m\, Y_{l+1}^{m-1} - F_l^m\, Y_{l-1}^{m-1} \tag{8}$$

$$\frac{\partial}{\partial z} R Y_l^m = A_l^m\, \alpha_l\, Y_{l+1}^m + B_l^m\, \beta_l\, Y_{l-1}^m \tag{9}$$

$$\partial^+ R\, Y_l^m \equiv \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) Y_l^m = -C_l^m\, \alpha_l\, Y_{l+1}^{m+1} + D_l^m\, \beta_l\, Y_{l-1}^{m+1} \tag{10}$$

$$\partial^- R\, Y_l^m \equiv \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) Y_l^m = E_l^m\, \beta_l\, Y_{l+1}^{m-1} - F_l^m\, \beta_l\, Y_{l-1}^{m-1} \tag{11}$$

where

$$A_l^m = \left[\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}$$

$$B_l^m = \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}$$

$$C_l^m = \left[\frac{(l+m+2)(l+m+1)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}$$

$$D_l^m = \left[\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}$$

$$E_l^m = \left[\frac{(l-m+2)(l-m+1)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}$$

$$F_l^m = \left[\frac{(l+m)(l+m-1)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}$$

$$G_l^m = \left[(l+m+1)(l-m)\right]^{\frac{1}{2}}$$

$$H_l^m = \left[(l+m)(l-m+1)\right]^{\frac{1}{2}}$$

$$\alpha_l = \left(\frac{d}{dr} - \frac{l}{r}\right) R$$

$$\beta_l = \left(\frac{d}{dr} + \frac{l+1}{r}\right) R \tag{12}$$

The first few spherical harmonics are given explicitly:

$$Y_0^0 = \frac{1}{(4\pi)^{\frac{1}{2}}}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$$

$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\, e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{4\pi}\right)^{\frac{1}{2}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_2^{\pm 2} = +\frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_3^0 = +\left(\frac{7}{4\pi}\right)^{1/2} \left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)$$

$$Y_3^{\pm 1} = \mp\frac{1}{4}\left(\frac{21}{4\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1)\, e^{\pm i\phi}$$

$$Y_3^{\pm 2} = +\frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2} \sin^2\theta\cos\theta\, e^{\pm 2i\phi}$$

$$Y_3^{\pm 3} = \mp\frac{1}{4}\left(\frac{35}{4\pi}\right)^{1/2} \sin^3\theta\, e^{\pm 3i\phi}$$

$$Y_4^0 = \left(\frac{9}{4\pi}\right)^{1/2} \left(\frac{35}{8}\cos^4\theta - \frac{15}{4}\cos^2\theta + \frac{3}{8}\right)$$

$$Y_4^{\pm 1} = \mp\frac{3}{4}\left(\frac{5}{4\pi}\right)^{1/2} (7\cos^3\theta - 3\cos\theta)\sin\theta\, e^{\pm i\phi}$$

$$Y_4^{\pm 2} = +\frac{3}{4}\left(\frac{5}{8\pi}\right)^{1/2} \sin^2\theta\,(7\cos^2\theta - 1)\, e^{\pm 2i\phi}$$

$$Y_4^{\pm 3} = \mp\frac{3}{4}\left(\frac{35}{4\pi}\right)^{1/2} \sin^3\theta\cos\theta\, e^{\pm 3i\phi}$$

$$Y_4^{\pm 4} = +\frac{3}{8}\left(\frac{35}{8\pi}\right)^{1/2} \sin^4\theta\, e^{\pm 4i\phi}$$

III. MAXWELL STRESS TENSOR

The force on a permeable body is found by integrating the Maxwell stress tensor over its surface:

$$\mathbf{F} = \iint \frac{\mathbf{BB} - \frac{1}{2}B^2\,\mathbf{1}}{\mu_0} \cdot d\mathbf{S} \qquad (13)$$

Given a perfect Meissner effect, $B_{normal} = 0$ at the surface

IV. EXPRESSION FOR THE MAGNETIC FIELD

The scalar potential Φ_A of the applied field B_A is expressed as an expansion in complete spherical harmonic functions about the center of the sphere as an origin (see Fig. 1 for the coordinate system):

$$-\mathbf{B}_A = \nabla\Phi_A = \nabla \sum_{l,m} b_{lm}\, r^l\, Y_l^m(\mu, \phi)$$

The induced field resulting from the Meissner effect can be written

$$-\mathbf{B}_I = \nabla \sum_{l,m} a_{lm}\, r^{-l-1}\, Y_l^m(\mu, \phi)$$

The induced field combines with the applied field to satisfy the boundary condition $B_{normal} = 0$ at the surface of the superconductor. Hence

$$-B_{normal} = \frac{\partial}{\partial r}\sum_l (b_{lm}\, r^l + a_{lm}\, r^{-l-1})\, Y_l^m(\mu, \phi) = 0 \qquad \text{at } r = R$$

which implies

$$a_{lm} = \frac{l}{l+1} R^{2l+1}\, b_{lm}$$

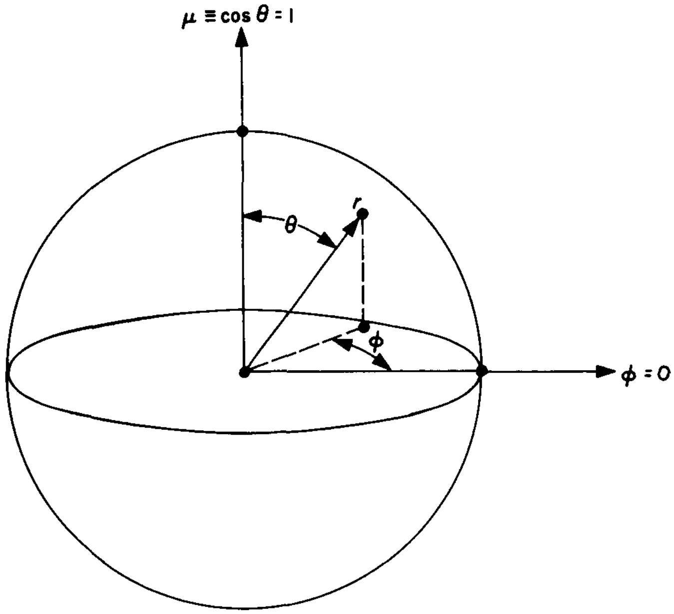

Fig. 1. Coordinates for analysis of magnetic force

Then the resultant scalar potential is

$$\Phi_T(r,\mu,\phi) = \sum_{l,m} b_{lm}\left(r^l + \frac{l}{l+1}\,\frac{R^{2l+1}}{r^{l+1}}\right) Y_l^m(\mu,\phi)$$

and the resultant field is

$$-\mathbf{B}_T = \nabla\Phi_T = \frac{\partial\Phi_T}{\partial r}\hat{r} + \frac{1}{r}\,\frac{\partial\Phi_T}{\partial\theta}\hat{\theta} + \frac{\csc\theta}{r}\,\frac{\partial\Phi_T}{\partial\phi}\hat{\phi}$$

At $r = R$:

$$-\mathbf{B}_T = \frac{1}{R}\,\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{\csc\theta}{R}\,\frac{\partial\Phi}{\partial\phi}\hat{\phi}$$

by virtue of the boundary condition. Hence the field squared on the sphere is given by

$$B_T^2 = \frac{1}{R^2}\left[\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \csc^2\theta\left(\frac{\partial\Phi}{\partial\phi}\right)^2\right]$$

where

$$\Phi = \sum_{l,m}\frac{2l+1}{l+1}\,b_{lm}\,R^l\,Y_l^m(\mu,\phi)$$

V. EXPANSION OF B_T^2 IN SPHERICAL HARMONICS

The principal difficulty in the calculation is to express B_T^2 in a series of products of spherical harmonics so as to be able to take advantage of the orthogonality of the functions in performing the integration for force. The following observation greatly facilitates this difficulty:

$$-L_+\Phi\cdot L_-\Phi - L_z\Phi\cdot L_z\Phi = \left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \cot^2\theta\left(\frac{\partial\Phi}{\partial\phi}\right)^2 + \left(\frac{\partial\Phi}{\partial\phi}\right)^2 = \left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \csc^2\theta\left(\frac{\partial\Phi}{\partial\phi}\right)^2 = R^2B_T^2$$

To obtain products of the form $Y_l^{m*}\ Y_p^q$, note that $\Phi = \Phi^*$, so that

$$R^2\,B_T^2 = -\,L_+\Phi^*\cdot L_-\Phi - L_z\Phi^*\cdot L_z\Phi$$

Further,

$$L_+\Phi^* = (L_+^*\Phi)^* = (-L_-\Phi)^* = -\sum_{l,m}\frac{2\,l+1}{l+1}\,b_{lm}^*R^l\,H_l^m\,Y_l^{m-1*}$$

$$L_-\Phi = \sum_{p,q}\frac{2\,p+1}{p+1}\,b_{pq}\,R^p\,H_p^q\,Y_p^{q-1}$$

$$L_z\Phi^* = (L_z^*\Phi)^* = (-L_z\Phi)^* = -\sum_{l,m}\frac{2\,l+1}{l+1}\,b_{lm}^*\,R^l m\,Y_l^{m*}$$

$$L_z\Phi = \sum_{p,q}\frac{2\,p+1}{p+1}\,b_{pq}\,R^p\,q\,Y_p^q$$

Consequently the field squared at the sphere's surface can be written

$$B_T^2 = \sum_{l,m,p,q} \frac{2l+1}{l+1} \frac{2p+1}{p+1} b_{lm}^* b_{pq} R^{l+p-2} (H_l^m H_p^q Y_l^{m-1*} Y_p^{q-1} + mq Y_l^{m*} Y_p^q) \quad (15)$$

VI. EVALUATION OF THE COMPONENTS OF FORCE

The z-component of force is computed from the integral

$$F_z = -\frac{1}{2\mu_0} \iint B_T^2 \cos\theta \, R^2 \, d\mu d\phi$$

$$= -\frac{1}{2\mu_0} \sum_{l,m,p,q} \frac{2l+1}{l+1} \frac{2p+1}{p+1} b_{lm}^* b_{pq} R^{l+p} \iint \left[H_l^m H_p^q Y_l^{m-1*} Y_p^{q-1} + m q Y_l^{m*} Y_p^q \right] \cos\theta \, d\mu d\phi$$

$$= -\frac{1}{2\mu_0} \sum_{l,m,p,q} \frac{2l+1}{l+1} \frac{2p+1}{p+1} b_{lm}^* b_{pq} R^{l+p} \iint \left[H_l^m H_p^q Y_l^{m-1*} (A_p^{q-1} Y_{p+1}^{q-1} + B_p^{q-1} Y_{p-1}^{q-1}) + m q Y_l^{m*} (A_p^q Y_{p+1}^q + B_p^q Y_{p-1}^q) \right] d\mu d\phi$$

where use has been made of Eq. (14), (15), and (6). The orthogonality relation (2) makes the integration trivial. After evaluating the constants from Eq. (12), and combining terms, the result is

$$F_z = -\frac{1}{2\mu_0} \sum_{l=-\infty}^{\infty} \sum_{m=-l}^{l} \frac{[(2l+1)(2l+3)]^{1/2}}{(l+1)(l+2)} R^{2l+1} b_{lm}^* b_{l+1,m} (l^2 + 2l + 2m) \left[(l+m+1)(l-m+1) \right]^{1/2} + \text{complex conjugate} \quad (16)$$

The x- and y-components of force are most readily evaluated by first computing the integrals

$$F_x + iF_y = -\frac{1}{2\mu_0} \iint B^2 \sin\theta \, e^{i\phi} R^2 \, d\mu d\phi$$

$$F_x - iF_y = -\frac{1}{2\mu_0} \iint B^2 \sin\theta \, e^{-i\phi} R^2 \, d\mu d\phi$$

in view of the relations (7) and (8).

After straightforward, though tedious, manipulation, the following result is obtained:

$$F_x = \frac{-1}{4\mu_0} \sum_{l,m} \frac{[(2l+1)(2l+3)]^{1/2}}{(l+1)(l+2)} R^{2l+1} b_{lm}^* \{ b_{l+1,m-1} (l^2 + 3l + 2m) \left[(l-m+2)(l-m+1) \right]^{1/2} - b_{l+1,m+1} (l^2 + l + 2m) \left[(l+m+2)(l+m+1) \right]^{1/2} \} + \text{complex conjugate} \quad (17)$$

$$F_y = \frac{i}{4\mu_0} \sum_{l,m} \frac{[(2l+1)(2l+3)]^{1/2}}{(l+1)(l+2)} R^{2l+1} b^*_{lm} \{ b_{l+1,m-1}(l^2+3l+2m)\left[(l-m+2)(l-m+1)\right]^{1/2} + b_{l+1,m+1}(l^2+l+2m)\left[(l+m+2)(l+m+1)\right]^{1/2} \} + \text{complex conjugate} \tag{18}$$

(If B_T^2 is written as $\Sigma\, \Lambda_{lm}\, Y^l_m\,(\mu,\phi)$, where $\Lambda_{lm} = \int\int B_T^2 Y_l^{m*}\, d\mu\, d\phi$, then the components of force are easily seen to be

$$F_z = -\frac{R^2}{2\mu_0}\left(\frac{4\pi}{3}\right)^{1/2}\Lambda_{10}$$

$$F_x + iF_y = -\frac{R^2}{2\mu_0}\left(\frac{8\pi}{3}\right)^{1/2}\Lambda_{11}$$

$$F_x - iF_y = \frac{-R^2}{2\mu_0}\left(\frac{8\pi}{3}\right)^{1/2}\Lambda_{1-1}\;)$$

VII. RESTORING FORCES

In assessing the stability of a magnetic suspension it is necessary to compute restoring forces for small displacements from an assumed equilibrium position. If the sphere is displaced relative to the applied field then it is necessary to express the field relative to the new origin in order to apply the formulas of Section VI. This transformation is performed by expanding the potential of the applied field in a Taylor series.

The potential at point P is

$$\begin{aligned}\Phi_A(x_0,y_0,z_0) &= \Phi_A(x+\delta x, y+\delta y, z+\delta z)\\ &= \Phi_A(x,y,z) + \nabla \Phi_A\big|_{x,y,z} \cdot (\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k})\end{aligned} \tag{19}$$

where x_0, y_0, z_0 are the coordinates of a field point P referred to the initial origin 0_0, while x, y, z are the coordinates of the same point referred to a subsequent position 0 of the center of the sphere, displaced by δx, δy, δz (see Fig. 2). In terms of spherical coordinates referenced to these origins, Eq. (19) is written

$$\begin{aligned}\Phi_A \text{ at } P &= \sum b_{lm}\, r_0^l\, Y_l^m(\mu_0,\phi_0) = \sum b_{lm}\, r^l\, Y_l^m(\mu,\phi)\\ &\quad + \sum b_{lm} \nabla \left[r^l Y_l^m(\mu,\phi)\right] \cdot (\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k})\end{aligned} \tag{19a}$$

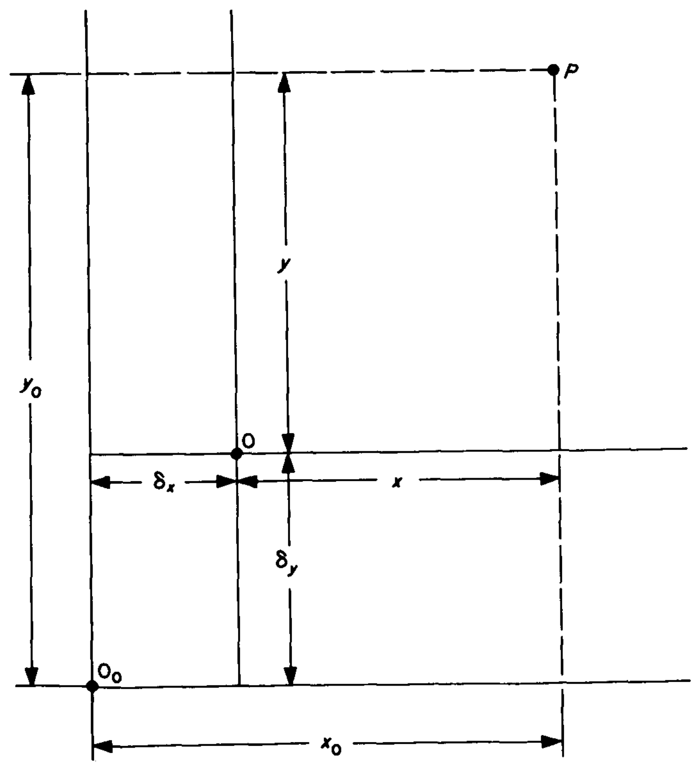

Fig. 2. Field coordinates relative to displaced origins

where

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The derivations are evaluated by reference to Eq. (9), (10), (11), and (12):

$$\frac{\partial}{\partial x} r^l Y_l^m = \frac{\partial^+ + \partial^-}{2} r^l Y_l^m = \frac{2l+1}{2}\left(D_l^m r^{l-1} Y_{l-1}^{m+1} - F_l^m r^{l-1} Y_{l-1}^{m-1}\right)$$

$$\frac{\partial}{\partial y} r^l Y_l^m = \frac{\partial^+ - \partial^-}{2i} r^l Y_l^m = \frac{2l+1}{2i}\left(D_l^m r^{l-1} Y_{l-1}^{m+1} + F_l^m r^{l-1} Y_{l-1}^{m-1}\right)$$

$$\frac{\partial}{\partial z} r^l Y_l^m = (2l+1)\, B_l^m r^{l-1} Y_{l-1}^m$$

Equation (19a) becomes

$$\Phi_A \text{ at } P = \sum_{l,m}\Big[b_{lm} r^l Y_l^m + \frac{2l+1}{2} b_{lm}\left(D_l^m r^{l-1} Y_{l-1}^{m+1} - F_l^m r^{l-1} Y_{l-1}^{m-1}\right)\delta x$$

$$+ \frac{2l+1}{2i} b_{lm}\left(D_l^m r^{l-1} Y_{l-1}^{m+1} + F_l^m r^{l-1} Y_{l-1}^{m-1}\right)\delta y + (2l+1)\, b_{lm}\left(B_{lm} r^{l-1} Y_{l-1}^m\right)\delta z\Big] \qquad (19b)$$

After evaluating constants and grouping terms, the above expression reduces to

$$\Phi_A = \sum_{l,m} \beta_{lm} r^l Y_l^m$$

where

$$\beta_{lm} = b_{lm} + \frac{1}{2}\left(\frac{2l+3}{2l+1}\right)^{1/2} b_{l+1,m-1}\left[(l-m+2)(l-m+1)\right]^{1/2} - b_{l+1,m+1}\left[(l+m+2)(l+m+1)\right]^{1/2}\delta x$$

$$+ \frac{1}{2i}\left(\frac{2l+3}{2l+1}\right)^{1/2}\left\{b_{l+1,m-1}\left[(l-m+2)(l-m+1)\right]^{1/2} + b_{l+1,m+1}\left[(l+m+2)(l+m+1)\right]^{1/2}\right\}\delta y$$

$$+ \left(\frac{2l+3}{2l+1}\right)^{1/2} b_{l+1,m}\left[(l+m+1)(l-m+1)\right]^{1/2}\delta z \qquad (20)$$

VIII. SPHERICAL HARMONIC REPRESENTATION OF THE FIELD DUE TO AN ARRAY OF CURRENT LOOPS

A frequently encountered situation in magnetic levitation involves an array of coils whose axes intersect at a common point. Smythe (Ref. 2) gives the expression for the vector potential generated by a single circular loop in terms of Legendre polynomials. By setting $-\nabla\Phi = \nabla \times \mathbf{A}$, the scalar potential is readily deduced as

$$\Phi_i = -\frac{\mu_0 I_i}{2}\sum_l \frac{\sin\gamma_i}{l}\left(\frac{r}{a_i}\right)^l P_l^1(\cos\gamma_i)\,P_l(\mu')$$

The resulting potential for an array of coils involving intersecting axes can be transformed to a single axis by means of the "addition theorem for spherical harmonics."

$$P_l(\mu') = \frac{4\pi}{2l+1}\sum_{-l}^{+l} Y_l^{m*}(\mu_i,\phi_i)\,Y_l^m(\mu,\phi)$$

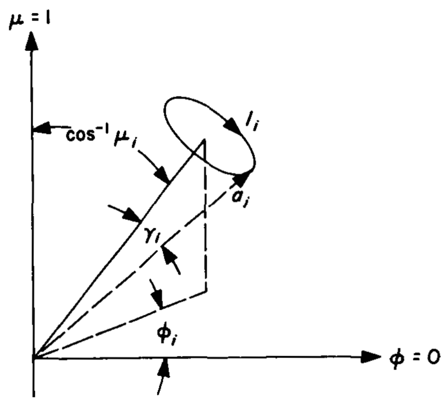

Fig. 3. Coil coordinates

Figure 3 depicts the coil coordinates. The resulting expression for the scalar potential is

$$\Phi_A = \sum_{l,m} b_{lm}\,r^l\,Y_l^m(\mu,\phi)$$

$$b_{lm} = -\sum_i \frac{2\pi\mu_0 I_i}{l(l+1)\,a_i^l}\sin\gamma_i\,P_l^1(\cos\gamma_i)\,Y_l^{m*}(\mu_i,\phi_i)$$

XI. EXAMPLE OF A TILTED GRADIENT FIELD

Consider the field given by

$$\Phi_A = \sum_{m=-2}^{2} b_{2m}\,r^2\,Y_2^m(\mu,\phi)$$

$$b_{2m} \equiv \left(\frac{4\pi}{5}\right)^{1/2} Y_2^{m*}(\cos\alpha, 0)$$

which, by direct comparison with the addition theorem, is merely the field

$$\Phi = r^2 Y_2^0(\mu) = \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2}(2z^2 - x^2 - y^2)$$

rotated about $\mu = 0$, $\phi = \pi/2$ (the y axis) by an angle α. This is the situation depicted in Fig. 4. That the origin is a position of zero force can be seen by examining Eq. (16), (17), and (18). Restoring forces are found by calculating the β_{lm} from Eq. (20):

$$\beta_{10} = \left(\frac{5}{3}\right)^{1/2}\left[3\sin\alpha\cos\alpha\,\delta x + (3\cos^2\alpha - 1)\,\delta z\right]$$

$$\beta_{11} = \left(\frac{5}{3}\right)^{1/2}\left\{\left[\frac{3}{2^{1/2}}\cos^2\alpha - 2^{1/2}\right]\delta x + \frac{1}{(-2)^{1/2}}\,\delta y - \frac{3}{2^{1/2}}\sin\alpha\cos\alpha\,\delta z\right\}$$

$$\beta_{1-1} = \left(\frac{5}{3}\right)^{1/2}\left\{\left[2^{1/2} - \frac{3}{2^{1/2}}\cos^2\alpha\right]\delta x + \frac{1}{(-2)^{1/2}}\,\delta y + \frac{3}{2^{1/2}}\sin\alpha\cos\alpha\,\delta z\right\}$$

$$\beta_{20} = \frac{1}{2}(3\cos^2\alpha - 1)$$

$$\beta_{21} = -\left(\frac{3}{2}\right)^{1/2}\sin\alpha\cos\alpha = -\beta_{2-1}$$

$$\beta_{22} = \left(\frac{3}{8}\right)^{1/2}\sin^2\alpha = \beta_{22}$$

Fig. 4. Stability of levitation in a titlted gradient field

then inserting these values into Eq. (16), (17), and (18), with the results:

$$F_z = -\frac{5}{2}\frac{R^3}{\mu_0}\left[(3\cos^2\alpha + 1)\,\delta z + 3\sin\alpha\cos\alpha\,\delta x\right]$$

$$F_x = -\frac{5}{2}\frac{R^3}{\mu_0}\left[(4 - 3\cos^2\alpha)\,\delta x + 3\sin\alpha\cos\alpha\,\delta z\right]$$

$$F_y = -\frac{5}{2}\frac{R^3}{\mu_0}\,\delta y$$

Since the virtual work function

$$-\mathbf{F}\cdot(\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k})$$

is positive definite (for any value of α), the gradient-field configuration produces a stable magnetic levitation.

X. EXAMPLE OF CROSSED GRADIENT FIELDS

Consider the field given by

$$\Phi_A = r^2 Y_2^0 (\mu, \phi) + \eta r^2 \left\{ \left[-\frac{1}{2} Y_2^0 (\mu, \phi) + \left(\frac{3}{8}\right)^{1/2} \left[Y_2^2 (\mu, \phi) + Y_2^{-2} (\mu, \phi) \right] \right] \right\}$$

$$= \frac{1}{2} \left(\frac{5}{4\pi}\right)^{1/2} \left[(2z^2 - x^2 - y^2) + \eta (2x^2 - y^2 - z^2) \right]$$

These are two Y_2^0 fields crossed at right angles. The origin is a position of zero force. The values for the β_{lm} are

$$\left(\frac{3}{5}\right)^{1/2} \beta_{10} = 2\left(1 - \frac{\eta}{2}\right) \delta z$$

$$\left(\frac{3}{5}\right)^{1/2} \beta_{11} = \frac{1}{2^{1/2}} (1 - 2\eta)\, \delta x + \frac{1}{(-2)^{1/2}} (1 + \eta)\, \delta y$$

$$\left(\frac{3}{5}\right)^{1/2} \beta_{1-1} = \frac{1}{2^{1/2}} (2\eta - 1)\, \delta x + \frac{1}{(-2)^{1/2}} (1 + \eta)\, \delta y$$

$$\beta_{20} = 1 - \frac{\eta}{2}$$

$$\beta_{21} = 0 = \beta_{2-1}$$

$$\beta_{22} = \left(\frac{3}{8}\right)^{1/2} \eta = \beta_{2-2}$$

and the restoring forces are

$$F_z = -\frac{10}{4} \frac{R^3}{\mu_0} (2 - \eta)^2\, \delta z$$

$$F_x = -\frac{10}{4} \frac{R^3}{\mu_0} (1 - 2\eta)^2\, \delta x$$

$$F_z = -\frac{10}{4} \frac{R^3}{\mu_0} (1 + \eta)^2\, \delta y$$

Such a field configuration is unstable only when $1 - 2\eta = 0$, $1 + \eta = 0$, or $2 - \eta = 0$; i.e., when the coefficients of x^2, y^2, z^2, respectively, are zero in the expression for Φ.

NOMENCLATURE[1]

μ_0 $4\pi 10^{-7}$

r, θ, ϕ spherical coordinates relative to the sphere

μ $\cos\theta$

R radius of sphere

$\mathbf{B}$ magnetic field (weber/m^2)

$\mathbf{B}_A$ applied field

Φ_A scalar potential of applied field

$\mathbf{B}_I$ induced field

Φ_I scalar potential of induced field

$\mathbf{B}_T$ $\mathbf{B}_A + \mathbf{B}_I$ = total or resultant field

Φ_T scalar potential of total field

$Y_l^m(\mu, \phi)$ normalized spherical harmonic function (SHF) defined in Eq. (1)

b_{lm} coefficient in SHF expansion of Φ_A

a_{lm} coefficient in SHF expansion of Φ_I

β_{lm} coefficient in SHF expansion of displaced Φ_A

Λ_{lm} coefficient in SHF expansion of B_T^2

L^2 the operator $-\dfrac{1}{\sin\theta}\left(\dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial}{\partial\theta} + \dfrac{1}{\sin\theta}\dfrac{\partial^2}{\partial\phi^2}\right)$

L_z the operator $-i\dfrac{\partial}{\partial\phi}$

L_+ the operator $e^{i\phi}\left(\dfrac{\partial}{\partial\theta} + i\cot\theta\dfrac{\partial}{\partial\phi}\right)$

L_- the operator $e^{-i\phi}\left(-\dfrac{\partial}{\partial\theta} + i\cot\theta\dfrac{\partial}{\partial\phi}\right)$

(L^2, L_z, L_+, and L_- are familiar in quantum mechanics as the total angular-momentum squared operator, the z-component of angular-momentum operator, the spin-raising operator, and the spin-lowering operator, respectively.)

I_i current in ith current loop

γ_i semi-angle subtended at origin by ith current loop (see Fig. 3)

μ_i, ϕ_i spherical coordinates of center of ith current loop

l, m, p, q integers used as summation indices

* denotes complex conjugates

[1]MKS units used throughout.

REFERENCES

1. Goertzel, Gerald, and Tralli, Nunzio, *Some Mathematical Methods of Physics*, McGraw-Hill Book Company, Inc., New York, 1960, p. 161.
2. Smythe, William R., *Static and Dynamic Electricity*, McGraw-Hill Book Company, Inc., New York, 1950, 2nd ed., p. 274.

N66-23859

JET PROPULSION LABORATORY California Institute of Technology • 4800 Oak Grove Drive, Pasadena, California 91103

May 2, 1966

Recipients of Jet Propulsion Laboratory
Technical Report No. 32-806

SUBJECT: Errata for TR 32-806

Gentlemen:

It is requested that a change as noted below be made in your copy of Jet Propulsion Laboratory Technical Report No. 32-806, entitled "Force on a Superconducting Sphere in a Magnet Field: The General Case," by John T. Harding, dated September 1, 1965.

On page 3, the right-hand column under the heading "III. Maxwell Stress Tensor" should read as follows:

of a superconductor, Eq. (13) reduces to

$$\mathbf{F} = -\frac{1}{2\mu_0}\iint B^2\, d\mathbf{S} \tag{14}$$

where $d\mathbf{S} = R^2 d\mu d\phi\, (\cos\theta k + \sin\theta\cos\phi i + \sin\theta\sin\phi j)$ for a sphere (k = the $\mu = 1$ direction, i = the $\mu = 1$, $\phi = 0$ direction, j = the $\mu = 1$, $\phi = \pi/2$ direction).

Very truly yours,

I. E. Newlan, Manager
Technical Information Section

IEN/VMC:ram