NEUTRINO ASTROPHYSICS;

SUPERMASSIVE STARS, QUASARS, AND EXTRAGALACTIC RADIO SOURCES*

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and

NUCLEAR ENERGY GENERATION IN SUPERMASSIVE STARS*

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LECTURE I

NEUTRINO ASTROPHYSICS, I

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INTRODUCTION

This lecture and the following lecture deal with the new scientific discipline, neutrino astrophysics. The designation, neutrino astrophysics, is used rather than neutrino astronomy since the practitioners in the field so far are mainly physicists and not astronomers. Although neutrino astrophysics is a relatively new field, large funds have already been spent on experiments and preparations for observations and much thought has been given to theory but few positive effects have been observed to date. In a sense it might be said that the whole business is "much ado about nothing".

The two lectures will cover solar, stellar, galactic and extragalactic neutrino astrophysics. The general theme will be the estimation of the quantity

$$\varepsilon_\nu = \frac{\text{energy emitted as } \nu \text{ or } \bar{\nu}}{\text{rest mass-energy of emitting system}} \quad . \tag{1}$$

This quantity has obvious cosmological implications and on conservative estimates is usually $\sim 10^{-4}$. In certain speculative cases the value is considerably greater. In terms of detection at the earth, one quantity calculated will be $(\varphi_\nu \sigma_\nu)$, where φ_ν is the expected neutrino flux at the earth and σ_ν is the neutrino absorption cross section for a given process. This can be converted into counts per day expected in a detection system involving a stated mass of detector.

The literature on the subject of neutrino astrophysics is already quite extensive and the reader is especially referred to the excellent series of articles by Professor J. N. Bahcall (1964 a,b,c,d; 1965), who has made many theoretical contributions to neutrino astrophysics.

SIGNIFICANCE OF NEUTRINO DETECTION

The primary significance observations on neutrinos derives from the extremely small intereaction cross section of these particles with matter. This cross section is of the order of 10^{-44} cm^2 per nucleon or electron. In an object of characteristic dimension R and density ρ, the number of nucleons along the neutrino line of passage is ~ 10^{24} ρR and the number of interactions per neutrino is ~ 10^{-20} ρR. For the earth $\rho \sim 10$, $R \sim 10^9$ so that ~ 10^{34} nucleons are passed in the passage of a neutrino through the earth but only 10^{-10} of the incident neutrinos interact. For the sun, the corresponding numbers are $\rho \sim 1$, $R \sim 10^{11}$, 10^{35} nucleons along the path and 10^{-9} of the neutrinos interact. For the universe, one has $\rho \sim 10^{-29}$, $R \sim 10^{28}$, 10^{23} nucleons along the path and 10^{-21} of the neutrinos interact. Thus neutrinos bear direct information unscathed from the center of the sun and other stars and from the depths of the universe.

It is paradoxical that, in spite of the low interaction cross sections, neutrinos can still be detected terrestrially. Consider a flux at the surface of the earth such as that for the B^8 neutrinos to be discussed in the next section. This flux is of the order of 10^7 neutrinos cm^{-2} sec^{-1} or 10^{12} neutrinos cm^{-2} day^{-1}. Consider a detector with $\rho \sim 1$ and mass measured in kilotons (~ 10^9 grams). The counting rate will be ~ $10^{12} \times 10^{-44} \times 10^{24} \times 10^9 \sim 10$ events per day per kiloton. Sophisticated anticoincidence techniques have reduced background counting rates to a small fraction of this value and thus enough of the elusive neutrinos which have emerged from the sun or traversed the universe can be detected with assurance and compelled to yield the secrets of their birth and the nature of their birthplace.

NEUTRINO EMISSION BY THE SUN

Fowler (1958) showed that the main nuclear processes in the sun are those in the pp-chain listed in Table I. All of the neutrinos indicated in Table I are those associated with electron capture or positron emission and are frequently designated by ν_e. Bahcall (1964b) has pointed out that electron capture processes, such as $B^8(e^-,\nu_e)Be^{8*}$ and $He^3(e^-,\nu_e)T^3$, will also occur but not with high probability except in the case of Be^7 as included in the table. It can also be shown that the reaction $He^3(p,e^+\nu_e)He^4$, either directly or through Li^4, must be relatively infrequent. Note that there are two branchings in the reaction chain but that in any case the overall result is the conversion of four protons into an alpha-particle plus two positrons and two neutrinos or, alternatively, the conversion of four protons and one electron into an alpha particle plus one positron and two neutrinos.

There are three main groups of neutrinos emitted in the pp-chain, namely $\nu(pp)$, $\nu(Be^7)$ and $\nu(B^8)$. Their energies and relative probabilities of emission depend on the results of experiments carried out in nuclear laboratories throughout the world. As an example, measurements made at the California Institute of Technology by Parker and Kavanagh (1963) on the cross section for the production of Be^7 in the reaction $He^3(\alpha,\gamma)Be^7$ are shown in Figure 1.

Measurements of the type illustrated in Figure 1 are employed to determine the cross section factors S(E) in the equation

$$\sigma = \frac{S(E)}{E} \exp\left[- 31.29\, Z_1 Z_0\, A^{\frac{1}{2}}\, E^{-\frac{1}{2}}\right] \text{barns} \qquad (2)$$

where E is the center of momentum reaction energy in KeV, $A = A_0A_1/(A_0 + A_1)$ is the reduced mass in atomic mass units ($C^{12} = 12$) and S(E) is measured in KeV-barns. Except in the case of low lying resonances, S(E) is found experimentally to be a slowly varying function of E which can be accurately

TABLE I

THE PP-CHAIN

$$H^1 + H^1 \rightarrow D^2 + e^+ + \nu$$

$$D^2 + H^1 \rightarrow He^3 + \gamma$$

$$He^3 + He^3 \rightarrow He^4 + H^1 + H^1$$

OR

$$He^3 + He^4 \rightarrow Be^7 + \gamma$$

$$Be^7 + e^- \rightarrow Li^7 + \nu$$

$$Li^7 + H^1 \rightarrow He^4 + He^4$$

OR

$$Be^7 + H^1 \rightarrow B^8 + \gamma$$

$$B^8 \rightarrow Be^{8*} + e^+ + \nu$$

$$Be^{8*} \rightarrow He^4 + He^4$$

OVERALL:

$$\boxed{\begin{array}{ll} \left.\begin{array}{rl} 4p & \rightarrow \alpha + 2e^+ + 2\nu \\ \text{OR}\quad 4p + e^- & \rightarrow \alpha + e^+ + 2\nu \end{array}\right\} & \text{NUCLEAR} \\ \quad 4H^1 \rightarrow He^4 + 2\nu & \text{ATOMIC} \end{array}}$$

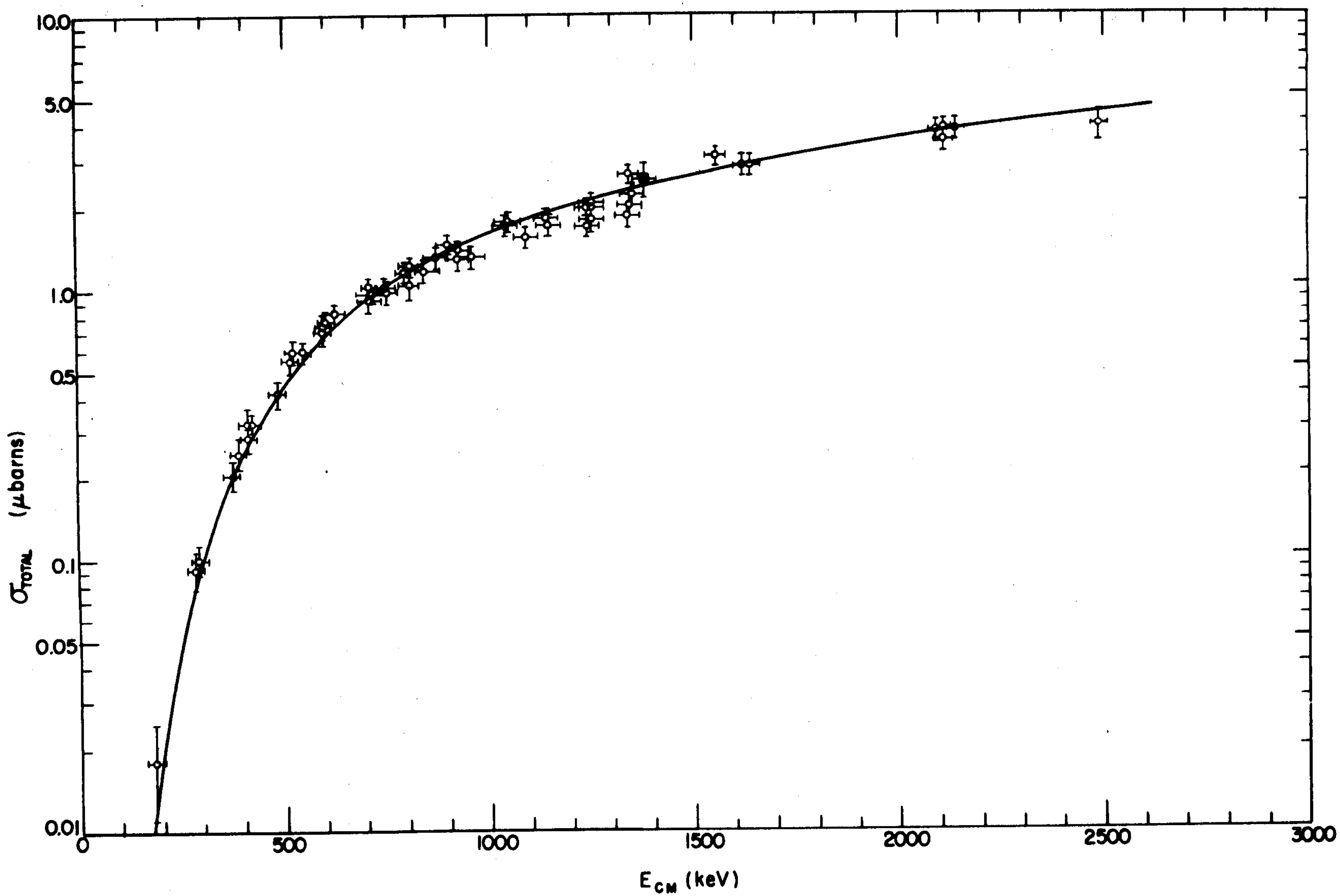


Fig. 1. Cross section versus alpha-particle laboratory energy for the $He^3(\alpha,\gamma)Be^7$ reaction after Parker and Kavanagh (1963).

extrapolated to the effective interaction energies in the sun and other stars. The effective interaction energy is given by

$$E_0 = 1.220\ (Z_0^2 Z_1^2 A\ T_6^2)^{\frac{1}{3}}\ \mathrm{KeV} \tag{3}$$

where $Z_0 Z_1$ are the charges of the interacting nuclei and $T_6 = T/10^6$ is the temperature in millions of degrees Kelvin. For the reactions of Table I, E_0 is the order of 10 KeV, whereas it is possible to measure S only down to 100 KeV in general. It is necessary to make a considerable extrapolation to obtain $S_0 = S(E_0)$ but this can usually be done with great accuracy. Experimental measurements on the pp-chain reactions are given in Table II. The quantity $f_0 \gtrsim 1$ is the electron shielding factor which corrects for the fact that electron shielding at stellar densities increases the reaction rates over that measured in the laboratory.

When translated into reaction rates the experimental results in Table II indicate that in the sun the reaction $He^3(He^3,2p)He^4$ occurs about 2/3 of the time, the reaction $He^3(\alpha,\gamma)Be^7$ about 1/3 of the time and the reaction $Be^7(p,\gamma)B^8$ about 1/2500 of the time. The greatest experimental uncertainties lie in the rates of the first and the third of these reactions and new measurements are now in progress.

Measurements of the mass-energy balance in the neutrino reactions of Table I can be summarized as follows:

$$\begin{array}{lll} \text{Continuum} & 0 < E_\nu(pp) < 0.42\ \mathrm{MeV} & \langle E_\nu(pp)\rangle = 0.25\ \mathrm{MeV} \\ \text{Lines} & E_\nu(Be^7) = 0.86\ \mathrm{MeV}\ \text{and}\ 0.38\ \mathrm{MeV} & \\ \text{Continuum} & 0 < E_\nu(B^8) < 14.1\ \mathrm{MeV} & \langle E_\nu(B^8)\rangle = 7.3\ \mathrm{MeV}. \end{array} \tag{4}$$

In connection with these results it is important to note that the cross section for the detection of neutrinos varies roughly as the square of the energy above the threshold energy for the detecting reaction.

TABLE II

Reactions of the proton-proton or pp chain (June, 1965).

The pp-Chain	Energy Release	S_o(keV-barns) or $\bar{\tau}$	Solar $f_o S_o$
$H^1 + H^1 \rightarrow D^2 + \beta^+ + \nu$	1.19 × 2 = 2.38 MeV	3.5×10^{-22}	3.8×10^{-22}
$D^2 + H^1 \rightarrow He^3 + \gamma$	5.49 × 2 = 10.98	3.0×10^{-4}	3.2×10^{-4}
$He^3 + He^3 \rightarrow He^4 + 2H^1$	12.86	1.1×10^3	1.4×10^3
	26.22 (2% ν-loss)		
or			
$He^3 + He^4 \rightarrow Be^7 + \gamma$	1.59	0.47	0.59
$Be^7 + e^- \rightarrow Li^7 + \nu + \gamma$	.05	$\bar{\tau}$ = 120 days (solar center)	
$Li^7 + H^1 \rightarrow 2He^4$	17.35	120 (non-res)	140
	25.67 (4% ν-loss)		
or			
$Be^7 + H^1 \rightarrow B^8 + \gamma$	0.13	30×10^{-3}	38×10^{-3}
$B^8 \rightarrow Be^{8*} + \beta^+ + \nu$	7.7	$\bar{\tau}$ = 1.1 sec	
$Be^{8*} \rightarrow 2He^4$	3.0	$\bar{\tau} = 10^{-16}$ sec	
	19.1 (29% ν-loss)		
$4H^1 \rightarrow He^4$	Total = 26.7313 MeV ± .0005		

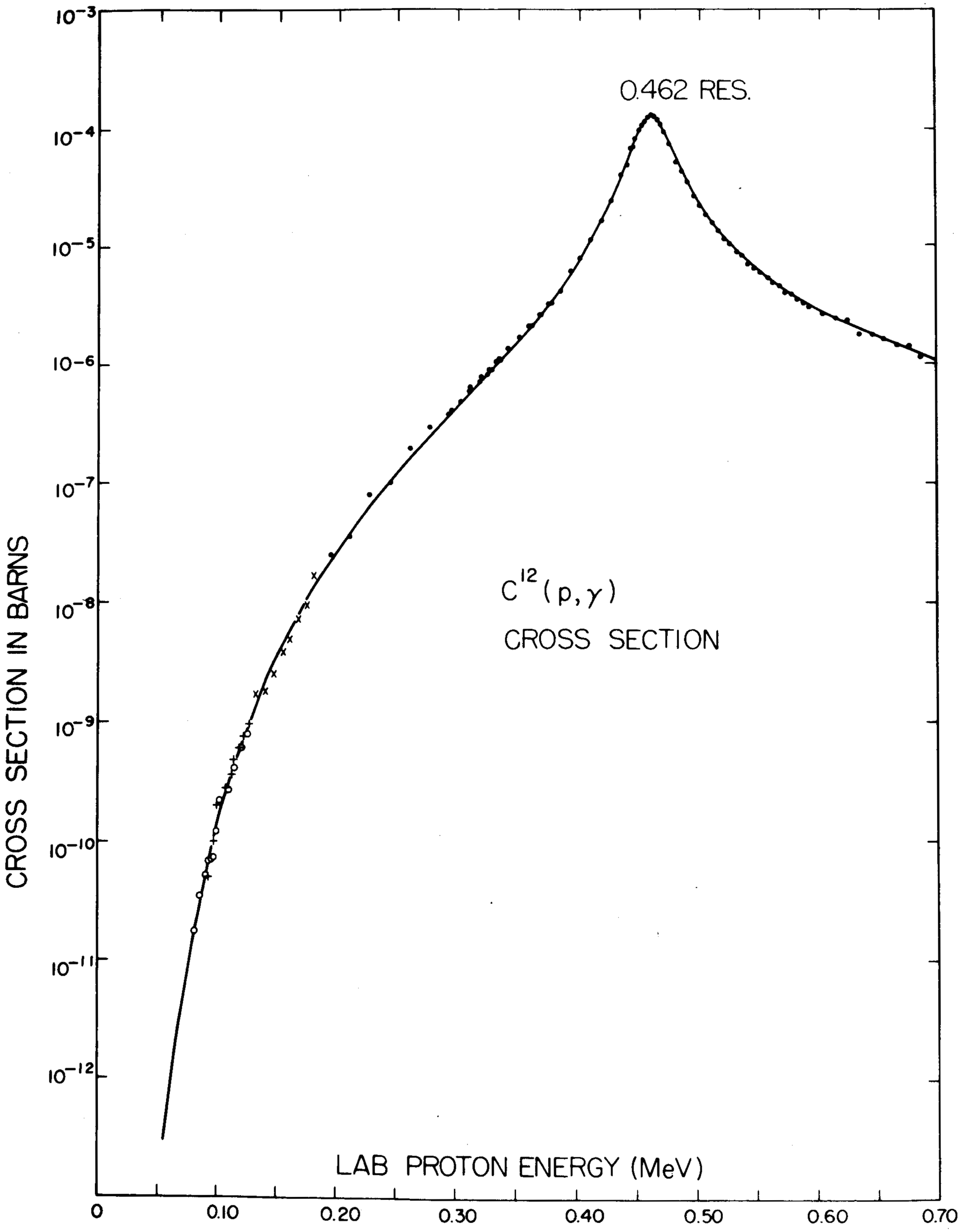


Fig. 2. Cross section versus proton laboratory energy for the $C^{12}(p,\gamma)N^{13}$ reaction after Vogl (1963).

The effective energy released in the pp-chain is 26.0 MeV with 0.7 MeV or 3 per cent emitted in the form of neutrinos. The total energy emission is 0.7 per cent of the rest mass energies involved and we can assume that about 1/3 of primordial hydrogen has been converted into helium during stellar evolution. Thus

$$\begin{aligned}\varepsilon_\nu &= \frac{1}{3} \times 0.03 \times 0.07 \\ &= 7 \times 10^{-5} \sim 10^{-4}\end{aligned} \tag{5}$$

A similar result is obtained for helium production in the "big bang" of evolutionary cosmology or for helium production in massive stars early in the formation of the Galaxy. For additional discussion see the end of Lecture II.

The CNO bi-cycle reactions also take place in the sun. As examples, cross section measurements for $C^{12}(p,\gamma)N^{13}$ and $C^{13}(p,\gamma)N^{14}$ made by Vogl (1963) and Seagrave (1951) are shown in Figures 2 and 3. Extrapolations of the cross-section factors are shown in Figure 4. The experimental results for the complete CNO bi-cycle are shown in Table III. In the sun the energy generation is primarily due to the pp-chain as illustrated in Figure 5. Nevertheless the rate of production of $\nu(N^{13})$ and $\nu(O^{15})$ can be calculated.

Since two neutrinos are emitted in the conversion of $4H^1 \rightarrow He^4$, the total neutrino flux at the earth can be easily computed from

$$\begin{aligned}\varphi_\nu(\text{total}) &= 2 \times \frac{\text{solar constant}}{(\text{Energy from } 4H_1 \rightarrow He^4)} \\ &= \frac{2 \times 1.34 \times 10^6}{26 \times 1.6 \times 10^{-6}} = 6.5 \times 10^{10}\ \nu\ \text{cm}^2\ \text{sec}^{-1},\end{aligned} \tag{6}$$

where the solar constant (Abbot and Fowle, 1911) has been taken as 2 cal cm^{-2} min^{-1} or 1.34×10^6 ergs cm^{-2} sec^{-1} and the energy release per pp-chain as 26.0 MeV or 4.17×10^{-5} ergs. The individual fluxes for the various types of neutrinos as calculated by Bahcall (1964d) are shown in Figure 6.

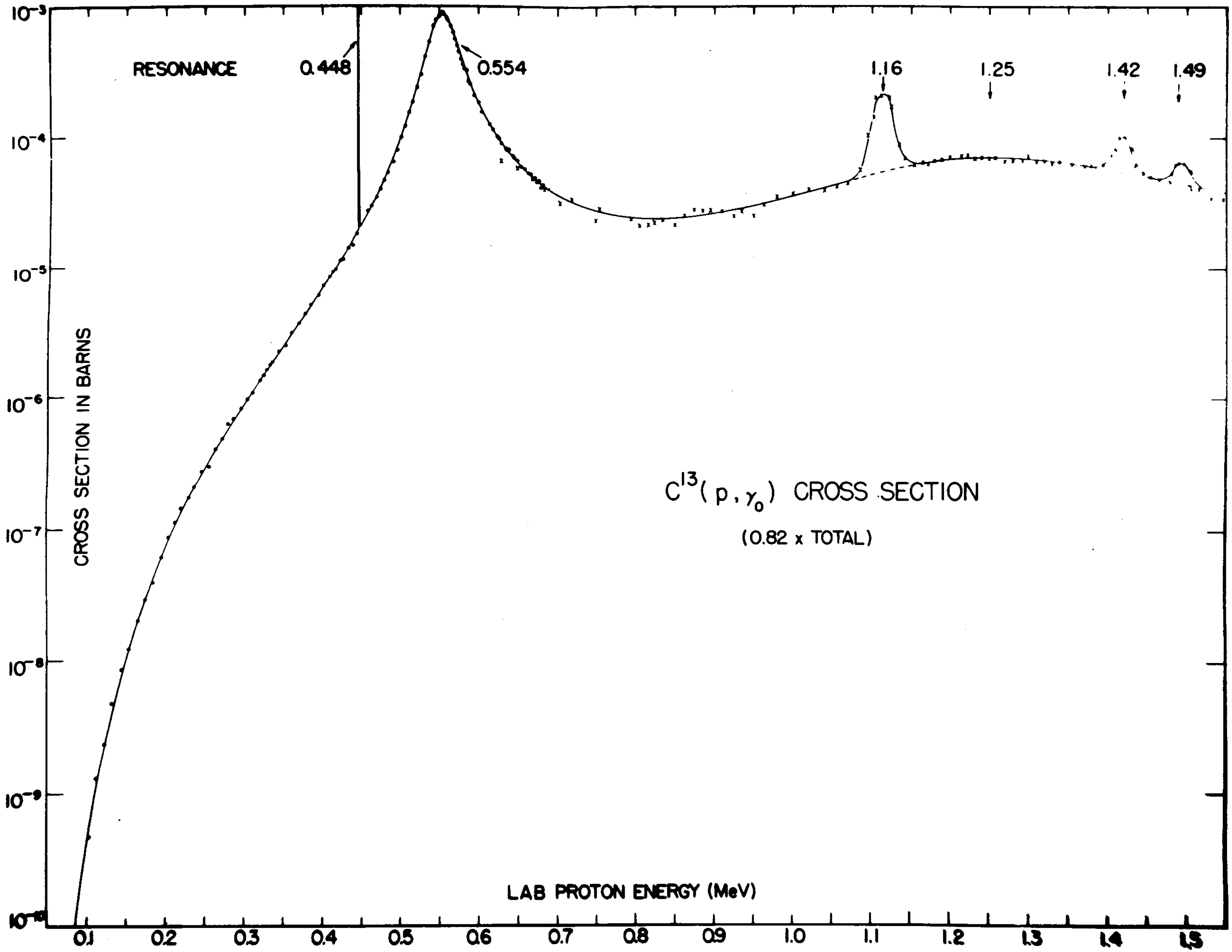


Fig. 3. Cross section versus proton laboratory energy for the $C^{13}(p,\gamma_o)N^{14}$ reaction after Vogl (1963) and Seagrave (1961). Divide the ordinate by 0.82 to obtain the total $C^{13}(p,\gamma)N^{14}$ cross section.

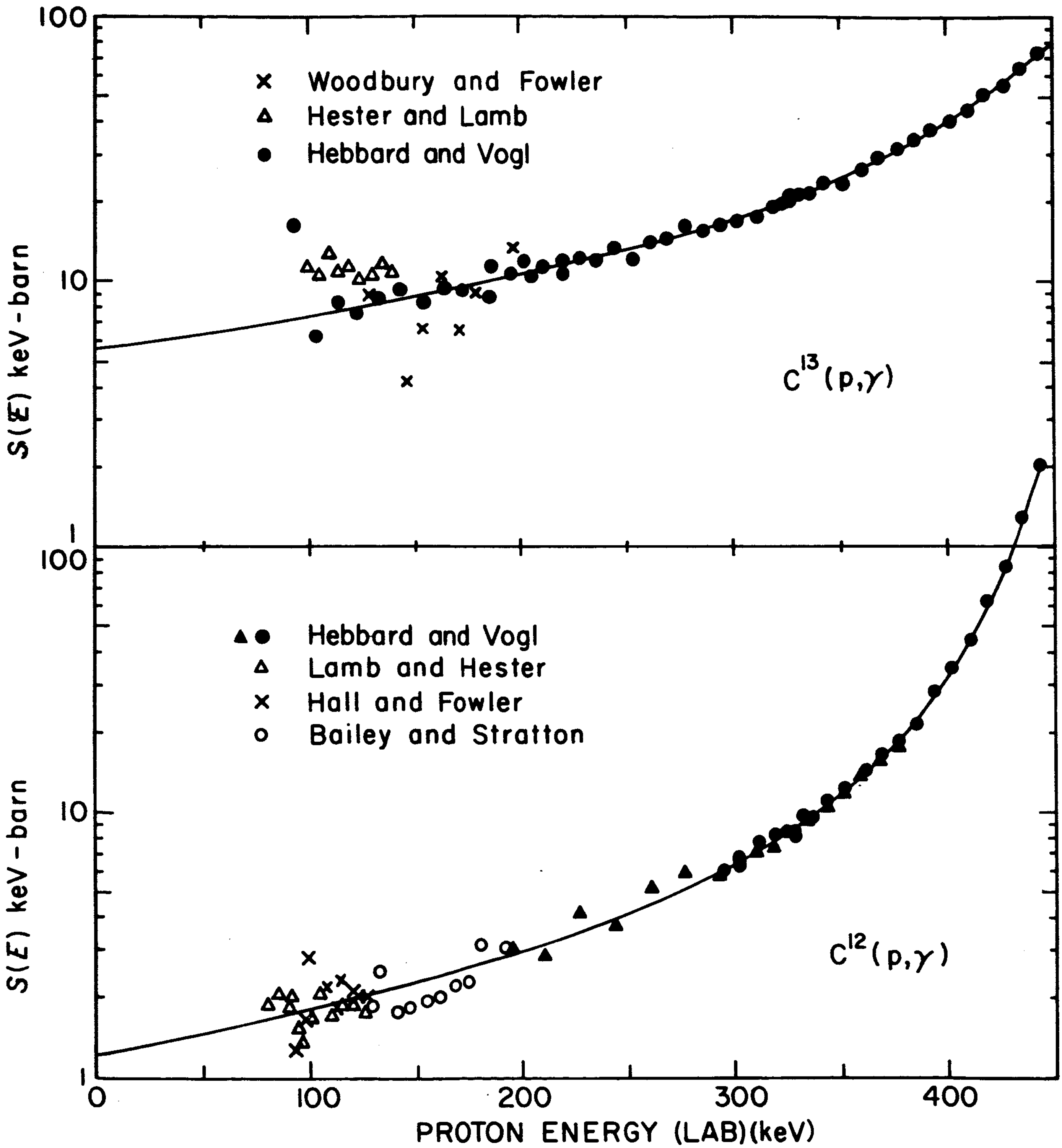


Fig. 4. The cross section factor, S(E), versus proton laboratory energy for the $C^{12}(p,\gamma)N^{13}$ and $C^{13}(p,\gamma)N^{14}$ reactions.

TABLE III

Reactions of the CNO-cycle (June, 1965).

The CNO-cycle		Energy Release	S_o(keV-barns) or $\bar{\tau}$	Solar $f_o S_o$
$C^{12} + H^1 \rightarrow N^{13} + \gamma$		1.94	1.53	2.2
$N^{13} \rightarrow C^{13} + \beta^+ + \nu$		1.50	$\bar{\tau}$ = 870 sec	
$C^{13} + H^1 \rightarrow N^{14} + \gamma$		7.55	5.9	8.4
$N^{14} + H^1 \rightarrow O^{15} + \gamma$		7.29	3.0	4.5
$O^{15} \rightarrow N^{15} + \beta^+ + \nu$		1.73	$\bar{\tau}$ = 178 sec	
$N^{15} + H^1 \rightarrow C^{12} + He^4$		4.96	7.5×10^4	1.1×10^5
or (1/2200)	(6% ν-loss)	24.97 MeV		
$N^{15} + H^1 \rightarrow O^{16} + \gamma$		12.13	32	48
$O^{16} + H^1 \rightarrow F^{17} + \gamma$		0.60	9.9	16
$F^{17} \rightarrow O^{17} + \beta^+ + \nu$		1.76	$\bar{\tau}$ = 95 sec	
$O^{17} + H^1 \rightarrow N^{14} + He^4$		1.19	10	16
	(1/2200)	15.68 MeV		
$4H^1 \rightarrow He^4$	Total =	26.7313 MeV ± .0005		

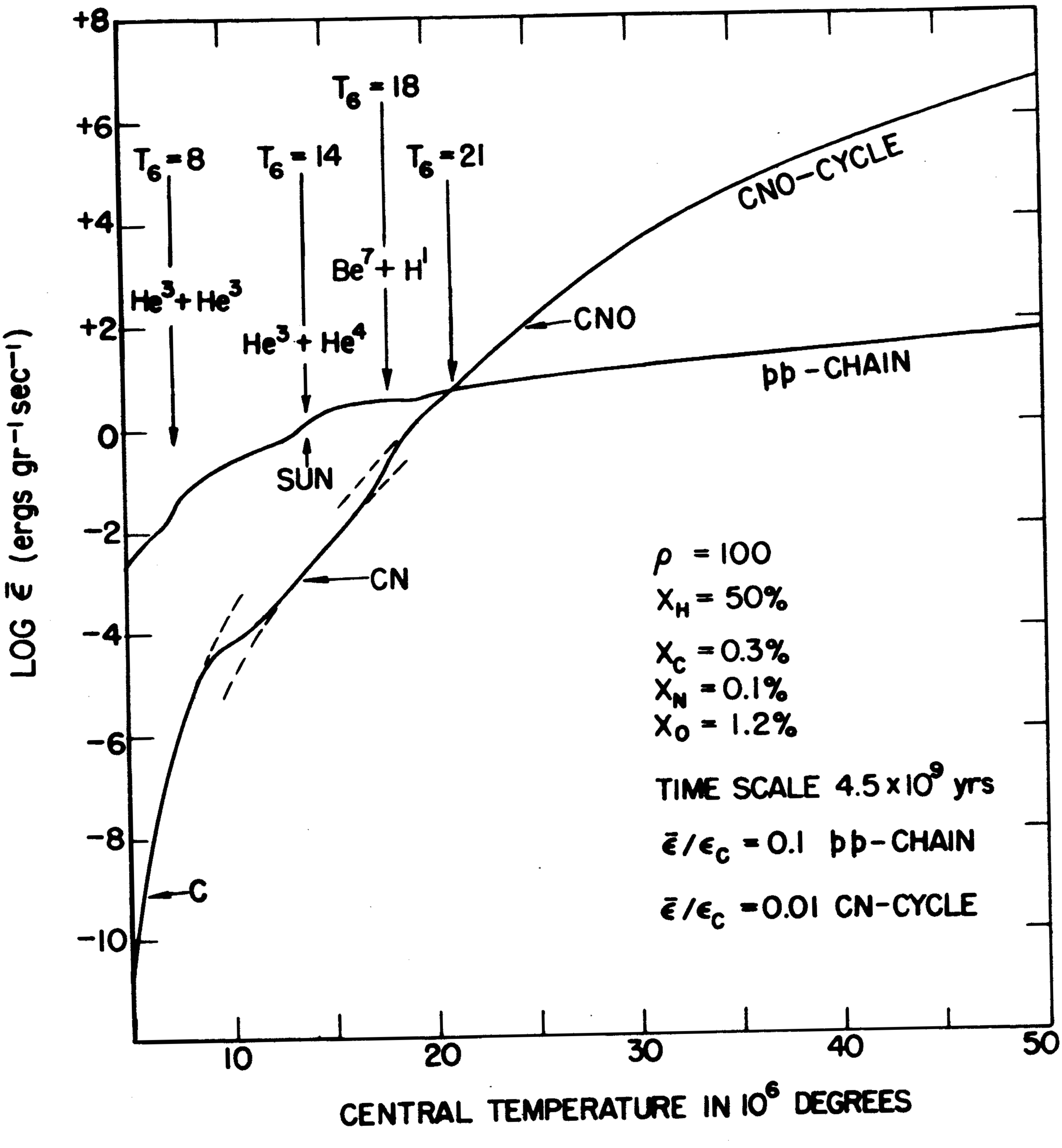


Fig. 5. Average energy generation throughout a star in ergs per gram-second as a function of central temperature for the p-p chain and the CNO cycle. The central density is taken as $\rho = 100$ g/cm^3, and the hydrogen concentration by weight as $X_H = 0.50$. Concentrations of C, N, and O by weight as given are those for a typical population I star. The age of the star is taken to be 4.5×10^9 years. The points of inflection in the p-p chain arise from the onset of the indicated interactions. Similarly C, N, and O are successively involved in the CNO cycle. Note that the sun and the cool stars operate on the p-p chain; hot stars operate on the CNO cycle.

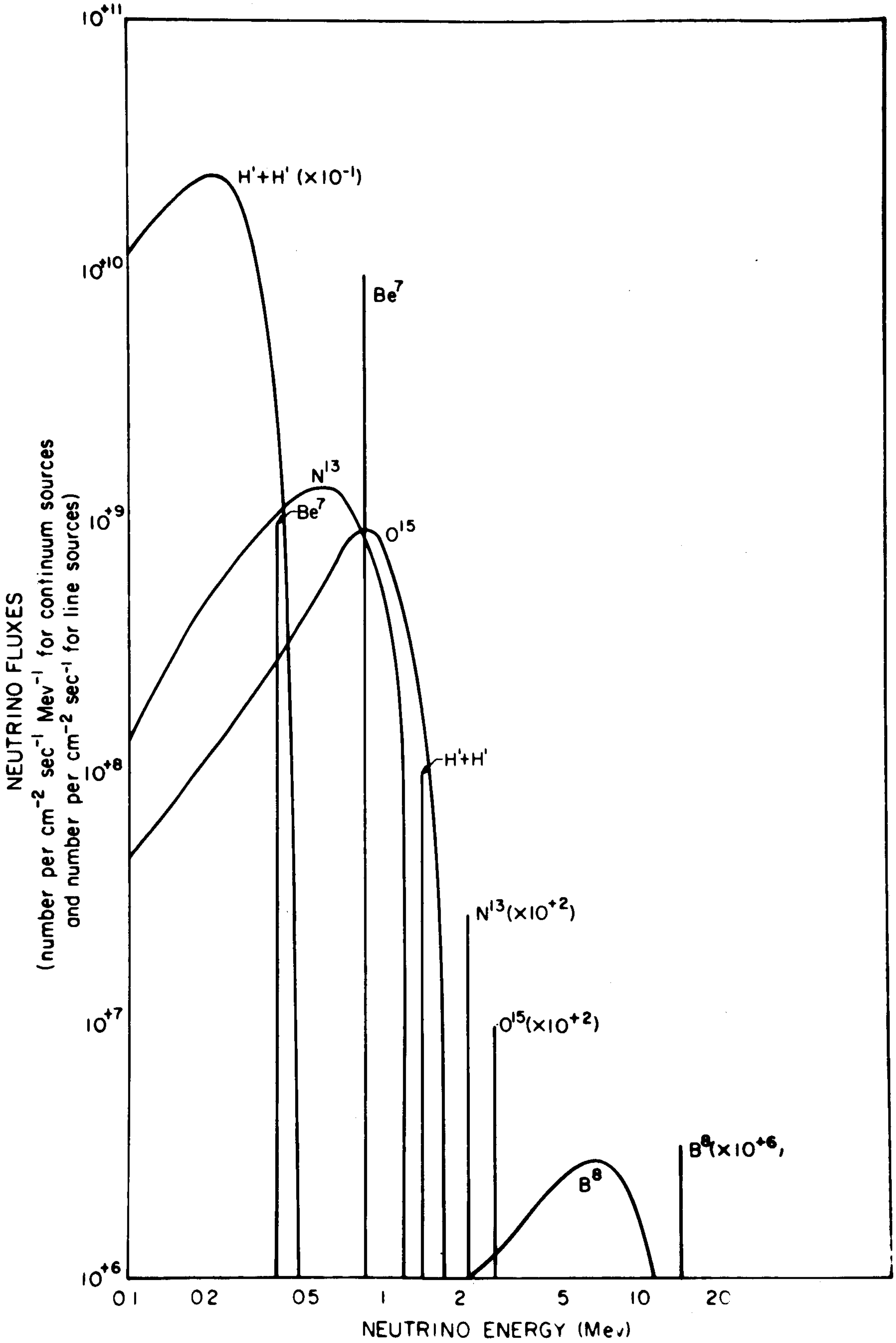


Fig. 6. Predicted neutrino spectrum from the sun after Bahcall (1965). Fluxes given are evaluated at the earth's surface. The neutrino lines are produced by the capture of free electrons; the small thermal widths ($\sim$ 1 keV) of these lines have been neglected in the figure.

DETECTION OF SOLAR NEUTRINOS

Davis (1955, 1964) of the Brookhaven National Laboratory has developed a technique for the detection of neutrinos based on a reaction suggested by Pontecorvo and elaborated upon by Alvarez, namely

$$Cl^{37}\left[J = \tfrac{3}{2}^{+},\ T = \tfrac{3}{2}\right] + \nu_e(\text{solar}) \rightarrow Ar^{37}\left[J = \tfrac{3}{2}^{+},\ T = \tfrac{5}{2}\right] + e^- - 0.814\ \text{MeV}, \qquad (7)$$

which is the reverse of the electron capture reaction which can be observed terrestrially

$$Ar^{37} + e^- \ (\text{orbital}) \rightarrow Cl^{37} + \nu_e\ , \qquad (8)$$

for which the measured half-life is 35.1 days. The Cl^{37} in the form of carbon tetrachloride, CCl_4, or of perchlorethylene, C_2Cl_4, is "exposed" to neutrinos with appropriate shielding in a deep mine and the rare gas Ar^{37} is collected by bubbling helium through the CCl_4 or C_2Cl_4. The Ar^{37} is then frozen out of the helium on to charcoal and is eventually deposited with carrier argon in a small, low background counter. Counting is made possible by the Auger electrons and X-rays emitted by the excited Cl^{37} atoms produced in the Ar^{37} decay.

Davis has already performed experiments using 1000 to 3000 gallons of CCl_4 or C_2Cl_4 under the reactor at Savannah River, Georgia and in a deep mine at Barberton, Ohio. The reactor emits antineutrinos and Davis has already shown that the reaction

$$Cl^{37} + \bar{\nu}_e\ (\text{reactor}) \rightarrow Ar^{37} + e^- \qquad (9)$$

has a cross section $\sigma(\bar{\nu}_e) < 0.02 \times 10^{-44}\ cm^2$, whereas $\sigma(\nu_e) = 1.2 \times 10^{-44}\ cm^2$ would be expected for neutrinos of the same energy spectrum. Thus he has shown that $\bar{\nu}_e$ and ν_e are not identical as postulated in some early theories of the weak interaction. (It is, of course, possible to argue that $\bar{\nu}_e$ and ν_e

are the right- and left-handed varieties of the <u>same</u> particle and still be in keeping with experiment if the mass of the neutrino is taken to be very small and the deviation from completeness of the violation of parity in the weak interactions is assigned a suitably small value.)

The cross section for the absorption of neutrinos in Cl^{37} to form the ground state of Ar^{37} is given by

$$\sigma(\nu_e) = \sigma_o \, p_e \, \omega_e \left(\frac{F}{2\pi\alpha Z}\right) , \tag{10}$$

where $F = F(\omega_e, Z)$ is the well known Fermi function and p_e and ω_e are the electron momentum in units $m_e c$ and the total electron energy in units $m_e c^2$ respectively. The cross section $\sigma_o = 1.9 \times 10^{-46}$ cm^2 can be calculated from the measured properties of the Ar^{37} decay. From the energetics of the reaction the outgoing electron energy is given in terms of the incident neutrino energy by

$$E_e = W_e - 0.511 \text{ MeV} = W_\nu - 0.814 \text{ MeV}$$

or $$\varepsilon_e = E_e/m_e c^2 = \omega_e - 1 = \omega_\nu - 1.59 \tag{11}$$

and $$p_e = (\omega_e - 1)^{\frac{1}{2}} .$$

The threshold occurs for $E_e = 0$ or $W_\nu = 0.814$ MeV. Thus the reaction is only capable of detecting the most energetic of the Be^7 neutrino lines and that part of the B^8 continuum above 0.814 MeV. Because of the phase space factor, $p_e \omega_e$, for the emitted electron the reaction is ~ 250 times more sensitive to the $\nu(B^8)$ than to the $\nu(Be^7)$. This makes up in part for a factor of ~ 500 in the relative fluxes favoring $\nu(Be^7)$.

Recently Bahcall (1964b) has suggested that the analogue state in Ar^{37} corresponding to the $J = 3/2$, $T = 3/2$ ground state of Cl^{37} should be produced with relatively high probability in the reaction

$$Cl^{37}\left[J = \frac{3}{2}^+,\ T = \frac{3}{2}\right] + \nu_e \rightarrow Ar^{37*}\left[J = \frac{3}{2}^+,\ T = \frac{3}{2}\right] + e^- - 5.95\ \mathrm{MeV}\ , \qquad (12)$$

since this is a super-allowed transition with a large matrix element for the transition. In this case $\sigma_0 \sim 10^{-44}\ \mathrm{cm}^2$ but only the B^8 neutrinos will produce the Ar^{37*}. This changes the overall cross-section factor for $\nu(B^8)/\nu(Be^7)$ to $\sim 10^4$. The threshold energy given above has been determined by experiments by McNally (1965) at the California Institute of Technology and elsewhere which isolated the T = 3/2 plate at 5.14 MeV excitation. The energy level diagram of Ar^{37} before and after the experiments stimulated by Bahcall's suggestion is shown in Figure 7.

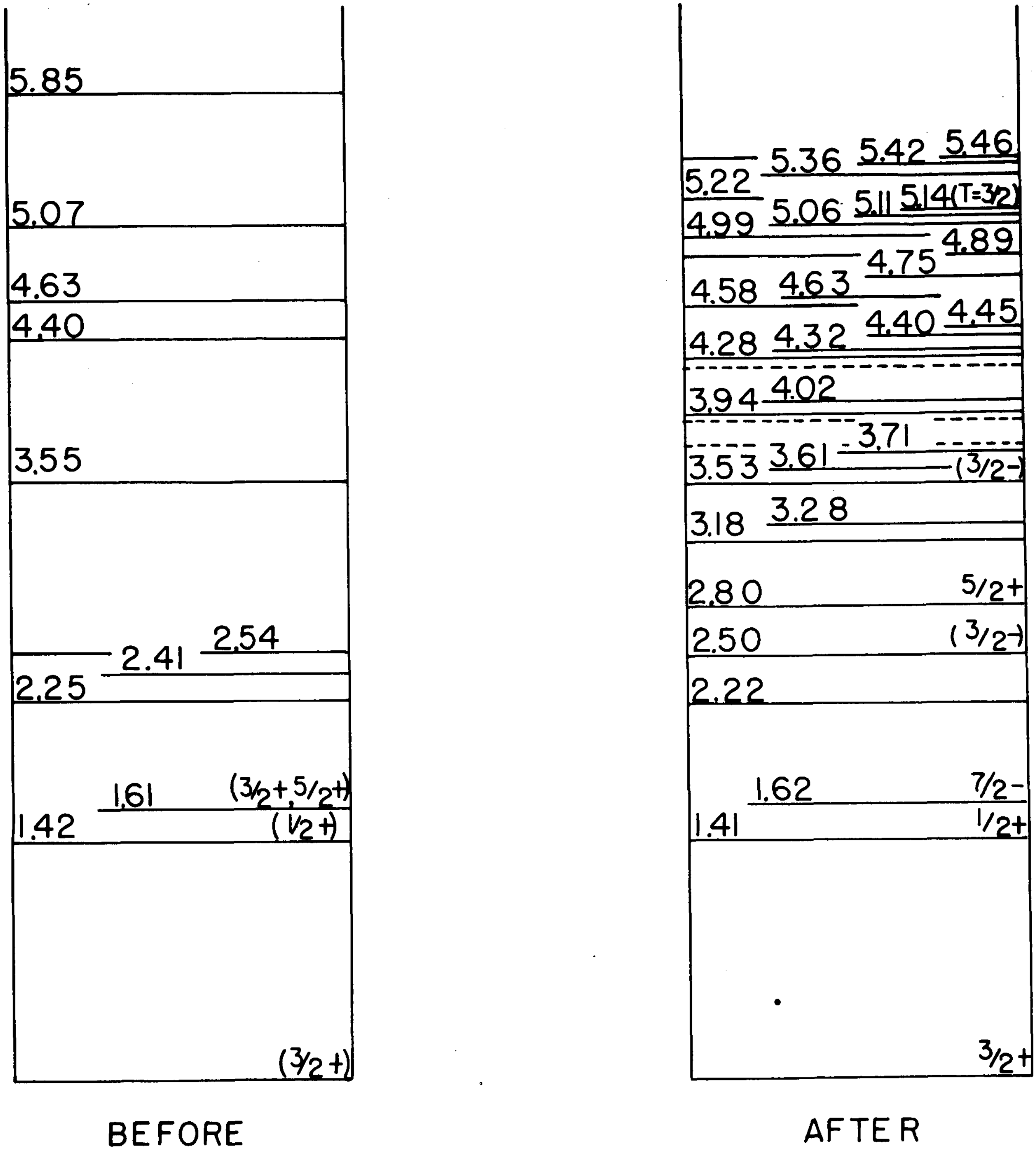
On the basis of calculations involving the ground state, the analogue state and other known states in Ar^{37}, Bahcall (1964d) has calculated the capture rates given in Table IV. The total flux-cross section product is

$$(\varphi\sigma)_\nu = (3.6 \pm 2) \times 10^{-35}\ \mathrm{sec}^{-1}\ \mathrm{Cl}^{-1} \qquad (13)$$

and the total expected counts per day in C_2Cl_4 is

$$\begin{array}{rcl} 100{,}000\ \mathrm{gal} & : & (5.7 \pm 2.3)\ \mathrm{counts\ day}^{-1} \\ 1{,}000\ \mathrm{gal} & : & 0.06\ \mathrm{counts\ day}^{-1} \quad . \end{array} \qquad (14)$$

Davis has already found an upper limit of < 0.3 counts day^{-1} with the 1000 gallon apparatus set up in a mine at Barberton, Ohio. This is within a factor of 0.3/0.06 = 5 of significant result. The upper limit is set by background produced in the reaction $Cl^{37}(p,n)Ar^{37}$ by protons produced in turn by cosmic ray muons which penetrate to the depth of the mine. Davis is now constructing a 100,000 gallon Neutrino Observatory in the Homestake Mine in South Dakota. Since this location is at a depth of 4,700 meter-water-equivalent



Ar^{37}

Fig. 7. The energy level diagram of Ar^{37} before and after experimental observations following the suggestion by Bahcall (1964b) that excited states of Ar^{37} play a role in the interaction of neutrinos with Cl^{37}.

TABLE IV

$$\nu + Cl^{37} \rightarrow Ar^{37} + e^-$$

Neutrino Source	φ_ν $cm^{-2}\ sec^{-1}$	State in Ar^{37}	σ cm^{-2}	Capture Rate per 10^5 gal. C_2Cl_4
Be^7 decay	$(1.2 \pm 0.5) \times 10^{10}$	Ground	2.9×10^{-46}	0.61 ± 0.25
B^8 decay	$(2.5 \pm 1.0) \times 10^{7}$	Ground	7.5×10^{-44}	0.32 ± 0.13
		1.4	18.3×10^{-44}	0.83 ± 0.32
		2.7	2.5×10^{-44}	0.13 ± 0.07
		5.1	84.0×10^{-44}	3.80 ± 1.52

Total neutrino capture rate in 10^5 gallons = 5.7 ± 2.3

he expects a background of < 0.2 counts day^{-1} which is less than the expected counting rate by a factor of ~ 30.

In his calculations Bahcall used stellar model calculations made by Sears (1964). As pointed out by Fowler (1958) the cross section for the $Be^7(p,\gamma)B^8$ reaction is very sensitive to the central temperature, T_c, of the sun being given by

$$\sigma_{p,\gamma}(Be^7) \sim T_c^{14} . \tag{15}$$

Thus the Davis apparatus is a very sensitive "solar thermometer" and Bahcall estimates that a measurement of the B^8 neutrino flux accurate to ± 50 per cent will determine the central temperature of the sun to ± 10 per cent. The preliminary 1000 gal experiment of Davis puts an upper limit of $T_c < 20 \times 10^6$ °K. The models of Sears (1964) indicate $T_c \sim 16 \times 10^6$ °K on the basis of currently accepted solar parameters and measured nuclear reaction rates. Thus neutrino astrophysics may soon provide us with direct information from the center of the sun which will serve as a check on current ideas of stellar structure and stellar energy generation.

ADDITIONAL OBSERVATIONAL TECHNIQUES

1. Reines and Kropp (1964) of the Case Institute of Technology have attempted to detect solar neutrinos using knock-on electrons produced in neutrino-electron scattering according to

$$\nu_e + e^- \rightarrow \nu_e + e^- \tag{16}$$

This scattering process has not yet been observed in the laboratory but it is predicted on the basis of universality in current theories of the weak interactions. The cross section should be of the order of 10^{-44} cm^2 for B^8 neutrinos.

The advantage of using this reaction for the observation of solar neutrinos is that it permits the measurement of the neutrino energy and direction in principle. The secondary electron is projected forward within a cone of $\pm 10^{\circ}$ with respect to the primary neutrino; the angle and energy determine the neutrino energy by kinematics.

A modest experiment was performed by Reines and Kropp which enabled them to set certain limits and to assess the necessary full-scale effort. The experiment consisted of looking for unaccompanied counts in a 200-liter liquid scintillator detector (5×10^{28} target electrons) which was surrounded by a large Cherenkov anticoincidence detector and located 2000 feet underground in a salt mine. In a counting time of 4500 hours ($\sim$ 200 days) only three events were observed in the energy range 9 to 15 MeV, unaccompanied by pulses in the anticoincidence guard. This sets an upper limit on the flux of B^8 neutrinos

$$\varphi_{\nu}(B^8) < 5 \times 10^8 \ \nu_e \ cm^{-2} \ sec^{-1} \tag{17}$$

which is 20 times that expected from the calculations given in Table IV.

Reines and Kropp estimated that a large detector with sensitive volume of 10^4 gallons would yield an expected rate $\sim$ 100 counts yr^{-1}. During the CERN neutrino conference in January 1965, the possibility of a CERN-Case-Turin collaboration was discussed. The present laboratory in the Mont Blanc tunnel under 2000 meters of rock or 5000 meters water equivalent was suggested as a suitable location. On the basis of a predicted rate of 5 events per year per ton of detector it was argued that at least 5 tons of detector would be required. It was felt that the problem of light collection should be studied in a model. No approval for this project has yet been obtained as of July 1965.

2. Reines and Woods (1965) have proposed the use of the inverse beta decay reactions

$$\nu_e + B^{11} \rightarrow C^{11} + e^- - 1.98 \text{ MeV} \tag{18}$$

which is related to

$$C^{11} \rightarrow B^{11} + e^{+} + \nu_e + 0.96 \text{ MeV} \tag{19}$$

for which log ft = 3.6 experimentally, and

$$\nu_e + Li^7 \rightarrow Be^7 + e^- - 0.86 \text{ MeV} \tag{20}$$

which is the inverse of

$$e^- + Be^7 \rightarrow Li^7 + \nu_e + 0.86 \text{ MeV} \tag{21}$$

for which log ft = 3.3. All energies are computed from nuclear, not atomic, mass differences. The cross sections for (18) and (20) can be calculated in terms of the ft-values of the related reactions.

In the suggested experimental arrangement thin, large-area slabs of target material containing Li or B are surrounded on both sides in layers by organic scintillator detectors. Several tons of Li and tens of tons of B are required to obtain counting rates of the order of 100 events per year. The angular correlation of the emitted electrons and incident neutrinos is not isotropic so the neutrino direction can be measured in principle. The advantage of this method over neutrino-electron scattering is that processes (18) and (20) are certain to occur while (16) is based on theoretical predictions. In the case (16), failure to observe events may mean that the weak interaction is not universal. The disadvantage is that reactions (18) and (20) have energy thresholds as indicated just as in the case of (7), whereas (16) does not.

3. Reines, Crouch, Jenkins, Kropp, Gurr, Smith, Sellschop, and Meyer (1965) are searching for high energy neutrinos with apparatus installed 10,492 feet below the surface in a gold mine in South Africa. The venture is a

collaborative effort of the Case Institute of Technology and the University of Witwatersrand. The idea of the experiment is to detect the energetic muons produced in neutrino interactions in the rock surrounding the mine tunnel by means of a large detector array located in the tunnel. Backgrounds are reduced by the large overburden and by utilizing the fact that the angular distribution of the unwanted residual muons from the earth's atmosphere is strongly peaked in the vertical direction at mine depth.

The detector array consists of two parallel vertical walls made up of 36 detector elements. The array is grouped into 6 "bays" with 3 elements, upper, middle, and lower, on each side. Each detector element is a lucite box containing 380 liters of liquid scintillator viewed at each end by two 5-inch photomultiplier tubes. Through coincidence counting the array constitutes a hodoscope which gives a rough measurement of the zenith angle of a charged particle passing through it. Only muons with zenith angle $> 45^{\circ}$ are detected on both sides of the array. In addition, each event is located along the detector axis by the ratio of the photomultiplier responses at the two ends of the triggered element.

In 563 bay days of operation corresponding to 14,200 m^2 days sr of effective detection, a total of 7 events involving coincidences in one element on each side were recorded. We turn our attention to possible sources of these events.

The most likely source is the earth's atmosphere in which neutrettos (ν_{μ}, $\bar{\nu}_{\mu}$) are produced by the decay of pions, kaons, and muons which in turn originate from collisions between primary cosmic particles with nuclei of the atmosphere. The angular distribution of the muons produced near the mine tunnel by neutretto interactions should show a slight peaking in the horizontal direction to which the coincidence arrangement of the detector array is most sensitive. This is in contrast as noted above to the angular distribution of muons coming directly from interactions in the atmosphere. Reines and his co-workers estimate 3 events from atmospheric neutrettos during the period of operation in which 7 events were recorded and state that there are reasons to believe that their estimate may be too low and that all the events can conceivably be from this source.

High energy, cosmic neutrinos and neutrettos can trigger the detector array. Bahcall and Frautschi (1964a,b) suggest that neutrinos and neutrettos may be emitted from supernovae, quasi-stellar objects and extended radio sources with energy comparable to that of the electrons which produce the radio noise from these objects.

The efficiency of detection for antineutrinos ($\bar{\nu}_e$) from these sources will be particularly high if a vector boson (W^-) serves as an intermediate stage in the reaction

$$\bar{\nu}_e(\text{cosmic}) + e^-(\text{terrestrial rock}) \rightarrow (W^-) \rightarrow \bar{\nu}_\mu + \mu^- \quad . \qquad (22)$$

Resonance in this reaction occurs at 4×10^{12} eV if the mass of the vector boson is 2×10^9 eV/c^2. The μ^- would be the detected particle in the mine array.

If the earth is used as a target, Bahcall and Frautschi estimate an antineutrino-induced counting rate at a depth of 1 km of 100 high-energy muons per m^2 per year from the direction of the Crab Nebula (supernova of July 4, 1054 A.D.). For a young radio source at the distance of Cygnus A, the counting rates would be 10^3 times those estimated for the Crab if all the $\bar{\nu}_e$ were emitted within the first 10^3 years of the lifetime of the source. Even so, highly directional sources of this nature would not contribute significantly to the counting rate expected in systems similar to those constructed by Reines and his co-workers. Furthermore, resonance effects can not be detected at excessive depths.

4. The Tata Institute of Fundamental Research, Bombay, the Durham University, U.K., and the Osaka University, Japan, are jointly running a neutrino experiment in a gold mine at Kolar, India, about 2,000 meters underground. Neutrinos react with nuclei in the rock and produce muons. The detector system has a collecting area of about 6 m^2 and consists of two outer banks of plastic scintillators and three banks of neon flash tubes with lead walls each of 2.5 cm between the flash tubes. The direction of individual particles can be estimated to an accuracy of about 1°. The detector has been in operation for about four months and 4 events have so far been observed, 3 of single particles and 1 of two particles. All 4 events made angles larger than 45° to the vertical, including the double particle event which arrived along the horizontal direction.

5. J. Keuffel (1965) of the University of Utah is planning the installation of large detectors at 2000 feet depth near Park City, Utah. The detector consists of a block of concrete 10 m × 10 m × 6 m high with four vertical slots one meter **wide** filled with water used as Cherenkov counters. An additional nine vertical slots are filled with trays of cylindrical spark counters in which timing of the audible discharge with simple microphones permits localization of particle tracks to a few millimeters. Counting rates of 5-10 events per year are expected from atmospheric neutrettos.

6. Cowan, Ryan, Acosta, Buckwalter, Carey, and Curtin (1965) are engaged in the observation of muons appearing in massive detectors under the influence of a neutral component of the cosmic rays. The muons are of intermediate energies, 10 to 150 MeV, as they must be produced in and must stop in the detectors. They are identified by means of their decay electron through its energy and its delay time relative to the appearance of the muon. The detector consists of a block of plastic scintillator 2' × 4' × 1'. This presents a target of about 230 kg to the incident neutral particles. It is surrounded on all sides except the small east and west ends by sheets of plastic scintillator three-quarters of an inch thick which, with their photomultipliers provide the anticoincidence shielding. The top and bottom sheets overhang the detector somewhat. Twelve photomultipliers look into the target through the open east and west ends. A considerable number of events have been recorded but Cowan and his co-workers do not believe that incident neutrinos or neutrettos satisfy all the conditions implied by their experimental results to date.

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LECTURE I

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LECTURE II*

NEUTRINO ASTROPHYSICS, II

N66 24328

INTRODUCTION

The question of the number and energy of neutrinos and antineutrinos emitted by a star during its evolutionary lifetime was raised in Lecture I. The answer to this question depends in part on the universality of the weak interactions. Is the coupling constant the same for all possible four-particle interactions between pairs of protons and neutrons, pairs of electrons and electron-neutrinos, and pairs of muons and muon-neutrinos (neutrettos) or does the observed value only hold for those cases on which direct observations have been made, namely beta-decay, muon-decay, and muon-capture?

Complete universality is implied by the conserved vector current theory of Feynman and Gell-Mann (1958a,b). This theory has been shown to make correct predictions in a number of experimental tests by Bardin, Barnes, Fowler, and Seeger (1960, 1962), Freeman, et al. (1962, 1964), Nordberg, Morinigo and Barnes (1960, 1962), Mayer-Kuckuk and Michel (1961, 1962) and Lee, Mo, and Wu (1963), but the completeness of the universality was not directly in question in these experiments. Recent evidence for the existence of the parity violating characteristic of the weak interaction in interactions involving only nucleons has been given by Boehm and Kankeleit (1964) and by Abov, Krupchitsky and Oratovsky (1964). These experiments are very suggestive but do not absolutely require that the extension to cases involving only leptons must

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be accepted as valid. From an analysis of the isotopic abundances in the iron group elements on the basis of nucleosynthesis of these elements in an equilibrium process, Fowler (1962, 1963) and Fowler and Hoyle (1964) have suggested that there is strong circumstantial evidence for the emission of neutrinos and antineutrinos in electron-positron pair annihilation at just the rate predicted on the basis of the universality of the weak interaction. In the second part of this lecture this suggestion is reviewed and in the third part a brief discussion of the extent of neutrino emission by astronomical objects is given on the basis that this suggestion is correct.

NEUTRINO EMISSION DURING THE EQUILIBRIUM PROCESS

In massive stars in the range $10\ M_\odot < M < 50\ M_\odot$, Fowler and Hoyle (1964) show that nuclear evolution involving charged particle reactions proceeds from hydrogen burning via the CNO bi-cycle, through helium burning with the production primarily of oxygen, to oxygen burning with the production primarily of silicon. At the termination of oxygen burning, photodisintegration into alpha-particles with subsequent capture of these particles, the so-called $\underline{\alpha}$-process, leads to the synthesis of iron-group nuclei with mass number $A \sim 50$ to 60 which have the greatest binding energy and stability of all nuclear species.

These charged-particle reactions proceed primarily through nuclei which on the average have an equal number of neutrons and protons, $\bar{N} = \bar{Z}$. When the $\underline{\alpha}$-process comes to an end at $T_9 = T/10^9 = 3.5$, energy loss by neutrino emission leads to a mild contraction of the stellar core and a slight rise in temperature and density. At this point beta-processes, positron emission and electron capture, begin to play a role in the transformation to nuclei which have a greater number of neutrons than protons, e.g., Fe^{56}, and which are more stable than those with equal numbers, e.g., Ni^{56}.

The pertinent question is this. In the time scale permitted by the neutrino losses, how far will the beta-processes go in producing nuclei with a neutron excess? In other words, in the "equilibrium" or e-process in which the iron-group abundances are finally determined, does the material come to the complete equilibrium corresponding to the ambient temperature and density or does the limited reaction rate of the beta-processes impose an additional constraint? It has been emphasized by Burbidge, Burbidge, Fowler and Hoyle (1957) and Hoyle and Fowler (1960) that the abundances of the iron-group nuclei found in the solar system (particularly, terrestrial isotopic abundances) show definite effects of such a rate limitation. Figure 1 is adapted from B^2FH (1957) and shows the excellent agreement between observed iron-group abundances and those calculated for equilibrium at $T_9 = 3.8$.

It is assumed that solar-system iron-group nuclei are typical of nuclei produced in the e-process just outside the imploding central regions of Type II supernovae. These nuclei reside in the material which is swept out by the explosion of the mantle and envelope of the star. This explosion occurs in such a short time interval that the quasi-equilibrium abundances reached before the implosion-explosion are essentially unchanged. In what follows a most significant connection will be found between iron-group abundances and the time scale set by neutrino losses during the stellar stage just prior to core implosion and mantle-envelope explosion. The neutrino losses are taken to arise from the formation at high temperature of electron-positron pairs which can then annihilate with the production of neutrino-antineutrino pairs according to

$$e^+ + e^- \rightarrow \nu + \bar{\nu} \qquad (1)$$

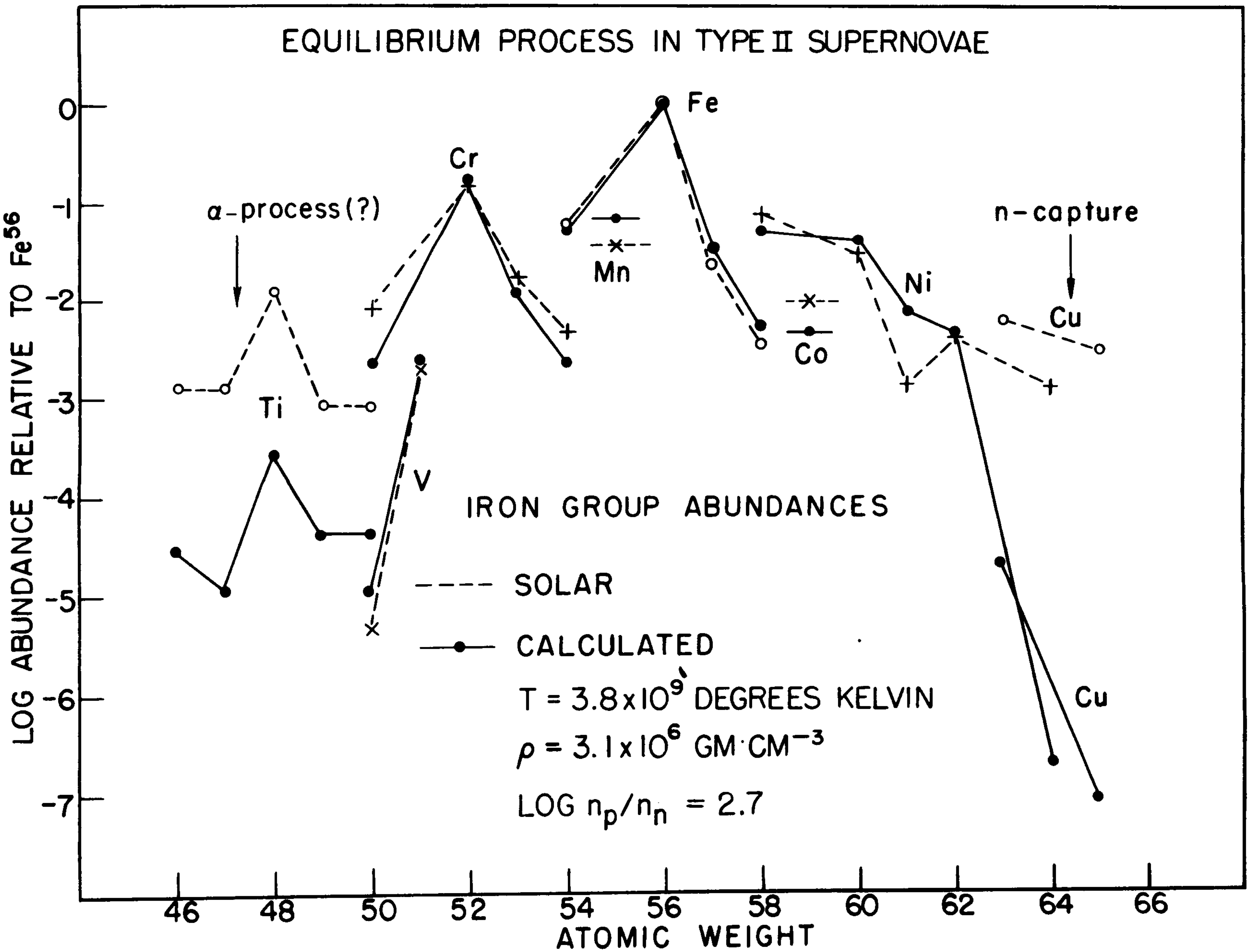

Fig. 1. The abundance relative to Fe^{56} of nuclei produced in the e-process.

This process follows straightforwardly from the conserved vector current theory of the weak interactions proposed by Feynman and Gell-Mann (1958a, 1958b).

The measure of beta-interaction rates appropriate for the present purposes is the rate of change of one-half the average neutron-proton difference per nucleus. This can be calculated from

$$\frac{1}{2}\frac{d(\overline{N}-\overline{Z})}{dt} = \frac{d\overline{N}}{dt} = -\frac{d\overline{Z}}{dt} = \frac{\Sigma \pm n(N,Z)/\tau(N,Z)}{\Sigma n(N,Z)} \tag{2}$$

where $\overline{N} = \Sigma N n(N,Z)/\Sigma n(N,Z)$, $\overline{Z} = \Sigma Z n(N,Z)/\Sigma n(N,Z)$, $n(N,Z)$ is the number of nuclei containing N neutrons and Z protons, and $\tau(N,Z)$ is the mean lifetime of these nuclei for beta-interactions. The positive sign is to be used for positron emission or electron capture and the negative sign for electron emission or positron capture. The problem at hand involves first of all the calculation of $n(N,Z)$ as a function of the ratio of protons to neutrons, $\overline{Z}/\overline{N}$. This is a task of considerable magnitude if temperature and density are also varied, and a computer program to accomplish the purpose has been undertaken by Clifford and Tayler (1964 ,1965) at Cambridge University. Here we will fix on a temperature and density using some of their results and will discuss only in a general way what is essentially the "approach" to equilibrium in stellar nuclear processes.

Since B^2FH (1957) found that equilibrium calculations at $T_9 = 3.8$ gave excellent agreement with solar-system iron-group abundances, and since this temperature is just slightly above that at which the pure $\underline{\alpha}$-process ends, we will use this value in what follows. Then in a stellar core with effective mass $M_c = 20\ M_\odot$ FH (1964) show that $\rho_6 = \rho/10^6 = 3.1$, $N_e = 4.8 \times 10^{23}$ electrons and positrons per gm, $n_e = 1.50 \times 10^{30}$ electrons and positrons per cm^3,

$N_- = 3.9 \times 10^{23}$ electrons per gm, $n_- = 1.22 \times 10^{30}$ electrons per cm^3, $N_+ = 0.9 \times 10^{23}$ positrons per gm, and $n_+ = 0.28 \times 10^{30}$ positrons per cm^3. The electron-positron numbers will change slightly as Ni^{56} changes to Fe^{56} as the dominant nucleus during the operation of the e-process.

The termination of the α-process at $T_9 = 3.5$ followed by a slight rise in temperature and density upon contraction brings the material to $T_9 = 3.8$ with $\overline{Z}/\overline{N} = 1$ and Ni^{56} the most abundant nucleus. Beta-processes will now lower $\overline{Z}/\overline{N}$. For substitution in equation (2) one thus needs relative values for n(N,Z) for a series of values for $\overline{Z}/\overline{N}$ at $T_9 = 3.8$. A fixed value for $\overline{Z}/\overline{N}$ serves as a constraint on the equilibrium process in the manner described by B^2FH (1957; see pp. 577, 578). Clifford (1964) has carried out special abundance calculations for $\overline{Z}/\overline{N} = 1.00$, 0.975, 0.950, 0.925, 0.900, 0.875, 0.8725, 0.870, 0.865, and 0.860 at $\rho_6 = 3.1$ and $T_9 = 3.5$. The interval between successive values corresponds to $\Delta\overline{N} = 0.4$ neutrons per nucleus when $\Delta(\overline{Z}/\overline{N}) = 0.025$. A total change of ~2 neutrons per nucleus is thus covered as expected for the typical case $_{28}Ni^{56}_{28} \rightarrow {}_{26}Fe^{56}_{30}$. Table 1 lists the principal components of the material for various values of $\overline{Z}/\overline{N}$.

Methods for calculation of the τ(N,Z) under stellar conditions have been described by FH (1964). It will be clear that electron capture and positron emission are the important beta-processes since the trend in stability is toward nuclei with a neutron excess. Under terrestrial laboratory conditions positron emission is more rapid than electron capture if sufficient energy is available in the nuclear transformation to produce the positron rest mass and give the positron kinetic energy at least comparable to its rest-mass equivalent energy. However, in dense stellar interiors the electron density at the nucleus is considerably greater than in the undisturbed atom so that the rate of electron capture is greatly enhanced. The result is that the proton-to-neutron change in radioactive nuclei which normally capture electrons or emit positrons is increased in rate and even stable nuclei, e.g., Ni^{58}, have fairly short lifetimes for capture of electrons having high energy in the tail of the thermal energy distribution.

Reference to Table 1 indicates that the nuclei which make important contributions in equation (2) are: Ni^{56} (2×10^3 sec), Ni^{57} (2×10^3 sec), Ni^{58} (5×10^4 sec), and Fe^{54} (4×10^4 sec). The proton (4×10^3 sec), Co^{55} (2×10^3 sec), and Fe^{55} (10^4 sec) also contribute. In general the transformation from $\overline{Z/N} = 1.00$ to smaller values can be followed in Figure 2. At $\overline{Z/N} = 1$ the principal constituents are Ni^{56}, Ni^{57}, and Co^{55}. These capture electrons or emit positrons to become Co^{56}, Co^{57}, and Fe^{55} respectively. The Co^{56} immediately becomes Fe^{54} and Ni^{58} through fast nuclear processes since $2\ Co^{56} \rightarrow Fe^{54} + Ni^{58} + 4.45$ MeV. Fe^{54} and Ni^{58} capture electrons to become Mn^{54} and Co^{58} which change by fast nuclear processes to Cr^{52}, Fe^{56}, and Ni^{60}. Fe^{55} and Co^{57} produce Mn^{55} and Fe^{57}. Eventually nuclear processes produce the equilibrium abundances which mainly reside in the stable nuclei which form the shaded "steps" in Figure 2, namely $Cr^{52,53,54}$, Mn^{55}, $Fe^{56,57,58}$, Co^{59}, and $Ni^{60,61,62}$ (the last two nuclei are not shown). Some material remains as stable Fe^{54} and Ni^{58} and also as stable Cr^{50} (not shown) and the other rare iron-group nuclei.

Table 1 gives the time intervals calculated using equation (2) for the changes through $\overline{Z/N} = 1.00, 0.975, \ldots,$ to 0.860, and the total time to each value. Table 1 also gives the quantity $\theta = \log_{10} n_p/n_n$, the logarithm to the base 10 of the ratio of densities of free protons to free neutrons external to the complex nuclei. As pointed out by B^2FH (1957) equilibrium calculations

TABLE 1

A. The Approach to Equilibrium at $T = 3.8 \times 10^9$, $\rho = 3.1 \times 10^6$, $M_c = 20\, M_\odot$, $M \approx 30\, M_\odot$

	$\frac{1}{2}(\bar{N} - \bar{Z})$	$\bar{Z}/\bar{N}$	$\log n_p/n_n$ θ		Time (10^4 sec)	
1.	0.0	1.000	8.62			0.0
2.	0.4	0.975	7.36		0.1	0.1
3.	0.8	0.950	6.61		0.1	0.2
4.	1:2	0.925	5.18		0.2	0.4
5.	1.6	0.900	4.04		0.8	1.2
6.	2.0	0.875	2.94		1.6	2.8
7.	2.04	0.8725	2.74		0.4	3.2
8.	2.08	0.870	2.48		0.5	3.7
9.	2.16	0.865	1.76	18.0	1.3	5.0
10.	2.24	0.860	1.17		3.0	8.0
11.	2.40	0.850	....		12.8	20.8

TABLE 1

B. ABUNDANCE IN PER CENT BY MASS

(Naturally Radioactive Nuclei in Parentheses)

Nucleus	(Co^{55})	(Ni^{56})	(Ni^{57})	Ni^{58}	Fe^{54}	(Fe^{55})	Fe^{56}	Fe^{57}	Fe^{58}
Product	$(Fe^{55})Mn^{55}$	$(Co^{56})Fe^{56}$	$(Co^{57})Fe^{57}$	(Co^{58})	(Mn^{54})	Mn^{55}	(Mn^{56})	....	....
Energy diff. (MeV)	3.46	2.10	3.24	−0.38	−0.69	0.23	−3.71	....	....
τ_{star} (sec)	2×10^{3}	2×10^{3}	2×10^{2}	5×10^{4}	4×10^{4}	10^{4}	10^{8}	....	....
1	3.3	89.1	2.9	0.7	1.7	3×10^{-3}	6×10^{-5}	....	....
2	8.7	54.3	7.5	7.9	19.3	0.2	1×10^{-2}	....	....
3	8.2	21.4	6.8	16.7	43.4	0.9	0.1	....	....
4	2.1	1.0	1.5	18.5	60.1	6.0	4.1	2×10^{-3}	....
5	0.3	3×10^{-2}	0.2	8.3	34.0	12.0	29.2	6×10^{-2}	4×10^{-3}
6	2×10^{-2}	5×10^{-4}	9×10^{-3}	1.2	6.8	7.9	62.9	0.4	0.1
7	8×10^{-3}	2×10^{-4}	5×10^{-3}	0.8	4.7	6.7	66.2	0.5	0.2
8	4×10^{-3}	....	2×10^{-3}	0.4	2.8	5.3	69.2	0.7	0.3
9	4×10^{-4}	....	2×10^{-4}	0.1	0.6	2.5	70.5	1.5	1.2
10	....	....	....	2×10^{-2}	0.2	1.2	64.5	2.7	4.0

Co^{58} which change by fast nuclear processes to Cr^{52}, Fe^{56}, and Ni^{60}. Fe^{55} and Co^{57} produce Mn^{55} and Fe^{57}. Eventually nuclear processes produce the equilibrium abundances which mainly reside in the stable nuclei which form the shaded "steps" in Figure 10, namely, $Cr^{52,53,54}$, Mn^{55}, $Fe^{56,57,58}$, Co^{59}, and $Ni^{60,61,62}$ (the last two nuclei are not shown). Some material remains as stable Fe^{54} and Ni^{58} and also as stable Cr^{50} (not shown) and the other rare iron-group nuclei.

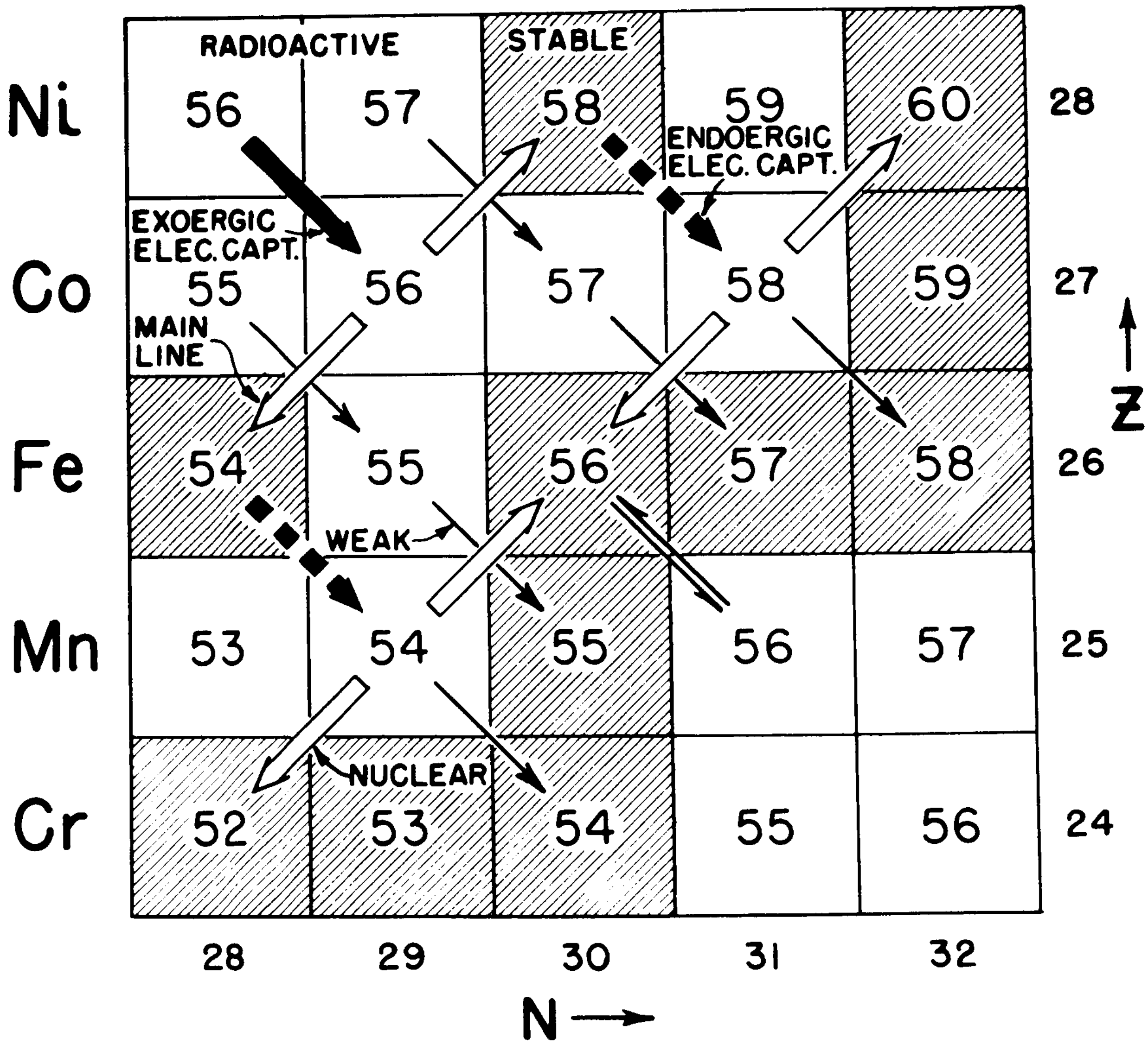

Fig. 2. The flow of nuclear material in the N,Z plane during the equilibrium or e-process showing the effects of the slow beta interactions and the rapid nuclear interactions. The α-process results mainly in the production of Ni^{56} with about 10 per cent Co^{55}, Ni^{57}, Fe^{54}, and Fe^{52} (not shown).

can be made quite simply using θ as a parameter. It will be clear, however, that $\overline{Z}/\overline{N}$ is the more significant parameter. The computer program of Clifford and Tayler (1964) essentially finds the values of θ which yield the chosen values of $\overline{Z}/\overline{N}$ and calculates the corresponding equilibrium abundances. It will be noted immediately that, as expected, very large ratios of free protons to free neutrons are required external to the complex nuclei to maintain the larger values for $\overline{Z}/\overline{N}$, e.g., $\theta = 8.62$ for $\overline{Z}/\overline{N} = 1.00$. In simple physical terms a dense atmosphere of protons is necessary to prevent the nuclei with $Z = N$ from decaying to the more stable nuclei with $Z < N$. The electrostatic repulsion between protons in the nucleus which leads to increased stability for $Z < N$ is seen to have a powerful effect.

B^2FH (1957) found the optimum correspondence between solar system iron-group abundances and the calculated values for the case $T_9 = 3.8$ and $\theta = 2.5$. We have already seen that $T_9 = 3.8$ is reached naturally in the stellar and nuclear evolution under discussion. The new calculations of Clifford and Tayler (1964,1965) yield optimum results at $\theta = 2.7$ which differs insignificantly from the B^2FH values. Correspondingly $\overline{Z}/\overline{N} = 0.872$ and $\frac{1}{2}\,\Delta(\overline{N}-\overline{Z}) \approx 2.0$ showing that the beta-processes changed approximately two neutrons into protons in the transformation from material with Ni^{56} the most abundant nucleus to material with Fe^{56} the most abundant.

The correspondence between the observations and the calculations of Clifford and Tayler (1964,1965) is illustrated for the stable iron isotopes $Fe^{54,56,57,58}$ in Table 2. The solar-terrestrial values are those found first by dividing the iron abundance by mass by the abundance of all the equilibrium process elements (V, Cr, Mn, Fe, Co, Ni) using the solar spectroscopic data given by Aller (1961). The resulting value 73 per cent was then divided among the iron isotopes according to the terrestrial isotopic abundance ratios.

TABLE 2

IRON ISOTOPES

PER CENT OF TOTAL E-PROCESS ABUNDANCE BY MASS

$M_c = 20\ M_\odot,\ M \approx 30\ M_\odot$

$\overline{Z}/\overline{N}$	$\log n_p/n_n$	Fe^{54}	Fe^{56}	Fe^{57}	Fe^{58}	ELECTRON CAPTURE TIME (10^4 SEC)
1.000	8.6	1.7	89.1	2.9	0.0	0.0
0.950	6.6	43.4	21.9	7.2	0.0	0.2
0.900	4.0	34.0	29.6	4.7	0.04	1.2
0.872*	2.7	4.3	66.6	2.5	0.23	3.2
0.860	1.2	0.2	64.5	3.0	4.0	8.0
0.850	--	--	--	--	--	20.8
SOLAR-TERR VALUES		4.2	67.2	1.6	0.25	

* Interpolated from calculated values at 0.8725 and 0.870.

The chondritic iron abundance given by Suess and Urey (1956) is somewhat higher than the solar value. This higher value can be obtained from the equilibrium process calculations by employing a slightly lower value for the temperature without changing the isotope ratios significantly. The calculated values in Table 2 have been obtained from the abundances of Clifford and Tayler (1965) given in part in Table 1 by assigning all of the material at mass 56 to Fe^{56}, for example, on the basis that if the equilibrium process terminated at a given value for $\overline{Z}/\overline{N}$ then Ni^{56} and Co^{56} would subsequently decay to Fe^{56} and so forth.

The table shows that almost exact correspondence is obtained at $\overline{Z}/\overline{N}$ = 0.872 or $\theta = \log n_p/n_n = 2.7$ as noted previously. The time required for the electron captures up to this point is seen to be 3.2×10^4 sec. This value holds for a star of mass $M = 30\ M_\odot$ with core mass $M_c = 20\ M_\odot$ where $T_9 = 3.8$ and $\rho_6 = 3.1$ are the assumed equilibrium conditions. The mass $M = 30\ M_\odot$ is taken as typical of the range $10\ M_\odot < M < 50\ M_\odot$ for Type II supernovae. In the calculations positron emission, electron emission, and positron capture have been neglected relative to electron capture. At still lower $\overline{Z}/\overline{N}$, as complete equilibrium is attained, all processes, in particular positron capture, must be considered. The time required for the fast nuclear reactions to re-establish equilibrium as the electron captures take place has also been neglected. This is justified since, for example, the lifetime of Co^{56} to $Co^{56}(\gamma,n)Co^{55}$ - 10.07 MeV is $\sim 10^{-4}$ sec at $T_9 = 3.8$, $\rho_6 = 3.1$.

It will be noted that the time required for a given change in $\overline{Z}/\overline{N}$ or in $\frac{1}{2}(\overline{N}-\overline{Z})$ rapidly increases after $\overline{Z}/\overline{N} = 0.872$. Table 1 shows that the change 0.875-0.850 requires more than ten times the interval required for the change 0.900-0.875. Table 2 shows that the total time from 1.000 to 0.850 is more than six times that required to reach 0.872. Thus we are in position to reach an answer to the question posed in the third paragraph of this paper. In the time scale permitted by the neutrino losses, how far will the beta-processes go in producing nuclei with a neutron excess?

To answer this question it is necessary to compute the neutrino time scale under the conditions of temperature and density which have been reached in a star with $M \cong 30\ M_{\odot}$ when the beta-processes operate to change Ni^{56} and other $Z = N$ nuclei produced in the $\underline{\alpha}$-process to nuclei such as Fe^{56} with $\frac{1}{2}(N-Z) = 2$. In the Ni^{56}-Fe^{56} transformation the energy release is 6.6 MeV or 1.13×10^{17} erg gm^{-1} which is reduced to $\sim 10^{17}$ erg gm^{-1} by direct neutrino losses. At $T_9 = 3.8$ and $\rho_6 = 3.1$, FH (1964) show that $dU_{\nu}/dt \sim 10^{14}$ erg gm^{-1} sec^{-1} so the time scale is $t_{\nu} \sim 10^{17}/10^{14} \sim 1000$ sec ~ 17 min. This calculation underestimates t_{ν}. Some Ni^{56} begins to decay as soon as it is first produced at the beginning of the $\underline{\alpha}$-process. Thus an upper limit for t_{ν} is the sum of the interval for the $\underline{\alpha}$-process, which FH (1964) estimate to be 4000 sec, plus that for the Ni^{56}-Fe^{56} transformation. This sum is 4000 sec + 1000 sec = 5000 sec. The intermediate value, $t_{\nu} \sim 3000$ sec, is tentatively adopted.

The value just adopted tentatively holds for the time scale at the center of the star. Since the neutrino loss decreases rapidly with decreasing temperature the time scale will be somewhat longer throughout the central region in which the Ni^{56}-Fe^{56} transformation is taking place. Rough calculations lead to the final choice, $t_{\nu} \sim 6000$ sec. The Ni^{56}-Fe^{56} transformation is relatively insensitive to temperature, and no correction is necessary.

Thus we find $t_{\nu} \sim 6000$ sec is considerably shorter than $t_e \sim 3 \times 10^4$ sec. However, it must be recalled that these calculations have been made for a particular example, $M = 30\ M_{\odot}$, of the type of stars which HF (1960) suggested would evolve to become Type II supernovae, namely, stars for which

$10\ M_{\odot} < M < 50\ M_{\odot}$. The core mass was taken to be $M_c = (2/3)\ M = 20\ M_{\odot}$. Similar calculations for $M_c = (1/3)\ M$, which may be more realistic, show excellent correspondence between t_{ν} and t_e. In addition the lower range of stellar masses, $10\ M_{\odot} < M < 30\ M_{\odot}$, may well have contributed relatively more e-process material than the higher range, $30\ M_{\odot} < M < 50\ M_{\odot}$.

Thus, it would seem that quite close correspondence in the time scales exists for $e^- + e^+ \rightarrow \nu + \bar{\nu}$ and for $Ni^{56} + 2e^- \rightarrow Fe^{56} + 2\nu$ with a universal Fermi interaction for these two types of beta-interactions if the stars in which solar system e-process material was produced had masses 10-50 $M_{\odot}$ as originally contemplated by HF (1960).

The point under discussion here can be sharpened by a consideration of the time scale if $e^- + e^+ \rightarrow \nu + \bar{\nu}$ was not operative. Photon losses in the interval $3 < T_9 < 4$ can be estimated to be $\sim 10^7$ erg gm^{-1} sec^{-1} rather than the value $\sim 10^{14}$ erg gm^{-1} sec^{-1} for dU_{ν}/dt. Thus the photon-loss time scale for Ni^{56}-Fe^{56} is $\sim 6 \times 10^{10}$ sec $\sim$ 2000 years or ample time for the beta-interactions to reduce $\overline{Z/N}$ well below the last values tabulated in Tables 1 and 2. The result, as shown in Table 2, would be, among other things, an enhancement in Fe^{58} and a decrease in Fe^{54} completely in variance with the terrestrial ratio. Clearly the time scale was not this long. Photon losses by the stellar material were not competent to decrease the time scale to the necessary value. On the other hand the neutrino time scale set by assigning the universal Fermi interaction strength to the process $e^+ + e^- \rightarrow \nu + \bar{\nu}$ in the pre-supernova stage of massive stars is closely that required to match the electron capture times involved in the formation of the Fe-isotopes and the other iron-group nuclei. The isotopic abundance ratios in any sample of terrestrial iron are circumstantial evidence for the universality of the beta-interactions.

This much can be asserted with some certainty: <u>The terrestrial iron group isotopic abundance ratios strongly indicate the operation in massive stars of an energy loss mechanism having a loss rate of the same order of magnitude as that calculated for</u> $e^+ + e^- \rightarrow \nu + \bar{\nu}$ <u>on the basis of the universal Fermi interaction strength</u>.

A comment on the ultimate values for $\overline{Z}/\overline{N}$ or $\Theta = \log n_p/n_n$ reached when the beta-interactions are in complete equilibrium is in order at this point. B^2FH (1957) estimated $\Theta = 1.4$ from a consideration of the equilibrium between free neutrons and free protons and electrons. This value is only an approximation at best. It does in principle cover the equilibrium between free protons and free neutrons and positrons. The difficulty involves the fact that neutrinos and antineutrinos escape and do not enter into reverse reactions once produced. This means, among other things, that energy must be supplied to maintain equilibrium at a given density and temperature. Granted this energy supply the equilibrium will depend more on the properties of the heavy nuclei than on those of the free neutrons and protons. The electron-positron ratio will be given as calculated by FH (1964) on the basis $\gamma \rightleftarrows e^+ + e^-$. Then when electron capture and positron emission are balanced exactly by positron capture and electron emission, equilibrium in the beta-interactions will have been reached. We found above that it was not necessary to carry the calculations this far. However, it is of considerable interest to know the value of Θ and $\overline{Z}/\overline{N}$ for the ultimate equilibrium. Calculations to determine these values are being made by Clifford (1964).

IMPLICATIONS

Neutrinos and antineutrinos obey Fermi statistics and in an expanding universe they form a degenerate Fermi sea at very low energies. The effects of this universal Fermi degeneracy in various cosmologies has been discussed in detail by Weinberg (1962) and the reader is referred to this paper for the basic treatment of neutrino cosmology. Neutrino and antineutrino effects in astronomy and astrophysics have been most extensively discussed by Bahcall (1964 a,b,c,d,1965) and by Bahcall and Frautschi (1964a,b). Here we will content ourselves with a discussion of the extent of neutrino and antineutrino emission by various astrophysical sources. Our discussion reviews and extends that given by Weinberg (1962) in Part IV of his paper. Our results will be expressed as the ratio of energy emitted as neutrinos or antineutrinos to the rest mass energy equivalent of the emitting system and are given in Table 3.

Nucleogenesis may occur through the creation of neutrons and antineutrons or through the creation of protons and antiprotons. In the former case beta decay probably occurs before other interactions but not, of course, in the latter case. In the former case the antineutrino or neutrino energy emitted is approximately 0.5 MeV or $\sim 5 \times 10^{-4}$ of the rest mass of the emitting particles.

Nucleosynthesis involving matter begins with the conversion of four protons into helium with the emission of two positrons and two neutrinos. The total energy emitted is 7×10^{-3} of the original rest mass; of this, 2% is emitted as neutrino energy in the pp-chain and 6% in the CNO bi-cycle. In the main-sequence and red-giant evolution of a star a minimum of one-tenth and a maximum of two-thirds of the hydrogen is converted into helium. An average value can be estimated on the basis that observationally one-third of the hydrogen of the Galaxy has been converted into helium. Taking all

TABLE 3

NEUTRINO ENERGY EMISSION

			NEUTRINO ENERGY/REST MASS ENERGY
NUCLEOGENESIS	p, e^-		0
	$n \to p + e^- + \bar{\nu}$		$\sim 5 \times 10^{-4}$ *
	$p, \bar{p}, n, \bar{n} \to e^{\pm}, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \gamma$		< 0.25 †
	$p, e^-, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$		arbitrary
NUCLEOSYNTHESIS			
HYDROGEN BURNING	$4p \to \alpha + 2e^+ + 2\nu_e$		$\sim 10^{-4}$
	$4p + e^- \to \alpha + e^+ + 2\nu_e$		
HELIUM BURNING	$3\ He^4 \to C^{12}$, $4\ He^4 \to O^{16}$		0
CARBON BURNING OR OXYGEN BURNING			
+ α- and e-PROCESSES	C^{12} or $O^{16} \to Fe^{56} + 2\nu_e$		$\sim 10^{-4}$
+ PAIR ANNIHILATION	$e^+ + e^- \to \nu + \bar{\nu}$		
NEUTRON s- and r-PROCESSES	$(A > 60)$		$\sim 3 \times 10^{-10}$
QUASI-STELLAR OBJECTS (HYDROGEN BURNING)			$\sim 10^{-4}$
RADIO GALAXIES **	$p + p \to p, n, \pi^{\pm}, \pi^0$		$\sim 10^{-4}$ (?)
	$\pi \to e^{\pm}, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \gamma$		
GEN. REL. COLLAPSE $\to$ INVISIBLE MASS			
	$e^+ + e^- \to \nu + \bar{\nu}$	$M \sim 10^3\ M_\odot$	~ 0.03
		$M \sim 10^6\ M_\odot$	$\sim 10^{-6}$
		AVERAGE WITHOUT ROTATION	$\sim 5 \times 10^{-3}$
		AVERAGE WITH ROTATION	$\sim 2 \times 10^{-2}$

* Continuous creation of neutrons excluded by cosmic X-ray observations at present time.

† This value depends upon the fraction of nucleons and antinucleons which avoid annihilation. Continuous creation of nucleons and antinucleons with some annihilation is excluded by cosmic X-ray observations.

** Note that muon neutrinos and antineutrinos are also emitted in this case and that high energy resonant scattering is possible for the electron neutrinos and antineutrinos (see references by Bahcall and by Bahcall and Frautschi).

of these factors into account it will be seen that to order of magnitude $\sim 10^{-4}$ of the rest mass energy is converted into neutrinos. A similar value holds for antineutrinos in nucleosynthesis involving antimatter.

Helium burning either to carbon via $3\ He^4 \rightarrow C^{12}$ or to oxygen via the additional reaction $C^{12} + He^4 \rightarrow O^{16} + \gamma$ does not involve neutrino or antineutrino emission directly. Furthermore helium burning takes place at such a low temperature that the pair annihilation process, equation (1), is not operative even if the universal strength of the weak interactions applies.

Fowler and Hoyle (1964) show that helium burning results in carbon production in stars with $M < \sim 10\ M_\odot$ and in oxygen production in stars with $M > \sim 10\ M_\odot$. There is considerable uncertainty in the critical mass because the rate of the $C^{12}(\alpha,\gamma)O^{16}$ reaction has not yielded to experimental measurements to date and only approximate theoretical estimates can be made. Carbon burning results in the production of nuclei near Mg^{24} with little direct neutrino-antineutrino emission and takes place at too low a temperature for pair annihilation to be effective. Oxygen burning results in the production of nuclei near Si^{28} with little direct neutrino-antineutrino emission but does take place at high enough temperature ($> 2 \times 10^9$ degrees) for pair annihilation to be effective and to dissipate practically all of the nuclear energy if the universal strength of the weak interactions applies to reaction (1).

The nuclear energy release is 5×10^{17} ergs gm^{-1} so the neutrino-antineutrino emission is $\sim 5 \times 10^{-4}$ of the rest mass. Subsequent to carbon or oxygen burning, the α- and e-processes occur in which heavier nuclei up to the iron group are produced. Pair annihilation is operative and again practically all of the available nuclear energy is radiated in the form of neutrinos and antineutrinos. The nuclear energy release is 3×10^{17} ergs gm^{-1} so that the neutrino-antineutrino energy emission is $\sim 3 \times 10^{-4}$ of the rest mass. Thus in stars with $M < \sim 10\ M_\odot$ nucleosynthesis beyond hydrogen burning releases $\sim 3 \times 10^{-4}$ of the rest mass energy in neutrinos and antineutrinos while for $M > \sim 10\ M_\odot$ the total is $\sim 8 \times 10^{-4}$. However, in the smaller mass range a larger fraction, say one-third, of the mass is completely evolved to the nuclear end-point than in the larger mass range where a fraction more like one-eighth can be conveniently taken as a good average. Thus in both cases the overall result is that $\sim 10^{-4}$ of the total rest mass is emitted as neutrino-antineutrino energy. If we include hydrogen burning then in the full run of nuclear evolution in matter, $\sim 1.5 \times 10^{-4}$ of the rest mass energy appears as neutrinos, 0.5×10^{-4} as antineutrinos and 2×10^{-4} in toto. For nuclear evolution in antimatter the roles of neutrinos and antineutrinos are reversed. The arguments given in this paragraph are those used to justify the private communication referred to by Weinberg (1962) in footnote 9, page 1461.

We have so far neglected the neutron capture s- and r-processes by which the heavier elements beyond $A \sim 60$ are produced. On the basis of solar system element abundances only $\sim 10^{-6}$ of the original mass has been through the s-process. For the heavy nuclei averaged over abundances the mean atomic weight is $A \sim 75$. About twenty neutrons have been added to the typical seed nucleus but only about seven or eight of these have been converted into protons with the emission of an antineutrino so that one additional antineutrino per 10 nucleons is emitted in the s-process. The antineutrino energy is of the order of 2 MeV compared to the nucleon rest mass equivalent $\sim 10^3$ MeV so the overall factor for antineutrino emission in the s-process is $\sim 2 \times 10^{-10}$. For the r-process, the abundance is $\sim 2 \times 10^{-7}$ according to recent estimates but the antineutrino energy is of the order of 6 MeV so that the overall factor is $\sim 1 \times 10^{-10}$. It is thus clear that these processes can indeed be neglected in comparison to nucleosynthesis up to the iron group elements.

The above discussion follows conventional ideas concerning nuclear evolution in stars and galaxies. We now turn to new and somewhat more speculative ideas which have been mainly put forward in attempts to understand the strong radio sources discovered in radio astronomy and the ultra-luminous quasi-stellar objects which exhibit large optical red shifts.

According to Hoyle and Fowler (1963a, 1963b) and to Fowler (1964,1965,1966) the energy generation in the quasi-stellar objects can be understood in terms of hydrogen burning so that the neutrino emission factor is $\sim 10^{-4}$. We see no problem in meeting the energy requirements of the quasi-stellar objects, which we identify as massive stars with $M \sim 10^{8\pm1} M_\odot$, by means of the nuclear resources of the cores of these stars. There is a problem in the mechanism by which they are stabilized for periods as high as $\sim 10^5$ years. If spherically symmetric and non-rotating, the hydrostatic equilibrium of these stars requires large inputs of energy (Fowler, 1964 and Iben, 1963) and is unstable to general relativistic collapse (Fowler, 1964, Chandrasekhar, 1964a,b,c and Gratton, 1964). However, rotation (Fowler, 1966) and magnetoturbulence (Bardeen and Anand, 1966) are two of numerous mechanisms which have been suggested as stabilizing agents.

The energy requirements for the strong radio emission from radio galaxies are so great according to some methods of calculation that Hoyle and Fowler (1963a, 1963b) pointed out that nuclear reactions in a galactic nucleus limited to less than 1% of the total mass would not suffice as the energy source. The release of gravitational energy during gravitational collapse was suggested as an alternative but serious difficulties arise in the energy emission cut-off due to the large red shifts which develop during collapse to the Schwarzschild limit. The problem is under extensive study in many places and until such a time as a solution is reached we are compelled to proceed on the basis of direct analysis of the observations. If the high energy electrons which yield the radio synchrotron emission are produced in proton-proton collisions (Burbidge, 1962) then electron neutrinos and antineutrinos plus muon neutrinos and antineutrons are produced with roughly equal energies to that of the high energy electrons. Take 10^{44} ergs sec^{-1} as a representative rate of emission over a maximum time scale of 5×10^8 years (Minkowski, 1964) to obtain $\sim 10^{60}$ ergs for the energy emitted by the high energy electrons and assign a similar value to the high energy neutrinos of all types. On the grounds that the radio galaxies do not show internal evidence for spectacular violent events let us assume that at most 1% of the total mass, most probably in the galactic nucleus, was at one time involved in the energy production. Sandage (1964) has suggested that radio galaxies are probably quite massive of the order of 10^{12} $M_\odot$. Thus we take $\sim 0.01 \times 10^{12} \times 10^{54} \sim 10^{64}$ ergs as the energy equivalent of the mass involved. Again the value $\sim 10^{-4}$ is obtained for the neutrino energy relative to this energy but here there is considerably greater uncertainty to be attached to this value than even in our previous estimates. The value could be as high as 10^{-2}.

A still more speculative source of neutrino energy lies in thé process of general relativistic collapse, in which a non-rotating, supermassive star implodes at approximately the rate of free fall, releasing gravitational energy until the Schwarzschild limiting radius is reached. At this radius the star becomes invisible since all forms of radiation, including particles, are red shifted to zero energy in the coordinates of a distant, external observer. The star becomes "hidden mass" except in so far as it exerts a static gravitational field. Hoyle, Fowler, Burbidge, and Burbidge (1964) showed that during the collapse of a supermassive star the loss of energy in the form of neutrino-antineutrino pairs from equation (1) was substantial even though not enough to reduce the externally observable mass to zero at the Schwarzschild limit.

Bardeen (1965) has employed the analysis of Zel'dovich and Podurets (1964) to express the total energy radiated in the form of neutrinos from a collapsing supermassive star as

$$E_\nu = \int_0^{\rho_{max}} \frac{du_\nu/dt}{(24\pi G\rho)^{\frac{1}{2}}} \frac{(1 - x)}{(1 - x/3)^3} \frac{d\rho}{\rho^2} \tag{3}$$

In this expression, du_ν/dt is the rate of neutrino energy loss and is given approximately by

$$\frac{du_\nu}{dt} \approx 4.58 \times 10^{15}\, T_9^{\;9} \text{ erg cm}^{-3} \text{ sec}^{-1} \qquad (T_9 > 3) \tag{4}$$

The free fall characteristic time is $(24\pi G\rho)^{-\frac{1}{2}}$ and the function involving $x = (\rho/\rho_{max})^{1/6}$ is a general relativistic correction which approaches unity for small x. This factor was ignored by HFB[2] (1964). The integration limit ρ_{max} is the density at the stage of contraction when neutrinos emitted from the central region just reach the surface at the later time when the outer radius equals the Schwarzschild limit. It can be shown that

$$\rho_{max} = \frac{4}{9}\rho_{sch} = 0.82 \times 10^{16} \left(\frac{M}{M_\odot}\right)^2 \text{ gm cm}^{-3} \tag{5}$$

where ρ_{sch} is the limiting Schwarzschild density.

The integral, equation (3), may be evaluated once the appropriate ρ,T relation has been established by the method used by HFB[2] (1964). An approximate solution for supermassive stars is

$$\varepsilon_\nu = \frac{E_\nu}{Mc^2} \sim 10^3 \left(\frac{M_\odot}{M}\right)^{3/2} \qquad (M > 10^3\, M_\odot) \tag{6}$$

Thus for $M = 10^6\, M_\odot$, $\varepsilon_\nu \sim 10^{-6}$ while for $M = 10^3\, M_\odot$, $\varepsilon_\nu \sim 0.03$. For smaller stars degeneracy leads to lower values of ε_ν. If an average is taken over a reasonable stellar population the following figure is obtained

$$\langle \varepsilon_\nu \rangle \sim 5 \times 10^{-3} \qquad \text{(without rotation)} \tag{7}$$

The above calculations assume that the collapsing mass is not rotating. Rotation leads to a longer time scale for collapse and to the value

$$\langle \varepsilon_\nu \rangle \sim 2 \times 10^{-2} \qquad \text{(with rotation)} \tag{8}$$

The estimates given in equations (7) and (8) are admittedly quite speculative but it will be noted that these values are the order of 100 times the neutrino losses which occur during the evolution of stars with $M < 100\ M_\odot$ or in the production of energy in radio sources. Furthermore it must be emphasized that these results depend on the assumption that the weak interaction is universal and that the coupling constant measured in beta decay, muon decay and muon capture applies to the annihilation of electron-positron pairs with neutrino emission.

These speculations lead to a maximum value for ε_ν of the order of a few percent. We conclude that processes during stellar evolution never develop neutrino energies comparable to the rest mass-energies of the emitting systems. For this reason if the neutrino energy density in space is to be of cosmological significance, its origin must lie in cosmological processes.

Weinberg's calculations show that on the basis of current ideas the degenerate neutrino sea will be observable, if at all, only in the oscillating cosmologies. However, his results serve to illustrate the main thesis of this lecture. The mass density equivalent of the neutrino energy density is related to the Fermi energy, E_F, by the well-known expression

$$\begin{aligned} \rho_\nu &= \frac{1}{8\pi^2}\left(\frac{E_F}{\hbar c}\right)^3 \frac{E_F}{c^2} \\ &\sim 3 \times 10^{-21}\, E_F^4 \ \text{gm cm}^{-3} \end{aligned} \tag{9}$$

for E_F in eV. E_F can be measured by observations on the behavior near the end point of the energy spectrum in beta decay and present results set an upper limit, $E_F < 200$ eV. This corresponds to an upper limit, $\rho_\nu < 5 \times 10^{-12}$ gm cm^{-3}, which is $\sim 10^{18}$ times the nucleon rest mass density of 10^{-29} gm cm^{-3} required in steady state cosmology which in turn may be one or two orders of magnitude greater than that observed in luminous stars. Alternatively it will be noted that $E_F \sim 0.01$ eV if the "missing" mass-energy is indeed in the form of neutrinos. Clearly this low value is impossible to detect experimentally in terrestrial laboratories with present techniques.

However, it is possible to set observational limits on the possible neutrino degeneracy in the universe from a consideration of the shape of the energy spectrum of the cosmic rays between 10^{10} and 10^{20} eV. Cowsik, Pal and Tandon (1964) have pointed out that the low energy neutrinos in a Fermi sea appear to have quite large energies to a high energy cosmic ray proton in its rest mass system. Proton-neutrino interactions, elastic and inelastic, increase in probability quite rapidly with energy but eventually flatten off. This will have the result that the cosmic ray energy spectrum should steepen at high energies and then eventually regain the original slope. Such an effect has been observed at 10^{15} to 10^{16} eV. If this is attributed entirely to proton-neutrino interactions then Cowsik, Pal and Tandon show that the height of the Fermi sea is at most a few eV.

Experiment, observation and theory in neutrino astrophysics will continue to be fruitful and productive in the quest for an understanding of astronomical systems and the Universe.

LECTURE II

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LECTURE III*

SUPERMASSIVE STARS, QUASARS, AND EXTRAGALACTIC RADIO SOURCES, I

N66 24329

INTRODUCTION

In this and the following lecture our concern will be the revolution which has occurred in astronomy during the past two decades. Radio astronomers throughout the world – in Australia, England, the Netherlands, the Soviet Union, Italy and the United States – have been the real heroes of this revolution. They have not only detected radio waves from extragalactic sources but have succeeded in pin pointing the location of these sources on the celestial sphere.

In these lectures the first purpose will be to discuss the observational work which makes possible the identification of radio sources with optical objects observable through large telescopes. The second purpose will be to consider various suggestions which have been made concerning the source of the prodigious energies involved in the radio objects. Background references are Hoyle, Fowler, Burbidge and Burbidge (1964) and Fowler (1964; 1965a,b; 1966).

The fundamental problem is this – what physical phenomenon is the source of the energy? Ordinary stars shine on nuclear energy. Are the nuclear resources in supermassive stars sufficient to meet the observed energy requirements in radio objects or must we turn to other mechanisms – annihilation, multi-supernovae, stellar collisions, gravitational collapse, or new and unknown phenomena – to explain radio "stars" and "galaxies". In case nuclear reactions in supermassive stars are effective, then we must ask whether these stars are stable or unstable during nuclear burning. After the exhaustion of nuclear

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fuel, what happens? In addition, the mechanisms of transfer of energy from the raw form in which it is produced to the exotic forms exhibited in the radio sources must be studied.

THE OPTICAL IDENTIFICATION OF RADIO SOURCES

The development in many places throughout the world of radio telescopes capable of determining positions to better than ten seconds in angle have led to a significant breakthrough in the observation and identification of radio sources. As one example, radio astronomers at the California Institute of Technology have constructed at the Owens Valley Radio Observatory in Bishop, California, an interferometer consisting of two 90-foot dishes which can be separated by distances up to 1000 meters, yielding a limiting angular resolution near 10^{-4}.

The precise determination of the position of a radio object makes possible an accurate comparison with the position of optical objects visible through large conventional telescopes which have very high angular resolution because of the short wave length of visible light. The ultimate objective is to make an "identification" of the radio source with an optical object. Radio astronomers at the California Institute of Technology have the unique advantage of being able to cooperate with staff members of the Mount Wilson and Palomar Observatories in using the 200-inch Hale Telescope on Mount Palomar for making position comparisons and identifications.

Early identifications, made before great precision had been reached in the radio observations, indicated that in some cases radio sources seemed to be associated with pairs of galaxies in close proximity and perhaps even in collision. This led naturally to the assumption that the energy freed in such a collision might be the source of the radio energy. It is now believed that collision energy is inadequate in this regard but more importantly the great majority of the more precise identifications for radio sources outside of our Galaxy, the Milky Way, are with single, isolated galaxies and not with pairs of galaxies.

Until recently there has been no way to determine directly the distance to the radio objects of interest although red shift measurements have been made on the 21-cm atomic hydrogen line from nearby objects and are being rapidly extended to more distant objects. On the other hand the distance to the galaxy can be calculated if optical red shift measurements have been made and if the red shift is assumed to be proportional to distance in accordance with Hubble's Law.

THE ENERGY REQUIREMENTS OF THE RADIO SOURCES

Identification with an optically red shifted galaxy thus makes it possible to determine the absolute luminosity of radio sources from the measured apparent luminosity, that is, the radio flux at the earth in erg cm^{-2} sec^{-1} can be translated into the total rate of energy emission in erg sec^{-1} at the source with the additional assumption of isotropic emission. (The possibility that the radio waves are directed at the earth is rightly given scant attention.)

The results are staggering. More than 50 radio sources have been listed by Matthews, Morgan and Schmidt (1964) with luminosities exceeding 10^{38} erg sec^{-1} and ranging up to 2×10^{45} erg sec^{-1}, the value for 3C 295 (the 295th object in the third Cambridge University catalogue of radio sources). The optical luminosity of the sun is 4×10^{33} erg sec^{-1} and that from the Galaxy is approximately 10^{44} erg sec^{-1}. Thus 3C 295 has a radio luminosity almost 10^{12} times that of the optical emission from the sun and more than ten times that from the Galaxy.

The total amounts of energy required to sustain these luminosities can be calculated in several ways. It is reasonable to assume that the minimum age to be assigned to the sources is that given by dividing the observed dimensions by the velocity of light. Actually the linear growth of the sources might well have taken place at considerably smaller velocities. Even so the ages fall in the range 10^5 to 10^6 years or 10^{13} seconds in order of magnitude and thus the cumulative emissions are at least as high as 2×10^{58} ergs.

Another method of determining the total energy involved in the radio sources is based on the assumption that the radio emission is synchrotron radiation from high energy electrons spiraling in a magnetic field extending throughout the object. This process is thought to be the most efficient for the generation of radio waves and accounts qualitatively at least for the polarization observed in many of the sources. The synchrotron theory implies that energy is <u>stored</u> in the radio objects in the form of magnetic field energy and relativistic electron energy. The magnetic field energy is proportional to the mean square of the field intensity (B) and to the volume. The total energy of the electrons is proportional to the rate at which they emit energy, the radio luminosity, divided by the three halves power of the field intensity and the square root of the characteristic radio frequency emitted. Thus for a given observed volume, luminosity and radio emission spectrum the total energy is equal to a term proportional to B^2 plus one proportional to $B^{-3/2}$. The field intensity is, of course, unknown but even so the total energy exhibits a minimum as a function of B and this minimum can be readily determined. The values for the minimum stored energy even exceed those of the minimum cumulative energies. In the case of the Hercules A source the minimum stored energy, if the only high energy particles are electrons, is approximately 10^{60} erg. The theory does not explicitly indicate the method by which the electrons are accelerated to high energy but it is reasonable to assume, as is the case in the cosmic radiation, that the nuclear component (mostly protons) of the neutral medium or plasma must have considerably greater total energy content than do the electrons. Upon taking this factor into account the stored energy in Hercules A, for example, is almost 10^{61} erg. Because of their greater mass the protons do not take part in the synchrotron emission.

In coming to a realistic estimate of the energy requirements in radio objects there remains the knotty problem concerning the efficiency with which the energy generated has been converted into relativistic particles and magnetic fields. Acceleration mechanisms employed in terrestrial laboratories are notoriously inefficient but this may well be due to the very small scale, astrophysically speaking, within which such mechanisms must operate. However, it is estimated that even in solar flares not more than a few per cent of the energy released is in the form of relativistic particles, the main energy release occurring in mass motions and electromagnetic radiation.

On the above basis, the figure 2×10^{62} ergs is frequently quoted as a representative value of the energy requirement in the larger radio sources and for the purposes of argument this figure will be accepted as the maximum value in what follows. Suggestions have been made which modify the simple synchrotron model in such a way as to reduce the energy requirements. The magnetic field can be imagined to have a "clumpy" structure such that the effective emitting volume, where the field is highest, is much smaller than the overall volume observed. The magnetic field energy is proportional to the emitting volume not the overall volume. The emission may come from groups of electrons radiating coherently and thus much more efficiently. Detailed studies of modifications along these lines will be necessary before the energy problem can be considered to be solved.

It will be noted that there is considerable disparity in the two estimates which it is possible to make for the energy requirements in the extended radio sources. On the one hand the cumulative emissions range up to 2×10^{58} ergs while the stored energies on the synchrotron model have been estimated to be as high as 2×10^{62} ergs.

SUPERMASSIVE STARS*

The immensity of 2×10^{62} ergs, can best be appreciated by a comparison with the equivalent rest mass energy of a single star, for example, the sun. The mass of the sun is 2×10^{33} grams and the square of the velocity of light is $(3 \times 10^{10})^2 \sim 10^{21}$ ergs per gram. Thus Einstein's relation between energy and mass

$$E = Mc^2 \tag{1}$$

becomes numerically

$$E \approx 2 \times 10^{54} \; (M/M_\odot) \text{ erg} \tag{2}$$

where $M/M_\odot$ is the stellar mass expressed in units of the solar mass. We see

*The designation supermassive applies throughout these lectures to stars with mass $M > 10^3 \; M_\odot$. The prefix super will frequently be omitted but the stars under discussion in this paper are not to be confused with stars with M between 30 $M_\odot$ and 100 $M_\odot$ which are frequently called massive stars.

then that the energy stored on the synchrotron theory in particles and magnetic fields in the invisible radio objects requires the original production of energy of the order of that obtained by the complete annihilation of the mass of one-hundred million suns, $10^8 M_\odot$. The problem can be taken in a quite literal sense on the grounds that the conversion of mass is the fundamental mechanism for the production of energy. On this basis the problem reduces to how, when and where did the conversion take place.

Before proceeding it is advisable to write Einstein's relation in a form more directly applicable to the problem under consideration as follows

$$\begin{aligned} \Delta E &= (M_o - M)c^2 \\ &= 2 \times 10^{54} (M_o - M)/M_\odot \text{ erg} \end{aligned} \tag{3}$$

where ΔE is the energy made available from a system of particles with total rest mass M_o when by some mechanism the mass, measured through gravitational or inertial effects by an external observer, has been reduced to M. The quantity ΔE is the energy store available for transformation at varying efficiencies into the various observable forms -- gamma ray, x-ray, optical, radio, neutrino and high energy particle emission.

In principle it is possible for M to decrease to zero but not to negative values and so the maximum available energy is indeed $M_o c^2$. One mechanism by which this can occur is through the annihilation of equal amounts of matter and antimatter. This mechanism has been discussed by E. Teller (1965).

The main problem has to do with the assembly of matter and antimatter in sufficient quantities on a time scale no greater than that associated with the assumed explosive origin of these objects. Although annihilation will not be discussed further in this paper, it may prove to be the ultimate solution to the problem.

The success of the idea of nuclear energy generation in stars led quite naturally to the extension of this idea to the radio sources. Hoyle and Fowler (1963a) investigated the possibility that a mass of the order of 10^8 $M_\odot$ has condensed into a single star in which the energy generation takes place. On this point of view, using the standard theory of stellar structure in Newtonian hydrostatic equilibrium, one immediately obtains optical luminosities of the order of 10^{46} erg sec^{-1} and lifetimes for nuclear energy generation of the order of 10^6 to 10^7 years so that the overall energy release is approximately 10^{60} ergs. (See Lecture IV for additional details.)

There is, of course, a basic limitation inherent in thermonuclear energy generation. The conversion of hydrogen into helium involves the transformation of only 0.7 per cent of the rest mass into energy and further nuclear burning leading to the most tightly bound nuclear species near iron brings this figure only to slightly less than one per cent. Thus M_o - M in equation (3) is at most equal to 0.01 M_o and the complete nuclear conversion of 10^8 solar masses of hydrogen into iron group elements leads to the release of 2×10^{60} ergs. In general

$$\Delta E_{nucl} < 2 \times 10^{52} M_o/M_\odot \text{ erg} \tag{4}$$

Equation (4) is expressed in terms of an upper limit for the following reason. In the observed stars with masses ranging approximately from 1 to 100 $M_\odot$ the conversion never seems to reach completion before steady mass loss or supernova explosion terminates the life of the star. Thus it is clear that the nuclear generation of 2×10^{62} ergs, the maximum value discussed above, involves at least 10^{10} $M_\odot$. This figure corresponds to the entire mass of a medium size galaxy! On the other hand, the nuclear generation of 2×10^{58} ergs, the minimum value discussed above, involves the order of $10^6\ M_\odot$. This figure corresponds to the mass of the larger globular clusters of stars in the halo of the Galaxy and in other galaxies. If globular clusters are involved in the energy production, it need not necessarily take place at the center of the galaxy.

In the massive galaxies associated with the strong radio sources there seemed to be no observational evidence for the abnormal heavy element concentration which would presumably follow from the nuclear conversion of $10^{10}\ M_\odot$. If the larger energy requirements are accepted, it can be argued that nuclear energy might prove inadequate and so Hoyle and Fowler (1963b) turned to another possibility, gravitational energy. On classical Newtonian theory the gravitational binding energy of a system of rest mass M_o with maximum radius R is approximately given by

$$\Omega = \frac{3}{5-n}\frac{GM_o^2}{R} \approx \frac{2GM_o^2}{R} \tag{5}$$

where n is the polytropic index which has been arbitrarily chosen equal to 3.5 in the final approximation given. If no energy is stored in the system which remains as "cold" gas or "dust" then Ω becomes ΔE, the energy freed by the system on condensing from the dispersed state in which the gravitational interaction can be neglected. If equation (5) is written

$$\Delta E_{grav} = \Omega \approx \left(\frac{2GM_o}{Rc^2}\right) M_o c^2 \tag{6}$$

it will be seen that the dimensionless quantity $2GM_o/Rc^2$ is just the fraction of the rest mass energy made available. Classical Newtonian theory places no limitation on $2GM_o/Rc^2$ but the theory of general relativity limits it to unity at the Schwarzschild limit. Thus

$$\begin{aligned}\Delta E_{grav} &\leq M_o c^2 \\ &\leq 2 \times 10^{54}\, M_o/M_\odot \text{ erg}\end{aligned} \tag{7}$$

in agreement with the statement made previously that M could not become negative.

In what way can use be made of the release of gravitational energy? We assume that in some way this energy is removed from the collapsing core and is either absorbed in the outer envelope or is completely lost by the star. In either case the hydrostatic balance in the envelope is destroyed and the envelope material is ejected with high velocity. The energy loss from the core may occur through photon or neutrino emission. Another possibility exists if the massive star is in rotation. After the exhaustion of nuclear energy the star will contract with the contraction of the core being much more rapid than that of the envelope. It is reasonable to suppose that the angular momentum of the core will be conserved once it has contracted away from the envelope and that eventually the core will become unstable to fission into two bodies rotating about each other as in a binary star. Such a system loses rotational energy by radiating gravitational waves.

All emission mechanisms suffer from the limiting effect of the gravitational red shift. In order for gravitational energy to be released from the core it is necessary that the core contract or that $(GM_o/Rc^2)_{core}$ increase. But the red shift in radiation is just proportional to this dimensionless quantity in first order. Radiation arrives at a distant observer with less energy than that calculated by a local observer where the radiation is emitted. This is true for all forms of energy transfer, by particles as well as radiation.

Thus the rate of any form of energy loss by the core is greatly reduced as $(GM_{\odot}/Rc^2)_{core}$ increases and, as a result, the energy loss is not complete as implied in equation (6) where it was assumed that no internal energy of motion or radiation remained in the star during contraction. As a matter of fact even the most optimistic calculations have not revealed mechanisms whereby a contracting massive star can transfer more than a few per cent of the gravitational energy of its core to the outer envelope. The gravitational release of energy may be somewhat more efficient than nuclear release but not by a large factor. Thus the release of 2×10^{62} ergs must, on just about any grounds, involve a mass of the order of 10^{10} $M_{\odot}$. The large elliptical galaxies associated with radio sources have total masses estimated at 10^{12} $M_{\odot}$. Thus if 2×10^{62} ergs is indeed the correct value for the energy requirement in radio galaxies, then of the order of one per cent of the mass of the galaxy has been involved in the generation of this energy.

QUASARS

It has been noted previously that Hoyle and Fowler (1963a) had obtained optical luminosities of the order of 10^{46} erg sec^{-1} for a massive star of $10^8 M_{\odot}$ in hydrostatic equilibrium and in fact it was found that the luminosity is just proportional to the mass for $M > 10^3$ $M_{\odot}$. These large optical luminosities did not seem to have any immediate connection with the extended radio sources since the problem concerning the transformation of the optically emitted energy into high energy electrons and magnetic fields remained unsolved.

However, at the same time that these calculations were being made, an observational discovery of great significance was made in Pasadena by Schmidt (1963) and was quickly confirmed by Oke (1963) and by Greenstein and Matthews (1963). It had been known for some time that certain of the radio sources were located in coincidence with star-like objects which apparently had diameters too small to be resolved by optical telescopes and which showed on photographic plates as diffraction images characteristic of the telescope. These objects were called "radio stars."

The Pasadena group pioneered in the use of the 200-inch Hale Telescope on Mount Palomar to investigate the spectroscopy of these "radio stars." For several years their investigations of four of these objects led nowhere; they were unable to understand the peculiar emission lines of the spectra which the telescope revealed. There the matter rested until Schmidt began studying the spectrum of a fifth object catalogued by Cambridge University radio astronomers as 3C 273. This time the Gordian knot was cut. Several of the emission lines from 3C 273 formed a simple harmonic pattern, with separation and intensity decreasing toward the ultra violet. The lines obviously belonged to a series of the type expected from hydrogen or any other atom that had been stripped of all electrons but one. Schmidt soon concluded that no atom gave the observed wave lengths. If he assumed, however, that the spectrum lines had been shifted toward the red by 16 per cent, the observed wave length agreed with those of hydrogen. Shortly thereafter Oke found the Hα-line in exactly the position predicted by the red-shift hypothesis and Greenstein and Matthews found an even greater red shift of 37 per cent in 3C 48 when they properly identified the lines observed as corresponding to well-known lines from the elements oxygen, neon and magnesium.

Greenstein and Schmidt (1964) soon showed that the red shifts could not be gravitational red shifts associated with large masses confined to regions of very small radius. The masses involved are found to be quite large but the radii of the emitting regions are so great that the gravitational red shift is negligible. They suggested that their "quasi-stellar" objects or "quasars" are extragalactic and that the red shifts arise from the general cosmological expansion of the universe. With this interpretation they were then able to determine the luminosity distance for the objects and to convert the observed apparent luminosities into absolute luminosities. The calculations indicated that the quasars have optical luminosities of the order of 10^{46} ergs sec^{-1} or more than one-hundred times the optical luminosity of our Galaxy. The quasars may or may not be located in galaxies, but if they are, they outshine the surrounding galaxy so that it is lost in the diffraction pattern of the quasar image.

The optical luminosities of the quasars are very high, $\sim 10^{46}$ ergs sec^{-1}, but there is no convincing evidence that these objects have lifetimes in excess of 10^5 to 10^6 years. Thus the cumulative optical emission is the order of 10^{59} ergs which is well within the nuclear resources of a star with $M = 10^8 M_\odot$. Only seven per cent of the hydrogen of such a massive star need be converted into helium to release this amount of energy. Because of the small volume the stored energies required are small.

It is now well established that the quasars exhibit variability in optical luminosity (Smith and Hoffleit 1963; Matthews and Sandage 1963; Sandage 1964; Sharov and Efremov 1963; Geyer 1964). In addition to luminous flashes with durations of the order of days or weeks, there is some evidence for cyclic variations with periods of the order of ten years. It is generally agreed that the occurrence of the cyclic variations is crucial to the question whether the primary radiation object is a single coherent massive star (10^4-10^8 $M_\odot$) as originally proposed by Hoyle and Fowler (1963a, 1963b) or a system of smaller stars (1-10^2 $M_\odot$) as discussed by numerous authors (Burbidge 1961; Hoyle and Fowler 1965; Gold, Axford and Ray 1965; Woltjer 1964; Ulam and Walden 1964; Field 1964). It is difficult on the basis of collisions or supernova outbursts in a system of many stellar objects to explain variations which exhibit a fairly regular periodicity. Thus, without prejudice to the problem of the reality of the cyclic variations since only additional and more precise observations will settle this matter, the possibility is investigated in what follows that such variations can arise from non-linear relaxation oscillations in a single massive star. The star is taken to have no rotation and to be spherically symmetric with all physical parameters depending only on the radial variable. Rotation or other mechanisms which destroy the spherical symmetry change the behavior of the star markedly and will be mentioned briefly at the end of the lecture and will be discussed in detail in Lecture IV.

It has also been suggested that the quasars are local. Terrell (1964, 1965) has proposed the hypothesis that the quasars were ejected at relativistic velocities in an explosive event at the center of the Galaxy some 10^7 to 10^8 years ago. Hoyle and Burbidge (1965, 1966) have suggested that a likely candidate to give rise to the quasars in our vicinity is NGC 5128 which is a powerful radio source in which at least two outbursts appeared to have occurred. In this case some objects with blue shifts may be expected.

On the local hypothesis the characteristic distances for the quasars are 1 to 10 megaparsecs rather than 10^3 to 10^4 megaparsecs as on the cosmological hypothesis. Terrell (1965) suggests that the quasar masses are $\sim 10^4\ M_\odot$ rather than the value 10^8 to $10^{10}\ M_\odot$ required if the observed red shifts are cosmological. <u>Thus on either the local or the cosmological hypothesis, supermassive stars, as defined in this lecture, are required</u>. Furthermore the original local outburst involved masses of the same order of magnitude as those attributed to the quasars themselves on the cosmological hypothesis. Consequently, in the remainder of the lecture and in Lecture IV reference to the quasars will be made on the basis that they are cosmological objects but this is not a necessary condition for the arguments put forward.

RELAXATION OSCILLATIONS IN NON-ROTATING SUPERMASSIVE STARS

The rapid generation of nuclear energy during the general relativistic collapse of a massive star is considered to be the triggering agent for the relaxation oscillations. From the standpoint of the model discussed in Hoyle and Fowler (1965) it is necessary to assume that the early fragmentation in the original gas cloud resulted in the formation of stars small enough ($< 10\ M_\odot$) that significant nuclear evolution (consumption of hydrogen) did not occur in the time scale ($\sim 3 \times 10^6$ years) in which stellar collisions reduced the system of stars once again to a single gaseous object. Thus the starting point is a massive star, say $M \sim 10^6\ M_\odot$, with a characteristic dimension of 10^{17} cm, central temperature of the order of 10^5 °K, with pressure support due almost entirely to radiation and a structure closely approximating that of a polytrope of index $n = 3$ (Fowler and Hoyle, 1963, 1964). The composition is the same as that of the original gas cloud, for example, $X = 0.75$, $Y = 0.22$ and $Z = 0.03$. After some exhaustion of hydrogen a representative composition for purposes of computation will be taken to be $X = 0.50$, $Y = 0.47$, $Z = 0.03$ with Z primarily made up of CNO-nuclei.

General relativistic considerations lead to dynamic instability in non-rotating massive stars when the radius falls below a certain critical value (Iben 1963; Fowler 1964; Chandrasekhar 1964a,b, 1965; McVittie 1964; Gratton 1964; Zel'dovich 1964). In what follows we make use of the post-Newtonian approximation in the notation of Fowler (1964). The significant results are illustrated in Figure 1 where the energy content of the star exclusive of the rest mass energy of the constituent particles is presented as a function of the radius and central temperature. The heavy solid curve represents the equilibrium binding energy in solar rest-mass energy equivalent units given by the post-Newtonian approximation. The decrease to a minimum at a certain outer radius and central temperature followed by a rise into the unbound region is in marked contrast to the linear decrease exhibited by the Newtonian term, $-3\beta GM/4Rc^2$. To the left of the minimum in the "classical" range an adiabatic perturbation toward smaller radii leads to more energy than that required for equilibrium and thus to more pressure than that necessary for hydrostatic equilibrium. Thus the contraction is opposed as will clearly also be the case for a perturbing expansion and thus the system is inherently stable. The same argument used to the right of the minimum (Fowler 1964) indicates that a contraction leads to less pressure than that needed for hydrostatic equilibrium while an expansion leads to more so that the system is dynamically unstable to <u>adiabatic</u> perturbations.

The equilibrium energy in the post-Newtonian approximation can be written for n = 3 as

$$\frac{E_{eq}}{Mc^2} = -\frac{3}{8}\beta\left(\frac{2GM}{Rc^2}\right) + \frac{3}{16}\left(\frac{3}{\pi}\right)^{\frac{1}{2}} R_3\left(\frac{2GM}{Rc^2}\right)^2 + \cdots \qquad (8)$$

where $R_3 = 6.897$ is one of the constants of integration for the polytropic equation for $n = 3$. From Eddington's quartic equation the ratio of gas pressure to total pressure is given for small values by Fowler and Hoyle (1964) as

$$\beta \approx 6\left(\Gamma_1 - 4/3\right) \approx \frac{4.28}{\mu}\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} << 1 \tag{9}$$

where μ is the mean molecular weight and $\Gamma_1 = d\,\ln p/d\,\ln \rho$ with p, the pressure and ρ, the density. For $M = 10^4$ to $10^8\ M_\odot$, $\beta \sim 10^{-1}$ to 10^{-3}. In massive stars the main pressure support is that due to radiation, β is small, the Newtonian term in (8) is small and the post-Newtonian term becomes significant for small values of $R_g/R = 2GM/Rc^2$ of the order of β. The limiting gravitational radius or Schwarzschild radius is designated by $R_g = 2GM/c^2 = 3 \times 10^5\ (M/M_\odot)$ cm.

The critical radius can be determined by setting the derivative of (8) equal to zero in which case

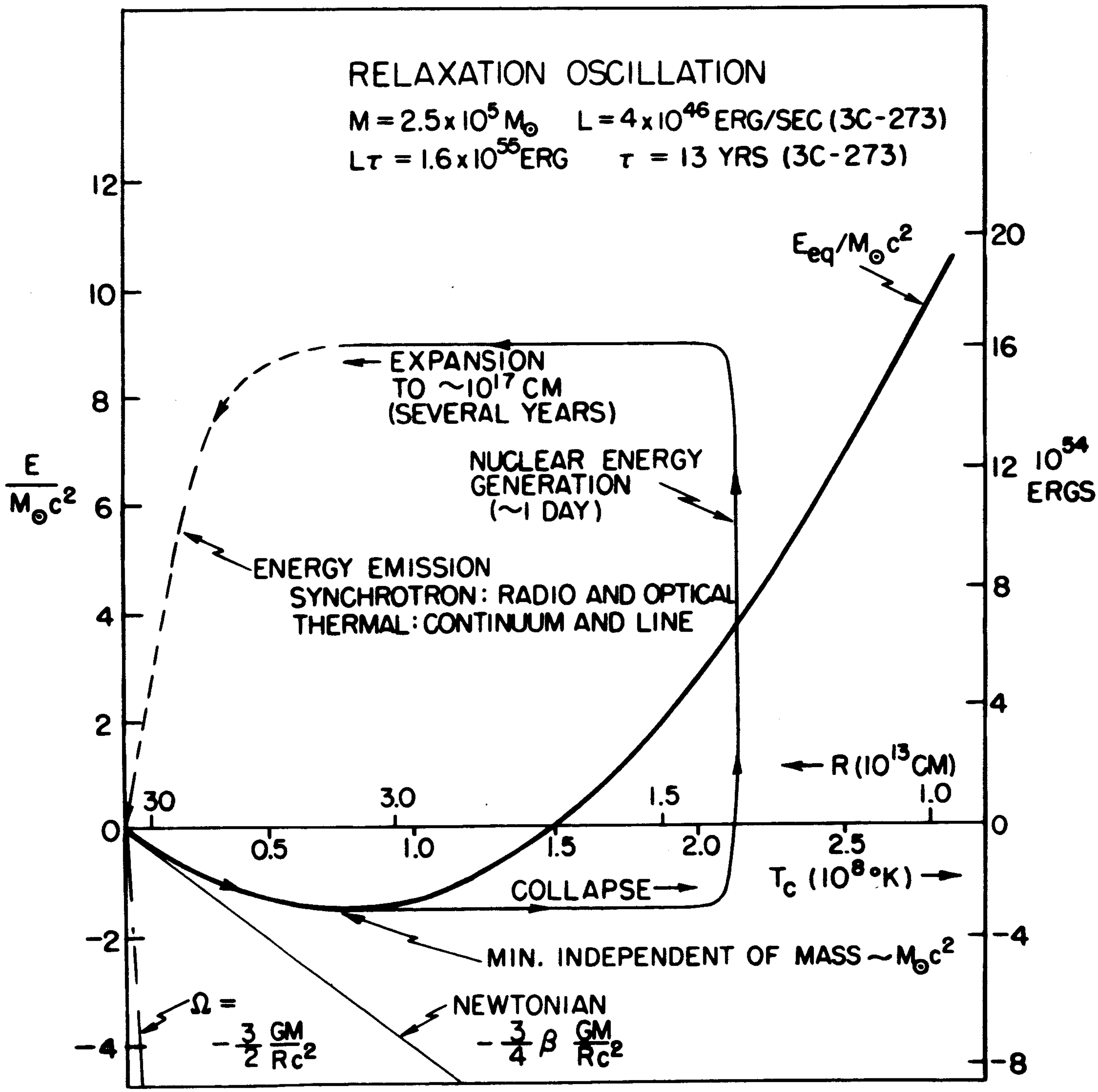
$$\frac{R_c}{R_g} = \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \frac{R_3}{\beta} = \frac{6.74}{\beta} \sim \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \sim 10^2 \text{ to } 10^4 \tag{10}$$

Numerically it is found that

$$R_{cr} = 3.4 \times 10^5\ (M/M_\odot)^{3/2} \text{ cm} \sim 10 \text{ to } 10^7 \text{ light seconds.} \tag{11}$$

At this point and in what follows it will be assumed that the gross internal structure of the star is insensitive in first order to static or dynamic changes in E/Mc^2 of the order of β. Thus the equations relating central temperature and density with the radius for a polytrope with $n = 3$ can be freely employed. In fact the second order term in equation (8) is correctly derived by using first order terms where required. The critical central

Fig. 1. The internal energy of a non-rotating massive star (2.5×10^5 $M_\odot$) in excess of the rest mass energy is shown as a function of radius and central temperature. The heavy curve shows the energy required for hydrostatic equilibrium when general relativistic considerations are taken into account. This curve deviates quadratically from the linear Newtonian term and reaches a minimum with absolute value the order of $M_\odot c^2$ at $R \sim 4 \times 10^{13}$ cm, $T_c \sim 0.7 \times 10^8$ ^oK. This minimum is reached before nuclear energy generation begins in the interior. A general relativistic collapse occurs which is stopped and reversed by hydrogen burning through the CNO bi-cycle in a time of approximately one day near $R \sim 10^{13}$ cm, $T_c \sim 2 \times 10^8$ ^oK. A radial shock wave is initiated and the resulting expansion extends to a radius of approximately 10^{17} cm in a time scale of the order of several years. Damping of the expansion occurs through radio and optical synchrotron emission and by non-equilibrium continuum and line emission. The overall process can best be described as a relaxation oscillation. The case illustrated employs the luminosity and period observed for the quasar in 3C-273B which is assumed to be at the cosmological distance corresponding to its red shift.



RELAXATION OSCILLATION
$M = 2.5 \times 10^5 M_\odot$ $L = 4 \times 10^{46}$ ERG/SEC (3C-273)
$L\tau = 1.6 \times 10^{55}$ ERG τ = 13 YRS (3C-273)
$E_{eq}/M_\odot c^2$
EXPANSION TO $\sim 10^{17}$ CM (SEVERAL YEARS)
NUCLEAR ENERGY GENERATION ($\sim$1 DAY)
ENERGY EMISSION
SYNCHROTRON: RADIO AND OPTICAL
THERMAL: CONTINUUM AND LINE
$\frac{E}{M_\odot c^2}$
10^{54} ERGS
R (10^{13} CM)
30
3.0
1.5
1.0
0.5
2.0
2.5
T_c (10^8 °K)
COLLAPSE
MIN. INDEPENDENT OF MASS $\sim M_\odot c^2$
$\Omega = -\frac{3}{2}\frac{GM}{Rc^2}$
NEWTONIAN $-\frac{3}{4}\beta\frac{GM}{Rc^2}$
12
10
8
6
4
2
0
−2
−4
20
16
12
8
4
0
−4
−8

temperature and density are thus

$$T_c = \frac{\Re c^2}{2\,\mu a^{\frac{1}{2}}\,G^{3/2}\,M} = \frac{1.25 \times 10^{13}}{\mu}\left(\frac{M_\odot}{M}\right)\ {}^{\circ}K$$

$$\sim 10^9 \text{ to } 10^5\ {}^{\circ}K \quad \text{for} \quad M \sim 10^4 \text{ to } 10^8\ M_\odot \tag{12}$$

$$\rho_c = \frac{2.54 \times 10^{17}}{\mu^3}\left(\frac{M_\odot}{M}\right)^{7/2} \text{gm cm}^{-3}$$

$$\sim 10^4 \text{ to } 10^{-10} \text{ gm cm}^{-3} \text{ for } M \sim 10^4 \text{ to } 10^8\ M_\odot \tag{13}$$

Moreover it is possible to show (Fowler 1964a) that equation (8) can be expanded in increasing powers of T with only the linear and quadratic terms retained in the post-Newtonian approximation.

The minimum equilibrium energy at R_c, T_c, ρ_c turns out to be independent of the stellar mass and is given by

$$E_{eq}^{min} = -\frac{3\,M_3\,\Re^2\,c^2}{4\,\mu^2\,R_3\,a^{\frac{1}{2}}\,G^{3/2}} \tag{14}$$

where $M_3 = 2.018$ is the second constant of integration for the polytropic equation for n = 3. Equation (14) can be rewritten by introducing $\Re^4/a = (15/\pi^2)(\hbar^3 c^3/M_u^{\ 4})$ and after evaluation of numerical factors becomes

$$E_{eq}^{min} = -\frac{0.27}{\mu^2}\left(\frac{\hbar c}{GM_u^{\ 2}}\right)^{3/2} M_u c^2 \sim -10^{57} \times 10^{-24}\ c^2 \tag{15}$$

where M_u is the atomic mass unit. It is well known that the dimensionless gravitational interaction constant $GM_u^{\ 2}/\hbar c$ is very small, of order 10^{-38}. The corresponding fine structure constant in electromagnetism is $e^2/\hbar c = 1/137$.

As indicated in equation (15) this leads to

$$E_{eq}^{min} \sim - M_{\odot}c^2 \sim - 2 \times 10^{54} \text{ ergs} \qquad (16)$$

which indicates that the maximum binding energy of a non-rotating massive star is the order of one solar mass-energy equivalent. More precisely

$$E_{eq}^{min} = - \frac{0.51}{\mu^2} M_{\odot}c^2 = - \frac{0.91 \times 10^{54}}{\mu^2} \text{ erg} \qquad (17)$$

In a first appraisal of the problem it is interesting to consider equation (16) in relation to the luminosity of the quasars which is of order $L \sim 10^{46}$ erg $\sec^{-1}$ (Greenstein and Schmidt 1964, and Oke 1965). The Helmholtz-Kelvin contraction time, E_{eq}^{min}/L, with this luminosity is thus of the order of several years. This will also be the cycle time if energy of the order of the binding energy is supplied by nuclear burning during an oscillation or pulsation. It is indeed just this general idea which is now explored in somewhat greater detail.

The stellar masses which will prove to be of greatest interest fall in the range 10^5 $M_{\odot}$ to 10^6 $M_{\odot}$. For this mass range equation (12) indicates that $T_c \sim 10^8$ to 10^7 °K. The rate of energy generation by the CNO bi-cycle at these temperatures and the corresponding low densities is considerably less than that required to maintain a luminosity of the order of 10^{46} erg $\sec^{-1}$. Thus when the star reaches the minimum energy in Figure 1, collapse will commence and will continue until temperatures are reached at which nuclear energy generation becomes adequate for stability. Collapse will in fact be halted when, over the appropriate time interval, the nuclear energy generation matches that required to supply $E_{eq} - E_{eq}^{min}$. Collapse will be reversed to expansion in approximately the same time interval so that the nuclear processes overshoot and deliver in each pulse or cycle the following amount of energy

$$\begin{aligned}
\mathcal{E}_{cyc} &= 2\,(E_{eq} - E_{eq}^{min}) \\
&= 2\,|E_{eq}^{min}|\,(x_n - 1)^2 \qquad\qquad (18) \\
&= \frac{1.82 \times 10^{54}}{\mu^2}\,(x_n - 1)^2 \text{ erg}
\end{aligned}$$

where $x_n = T_n/T_c$ and T_n is the temperature at which the nuclear burning takes place while T_c is the critical temperature for the minimum in E_{eq}. The factor of 2 also follows directly from the conservation of momentum during the stopping and reversal of the collapse. The quadratic dependence on the temperature term in x_n follows directly from the fact that in equation (8) for E_{eq} only linear and quadratic terms in $R^{-1} \propto T$ are retained. In $E_{eq} - E_{eq}^{min}$ only the quadratic term remains. To determine T_n requires knowledge of the time scale for collapse and of the rate of energy generation in the CNO bi-cycle at elevated temperatures.

The time scale for nuclear burning during general relativistic collapse can be calculated with sufficient accuracy using the post-Newtonian approximation for the acceleration which can be written (see Lecture IV) as follows:

$$\ddot{r}(1 + \cdots\cdot) = -\frac{dp}{dr}\,\frac{(1 + v^2/c^2 - 2GM_r/rc^2)}{(\rho + p/c^2)} - \frac{GM_r}{r^2}\left(1 + \frac{4\pi p r^3}{M_r c^2}\right) \qquad \text{(all } r\text{)}$$

$$\begin{aligned}
&\approx \frac{1}{\rho_o}\,\frac{dp}{dr}\,\left(1 - \frac{u}{\rho_o c^2} - \frac{p}{\rho_o c^2} + \cdots\cdot\right) \\
&\quad - \frac{4}{3}\,\pi G \rho_o r\,\left(1 + \frac{u}{\rho_o c^2} + \frac{3p}{\rho_o c^2} + \cdots\cdot\right) \qquad \text{(post-Newtonian, center)} \qquad (19)
\end{aligned}$$

$$\approx \frac{1}{\rho_o}\,\frac{dp}{dr} - \frac{4}{3}\,\pi G \rho_o r \qquad \text{(Newtonian, center)}$$

In equation (19), r is the Lagrangian radial coordinate, p is the pressure, u is the internal energy density, ρ_o is the mass-density, ρ is the mass-energy density in mass units and M_r is the mass-energy interior to r. Relativistic terms have not been explicitly indicated on the left-hand side of equation (19). The first form on the right-hand side is relativistically exact, the second form is the post-Newtonian approximation at the center of the star, while the third form is the customary Newtonian approximation at the center. In this Newtonian approximation at hydrostatic equilibrium, $\ddot{r} = 0$. Thus to first order $\ddot{r}$ is just the difference of the first post-Newtonian terms on the right-hand side and the post-Newtonian terms on the left-hand side are not needed and have not been explicitly presented. It will be noted that the difference in the post-Newtonian terms is proportional to $(2u + 4p)/\rho_o c^2 \sim 10p/\rho_o c^2$ for $u \sim 3p$ as is the case in massive stars where radiation pressure dominates.

Standard methods of integration applied to equation (19) with $\ddot{r} = \dot{r} = 0$ at the initial critical conditions lead, for small $(p/\rho_o c^2)_c \approx (aT^4/3\rho_o c^2)_c$, to

$$\frac{\dot{T}}{T} \approx -\frac{\dot{r}}{r} \approx \left(\frac{40\pi aG}{9c^2}\right)^{\frac{1}{2}} T_{cr}^2 x(x^2 - 2x \ln x - 1)^{\frac{1}{2}} \tag{20}$$

$$\approx \left(\frac{40\pi aG}{27c^2}\right)^{\frac{1}{2}} T_{cr}^2 x(x - 1)^{3/2} \qquad \text{for } 1 < x < 2 \tag{21}$$

$$\sim \left(\frac{aG}{c^2}\right)^{\frac{1}{2}} T_{cr}^2 x^2 (x - 1)^{\frac{1}{2}} \qquad \text{for } 2 < x < 10 \tag{22}$$

where $x = T/T_c$. Solving for the e-folding time in T or r one finds numerically for equation (22), which is the case of primary interest, that

$$\tau_{gc} = T/\dot{T} = \frac{dt}{d \ln T} \sim \frac{1.3 \times 10^5}{T_8^2 (x - 1)^{\frac{1}{2}}} \text{ sec} \tag{23}$$

Thus the e-folding time in temperature or radius for general relativistic gravitational collapse (gc) at hydrogen burning temperatures, $(T_n)_8 \sim 2$, in massive stars for which $(T_{cr})_8 \sim 1$ is somewhat less than one day. This is considerably greater than the classical free fall time which is $\tau_{ff} = (8\pi G\rho/3)^{-\frac{1}{2}}$ $= 1340\ \rho^{-\frac{1}{2}}$ sec $\sim 10^3$ sec. However it is the shortness of τ_{gc} relative to the overall period of order 10 years which illustrates the extreme non-linearity of the oscillations under consideration.

In the discussion in Hoyle and Fowler (1965) of the behavior of CNO-burning of hydrogen in massive stars it was noted that the proton capture reaction by nuclei such as N^{13} proceed at a rate comparable to the beta-decay of N^{13} and that alpha-particle reactions lead to some transmutation of the CNO-nuclei into heavier nuclei. However these are not serious effects in the cases of primary interest in this paper and it is sufficiently accurate to make the assumption that all CNO-nuclei actively participate and remain as catalysts, mostly as N^{14} ($\sim 0.9Z$), and that the rate of energy generation is primarily determined by the $N^{14}(p,\gamma)$ reaction for which Hebbard and Bailey (1963) give the empirical parameters, $S_0 = 2.75 \pm 0.50$ keV-barns and $\langle dS/dE \rangle \approx 0$. This leads to a slight modification of the results of Caughlan and Fowler (1962). When expressed as a power law in temperature near $T_8 \sim 2$ the nuclear energy generation rate for the HCNO-burning is

$$\begin{aligned} \varepsilon &\approx 3.7 \times 10^{12}\ \rho X Z T_8{}^8 && \text{erg gm}^{-1}\ \text{sec}^{-1} \\ &\approx 5.6 \times 10^{10}\ \rho T_8{}^8 && X = 0.50,\ Z = 0.03 \end{aligned} \tag{24}$$

In a massive star with polytropic index n = 3, Fowler and Hoyle (1964) express the density in their equation (B120) as

$$\rho \approx 130 \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_8{}^3 \qquad \text{gm cm}^{-3} \tag{25}$$

so that

$$\varepsilon \approx 7.3 \times 10^{12} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_8^{11} \quad \text{erg gm}^{-1} \text{ sec}^{-1} \tag{26}$$

Fowler and Hoyle (1964) also give the energy generation averaged over the star and using their equation (C84) one finds

$$\bar{\varepsilon} \approx 4.4 \times 10^{11} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_8^{11} \quad \text{erg gm}^{-1} \text{ sec}^{-1} \tag{27}$$

This is still a quantity effectively representative of the central region of the stellar interior. As noted above, equation (27) must not be used when the burning becomes rapid enough that beta-decay processes limit the rate of energy generation. The limit comes when the time for the conversion of four protons into helium is just the sum of the mean lifetimes for proton capture by O^{14} (100 sec) and O^{15} (180 sec) and is given by

$$\begin{aligned} \bar{\varepsilon} &\sim \left(\frac{4}{14.5} \times \frac{6.0 \times 10^{18}}{280}\right) Z \sim 5.9 \times 10^{15}\ Z \quad \text{erg gm sec}^{-1} \\ &\sim 1.8 \times 10^{14} \qquad\qquad Z = 0.03 \end{aligned} \tag{28}$$

The limiting temperature given by combining equations (27) and (28) is

$$\begin{aligned} T_8 &< 1.7 \left(\frac{M}{M_\odot}\right)^{1/22} \\ &< \sim 3 \qquad\qquad \text{for } M = 10^5 \text{ to } 10^6\ M_\odot \end{aligned} \tag{29}$$

The simplest procedure is now to equate $\bar{\varepsilon}$ multiplied by the stellar mass and by the effective time for nuclear burning to $\mathcal{E}_{cyc}$ given by equation (18). The effective time can be estimated as follows. The quantity $\bar{\varepsilon}\tau_{gc}$ varies approximately as T^9. The e-folding time for T^9 is 1/9 that for T. However the velocity is reduced from its initial value to zero and is then reversed by the energy generation so that the effective time for nuclear burning during each cycle is $(4/9)\,\tau_{gc}$. Thus

$$\mathcal{E}_{cyc} = \frac{4}{9}\,\bar{\varepsilon}\tau_{gc}\,M \tag{30}$$

Equations (12), (18), (23), (27) and (30) can be combined to yield

$$\frac{M}{M_\odot} \approx 1.0 \times 10^5 \frac{(x_n)^{18/17}}{(x_n - 1)^{5/17}} \sim 10^5\,(x_n)^{13/17} \tag{31}$$

where $x_n = T_n/T_c$ as before. T_c is the critical temperature at which collapse begins and T_n is now the temperature at which the hydrogen burning generates $\mathcal{E}_{cyc}$ in the available time determined by the reversal of the collapse. Equations (18) and (31) then yield (for $x_n \sim 3$ as found below)

$$\mathcal{E}_{cyc} \sim 10^{41}\left(\frac{M}{M_\odot}\right)^{34/13} \quad \text{erg} \tag{32}$$

In this equation $\mu = 0.73$ has been used corresponding to $X = 0.50$, $Y = 0.47$, $Z = 0.03$.

We have now arrived at the nuclear energy generated in the pulse which triggers each relaxation oscillation or cycle. This must be equal to the total luminosity L for all forms of radiation multiplied by the cycle period τ_{cyc} so that

$$L\,\tau_{cyc} \sim 10^{41}\left(\frac{M}{M_\odot}\right)^{34/13} \quad \text{erg} \tag{33}$$

or

$$\frac{M}{M_\odot} \sim 2 \times 10^{-16}\,(L\,\tau_{cyc})^{13/34} \tag{34}$$

In the case of the quasar 3C 273 the observations indicate $L \sim 4 \times 10^{46}$ erg sec^{-1} (Oke 1965) and $\tau_{cyc} \sim 13$ yr $\sim 4 \times 10^8$ sec (Smith and Hoffleit 1963) so that

$L\,\tau_{cyc} \sim 1.6 \times 10^{55}$ erg and

$$\frac{M}{M_\odot} \sim 2.5 \times 10^5 \tag{35}$$

Corresponding to this value for $M/M_\odot$ it is found from equation (12) that $(T_{cr})_8 \sim 0.7$ and from equation (31) that $x_n \sim 3$ so that $(T_n)_8 \sim 2$. These values are illustrated in Figure 1. At $(T_{cr})_8 \sim 0.7$ the stellar radius is $\sim 4 \times 10^{13}$ cm while at $(T_n)_8 \sim 2$ the radius is $\sim 1.3 \times 10^{13}$ cm. The Schwarzschild limiting radius is $\sim 8 \times 10^{10}$ cm.

It will be noted in equation (32) that ε_{cyc} varies as a fairly high power, ~ 2.6, of the stellar mass. Thus ε_{cyc} rapidly approaches the total nuclear energy content of the star. At this point the nuclear energy would suffice for only one pulse of energy generation. These considerations lead to the conclusion that relaxation oscillations without serious overshooting due to excessive energy generation can only take place in <u>non-rotating</u> stars with mass not exceeding $\sim 10^6\ M_\odot$. It may prove significant that this is the order of magnitude of the mass of the larger globular clusters.

The above discussion has treated the relaxation oscillations almost solely in terms of energy considerations. Damping and stabilizing mechanisms will be discussed in the sequel but an important point in this connection should be noted at this time. The time spent during the oscillation with $R < R_{cr}$ when the star is dynamically unstable is very short compared to the overall period being of the order of one day. Thus practically the entire oscillation occurs with $R > R_{cr}$ during which the star is dynamically stable as in the classical, non-relativistic case. The maximum radius reached during the oscillation will be the order of the ratio of 10 years to 1 day multiplied by R_{cr} or approximately 10^{17} cm. The classical period for linear oscillations is not strictly applicable to the very large amplitude oscillations under discussion. This period is given by $\Pi \sim (\beta\bar{\rho}G)^{-\frac{1}{2}}$ and is the order of 10 years for the mean density in a star with $M \sim 2.5 \times 10^5\ M_\odot$ (3C-273) at an intermediate stage between the minimum ($R_n \sim 10^{13}$ cm) and maximum ($R_{max} \sim 10^{17}$ cm) excursions of the relaxation oscillations.

Equation (34) yields the mass of a non-rotating central stellar object having properties consistent with the product of the luminosity and period of a variable quasar such as 3C-273 if the period is that for relaxation oscillations in the star maintained by energy generation through HCNO-burning. The actual period will be determined by other considerations to be discussed later. At this point we turn our attention to the cumulative emissions from 3C-273.

The mass given by equation (35) can only satisfy the cumulative energy requirement for 3C-273 if the lifetime is relatively short. The nuclear energy resources for $M = 2 \times 10^5\ M_\odot$ are at most 4×10^{57} erg from equation (4). For a luminosity equal to 4×10^{46} erg sec^{-1} this yields a lifetime of 10^{11} sec or 3000 years. The number of relaxation oscillations is approximately 250. Greenstein and Schmidt (1964) discuss a model for the quasar in 3C-273 with lifetime equal to 10^3 years. They emphasize that on this basis the quasar is considerably younger than the associated objects in 3C-273, namely the radio halo surrounding the quasar and the optical jet. The quasar has then to be taken as a later event unassociated with the origin of the older, large scale components of 3C-273.

In the next lecture we will present in detail the modifications to the discussion presented here which are necessary if the value obtained for the mass from an equation such as (34) is to be substantially increased. Suffice it to note that the introduction of rotation or turbulent kinetic energy leads to a substantial increase in the mass in which the nuclear burning can take place without excessive overshooting during the relaxation oscillations. This lecture will be concluded with a brief enumeration of considerations connected with the energizing and damping of relaxation oscillations in such a way that stable pulsations are possible. We lean heavily on the quasar models discussed by Greenstein and Schmidt (1964) and by Oke (1965) in this enumeration. These considerations are illustrated schematically in Figure 2.

(1) Ledoux (1941) has shown that the relative radial displacement at the center of a pulsating massive star in which $\Gamma_1 \sim 4/3$ can be comparable in magnitude to that throughout the star and particularly at the surface. This means that nuclear energy generation at the center is extremely effective in triggering pulsations in the massive stars under discussion in this paper.

(2) Ledoux (1941) and Schwarzschild and Härm (1959) have emphasized that the problem of stability in massive stars depends critically on the mechanism of heat leakage in the envelope which serves to damp the oscillations energized in the core. They show that <u>pulsational instability</u> is to be expected for stellar masses above a critical value of the order of $\sim 10^2 M_\odot$, <u>if the only processes of heat transfer and loss are ordinary convection and radiation</u>. This can be understood on the basis that the radiative luminosity is proportional to R^2T^4 which in turn is proportional to R^{-2} so that at the large radii occurring during the expansion, $L_{rad} \propto R^{-2}$ is ineffective as a damping mechanism.

(3) What is required are damping mechanisms which are effective at large radii and low surface densities. It is therefore suggested that the extraordinary modes of energy emission evidenced by the quasars, namely, radio synchrotron emission, optical synchrotron emission as well as non-equilibrium continuum and line emission serve as the damping agents in stabilizing the pulsations. It is the overall rate of these emissions relative to the nuclear

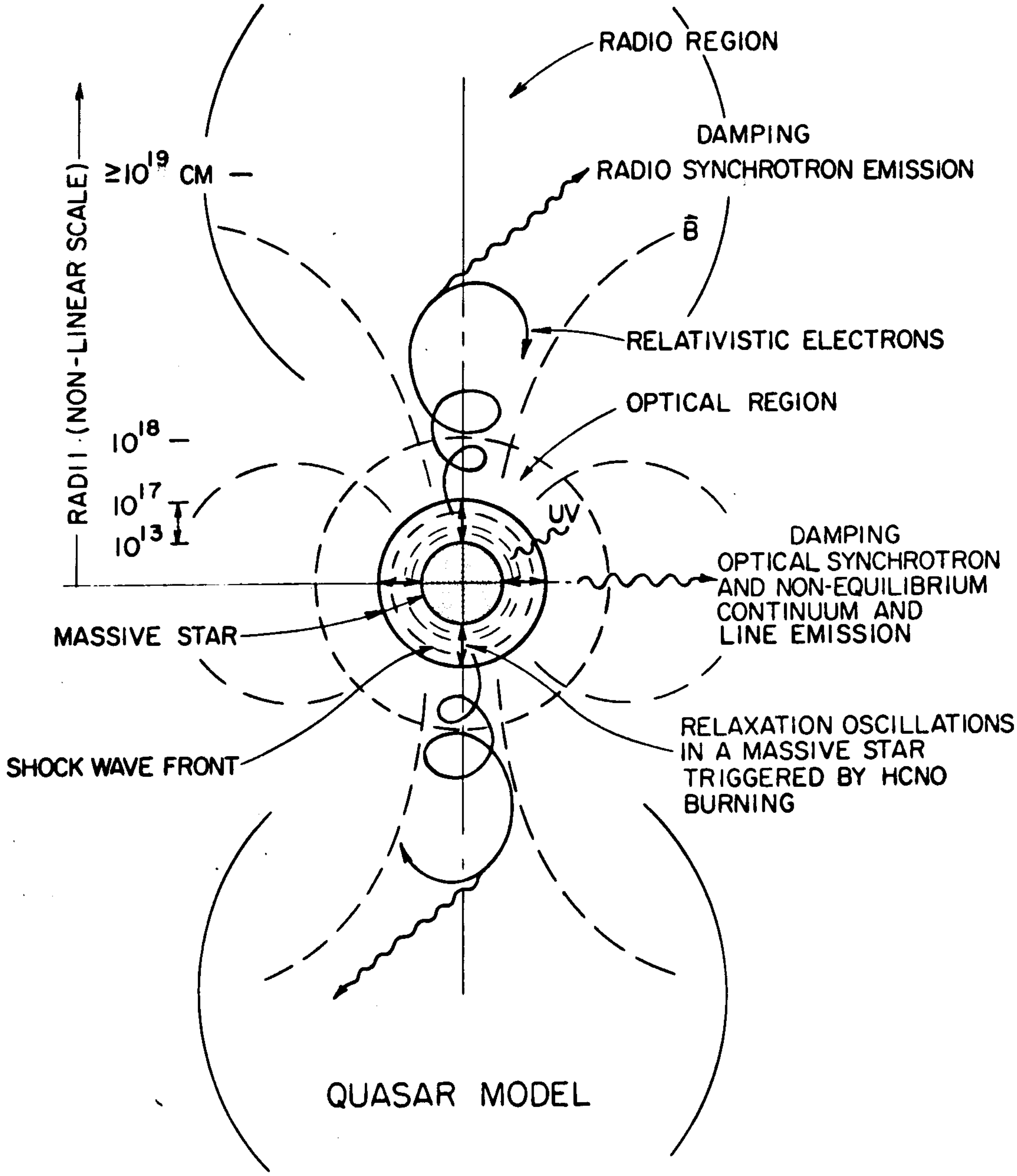


Fig. 2. Schematic model for a quasar. Large amplitude relaxation oscillations between radii of 10^{13} and 10^{17} cm are energized in a massive star by HCNO burning at a temperature near 2×10^8 °K. Shock waves transmit the energy to the tenuous outer envelope from which relativistic particles are ejected into the region surrounding the star. An associated dipole magnetic field channels the relativistic particles into two large scale regions ($\geq 10^{19}$ cm) in which radio synchrotron emission occurs. Optical synchrotron radiation is emitted from the region immediately surrounding the star ($\sim 10^{18}$ cm). Non-equilibrium continuum and line emission are also stimulated in this region by ultraviolet radiation from the star. It is this region which is visible and not the star itself. If the quasars are local all dimensions should be reduced by a factor ~ 100.

energy generation per pulse which determines the period of the oscillations. As discussed previously these emissions take place predominantly while $R > R_c$ during which the star is dynamically stable.

(4) As noted in the discussion of relaxation oscillations the nuclear energy generation takes place in the period of the order of a day which is very short compared to the observed overall periods of approximately 10 years. This nuclear pulse will lead to the propagation of a radial shock wave outward from the center of the star. From the work of Ôno, Sakashita and Ohyama (1961), Ohyama (1963) and Colgate and White (1964) it is known that such a shock wave will reach relativistic velocities in the tenuous outer envelope of the star and will there generate relativistic particles which are then ejected into the region surrounding the star. This high energy process becomes an especially effective damping agent during the latter stages of expansion when surface densities are low. It is generally believed that shock wave acceleration results in the production of relativistic particles with total energies comparable to that for non-relativistic particles. This seems to be required by the quasar observations.

(5) The ejection of relativistic particles leads to the formation of the region with dimensions of the order of 10^{18} cm in which an optical synchrotron continuum can be generated in the presence of an associated magnetic field. This region is in fact relatively transparent to high energy particles which can leak out to form a much more extended region with dimensions of the order of 10^{19} cm or even greater in which radio synchrotron emission takes place. The reader is referred to Greenstein and Schmidt (1964) for detailed description of the regions under discussion. If the quasars are local these dimensions must be reduced by a factor ~ 100.

(6) If the overall magnetic field has dipole structure then the ejection of the relativistic particles will tend to occur parallel to the dipole axis and to result in the formation of a two-component radio source as is frequently observed. For the field strengths required by synchrotron theory, the Larmor radii of the relativistic particles are quite small compared to the dimensions

of the radio sources. On this picture the line of centers of the radio components would lie along the axis of rotation even for an inclined magnetic dipole. For the line of centers to be perpendicular to the axis of rotation it is necessary to consider other possibilities such as the fission mechanism discussed by Hoyle and Fowler (1965) and Fowler (1964).

(7) From the original work of Hoyle and Fowler (1963a) on supermassive stars the <u>surface</u> temperature is estimated to be the order of 10^5 $^{\circ}K$ during the hydrogen burning stage in the <u>interior</u>. Intense ultraviolet emission at this temperature will amply suffice to excite non-equilibrium continuum and line emission from the 10^{18} cm region in which optical synchrotron radiation is also generated. At the same time the high opacity presented to the ultraviolet radiations would make observation of the embedded supermassive star impossible.

(8) Upon the exhaustion of nuclear energy resources, gravitational collapse occurs in a non-rotating massive star. For a rotating star collapse can also occur if mechanisms for the transfer of angular momentum are effective. In the case of collapse, gravitational energy becomes available and the evolution of a quasar into an extended radio source may become possible as discussed by Fowler (1964).

(9) Lynds and Sandage (1963) have shown that the peculiar optical and radio galaxy M82, shown in Figure 3, exhibits a complex filamentary system which contains a mass of expanding material which may be as large as 5.6×10^6 $M_{\odot}$ moving with kinetic energy equal to 2.4×10^{55} ergs. The expansion velocities are determined to be proportional to the distance from the galactic center and the data suggest that the primary explosive event took place 1.5×10^6 years ago. On the basis of the ideas presented in this lecture it seems reasonable to suggest that M82 was the site of the formation of a supermassive star with $M > 10^6$ $M_{\odot}$ which was not stabilized by rotation or internal turbulent energy.

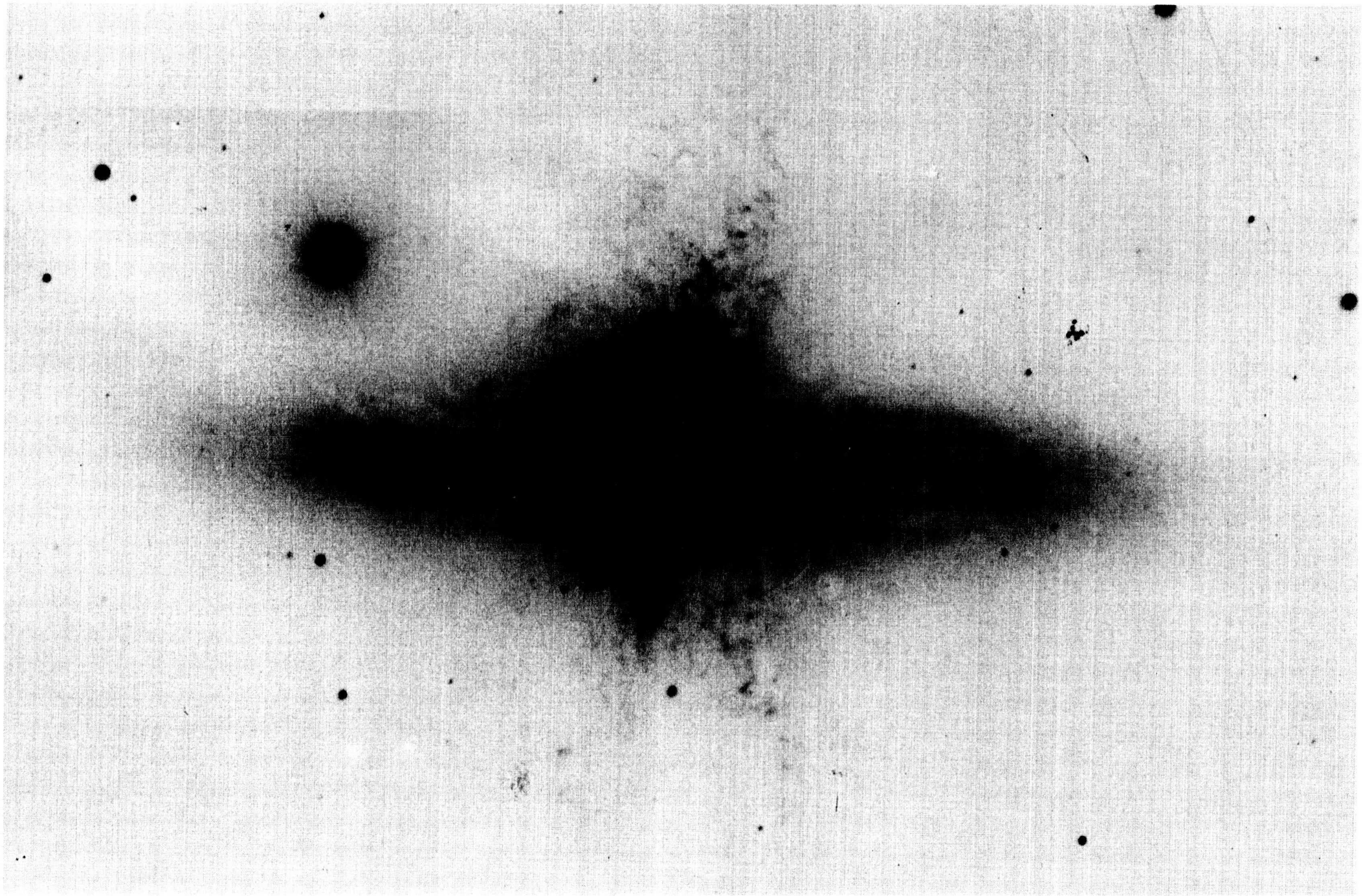


Fig. 3. Photograph of the galaxy M82 taken with the 200-inch Hale telescope on Mt. Palomar in Hα light. Note the complex filamentary structure expanding from the center of the galaxy. Lynd and Sandage (1963) point out that the total mass of the expanding material could be as great as $5.6 \times 10^6 M_{\odot}$. This exceeds the mass which can be stabilized during hydrogen burning by ordinary pressure forces because of general relativistic effects. It is suggested that the explosion is the result of the general relativistic instability discussed in the text. Photograph by Mt. Wilson and Palomar Observatories.

Thus effective damping of nuclear energy generation failed to occur. The burning of $\sim 3 \times 10^{-4}$ of the original hydrogen sufficed to supply the observed expansion energy. The burning in toto of $\sim 10^{-3}$ of the original hydrogen sufficed to supply all the observed energies, kinetic, magnetic, luminous and that stored in ionization and high energy electrons.

CONCLUSION

The work described in this lecture constitutes a return to the early point of view of Hoyle and Fowler (1963a,b) that supermassive stars can meet the energy requirements in radio sources, specifically in the quasars. General relativity leads to dynamic instability in non-rotating massive stars but the result is relaxation oscillations energized by hydrogen burning rather than catastrophic collapse at least for masses not exceeding $10^6\ M_\odot$. It is noted that the introduction of rotation or turbulent kinetic energy raises this limit by several orders of magnitude and that the next lecture will treat this matter. It is emphasized that the exotic forms of energy emission observed in the quasars can serve to damp the relaxation oscillations in such a way that stable pulsations result. It is thus suggested that quasars consist of pulsating supermassive stars, energized by nuclear reactions, with radio and optical emissions from extended surrounding regions which the star excites with ultraviolet radiation and relativistic particles. With the exhaustion of nuclear energy, gravitational energy may become available and the evolution of a quasar into an extended radio source becomes possible. It is suggested that exploding galaxies such as M82 developed a supermassive star with $M > 10^6\ M_\odot$ which was not stabilized by rotation or internal turbulent energy so that effective damping of nuclear burning failed to occur.

LECTURE III

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LECTURE IV*

SUPERMASSIVE STARS, QUASARS AND EXTRAGALACTIC RADIO SOURCES, II

N66 24330

INTRODUCTION

In the preceding lecture an attempt was made to understand the source of the energy requirements in quasars and extragalactic radio sources in terms of nuclear and gravitational energy release in supermassive stars. It was found that the conversion of hydrogen into helium could take place in stars with mass up to $10^6\ M_\odot$ during stable regimes of reasonable duration characterized by non-linear relaxation oscillations with periods similar to those observed. It was emphasized that the damping of these oscillations required mechanisms for energy transfer and emission other than radiation and convection. It was suggested that the necessary requirements were met by transfer through relativistic shock waves, by the development of magnetic fields, by the acceleration of electrons to high energies, and by the production of radio and optical synchrotron emission.

Because the onset of general relativistic instability occurs in stars with $M > 10^6\ M_\odot$ at temperatures far below those necessary for nuclear burning, damping cannot be effective once the nuclear processes are triggered. In this lecture it will be shown that the general relativistic instability is removed by rotation in stars with mass at least as high as $10^8\ M_\odot$ and perhaps as high as $10^9\ M_\odot$. Other stabilizing mechanisms are briefly mentioned.

Hoyle and Fowler (1963a) showed that the radiative luminosity of a stable supermassive star ($M > 10^3\ M_\odot$) is proportional to the mass according to the approximate relation

$$L \approx 2 \times 10^{38}\ (M/M_\odot)\ \text{erg sec}^{-1} \qquad (1)$$

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where M is the mass of the star and $M_\odot$ is the mass of the sun. On the assumption that one-half of the hydrogen in the star is processed to helium the available energy is

$$Q \approx \tfrac{1}{2} \times 7 \times 10^{-3} Mc^2 \approx 6 \times 10^{51} (M/M_\odot) \text{ erg}, \qquad (2)$$

so that the lifetime for the main-sequence stage of a supermassive star is

$$\tau \approx Q/L \approx 3 \times 10^{13} \text{ sec} \approx 10^6 \text{ yr} \qquad (3)$$

independent of mass. It was also found that hydrogen burning through the CNO bi-cycle takes place at a central temperature near 8×10^7 $^\circ$K and that the effective surface temperature during hydrogen burning is approximately 7×10^4 $^\circ$K indicating strong emission in the ultra-violet. A major unsolved problem concerned the mechanism by which the optical energy output is transformed into the high energy particles and magnetic field necessary to produce the radio emission on the basis of current synchrotron theory.

The discovery (Schmidt 1963; Oke 1963; Greenstein and Matthews 1963) and subsequent investigation (Greenstein and Schmidt 1964; Oke 1965) of the quasi-stellar radio sources or quasars shows that star-like objects associated with certain radio sources do indeed have very large luminosities in the optical range. The observed luminosities are claimed to be of the order of 10^{46} erg sec^{-1} which is that expected from equation (1) for $M \sim 10^8 M_\odot$. Lifetimes (Greenstein and Schmidt 1964) of the quasars fall in the range 10^3 to 10^6 years on various models. Thus it is tempting to associate the source of energy in the quasars with nuclear burning in massive stars. Subsequent gravitational energy release and possible connections with the extended radio sources are left aside for the time being. In fact, the association with quasars and radio galaxies is not the only motivation for this lecture. The stability of massive stars is a problem of interest and significance *per se*.

Support for the massive star model is given by the observed variability (Smith and Hoffleit 1963; Matthews and Sandage 1963; Sharov and Efremov 1963; Sandage 1964; Geyer 1964) of the optical radiation from the quasars. In addition to luminous flashes with durations of the order of days or weeks, there is evidence for cyclic variations with periods of the order of ten years. There is now good evidence for short period radio variations (Dent 1965; Maltby and Moffet 1965) but it is not yet possible to decide whether or not these are cyclic. It is generally agreed that the occurrence of the cyclic variations is crucial to the question whether the primary radiating object is a single coherent massive star (Hoyle and Fowler 1963a,b) or a system of smaller stars as discussed by numerous authors (Burbidge 1961; Woltjer 1964; Ulam and Walden 1964; Field 1964; Hoyle and Fowler 1965; Gold, Axford and Ray 1965). It is difficult on the basis of random collisions or supernova outbursts in a system of many stellar objects to explain variations which exhibit a regular periodicity. Furthermore, if the quasar dimensions are small enough, as would now seem to be the case, then collisions become very frequent and lead in a short time to a continuous medium which condenses into a single star.

Thus, without prejudice to the problem of the reality of the cyclic variations since only additional and more precise observations will settle this matter, the possibility is discussed in this lecture that such variations can arise from pulsations in a single massive star. In the major conclusion of the lecture it is shown that the general relativistic instability which occurs in non-rotating stars is removed during nuclear burning by a relatively small amount of rotation especially if differential rotation is taken into account.

An elegant treatment of the stability of supermassive stars using the exact equations of general relativity has been given by Chandrasekhar (1964a,b; 1965a) and applications to polytropic gas spheres have been made by Tooper (1964) and Gratton (1964). An analysis of the binding energy has been given by Iben (1963) and a general discussion of the binding energy and the various modes of oscillation has been given by Bardeen (1965). In the interest of simplicity and some gain in physical insight the following discussion will be restricted to the post-Newtonian approximation (Fowler 1964a) to the relativistic equations.

This restriction can be justified on the grounds that only the Newtonian and post-Newtonian terms in the Schwarzschild line element have been verified in the three so-called crucial tests of general relativity. There is even some question concerning the correspondence between observation and theory in the advance of the perihelion of Mercury which constitutes a test of the coefficient of the post-Newtonian term in the line element.

In determining the post-Newtonian terms a further approximation is made in that these terms are evaluated using the equilibrium configurations given by the Newtonian approximation. It must be emphasized that this cannot be justified without recourse to the detailed analysis of the exact formal solutions and the post-Newtonian approximation as given by Chandrasekhar. Only by such a detailed analysis can the conditions be determined under which this procedure gives a fair approximation to the correct results.

Even though it has important effects, rotation can be taken to be small and need only be treated in the Newtonian approximation and only for the case where distortion from spherical symmetry can be neglected. The two starting points will be (1) the equation for the binding energy of a star in hydrostatic equilibrium and (2) the radial equation for dynamic equilibrium throughout the star. The object is to derive useful relations for the binding energy and for the frequency of the fundamental mode of radial oscillation and to exhibit the connection between these two quantities. Because of the order of approximation to which the derivations are restricted, the results are applicable only to supermassive stars ($M > 10^3 M_\odot$) in which the ratio β of gas pressure to gas plus radiation pressure is small ($\beta < 0.1$) and can be approximated by equation (A21) in the appendix.

BINDING ENERGY OF A SUPERMASSIVE STAR IN HYDROSTATIC EQUILIBRIUM

Neglecting rotation for the time being and assuming the star to be spherically symmetric it is possible to define three masses in exact general relativistic terms. The total mass which determines the star's overall gravitational and inertial properties is given by

$$M = \int dM_r = \int \rho dV = 4\pi \int \rho r^2 dr \qquad (4a)$$

An observer at a distance from the star large compared to its radius would observe the Newtonian gravitational attraction exerted by the star on a test mass to be proportional to M. In equation (4a), dV is the Schwarzschild <u>coordinate</u> volume, $\rho = \rho_o + u/c^2$ is the total mass-energy density in mass units per unit <u>coordinate</u> volume, ρ_o is the rest mass density of nuclei and ionization electrons and u is the internal energy density of gas and radiation, and includes the rest mass-energy of particles created in the medium at elevated temperatures such as electron-positron pairs (see the appendix for further discussion). The radial variable in Schwarzschild coordinates, r, has the property that it automatically includes internal gravitational energy when an equation with the simplicity of (4a) is used to sum over the ρ measured by a local observer at r. The integration is taken from zero to R, the <u>coordinate</u> radius of the star, which is not equal to the <u>proper</u> radius of the star. M_r is the total mass-energy in mass units internal to r.

The rest mass is found by integrating ρ_o over <u>proper</u> volume elements according to the equation

$$M_o = \int \rho_o \left(1 - \frac{2GM_r}{rc^2}\right)^{-\frac{1}{2}} dV \qquad (4b)$$

where G is the gravitational constant and the square root term converts <u>coordinate</u> volume to <u>proper</u> volume. M_o can be determined by dispersing the constituent material at any time to infinity at zero temperature. Because of atomic and nuclear processes ρ_o and thus M_o at one time may not be the same as at another time. When hydrogen is converted into helium the rest mass per nucleon changes.

Unlike the creation and annihilation of pairs such nuclear changes may be irreversible during contraction and re-expansion. In the circumstances under discussion the number of nucleons remains invariant and it is only necessary to exercise care at a given time in stipulating the nuclear characteristics of the stellar material, i.e., the "composition" throughout the star.

The proper mass of the star exclusive of gravitational energy is given by

$$M_p = \int \rho \left(1 - \frac{2GM_r}{rc^2}\right)^{-\frac{1}{2}} dV \tag{4c}$$

where again the conversion to a <u>proper</u> volume element has been made. Using these three masses two binding energies can be defined. The gravitational binding energy, Ω, of the star which is taken to be positive and thus opposite in sign to the negative gravitational energy is given by

$$\begin{aligned} \Omega &= (M_p - M)\, c^2 \\ &= \int \rho c^2 \left[\left(1 - \frac{2GM_r}{rc^2}\right)^{-\frac{1}{2}} - 1\right] dV \qquad (5) \\ &\rightarrow \int \frac{GM_r}{r} \rho dV \qquad \text{(Newtonian approximation)} \end{aligned}$$

The binding energy, E_b, of the star is equal but opposite in sign to the total energy, E, exclusive of the rest mass energy and is given by

$$-E_b = E = (M - M_o)\, c^2 \tag{6}$$

In equation (6) nuclear binding has been excluded from E_b or E in the sense that it is included in M_o. This choice is arbitrary but is found to be the most convenient when treating the conversion of nuclear energy into internal energy. Thus E can increase when M_o decreases as is the case when hydrogen is converted into helium. At the same time E can decrease as M decreases as is the case when energy is radiated away by the star.

Since M_r is related to ρ and not to ρ_o it is convenient to retain ρ and u in expressing E so that

$$E = \int u \left(1 - \frac{2GM_r}{rc^2}\right)^{-\frac{1}{2}} dV + \int \rho c^2 \left[1 - \left(1 - \frac{2GM_r}{rc^2}\right)^{-\frac{1}{2}}\right] dV \tag{7}$$

$$= H - \Omega. \tag{8}$$

The first term in equation (7) is the proper internal energy of the star, which we designate by H in equation (8). The second term in equation (7) is the mass-energy of the star minus the proper mass-energy. If the sign is reversed this is just the gravitational binding energy (taken positive), which we designated by Ω in equation (5).

It is now appropriate to expand H and Ω, to retain only the Newtonian (subscript 0) and post-Newtonian (subscript 1) terms and to introduce the Newtonian term for the rotational energy which we designate by Ψ_0. The result is

$$E \approx H_0 - \Omega_0 + \Psi_0 + H_1 - \Omega_1 \tag{9}$$

$$\approx \int u dV - \int \frac{GM_r}{r} \rho dV + \frac{1}{2} \int r^2 \omega^2 \sin^2\theta \rho dV + \int \frac{GM_r}{rc^2} u dV - \frac{3}{2} \int \frac{G^2 M_r^2}{r^2 c^2} \rho dV. \tag{10}$$

The definition of the various terms in equation (9) will be obvious from the order of the terms in (10). In equation (10) ω is the angular velocity and θ is the polar angle measured from the axis of rotation. It will develop that $H_0 - \Omega_0$ is proportional to β and is thus small and comparable to $H_1 - \Omega_1$. We discuss only cases where Ψ_0 is comparable within a factor of ten to these two differences in the internal and gravitational energy terms.

EQUATION OF DYNAMIC EQUILIBRIUM

Again neglecting rotation for the moment, the exact general relativistic equation for dynamic equilibrium in the spherically symmetric case has been written by Misner and Sharp (1964) and others as

$$y^2 \ddot{r} + y \dot{r}^2 \frac{dy}{dr} = -\frac{1}{\rho} \frac{dp}{dr} \left(\frac{1 + y^2 \dot{r}^2/c^2 - 2GM_r/rc^2}{1 + p/\rho c^2} \right) - \frac{GM_r}{r^2} - \frac{4\pi G p r}{c^2} \tag{11}$$

where $$y = \frac{\rho + p/c^2}{\rho_o} = 1 + \frac{u}{\rho_o c^2} + \frac{p}{\rho_o c^2} \, . \qquad (12)$$

It will be noted that the left-hand side of equation (9) can be written in the more compact form $yd(y\dot{r})/dt$.

We now proceed to write equation (11) in the post-Newtonian approximation and to apply it to small perturbations (δ) about hydrostatic equilibrium. Conditions at equilibrium will be designated by the subscript eq. It will be clear that the two terms containing $\dot{r}^2$ can be neglected since $\dot{r} = \dot{r}_{eq} = 0$ at hydrostatic equilibrium and $\delta\dot{r}^2 = 2\,\dot{r}_{eq}\delta\dot{r}_{eq} = 0$ to first order. This leaves $y^2\ddot{r}$ on the left-hand side of equation (11) where the Newtonian term in y is unity and the post-Newtonian terms are much smaller than unity in all applications made in this lecture. After the manipulations on equation (11) which follow, it will develop that the Newtonian term on the right-hand side is small and comparable to the post-Newtonian term. Thus it is unnecessary to retain second order terms in the factor y^2 and in equation (11) we replace y^2 in $y^2\ddot{r}$ by unity.

Since the left-hand side of equation (11) has now been reduced to the classical Newtonian acceleration, $\ddot{r}$, with no ambiguities in space-time measurements, it will be clear that small rotational effects can be introduced in the approximation of the Newtonian centrifugal acceleration, $r\omega^2\sin^2\theta$. Thus the post-Newtonian equivalent of (11) for small rotation in supermassive stars is

$$\ddot{r} \approx r\omega^2\sin^2\theta - \frac{1}{\rho}\frac{dp}{dr}\left(1 - \frac{p}{\rho c^2} - \frac{2GM_r}{rc^2} + \cdots\right) - \frac{GM_r}{r^2} - \frac{4\pi G p r}{c^2} \, . \qquad (13)$$

Since $dp/dr = \rho GM_r/r^2$ in Newtonian hydrostatic equilibrium with no rotation, equation (13) can be written, to the order of the approximations being made in this discussion, as

$$\ddot{r} \approx r\omega^2\sin^2\theta - \frac{1}{\rho}\frac{dp}{dr} - \frac{GM_r}{r^2}\left(1 + \frac{p}{\rho c^2} + \frac{2GM_r}{rc^2} + \cdots\right) - \frac{4\pi Gpr}{c^2} \quad . \tag{14}$$

Multiply equation (14) by $r\rho dV$ and integrate over the entire star. The result is

$$\begin{aligned} \int r\ddot{r}\rho dV \approx & - \int 4\pi r^3 dp - \int \frac{GM_r}{r}\rho dV + \int r^2\omega^2\sin^2\theta\rho dV \\ & - \int \frac{GM_r}{rc^2} p dV - 2\int \frac{G^2M_r^2}{r^2c^2}\rho dV - 4\pi\int \frac{Gpr^2}{c^2}\rho dV. \end{aligned} \tag{15}$$

The first and last terms on the right-hand side can be integrated by parts from $r = 0$ where $M_r = 0$ to $r = R$ where $p = 0$ to yield

$$\int r\ddot{r}\rho dV \approx 3\int p dV - \Omega_0 + 2\,\Psi_0 + \int \frac{GM_r}{rc^2} p dV - 3\int \frac{G^2M_r^2}{r^2c^2}\rho dV. \tag{16}$$

From the discussion in the appendix $p = (\Gamma_4 - 1)u \approx 1/3\,(1 + \beta/2)u$ so that $p \approx 1/3\,u$ when β is small and it is natural to define a mean value of Γ_4 such that $\int p dV = (\bar{\Gamma}_4 - 1)\int u dV$. To the approximation of interest we can use this same $\bar{\Gamma}_4$ in the fourth term on the right-hand side of equation (16). The result is the virial equation

$$\int r\ddot{r}\rho dV \approx 3\,(\bar{\Gamma}_4 - 1)\,H_0 - \Omega_0 + 2\,\Psi_0 + (\bar{\Gamma}_4 - 1)\,H_1 - 2\Omega_1. \tag{17}$$

Under conditions of hydrostatic equilibrium, $\ddot{r} = 0$ everywhere and a simple virial relation is obtained between H_0, Ω_0, etc. For numerical calculations of the binding energy it is most convenient to eliminate H_0 in substituting into equation (9) and the result is

$$E_{eq} \approx -\frac{3\,\bar{\Gamma}_4 - 4}{3\,(\bar{\Gamma}_4 - 1)}\,\Omega_0 - \frac{5 - 3\,\bar{\Gamma}_4}{3\,(\bar{\Gamma}_4 - 1)}\,\Psi_0 + \frac{2}{3}\,H_1 + \frac{5 - 3\,\bar{\Gamma}_4}{3\,(\bar{\Gamma}_4 - 1)}\,\Omega_1. \tag{18}$$

Equation (A19) then yields

$$E_{eq} \approx -\frac{\overline{\beta}}{2}\Omega_0 - (1-\overline{\beta})\,\Psi_0 + \frac{2}{3}H_1 + (1-\overline{\beta})\,\Omega_1 \,. \tag{19}$$

For small $\overline{\beta}$ in massive stars

$$E_{eq} \approx -\frac{\overline{\beta}}{2}\Omega_0 - \Psi_0 + \frac{2}{3}H_1 + \Omega_1 \tag{20}$$

where, in recapitulation

$$\Omega_0 = \int \frac{GM_r}{r}\rho dV = 4\pi G\int \rho r M_r dr \tag{21}$$

$$\Psi_0 = \frac{1}{2}\int r^2\omega^2\sin^2\theta\rho dV = \pi\int \rho r^4\omega^2\sin^3\theta dr d\theta \tag{22}$$

$$H_1 = \int \frac{GM_r}{rc^2}u dV = \frac{4\pi G}{c^2}\int u r M_r dr \approx \frac{12\pi G}{c^2}\int p r M_r dr \tag{23}$$

$$\Omega_1 = \frac{3}{2}\int \frac{G^2M_r^2}{r^2c^2}\rho dV = \frac{6\pi G^2}{c^2}\int \rho M_r{}^2 dr. \tag{24}$$

In the last approximation in equation (23) we have used $p = (\Gamma_4 - 1)u \approx u/3$ for massive stars. In equation (20) it will be noted that all terms are <u>small</u> when this equation is applied to <u>slowly rotating</u>, <u>massive stars</u>. This circumstance arises from the fact that $H_0 - \Omega_0$ in equation (9) becomes proportional to $\overline{\beta}$ through equations (17) and (A19).

ADIABATIC RADIAL PULSATION

In order to determine the angular frequency, σ_R, of the fundamental mode of radial oscillation equation (17) is applied to a perturbation of the form

$$\frac{\delta r}{r} = \frac{\delta R}{R} \exp(-i\sigma_R t). \tag{25}$$

The result is

$$-\sigma_R^2 I \frac{\delta R}{R} \approx 3(\bar{\Gamma}_1 - 1)\delta H_0 - \delta\Omega_0 + 2\delta\Psi_0 + (\bar{\Gamma}_1 - 1)\delta H_1 - 2\delta\Omega_1 \tag{26}$$

where

$$I = \int r^2 \rho dV \tag{27}$$

is the moment of inertia of the star about the origin of coordinates. I is equal to 3/2 the usual moment of inertia about the axis of rotation if the distortion from spherical symmetry is ignored. In deriving equation (26) use has been made of equation (A31) in the appendix. Again we overlook the fact that the average Γ_1 in the coefficient of δH_1 is not quite the same as that in the coefficient of δH_0. If the oscillation is adiabatic the energy equation becomes

$$\delta E = \delta H_0 - \delta\Omega_0 + \delta\Psi_0 + \delta H_1 - \delta\Omega_1 = 0. \tag{28}$$

If equation (28) is employed to eliminate δH_0 in equation (26) the result is

$$-\sigma_R^2 I \frac{\delta R}{R} = (3\bar{\Gamma}_1 - 4)\delta\Omega_0 + (5 - 3\bar{\Gamma}_1)\delta\Psi_0 - 2(\bar{\Gamma}_1 - 1)\delta H_1 - (5 - 3\bar{\Gamma}_1)\delta\Omega_1. \tag{29}$$

APPLICATIONS TO POLYTROPIC MODELS

Within the approximations which have been carefully specified, equations (18) and (29) are quite general. Further elucidation requires that Ω_0, etc., be specified as functions of the stellar radius R and mass M and that $\delta\Omega_0$, etc. be related to δR through these quantities. This can only be done for specific stellar models. For our purposes polytropic models specified by the index $\underline{n}$ in the relation $pV^{1+1/n}$ = const or p = const $\rho_o^{1+1/n}$ are of sufficient diversity and accuracy.

Considerable simplification arises from the fact that our interest is concentrated on <u>slowly rotating</u>, <u>massive stars</u> in which the Newtonian terms in equations (18) and (29) are small and of the same order of magnitude as the post-Newtonian terms. This means that the integrals for Ω_0, Ψ_0, H_1 and Ω_1 can be evaluated using the run of the variables throughout the star given by the solution of the classical Lane-Emden polytropic equations without rotation. In particular it is not necessary to distinguish between ρ_o and ρ nor between M_o and M in keeping with the general presumption that $M_o - M$ is small compared to either one of them. Only one new physical concept must be introduced — namely that for an <u>isolated</u> star, angular momentum must be conserved through all stages of contraction or of oscillation.

The Newtonian gravitational binding energy in units of Mc^2 can be expressed in terms of the convenient dimensionless parameter $2GM/Rc^2$ as

$$\frac{\Omega_0}{Mc^2} = \frac{3}{2(5-n)}\left(\frac{2GM}{Rc^2}\right) \tag{30}$$

so

$$\frac{\delta\Omega_0}{\Omega_0} = -\frac{\delta R}{R} \, . \tag{31}$$

Rotational terms in Ω_0 result in terms of order $\beta\omega^2$ in E or $\sigma_R{}^2$ and can be neglected when both β and ω^2 are small. The Newtonian rotational energy is given in terms of the conserved angular momentum, Φ, by

$$\frac{\Psi_0}{Mc^2} = \frac{\Phi^2}{2(ckMR)^2} \tag{32}$$

where for uniform rotation $k = (2I/3MR^2)^{\frac{1}{2}}$ is the radius of gyration in units of R and $\Phi = k^2MR^2\omega$ = const. Differential rotation will be discussed in what follows. Once established under the conservation of angular momentum for all mass elements in a star, differential rotation requires $\Psi_0 \propto R^{-2}$ just as for uniform rotation so that in any case

$$\frac{\delta\Psi_0}{\Psi_0} = -2\,\frac{\delta R}{R} \tag{33}$$

It has been shown (Fowler 1964a,b) that the integrals for H_1 and Ω_1 in units of Mc^2 involve the dimensionless parameter $(2GM/Rc^2)$ to the second power as might be expected on general grounds. Numerical coefficients can be derived analytically for some polytropes and can be evaluated numerically for others. For the quantities of greatest interest, the result can be expressed as

$$\frac{H_1}{Mc^2} = \zeta_n'\left(\frac{2GM}{Rc^2}\right)^2 \tag{34}$$

$$\frac{\Omega_1}{Mc^2} = \zeta_n''\left(\frac{2GM}{Rc^2}\right)^2 \tag{35}$$

and

$$\frac{2}{3}\frac{H_1}{Mc^2} + \frac{\Omega_1}{Mc^2} = \zeta_n\left(\frac{2GM}{Rc^2}\right)^2 \tag{36}$$

where, for example, $\zeta_0' = 0.064$, $\zeta_0'' = 0.161$, $\zeta_0 = 0.204$, $\zeta_1' = 0.116$, $\zeta_1'' = 0.241$, $\zeta_1 = 0.318$, $\zeta_2' = 0.219$, $\zeta_2'' = 0.417$, $\zeta_2 = 0.563$, $\zeta_3' = 0.513$, $\zeta_3'' = 0.923$, $\zeta_3 = 1.265$, $\zeta_4' = 2.12$, $\zeta_4'' = 3.66$, and $\zeta_4 = 5.07$. Bardeen and Anand (1966) have shown that $\zeta_n \approx 5.07/(5-n)^2$.

In any case

$$\frac{\delta H_1}{H_1} = \frac{\delta \Omega_1}{\Omega_1} = -2\frac{\delta R}{R} . \tag{37}$$

Thus equation (29) becomes

$$\sigma_R^2 I \approx (3\bar{\Gamma}_1 - 4)\,\Omega_0 + 2(5-3\bar{\Gamma}_1)\,\Psi_0 - 4(\bar{\Gamma}_1 - 1)\,H_1 - 2(5-3\bar{\Gamma}_1)\,\Omega_1 . \tag{38}$$

The Newtonian terms in this equation are identical to those given by Chandrasekhar and Lebovitz (1962) in their equation (111). For $\bar{\Gamma}_1 \approx 4/3 + \bar{\beta}/6$ and $\bar{\beta}$ small as in supermassive stars

$$\sigma_R^2 I \approx \frac{\bar{\beta}}{2}\,\Omega_0 + 2\Psi_0 - \frac{4}{3}H_1 - 2\Omega_1 . \tag{39}$$

In those cases where $\bar{\Gamma}_1$ and $\bar{\Gamma}_4$ can be taken to be equal, as for example when $\bar{\beta}$ is small and $\bar{\Gamma}_1 \approx \bar{\Gamma}_4 \approx 4/3 + \bar{\beta}/6$, then it will be clear from equations (18), (30), (32), (34), (35) and (38) that

$$\sigma_R^2 \approx 3(\bar{\Gamma}_1 - 1)\,\frac{R}{I}\,\frac{dE_{eq}}{dR} \qquad \bar{\Gamma}_1 \approx \bar{\Gamma}_4 \tag{40}$$

$$\approx \frac{R}{I}\,\frac{dE_{eq}}{dR} \qquad \beta \ll 1. \tag{41}$$

This important relation has been previously (Fowler 1964a) used in the case of non-rotating massive stars and will be discussed further in what follows. A circumstance under which equations (40) and (41) do not hold will be noted near the conclusion of this paper.

In order to make the analysis which follows as transparent as possible it will prove expedient to specify a particular polytropic index. For massive stars it is well known that the case $n = 3$, for which β = constant, yields a fairly accurate representation for the internal structure. For $n = 3$, equations (20) and (39) become

$$\frac{E_{eq}}{Mc^2} \approx -\frac{3}{8}\beta\left(\frac{2GM}{Rc^2}\right) + 1.265\left(\frac{2GM}{Rc^2}\right)^2 - \frac{1}{2}\left(\frac{\Phi}{ckMR}\right)^2 \qquad n = 3 \qquad (42)$$

$$\sigma_R^2 \approx \frac{Mc^2}{I}\left[\frac{3}{8}\beta\left(\frac{2GM}{Rc^2}\right) - 2.53\left(\frac{2GM}{Rc^2}\right)^2 + \left(\frac{\Phi}{ckMR}\right)^2\right] \qquad n = 3 \qquad (43)$$

These equations display the Newtonian gravitational term in $1/R$, the Newtonian rotational term in $1/R^2$ and the general relativistic post-Newtonian term in $1/R^2$. The dependence on powers of $1/R$ can be replaced by dependence on powers of the central temperature, T_c, by use of the relation (Fowler 1964)

$$T_c = \frac{5.83 \times 10^{18}}{R}\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \qquad n = 3 \qquad (44)$$

ROTATIONAL STABILITY VS. GENERAL RELATIVISTIC INSTABILITY

The fundamental mode of radial oscillation becomes dynamically unstable when $\sigma_R^2 < 0$ or σ_R becomes imaginary in equation (25). In the case of no rotation, $\Phi = 0$, it has been noted (Chandrasekhar 1964a,b; Fowler 1964) that instability sets in for contraction below a critical radius given for $\sigma_R^2 \le 0$ in equation (43) by

$$R_{cr} = \frac{6.74}{\beta} R_g = 3.4 \times 10^5 \left(\frac{M}{M_\odot}\right)^{3/2} \text{cm} \qquad \Phi = 0,\ n = 3 \qquad (45)$$

where $R_g = 2GM/c^2$ is the limiting gravitational radius or Schwarzschild coordinate radius and β has been evaluated using $\mu = 0.73$ for a representative mixture of 50 per cent hydrogen, 47 per cent helium and 3 per cent heavy elements by mass. From equations (44) and (45) the critical central temperature, above which instability sets in, is

$$T_{cr} = 1.7 \times 10^{13}\ (M_\odot/M) \quad {}^\circ K \qquad \Phi = 0,\ n = 3 \qquad (46)$$

At the critical radius and central temperature, E_{eq} reaches a minimum value and the binding energy reaches a maximum value as indicated by equation (41). This is illustrated in Figure 1 for $M = 10^6\ M_\odot$ where $E_{eq}/M_\odot c^2$, σ_R and the period $\Pi = 2\pi/\sigma_R$ are shown as functions of R and T_c. In the calculations $I = 0.113\ MR^2$ for a polytrope of index n = 3 has been used. The situation can be understood physically in the following way. To the left of the minimum in E_{eq} in the Newtonian range an adiabatic perturbation (constant E) toward smaller radii leads to more energy than that required for equilibrium and thus, for any physically reasonable equation of state, to more pressure than that necessary for hydrostatic equilibrium. Thus the contraction is opposed. An adiabatic perturbation toward larger radii leads to less energy and less pressure than that required for hydrostatic equilibrium and thus expansion is opposed. The same argument used to the right of the minimum indicates that a contraction leads to less pressure than that needed for hydrostatic equilibrium while an expansion leads to more so that the system is dynamically unstable to adiabatic perturbations. It will be noted that the minimum E_{eq} given by $-9\bar{\beta}^2 Mc^2/64\zeta_n(5-n)^2$ has magnitude $0.028\ \bar{\beta}^2 Mc^2 \sim M_\odot c^2$, which is independent of the mass M, and that the minimum period during stable contraction is of order of one year. In general the minimum period is given by

$$\Pi_{min} = 1.7 \times 10^{-12}\ (M/M_\odot)^2\ \text{yr} \qquad \Phi = 0,\ n = 3 \qquad (47)$$

The critical temperature is only $1.7 \times 10^7\ {}^\circ K$ for $M = 10^6\ M_\odot$ and this is considerably below the temperature of $8 \times 10^7\ {}^\circ K$ which, it has been previously noted (Hoyle and Fowler 1963a) is necessary for hydrogen burning through the CNO bi-cycle. This means that there is no source of the energy required for hydrostatic equilibrium above $1.7 \times 10^7\ {}^\circ K$ or for contraction below 3.5×10^{14} cm so that the instability results in gravitational collapse until the onset of

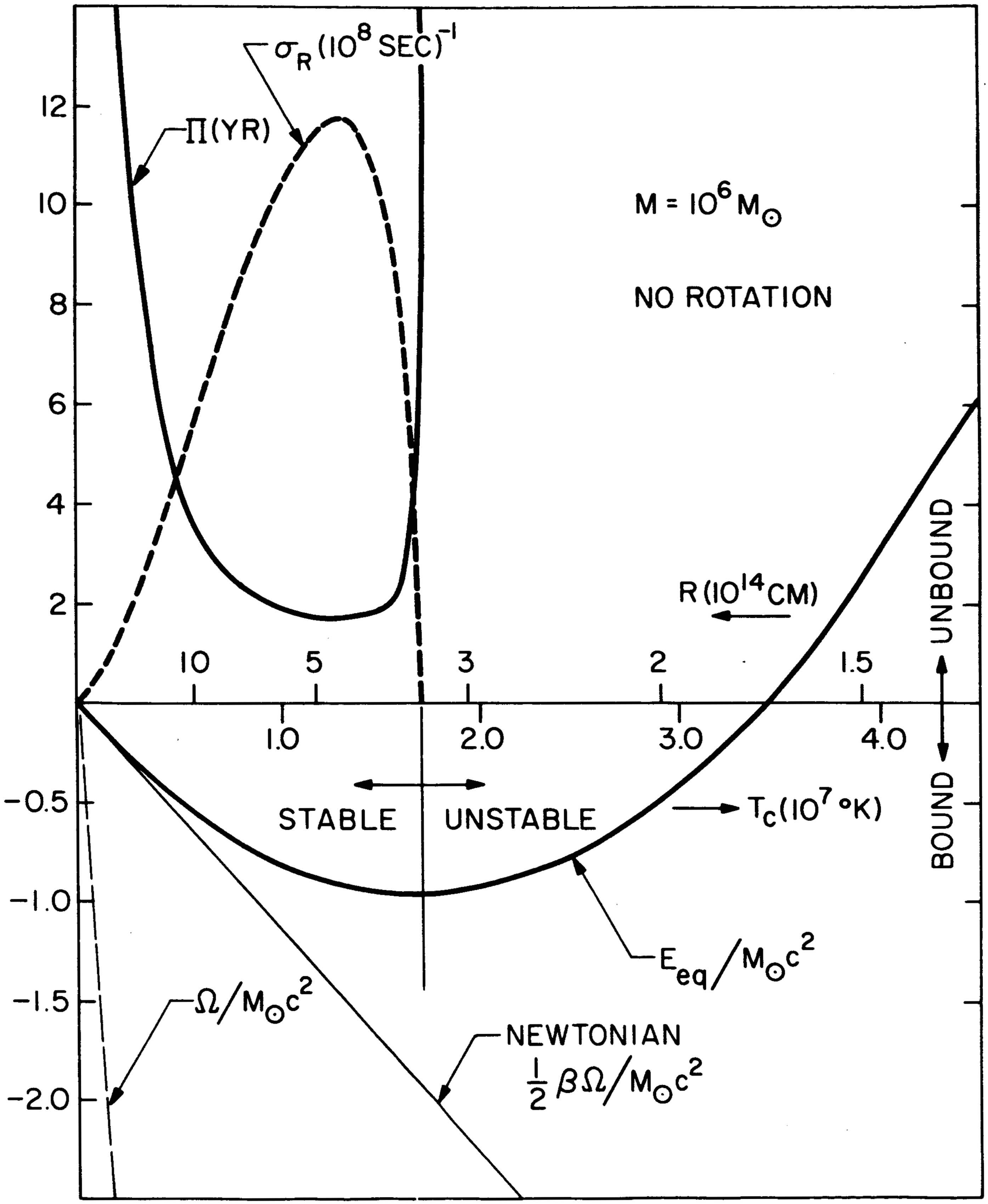

Fig. 1. The binding energy and the frequency and period of the fundamental mode of radial oscillation in a non-rotating star with mass equal to $10^6\ M_\odot$.

nuclear burning. The resulting relaxation oscillations for $M \leq 10^6\ M_\odot$ have been discussed in the previous lecture. For $M \geq 10^6\ M_\odot$, the onset of hydrogen burning is not sufficient to prevent continued gravitational collapse in a non-rotating star. This has placed a serious limitation on the energy available in that model which depicts quasars as non-rotating massive stars undergoing relaxation oscillations since hydrogen burning in a star with $M = 10^6\ M_\odot$ yields only $\sim 10^{58}$ ergs and the required energies are in some cases of the order of 100 times this figure.

In equation (43) the general relativistic term which leads to instability varies as R^{-2} and is negative. For constant angular momentum, Φ, the rotational term also varies as R^{-2} but is positive. Thus for large enough Φ, the general relativistic instability discussed above is removed by rotation. In physical terms the rotation prevents the gravitational collapse which would otherwise result from the general relativistic instability. Relative to the magnitude of the angular momentum common to astronomical systems the required Φ is quite small. For the rotational and general relativistic terms in equation (43) to cancel, the critical angular momentum for stability is given by

$$\Phi_{cr} = (2k^2 \zeta_n)^{\frac{1}{2}} \frac{2GM^2}{c} \tag{48}$$

where we have generalized to any n. Since the angular momentum is conserved it is simplest to calculate Φ_{cr} at the stage where the stellar mass is dispersed uniformly as a gaseous cloud. In this case $n = 0$, $k^2 = 2/5$ and $\zeta_0 = 0.204$ so that

$$\frac{\Phi_{cr}}{M} = 3.6 \times 10^{15} \left(\frac{M}{M_\odot}\right) \text{cm}^2 \text{ sec}^{-1}. \tag{49}$$

Even for $M = 10^8\ M_\odot$ this angular momentum per unit mass is very small compared to the typical value, 10^{30} cm^2 sec^{-1}, which applies to the rotation of the solar system in the Galaxy.

The rotational effects are illustrated for a star with mass $M = 10^8 M_\odot$ in Figures 2 and 3. Figure 2 exhibits the dependence of $E_{eq}/M_\odot c^2$ on R and T_c while Figure 3 shows the dependence of the period Π on these same quantities. The curves have been calculated for f = 0, 0.99, 1 and 2 where f is the ratio of the rotational energy to the "general relativistic" energy represented by the post-Newtonian terms in equations (20) and (36). For a given angular momentum f remains constant during homologous contraction. The calculations have been made for polytropic index n = 3.

It will be noted that dynamic stability at the temperature required for hydrogen burning through the CNO bi-cycle sets a lower limit on f equal to unity. For reasons to be discussed in the next section large values of f are irrelevant since angular momentum loss occurs if the original angular momentum is very large. The period of the fundamental radial oscillations at hydrogen burning varies rapidly with f being of the order of 1 year for f = 1 and 10 days for f = 2. It is extremely doubtful, however, that small amplitude, linear oscillations characterized by exactly these periods will occur. From the work of Ledoux (1941) and of Schwarzschild and Härm (1959) it is more probable that large amplitude, non-linear pulsations will be set up at the onset of nuclear burning. The energy generation will indeed take place in a relatively short period followed by a longer period of expansion to large radius and then recontraction during which the energy is transmitted to the surface of the star and radiated away. Relaxation oscillations of this nature in supermassive stars have been previously discussed and the possible connections with the periodicity and exotic forms of energy emission in quasi-stellar objects have been pointed out. Only one point need be added to that discussion: variations in the magnetic field which accompany the oscillations will accelerate electrons to relativistic energies through the betatron mechanism.

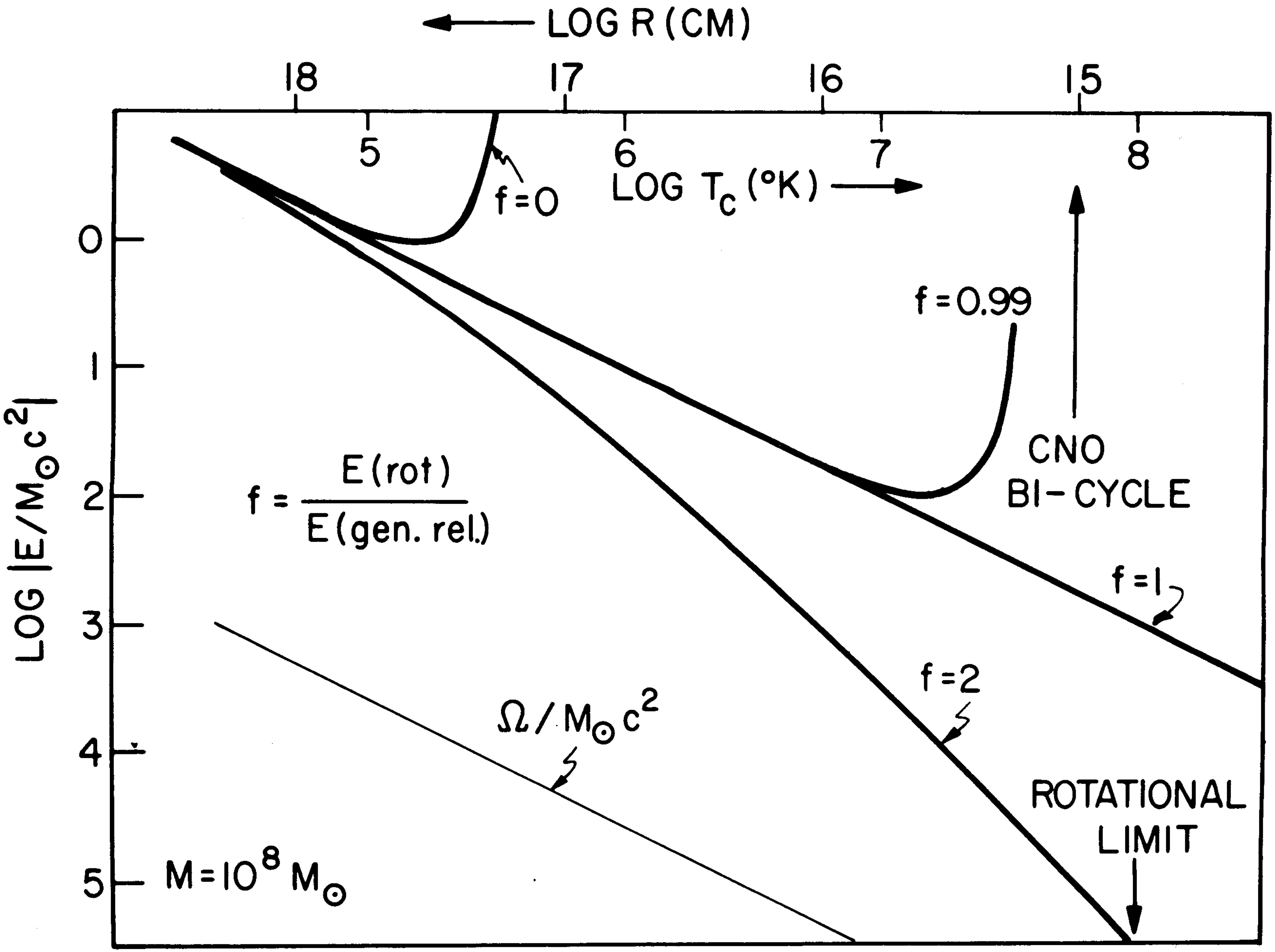

Fig. 2. The binding energy of a rotating star with mass equal to 10^8 $M_\odot$.

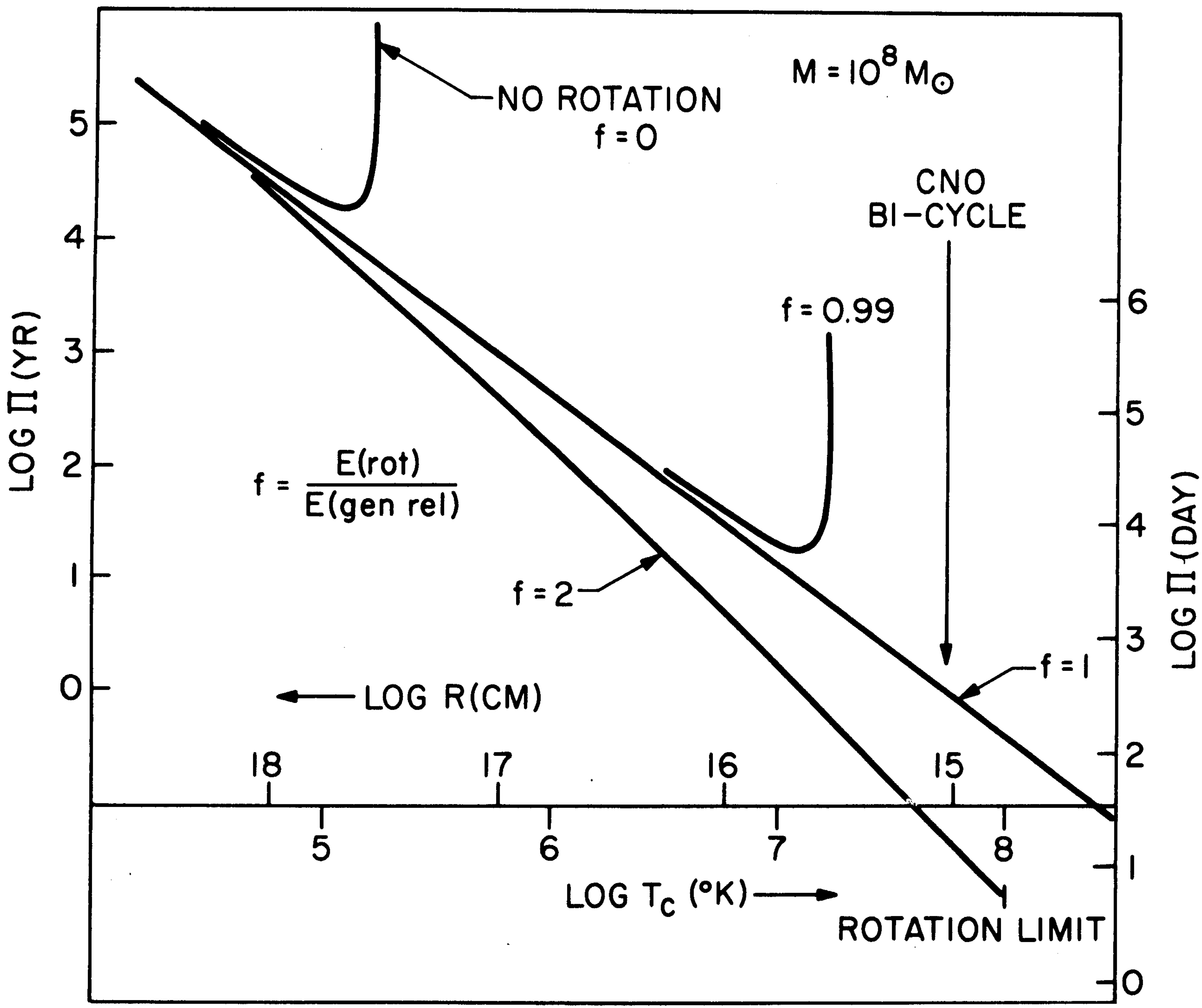

Fig. 3. The period of the fundamental mode of radial oscillation of a rotating star with mass equal to 10^8 $M_\odot$.

Here we emphasize that rotation extends the mass range in which stable relaxation oscillations triggered by hydrogen burning can occur up to masses of the order of 10^8 $M_\odot$ or somewhat more. This extends the available nuclear energy in such objects to at least 10^{60} ergs. These limits were 10^6 $M_\odot$ and 10^{58} ergs without rotation. With rotation as the stabilizing agent, a star of mass 10^8 $M_\odot$ can serve as the energy source in a quasar with total luminosity equal to 2×10^{46} erg sec^{-1} for a period as long as 10^6 years as noted in the Introduction.

For the record we note the period in supermassive stars with $f = 1$, $n = 3$, $k^2 = 0.075$, $\mu = 0.73$:

$$\Pi = \left(\frac{8\pi^2 k^2 R^3}{\bar{\beta} G M}\right)^{\frac{1}{2}} = \left(\frac{6\pi k^2}{\bar{\beta} G \ \bar{\rho}}\right)^{\frac{1}{2}} \tag{50}$$

$$= 2.8 \times 10^{-21} \ R^{\frac{3}{2}} \left(\frac{M_\odot}{M}\right)^{\frac{1}{4}} \text{yr} \qquad n = 3 \tag{51}$$

$$= 3.9 \times 10^{7} \ T_c^{-\frac{3}{2}} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \text{yr.} \qquad n = 3 \tag{52}$$

This is just the Newtonian period without rotation for small β. In equation (50), $\bar{\rho}$ is the mean density of the stellar matter. Note that for $f = 1$ the rotational energy just cancels the post-Newtonian general relativistic energy in equation (43). Equations (50) to (52) are derived from the Newtonian term in equation (43). Figure 3 illustrates the rapid decrease in Π as f is increased.

In concluding this section it can be pointed out that any physical phenomenon which leads to a positive term proportional to $1/R^2$ in the binding energy equation (42) will, if large enough, remove the general relativistic instability in supermassive stars. Thus turbulent kinetic energy associated with convection or internal magnetic disturbances scales as $1/R^2$ and will be effective in this regard. This has been discussed by Bardeen and Anand (1966).

THE LIMIT OF ROTATIONAL STABILITY

Even though the rotation required to remove the general relativistic instability is quite clearly available under typical astronomical circumstances as discussed in the previous section, the question arises whether the required angular momentum will lead to equatorial instability before sufficient contraction and high enough central temperature for hydrogen burning is reached.

It is first necessary to prescribe somewhat more precisely the central temperature required for hydrogen burning through the CNO bi-cycle. Equation (27) of the preceding lecture gave the average energy generation per gm per sec, $\bar{\varepsilon}$, throughout the star and, when multiplied by the mass, this yields the nuclear energy generation rate as

$$M\bar{\varepsilon} \approx 8.8 \times 10^{-44} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} T_c^{11} \text{ erg sec}^{-1}. \tag{53}$$

When $M\bar{\varepsilon}$ from equation (53) is equated to L from equation (1) it is found that the central temperature, T_{cn}, required for nuclear energy generation through the CNO bi-cycle is

$$T_{cn} \approx 2.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^{1/22} {}^\circ K \qquad \text{[CNO bi-cycle]} \tag{54}$$

so that $T_{cn} \approx 6 \times 10^7\ {}^\circ K$ for $M = 10^8\ M_\odot$. This is lower than the estimate, $T_{cn} \sim 8 \times 10^7\ {}^\circ K$, found originally by Hoyle and Fowler (1963a) but is somewhat more precise. It will be noted that the critical central temperature, equation (46), for general relativistic instability is less than that required for hydrogen burning, equation (54), for all masses $M \gtrsim 4 \times 10^5\ M_\odot$. This emphasizes the limitation on non-rotating models for supermassive stars.

With the required temperature in hand it is now necessary to ascertain the limiting central temperature at which rotation, governed by the conservation of angular momentum, leads to the equatorial instability characterized by loss of mass at the equator. It is probably true that a star survives this instability and that nuclear energy generation at the center is not terminated by the loss of mass at the surface but none-the-less this limitation is well worth investigating in some detail. The analysis which has been made to the present point in this paper has been limited to spherical symmetry in the post-Newtonian approximation. Thus the conclusions to follow require that the angular momenta considered be smaller than the critical angular momentum at which distortion from spherical symmetry is large.

The problem is best discussed in terms of angular velocity rather than angular momentum since the critical limiting angular velocity is given quite simply by equating centrifugal force to gravitational force. In terms of angular velocity the rotational energy can be written as

$$\Psi_0 = \tfrac{1}{2} K^2 MR^2 \omega_R^{\,2}. \tag{55}$$

Equation (55) has been written to include the case of differential rotation; $\omega_R = \omega(R)$ is the angular velocity at the equatorial radius and K is a constant which can be determined when $\omega = \omega(r)$ is specified as a function of the radius. For uniform rotation $\omega = \omega_R$ and $K = k$, the radius of gyration in units of R.

When differential rotation is considered the rotational instability may first occur at an arbitrary radius in the equatorial plane. It will be sufficient for our purposes to consider instability at the periphery and at the center. In the first case the critical angular velocity for instability is given by

$$\omega_{CR}^{\,2} = \frac{GM}{R^3} = \frac{4}{3}\pi G\bar{\rho}, \qquad \text{at } r = R, \tag{56}$$

where $\bar{\rho}$ is the mean density of the stellar material. In the second case the critical angular velocity is given by

$$\omega_{cr}^{\,2} = \left(\frac{GM_r}{r^3}\right)_c = \frac{4}{3}\pi G\rho_c\,, \text{ at } r = 0, \tag{57}$$

where ρ_c is the central density. Note that

$$\frac{\omega_{cr}}{\omega_{CR}} = (\rho_c/\bar{\rho})^{\frac{1}{2}}$$

$$= 7.37 \qquad\qquad n = 3 \tag{58}$$

Equations (55) through (58) can be manipulated to yield

$$\frac{\Psi_0}{Mc^2} = \frac{1}{4} K^2\alpha^2\left(\frac{2GM}{Rc^2}\right) \propto \frac{1}{R}, \tag{59}$$

where
$$\alpha = \omega_R/\omega_{CR}, \tag{60}$$

which is convenient when the instability first occurs at the periphery, or

$$\alpha = \left(\frac{\rho_c}{\bar{\rho}}\right)^{\frac{1}{2}}\left(\frac{\omega_R}{\omega_c}\right)\left(\frac{\omega_c}{\omega_{cr}}\right) \tag{61}$$

which is convenient when the instability first occurs at the center. The angular velocity at the center is designated by $\omega_c = \omega(0)$. For uniform rotation $\omega_c = \omega_R$. In this case instability first occurs at the periphery and equation (60) should be used although (61) is formally correct.

For a given type of differential rotation ω_R/ω_c is a fixed constant. It will be assumed in what follows that equatorial instability sets in for $\omega_R/\omega_{CR} = 1$ or $\omega_c/\omega_{cr} = 1$ and that angular momentum transfer to a small amount of mass lost by the star keeps the appropriate ratio constant thereafter. When this is the case Ψ_0/Mc^2 is proportional to R^{-1} and not to R^{-2} as was the case before the onset of equatorial instability.

It should be noted that equation (59) should not be taken to imply that the factor 2 does not appear on the right hand side of equation (33). Equation (59) applies to the relatively slow changes between equilibrium states. During the faster changes which occur during small radial oscillations it would seem reasonable to assume that angular momentum is conserved. Then equations (33) and (39) can be employed as written with Ψ_0 evaluated from (59) with α given by equation (60) or (61). Under the circumstances it will be clear that equation (41) no longer holds and that dynamical instability ($\sigma_R^2 = 0$) no longer sets in at the minimum in the equilibrium energy curve.

In order to illustrate these points in the simplest possible manner, consider only stars for which $K^2\alpha^2 \gg \bar{\beta}$ so that $\bar{\beta}\Omega_0/2$ can be neglected in comparison with Ψ_0 in equation (20) and with $2\Psi_0$ in equation (39). Then from equations (20), (36) and (59) one has

$$- \frac{E_b}{Mc^2} = \frac{E_{eq}}{Mc^2} \approx -\frac{1}{4} K^2\alpha^2 \left(\frac{2GM}{Rc^2}\right) + \zeta_n \left(\frac{2GM}{Rc^2}\right)^2 \tag{62}$$

When this is differentiated with respect to R with all coefficients held constant the maximum binding energy $E_b^{max} = |E_{eq}^{min}|$ is found to be

$$\frac{E_b^{max}}{Mc^2} \approx \frac{K^4\alpha^4}{64\zeta_n} \tag{63}$$

and occurs at the radius

$$\begin{aligned} R(E_b^{max}) &\approx \frac{8\zeta_n}{K^2\alpha^2}\left(\frac{2GM}{c^2}\right) \\ &\approx \frac{3.0\times 10^6}{K^2\alpha^2}\left(\frac{M}{M_\odot}\right) \text{cm} \qquad n = 3 \end{aligned} \tag{64}$$

and at the central temperature

$$T_c(E_b^{max}) \approx 1.95 \times 10^{12}\, K^2\alpha^2 \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} {}^\circ\text{K} \qquad n = 3 \tag{65}$$

It will be noted that E_b^{max} is independent of the original angular momentum Φ possessed by the star before the onset of angular momentum transfer. It can be shown that this is only true if the original angular momentum was large enough that the original rotational energy was equal to or greater than twice the post-Newtonian relativistic term. If this is looked at from another point of view it becomes clear that angular momentum transfer or loss will automatically reduce an originally large angular momentum to the point where the rotational energy is given by equation (59) and the maximum binding energy by equation (63). The consequences in terms of gravitational and nuclear energy release will be emphasized in what follows. A mechanism for the loss of large amounts of angular momentum has been discussed in the case of the sun by Hoyle (1960). It need only be argued that such a mechanism can be effective for supermassive stars as well as has clearly been the case for the sun and other stars.

When $\bar{\beta}\Omega_0/2$ is neglected in equation (39), it becomes

$$\begin{aligned}\sigma_R^2 &\approx \frac{Mc^2}{I}\left[\frac{1}{2}K^2\alpha^2\left(\frac{2GM}{Rc^2}\right)-2\zeta_n\left(\frac{2GM}{Rc^2}\right)^2\right] \approx \frac{2E_b}{I} \\ &\approx \frac{1}{3}\left(\frac{c^3}{2GMk}\right)^2\left[K^2\alpha^2\left(\frac{2GM}{Rc^2}\right)^3 - 4\zeta_n\left(\frac{2GM}{Rc^2}\right)^4\right]\end{aligned} \tag{66}$$

Thus instability sets in with $\sigma_R^2 = 0$ at one-half the radius given by equation (64) and at twice the temperature given by equation (65). Thus the critical radius for instability is

$$R_{cr} = R(\sigma_R^2 = 0) \approx \frac{1.5\times10^6}{K^2\alpha^2}\left(\frac{M}{M_\odot}\right) \text{ cm} \qquad n = 3 \tag{67}$$

and the critical central temperature is

$$T_{cr} = T_c(\sigma_R^2 = 0) \approx 3.9\times10^{12}\, K^2\alpha^2\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} {}^\circ\text{K} \qquad n = 3 \tag{68}$$

These equations are to be compared respectively with equations (11) and (12) of the previous lecture. It will be apparent from equations (20), (39) and (62) that the binding energy is zero where $\sigma_R^2 = 0$. The maximum value for σ_R or the minimum period occurs when the binding energy is a maximum. When $\bar{\beta}\Omega_0/2$ is not neglected, $\sigma_R^2 = 0$ occurs between E_{eq}^{min} and $E_{eq} = 0$.

For uniform rotation $K^2 = k^2 = 0.075$ for a polytrope of index 3 and thus $T_c(E_b^{max})$ is only 1.5×10^7 °K and T_{cr} is only 3×10^7 °K for a star with $M = 10^8 M_\odot$ even when the maximum $\alpha = \omega_R/\omega_{CR} = 1$ is used in equation (67). This is not sufficient for hydrogen burning since 6×10^7 °K is required by equation (54). The limiting mass which can be stabilized by uniform rotation during hydrogen burning is approximately $10^7 M_\odot$.

Differential rotation with an increase in angular velocity toward the center of the star results in a marked increase in K^2 and thus in T_{max}. Two models with differential rotation have been considered. In the first model the massive star is assumed to contract from a cloud with polytropic index $n = 0$ to a structure with index $\underline{n}$ in such a way that each <u>spherical</u> shell retains its angular momentum. This model is not self-consistent in that the Newtonian equation for hydrostatic equilibrium cannot be satisfied by a polytropic relation between p and ρ when the centrifugal forces are not neglected. The second model is that of Stoeckly (1965) in which the star contracts in such a way that the angular momentum is conserved in each <u>cylindrical</u> shell (but not each ring) parallel to the axis of rotation. In this model the polytropic relation may be employed when centrifugal forces are included in the equation for hydrostatic equilibrium. The results for the two models are fortunately very similar as will be noted in the following tabulation:

n	0	1	2	3	4
K^2 (spherical model)	0.400	0.629	1.14	2.61	10.8
K^2 (cylindrical model)	0.400	0.624	1.10	2.47	9.8

In the spherical contraction model the angular velocity throughout the star relative to that at the periphery is given by

$$\frac{\omega_r}{\omega_R} = \left(\frac{M_r}{M}\right)^{2/3} \left(\frac{R}{r}\right)^2$$
$$= (\bar{\rho}_r/\bar{\rho})^{2/3} \tag{69}$$

where $\bar{\rho}_r$ is the mean density internal to r and $\bar{\rho}$, as before, is the mean density for the entire star. The maximum angular velocity occurs at the center and is given by

$$\frac{\omega_c}{\omega_R} = (\rho_c/\bar{\rho})^{2/3}$$
$$= (54.18)^{2/3} = 14.3 \qquad n = 3 \tag{70}$$

where ρ_c is the central density. Equation (70) has been evaluated for a polytrope, in this case n = 3, in spite of the lack of self-consistency noted above. For the case of cylindrical contraction, for which the polytropic model can be employed without difficulty, the numerical value in equation (70) becomes 10.9 instead of 14.3.

It will now be clear from equations (58) and (70) that rotational instability when centrifugal forces match gravitational forces occurs first at the center rather than at the periphery in these cases of differential rotation. Thus equation (61) is to be employed at this point rather than equation (60) in determining the consequences of differential rotation.

The critical temperature given by equation (68) occurs at zero binding energy and requires that a large supply of energy become available after the maximum binding energy or the minimum total energy is reached at one-half the critical temperature. Thus it would appear that nuclear energy generation must at least start at $T_c(E_b^{max})$ and so this temperature will now be computed. For the contraction with angular momentum conservation in each cylindrical shell up to the point of rotational instability in a polytrope of index n = 3 it is found that $K^2 = 2.47$ and $\alpha^2 = 0.456\ (\omega_c/\omega_{cr})^2$. Thus

$$T_c(E_b^{max}) \approx 2.20 \times 10^{12} \left(\frac{\omega_c}{\omega_{cr}}\right)^2 \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} \tag{71}$$

If this temperature is equated to T_{cn} from equation (54), the solution will yield the maximum mass in which nuclear energy generation is triggered before the maximum binding energy is reached. The result is

$$\frac{M}{M_\odot} \leq 10^9 \left(\frac{\omega_c}{\omega_{cr}}\right)^{11/3} \tag{72}$$

Equation (72) indicates that the limiting mass is quite sensitive to the choice of ω_c/ω_{cr}, i.e., to the value of the angular velocity ω_c at which centrifugal forces may tend to disrupt the star rather than lead to stable angular momentum transfer and small mass loss. If $\omega_c = \omega_{cr}$ then $M \leq 10^9\ M_\odot$ but a more conservative choice would seem to be $M \leq 10^8\ M_\odot$. In Figures 2 and 3 which are drawn for $M = 10^8\ M_\odot$ the rotational limit indicated is for $(\omega_c/\omega_{cr})^2 = \frac{1}{2}$. This limit has not been reached at $6 \times 10^7\ {}^\circ K$ at which hydrogen burning takes place.

It is of interest to calculate the maximum binding energy, equation (63), for the example discussed above with $K^2 = 2.47$, $\alpha^2 = 0.456\ (\omega_c/\omega_{cr})^2$ and $\zeta_3 = 1.265$. The calculation yields

$$\frac{E_b^{max}}{Mc^2} \approx 0.016 \left(\frac{\omega_c}{\omega_{cr}}\right)^4 \tag{73}$$

Again the result is quite sensitive to the choice of ω_c/ω_{cr} but for the maximum reasonable choice of unity it is seen that E_b^{max} can be 1.6 per cent of Mc^2 which is about 5 times that from the burning of one-half the hydrogen of the star. Since this energy must be lost during contraction it is another source for the observed energy emissions in quasars. It will be noted this arises fundamentally because the coefficient of Ψ_0 in equation (19) is $1-\overline{\beta}$ and $\overline{\beta}$ is small in supermassive stars. In ordinary stars $\overline{\beta} = 1$ and the term in Ψ_0 in equation (19) vanishes; the binding energy is $\frac{1}{2}\Omega_0$ with or without rotation since rotational energy and internal kinetic energy enter into the virial theorem in the same manner.

DYNAMIC AND ROTATIONAL PERIODS

The dynamic period of the fundamental mode of radial oscillation has been given in equations (50) through (52) for the case in which rotation is just large enough to cancel the post-Newtonian term in equation (43). If rotation is limited only by equatorial instability the rotational term is larger than the term in β in equation (43) and is also larger than the post-Newtonian term in equation (66) during the early stages of contraction. Hydrogen burning occurs during this early stage except for $M \gtrsim 10^9\ M_\odot$. Thus a useful approximation for the dynamical period with large rotation is

$$\Pi \approx \left(\frac{6\pi^2 k^2 R^3}{K^2 \alpha^2\ GM}\right)^{\frac{1}{2}} \tag{74}$$

$$\approx 5.5 \times 10^{-21}\ R^{\frac{3}{2}} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}}\ \text{yr} \qquad n = 3 \tag{75}$$

$$\approx 7.7 \times 10^{7}\ T_c^{-\frac{3}{2}} \left(\frac{M}{M_\odot}\right)^{\frac{1}{4}}\ \text{yr} \qquad n = 3 \tag{76}$$

In deriving (74), $I = \frac{2}{3}\ k^2 MR^2$ has been used. The numerical values are for $n = 3$, $k^2 = 0.075$, $\zeta_3 = 1.268$, $K^2 = 2.47$, $\alpha^2 = 0.456$. When the temperature required for hydrogen burning, equation (54), is substituted into (76) the result is

$$\Pi(T_{cn}) \approx 6.1 \times 10^{-4} \left(\frac{M}{M_\odot}\right)^{\frac{2}{11}}\ \text{yr} \qquad n = 3 \tag{77}$$

$$\approx 0.22 \left(\frac{M}{M_\odot}\right)^{\frac{2}{11}}\ \text{day}$$

so that the periods during nuclear energy generation range from ~ 1 to 6 days for $M = 10^3$ to $10^8\ M_\odot$.

A minimum is eventually reached in the period as the post-Newtonian general relativistic term becomes important. This minimum period is given by

$$\Pi_{min} \approx \frac{256\pi k \zeta_n^{\frac{3}{2}}}{3K^4\alpha^4}\left(\frac{2GM}{c^3}\right) \tag{78}$$

$$\approx 2.6 \times 10^{-11}\left(\frac{M}{M_\odot}\right) \text{yr} \qquad n = 3 \tag{79}$$

$$\approx 1.0 \times 10^{-8}\left(\frac{M}{M_\odot}\right) \text{day} \qquad n = 3 \tag{80}$$

The minimum period is very short being the order of only one day even for $M = 10^8\ M_\odot$. The periods given in (74) through (80) are quite sensitive to the choice for ω_c/ω_{cr} in α and could be an order of magnitude greater for a more conservative choice than the $\omega_c/\omega_{cr} = 1$ used here.

If equations (56) and (60) are employed it can be shown that in the early stage of contraction the rotational period at the periphery is given by

$$P_R = \frac{2\pi}{\omega_R} \approx \left(\frac{4\pi^2 R^3}{\alpha^2\ GM}\right)^{\frac{1}{2}} \tag{81}$$

$$\approx \left(\frac{2K^2}{3k^2}\right)^{\frac{1}{2}} \Pi = 4.7\Pi \qquad n = 3 \tag{82}$$

The central rotational period is an order of magnitude shorter. As contraction proceeds Π approaches a minimum while P_R does not. At the minimum in Π it is found that

$$P_R(\Pi_{min}) \approx \left(\frac{K^2}{6k^2}\right)^{\frac{1}{2}} \Pi_{min} = 2.3\ \Pi_{min} \tag{83}$$

The peripheral velocity is given by

$$v_R = \omega_R R \approx \left(\frac{\alpha^2 GM}{R}\right)^{\frac{1}{2}} \tag{84}$$

so that

$$\frac{v_R}{c} \approx 10^{-7} \left(\frac{M}{M_\odot}\right)^{\frac{1}{4}} T_c^{\frac{1}{2}} \qquad n = 3 \qquad (85)$$

For $T_c = 10^8$ °K and $M = 10^8 M_\odot$ the peripheral velocity is 10% of the velocity of light.

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CONCLUSION

The results presented in this lecture constitute an extension of the previous lecture material on relaxation oscillations in supermassive stars. It has been shown that a relatively small amount of rotation is sufficient to remove the general relativistic instability which arises in such stars when rotation is absent. The post-Newtonian equations for the binding energy and for the frequency of the fundamental mode of radial oscillations have been derived and close connection between these two quantities has been exhibited. The equatorial instability associated with contraction under rotation has been investigated and the results used to estimate the limiting mass in which hydrogen burning can be effective as a source of energy during relaxation oscillations. For differential rotation this limit is found to be at least $10^8 M_\odot$ and perhaps as high as $10^9 M_\odot$ whereas, without rotation, the limit arising from general relativistic considerations is $10^6 M_\odot$. For uniform rotation the limit is $10^7 M_\odot$.

Author

APPENDIX

In this appendix the relations used in the main text between the internal energy density u, and pressure p, both in erg cm^{-3}, are discussed and the use of various expressions for the "effective ratio of specific heats" is clarified. It is sufficiently general for our purposes to consider the medium to be approximately non-degenerate and to be made up of nuclei, ionization electrons, electron-positron pairs and radiation as treated in detail by Fowler and Hoyle (1964). Then from equations (B62) and (B43) of this reference one has

$$u = xnkT + 2n_+ m_e c^2 + aT^4 \tag{A1}$$

$$p = qnkT + \frac{1}{3} aT^4 \tag{A2}$$

where $n = n_0 + n_N + 2n_+$ is the number density of all particles -- ionization electrons, nuclei and electron-positron pairs, x is the mean kinetic energy per particle in units of kT and q is a factor, close to unity in value, which incorporates the deviations from Boyle's Law in the gas. The internal energy density includes everything except the rest mass-energy density, $\rho_0 c^2$, of the nuclei and the associated ionization electrons.

The number density of paired electrons and positrons is

$$2n_+ \approx (n_0^2 + 4n_1^2)^{\frac{1}{2}} - n_0 \tag{A3}$$

where

$$n_0 = Zn_N = \frac{\rho_0 Z}{M_u A} = 6.02 \times 10^{23} \rho_0 \frac{Z}{A} \tag{A4}$$

is the original number of ionization electrons, n_N is the number density of nuclei, ρ_0 is the rest mass density, M_u is the atomic mass unit, Z is the mean number of free electrons per nucleus, A is the mean nuclear mass plus that of associated electrons in atomic mass units, and

$$n_1 = \frac{1}{\pi^2 z}\left(\frac{m_e c}{\hbar}\right)^3 K_2(z) \tag{A5}$$

In (A5), $z = m_e c^2/kT$ and $K_2(z)$ is the modified Bessel function of second order. Numerically one has

$$n_1 \approx 1.521 \times 10^{29}\, T_9^{3/2} \exp(-5.93/T_9)\ \mathrm{cm}^{-3} \qquad T_9 < 3 \tag{A6}$$

$$\approx 1.688 \times 10^{28}\, T_9^{3}\ \mathrm{cm}^{-3} \qquad T_9 > 3 \tag{A7}$$

Because of the low density in massive stars for a given temperature, the number of positrons and paired electrons becomes comparable to the number of ionization electrons at relatively low temperatures, e.g. at $5 \times 10^8\ ^\circ K$ in a star with $M = 10^8\ M_\odot$. This is above the temperature for hydrogen to helium conversion however.

The factor x in equation (A1) is equal to 3/2 for non-relativistic, non-degenerate electrons and nuclei and has been tabulated for relativistic, non-degenerate electrons by Chandrasekhar (1939) as U/PV in Table 24, p. 397. The entries in this table also apply to the pair positrons under relativistic non-degenerate conditions. Although the entries in the table range from $x = 3/2$ up to maximum value, $x = 3$, there are circumstances (Fowler and Hoyle 1964) under which x can be as high as 3.15, in which case $q = 1.05$. At low temperatures pairs can be neglected, the electrons and nuclei may recombine into atoms and molecules and in any case x can be found in terms of the specific heat at constant volume c_V or the ratio of specific heats γ from

$$\frac{d(xT)}{dT} = c_V = \frac{1}{\gamma - 1} \tag{A8}$$

When x is constant, one has

$$x \approx c_V = \frac{1}{\gamma - 1} \tag{A9}$$

Under the circumstances of major interest in this paper, the nuclei are ionized, the electrons are non-relativistic and non-degenerate and pairs can be neglected. Then $\gamma = 5/3$ and $x = c_V = 3/2$.

If $\beta = qnkT/p$ is introduced as the ratio of gas pressure to total pressure and $1-\beta = aT^4/3p$ as the ratio of radiation pressure to total pressure, then from equations (A1) and (A2) the dimensionless ratio of internal energy density to pressure is given by

$$\frac{u}{p} = 3 - (\beta/q)\,[3q - x - z(2n_+/n)] \qquad \text{(A10)}$$

As is required relativistically this ratio approaches 3 at very high temperatures independent of β since then $kT \gg m_e c^2$, $z \to 0$, $x/q \to 3$ and $2n_+ \to n$. When pairs are first copiously produced this ratio can exceed 3 under certain circumstances. The relativistic behavior for β is discussed in detail by Fowler and Hoyle (1964); it passes through a minimum near zero in massive stars but increases to a limiting value, $\beta = 7/11$, at high temperatures when pairs become copious.

The customary non-relativistic expression for u/p is found by setting $q = 1$, $x = (\gamma-1)^{-1}$ and $n_+ = 0$ so that

$$\frac{u}{p} \approx \frac{\beta}{\gamma-1} + 3(1-\beta) = \frac{3(\gamma-1) - \beta(3\gamma-4)}{\gamma-1} \qquad \text{NR} \qquad \text{(A11)}$$

It is convenient at this point to introduce a quantity which is very similar to the adiabatic coefficients Γ_1, Γ_2 and Γ_3 defined by Chandrasekhar (1939) pp. 57 and 58. We denote this quantity by Γ_4 and define it by

$$\Gamma_4 - 1 \equiv p/u$$

$$= \frac{1}{3 - (\beta/q)[3q - x - z(2n_+/n)]} \qquad \text{(A12)}$$

$$\approx \frac{\gamma-1}{3(\gamma-1) - \beta(3\gamma-4)} = \frac{\gamma-1}{1 + (1-\beta)(3\gamma-4)} \qquad \text{NR} \qquad \text{(A13)}$$

Thus
$$\Gamma_4 = \frac{4 - (\beta/q)[3q - x - z(2n_+/n)]}{3 - (\beta/q)[3q - x - z(2n_+/n)]} \tag{A14}$$

$$= \frac{4}{3} + \frac{(\beta/q)[3q - x - z(2n_+/n)]}{9 - 3(\beta/q)[3q - x - z(2n_+/n)]} \tag{A15}$$

$$\approx \frac{4(\gamma-1) - \beta(3\gamma-4)}{3(\gamma-1) - \beta(3\gamma-4)} \qquad \text{NR} \tag{A16}$$

$$\approx \frac{4}{3} + \frac{\beta(3\gamma-4)}{9(\gamma-1) - 3\beta(3\gamma-4)} \qquad \text{NR} \tag{A17}$$

It will be clear from the definition of Γ_4 that averaging over the entire volume of the star yields

$$\int p dV = \int(\Gamma_4-1)u dV = (\bar{\Gamma}_4-1)\int u dV \tag{A18}$$

The appropriate mean value for Γ_4 is that obtained by averaging over each element of internal energy, udV.

Extreme relativistic conditions arise when $x = 3q$ and $z = 0$ in (A14) or (A15) in which case $\Gamma_4 = 4/3$ as expected. Under intermediate circumstances Γ_4 can be found by using (A14). However, under the circumstances of major interest in this paper, (A16) with $\gamma = 5/3$ is applicable and

$$\Gamma_4 \approx \frac{8 - 3\beta}{6 - 3\beta} \qquad \gamma = 5/3 \qquad \text{NR} \tag{A19}$$

$$\approx \frac{4}{3} + \frac{\beta}{6} \qquad \beta << 1 \qquad \text{NR} \tag{A20}$$

where the second approximation holds for small β. This is the same approximation that holds for the first of Chandrasekhar's adiabatic coefficients, $\Gamma_1 \equiv - d \ln p/d \ln V$ when β is small. As a matter of fact in massive stars during hydrogen burning β is quite small being given by (Fowler and Hoyle 1964)

$$\beta \approx \frac{4.28}{\mu}\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} \qquad M > 10^4 M_\odot \qquad \text{NR} \tag{A21}$$

where $\mu = A/(Z+1)$ is the mean "molecular" weight. Note that $\beta \sim 10^{-3}$ for $M = 10^8 M_\odot$ and $\mu = 1/2$ (hydrogen). As discussed in the main text it is the smallness of β and the closeness of Γ_1 and Γ_4 to 4/3 which makes the Newtonian terms in the binding energy and pulsation frequency correspondingly small and thus brings the rotational and general relativistic terms into prominence in these quantities. It will be noted that Γ_1 and Γ_4 are effective ratios of specific heats under appropriate circumstances.

In the above analysis the ratio p/u at a given time and position in a star has been equated to Γ_4-1. In addition it is necessary to establish relationships between δp and δu and between $\delta\int p dV$ and $\delta\int u dV$ when the star is subject to an adiabatic perturbation at all points. The general relativistic adiabatic relation is

$$\delta Q = \delta(\rho c^2 V) + p\delta V = 0 \tag{A22}$$

where it is required that the volume V apply to a fixed number of baryons throughout the adiabatic change. This requirement follows from the generally accepted physical law of the conservation of baryons. Under the conditions of interest in this paper the only baryons involved are protons and neutrons, free or incorporated in nuclei as nucleons.

In order to proceed it is necessary to recall once again the relation

$$\frac{\delta p}{p} = -\Gamma_1 \frac{\delta V}{V} \tag{A23}$$

by which Chandrasekhar's first adiabatic coefficient is defined. If equations (A22) and (A23) are appropriately manipulated it is found that

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta\rho}{\rho + p/c^2} \tag{A24}$$

and

$$\frac{\delta(pV)}{\delta(\rho c^2 V)} = \Gamma_1 - 1 \tag{A25}$$

Under some circumstances it is of interest to consider adiabatic changes during which no nuclear or atomic processes occur so that the rest mass associated with a given number of baryons (or nucleons) does not change. Under these circumstances $\rho_o V$ is an invariant and $\delta(\rho c^2 V) = \delta(\rho_o c^2 V) + \delta(uV)$ becomes just equal to $\delta(uV)$. Then

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta\rho_o}{\rho_o} \qquad (\rho_o V = \text{const}) \tag{A26}$$

$$\frac{\delta(pV)}{\delta(uV)} = \frac{\delta(p/\rho_o)}{\delta(u/\rho_o)} = \Gamma_1 - 1 \qquad (\rho_o V = \text{const}) \tag{A27}$$

and

$$\frac{\delta p}{\delta u} = \frac{\Gamma_1}{\Gamma_4}\,\frac{p}{u} = \frac{\Gamma_1}{\Gamma_4}(\Gamma_4 - 1) \qquad (\rho_o V = \text{const}). \tag{A28}$$

Now consider the variations $\delta\int pdV$ and $\delta\int udV$ corresponding to adiabatic changes made throughout the entire star. These can be written respectively as

$$\delta\int pdV = \delta\int(pV)\frac{dV}{V} = \int\delta(pV)\frac{dV}{V} = \int(\Gamma_1 - 1)\,\delta(uV)\frac{dV}{V} \tag{A29}$$

and

$$\delta\int udV = \delta\int(uV)\frac{dV}{V} = \int\delta(uV)\frac{dV}{V}\,. \tag{A30}$$

The second equality in each of these equations derives from the fact that dV and V must each apply to a fixed number of baryons during any perturbation. Thus dV/V is replaceable by dN_B/N_B where N_B is the total number of baryons in the star and is therefore clearly invariant to any perturbation under consideration. In the last equality in (A29), equation (A27) has been used. Then, since Γ_1 and its average $\bar{\Gamma}_1$ over udV are constant to first order during any perturbation, it ultimately follows that

$$\begin{aligned} \delta\int pdV &= \delta\int(\Gamma_4 - 1)udV \\ &= (\bar{\Gamma}_1 - 1)\,\delta\int udV \qquad (\rho_o V = \text{const}). \end{aligned} \tag{A31}$$

It will be clear that Γ_4 is not constant during an adiabatic perturbation and, in fact, it can be shown that

$$\frac{\delta\Gamma_4}{\Gamma_4 - 1} = (\Gamma_4 - \Gamma_1)\frac{\delta V}{V} \tag{A32}$$

Comparison of (A31) with (A18) indicates that $\bar{\Gamma}_1$ replaces $\bar{\Gamma}_4$ in relations involving adiabatic perturbations. To first order in small β, Γ_1 and Γ_4 and hence $\bar{\Gamma}_1$ and $\bar{\Gamma}_4$ are equal. This can be seen from Chandrasekhar's non-relativistic expression for Γ_1 which corresponds to (A16). This expression is

$$\Gamma_1 \approx \beta + \frac{(4-3\beta)^2\,(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)} \qquad \text{(NR)} \qquad \text{(A33)}$$

$$\approx \frac{4}{3} + \frac{(4\beta-3\beta^2)(3\gamma-4)}{36(\gamma-1) - 3\beta(12\gamma-13)} \qquad \text{(NR)} \qquad \text{(A34)}$$

$$\approx \frac{32 - 24\beta - 3\beta^2}{24-21\beta} \qquad \gamma = 5/3 \qquad \text{(NR)} \qquad \text{(A35)}$$

$$\approx \frac{4}{3} + \frac{\beta}{6} \approx \Gamma_4 \qquad \beta << 1 \qquad \text{(NR)} \qquad \text{(A36)}$$

Fowler and Hoyle (1964) p. 289, give the relativistic expression for Γ_1 when pairs are included. Actually Γ_1 does not differ greatly from Γ_4 over the range $0 \le \beta \le 1$ as illustrated in Table A1. The two are equal at the two extremes of this range with $\Gamma_1 = \Gamma_4 = 4/3$ at $\beta = 0$ and $\Gamma_1 = \Gamma_4 = 5/3$ at $\beta = 1$ for $\gamma = 5/3$. In addition for small β, Γ_1 and Γ_4 are equal for any γ since

$$\Gamma_1 \approx \Gamma_4 \approx \frac{4}{3} + \frac{\beta}{3}\,\frac{\gamma - 4/3}{\gamma - 1} \qquad \beta < 1 \qquad \text{(NR)} \qquad \text{(A37)}$$

For convective stability it is necessary that

$$\frac{d\,\ln p}{dr} > \Gamma_1 \frac{d\,\ln \rho}{dr} \qquad \text{(CONV. STAB.)} \qquad \text{(A38)}$$

Chandrasekhar (1965b) has shown that this is a necessary and sufficient condition in general relativity except in very special physical situations involving only small regions in a star where the effect of general relativistic modifications

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TABLE A1

VALUES OF Γ_1 AND Γ_4 FOR $\gamma = 5/3$

β	0	0.01	0.1	0.2	0.3	0.4
Γ_1	1.333	1.335002	1.350	1.368	1.386	1.405
Γ_4	1.333	1.335008	1.351	1.370	1.392	1.417
β	0.5	0.6	0.7	0.8	0.9	1.0
Γ_1	1.426	1.449	1.476	1.511	1.563	1.667
Γ_4	1.444	1.476	1.512	1.556	1.606	1.667

is not of crucial importance. Since p and ρ usually decrease with r it is often convenient to use (A38) rewritten as

$$\left|\frac{d\, \ell n\, p}{dr}\right| < \Gamma_1 \left|\frac{d\, \ell n\, \rho}{dr}\right| \qquad \text{(CONV. STAB.)} \tag{A39}$$

For a polytrope at index n with $p \propto \rho^{1+1/n}$ this yields

$$\left(1 + \frac{1}{n}\right) < \Gamma_1$$

or

$$n > \frac{1}{\Gamma_1 - 1} \qquad \text{(CONV. STAB.)} \tag{A40}$$

$$> 3(1 - \beta/2) \qquad T_9 < 1$$

Thus the polytrope $n = 3$ which has been used extensively throughout this lecture is convectively stable except when electron-positron pair formation reduces Γ_1 below 4/3 in the range $1 < T_9 < 3$. Formation of other particles will reduce Γ_1 below 4/3 in additional ranges at still higher temperatures.

An important quantity in the considerations discussed in this lecture is $\bar{\beta}$, the ratio of gas pressure to total pressure averaged over the internal energy distribution in the star. See, for example, equations (17) and (19). It can be shown from the analysis of Fowler (1964) and Fowler and Hoyle (1964) that, for massive stars,

$$\frac{\bar{\beta}}{\beta_c} = \frac{\int \theta^{\frac{5n+1}{4}} \xi^2\, d\xi}{\int \theta^{n+1} \xi^2\, d\xi} \tag{A41}$$

where θ and ξ are the customary dimensionless variables in the Lane-Emden equation for the polytrope of index n and

$$\mu\beta_c = \left(\frac{3}{4\pi} (n+1)^3 \frac{\Re^4 M_n^2}{aG^3 M_\odot^2}\right)^{\frac{1}{4}} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} \tag{A42}$$

Numerical values for $\mu\beta_c(M/M_\odot)^{\frac{1}{2}}$ and $\mu\bar{\beta}(M/M_\odot)^{\frac{1}{2}}$ are tabulated in Table A2. Note that the latter quantity is approximately independent of n. In Table A2 R_n and M_n are the constants of integration corresponding to radius and mass respectively for the Lane-Emden equation.

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TABLE A2

n	R_n	M_n	$\rho_c/\bar{\rho}$	$\bar{\beta}/\beta_c$	$\mu\beta_c(M/M_\odot)^{\frac{1}{2}}$	$\mu\bar{\beta}(M/M_\odot)^{\frac{1}{2}}$
0	2.4494	4.8988	1.0000	1.8729	2.3569	4.4142
0.5	2.7528	3.7871	1.8361	1.5525	2.8088	4.3607
1.0	3.1416	3.1416	3.2899	1.3634	3.1743	4.3278
1.5	3.6538	2.7141	5.9907	1.2343	3.4879	4.3051
2.0	4.3529	2.4111	11.4025	1.1383	3.7691	4.2900
2.5	5.3553	2.1872	23.4065	1.0625	4.0299	4.2817
3.0	6.8969	2.0182	54.1825	1.0000	4.2788	4.2788
3.5	9.5358	1.8906	152.884	0.9465	4.5237	4.2817
4.0	14.9716	1.7972	622.408	0.8992	4.7734	4.2922
4.5	31.8365	1.7378	6189.47	0.8558	5.0416	4.3146
5.0	∞	1.7321	∞	0.8136	5.3727	4.3712

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Nuclear Energy Generation in Supermassive Stars*

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I. INTRODUCTION

Since the pioneer work of Hans Bethe[1] it has been known that the reactions of carbon and nitrogen with protons are the most important source of energy in main sequence stars with mass greater than a value approximately equal to that of the sun. Detailed analysis of the experimental reaction rates in the CN cycle and the proton-proton chain has indicated[2,3] that the cycle supplants the pp chain as the main source of energy in Population I stars at a central temperature near 2×10^7 $^{\circ}$K and at a somewhat higher temperature in Population II stars. This is illustrated for Population I stars in Fig. 1. Since the central temperature in the sun[4] is 1.6×10^7 $^{\circ}$K and since central temperature increases with mass, it follows that the crossover point occurs in stars with mass definitely in excess of the solar mass. Experimental studies of the rates of the reactions of the oxygen isotopes, O^{16} and O^{17}, with protons has shown that these isotopes react rapidly[5] at elevated temperatures with the cyclic production of N^{14}. The overall result is the CNO bi-cycle which is depicted in Table I.

At the present time there is some interest in the operation of the CNO bi-cycle in stars with mass in excess of 10^3 solar masses ($M \geq 10^3$ $M_{\odot}$) which have been termed supermassive stars. It has been suggested[6-13] that nuclear and gravitational energy release in supermassive stars is sufficient to meet

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Reactions of the CNO-cycle (June, 1965).

The CNO-cycle		Energy Release	S_o(keV-barns) or $\bar{\tau}$	Solar $f_o S_o$
$C^{12} + H^1 \rightarrow N^{13} + \gamma$		1.94	1.53	2.2
$N^{13} \rightarrow C^{13} + \beta^+ + \nu$		1.50	$\bar{\tau}$ = 870 sec	
$C^{13} + H^1 \rightarrow N^{14} + \gamma$		7.55	5.9	8.4
$N^{14} + H^1 \rightarrow O^{15} + \gamma$		7.29	3.0	4.5
$O^{15} \rightarrow N^{15} + \beta^+ + \nu$		1.73	$\bar{\tau}$ = 178 sec	
$N^{15} + H^1 \rightarrow C^{12} + He^4$		4.96	7.5×10^4	1.1×10^5
or (1/2200)	(6% ν-loss)	24.97 MeV		
$N^{15} + H^1 \rightarrow O^{16} + \gamma$		12.13	32	48
$O^{16} + H^1 \rightarrow F^{17} + \gamma$		0.60	9.9	16
$F^{17} \rightarrow O^{17} + \beta^+ + \nu$		1.76	$\bar{\tau}$ = 95 sec	
$O^{17} + H^1 \rightarrow N^{14} + He^4$		1.19	10	16
	(1/2200)	15.68 MeV		
$4H^1 \rightarrow He^4$	Total =	26.7313 MeV ± .0005		

the energy requirements in quasi-stellar objects or quasars and in extragalactic radio sources. This suggestion has been discussed in detail in the references just cited and will not be further elaborated upon in this paper. It will suffice in Part II to establish an average energy generation per unit mass and per unit time in supermassive stars which is demanded by this suggestion. Nuclear energy generation in supermassive stars is of interest per se.

The paper then continues with an analysis in Part III of energy generation through the CNO bi-cycle using the latest nuclear reaction rate data. The central temperature at which the required energy generation is met is determined.

In Part IV the general relativistic instability which arises at very low central temperatures during the early stages of contraction in supermassive stars is introduced. The effects of uniform and differential rotation are discussed and a comparison made between the limiting temperature for dynamic stability and the required nuclear temperature. The limiting masses for operation of the CNO bi-cycle under stable conditions are found to be approximately $10^6\ M_\odot$, $10^7\ M_\odot$ and $10^9\ M_\odot$ for no rotation, maximum uniform rotation and maximum differential rotation respectively.

II. AVERAGE ENERGY GENERATION REQUIRED IN SUPERMASSIVE STARS

We first seek the average energy generation required in stable supermassive stars. On the basis that such stars are largely convective polytropes of index $n = 3$ except near the surface where the flux transport is entirely radiative, and that radiative pressure is large compared to gas pressure, it is found[6] that the luminosity L is proportional to the mass M according to the relation

$$L \approx 4\pi c G M \kappa^{-1} \quad (1)$$

where c is the velocity of light, G is the gravitational constant and κ is the opacity. The effective surface temperature is found to be so high ($\sim 7 \times 10^4\ {}^oK$) that hydrogen and helium are effectively wholly ionized at the photosphere and the opacity is primarily due to electron scattering, $\kappa = 0.19\ (1 + x_H)$, where x_H is the fraction by mass of hydrogen. Numerically Eq. (1) becomes

$$L \approx \frac{2.6 \times 10^{38}}{1 + x_H} \frac{M}{M_\odot} \quad \text{ergs sec}^{-1} \tag{2}$$

or

$$\frac{L}{M} \approx \frac{1.3 \times 10^5}{1 + x_H} \quad \text{ergs gm}^{-1}\ \text{sec}^{-1} \tag{3}$$

The luminosity-mass ratio given by Eq. (3) yields the average energy generation per unit mass and per unit time in a supermassive star. Since x_H is less than unity we have to order of magnitude

$$\bar{\varepsilon}_{SMS} \approx 10^5 \qquad \text{ergs gm}^{-1}\ \text{sec}^{-1} \tag{4}$$

The energy released in the conversion of hydrogen into helium is 6×10^{18} ergs gm^{-1}. On the assumption that one-half of the mass of the star is eventually converted from hydrogen into helium, the lifetime for the main sequence stage of a supermassive star is found to be

$$\tau \approx 3 \times 10^{13}\ \text{sec} \approx 10^6\ \text{yr} \tag{5}$$

independent of mass.

III. ENERGY GENERATION IN THE CNO BI-CYCLE

A detailed analysis of the reaction rates found experimentally for the interaction of the carbon, nitrogen and oxygen isotopes with protons has been made by Caughlan and Fowler.[5] They find that the oxygen isotopes have lifetimes considerably shorter than 10^6 yr for temperatures above $0.5 \times 10^8\ {}^oK$ which is the lower limit of temperatures relevant to hydrogen burning in supermassive stars. In what follows it will be found that the relevant range falls in the interval 0.5 to 0.8 x 10^8 ^oK. Thus the bi-cycle, Table I, is fully operative. They also find that the $N^{14}(p,\gamma)O^{15}$ reaction is the slowest among the carbon-nitrogen isotopes and that the overall rate of the bi-cycle is essentially determined by the rate of this reaction. When all the reactions are in equilibrium, their results yield

$$\varepsilon_{HCNO} = 1.23 \times 10^{26}\ \rho x_H\ x_{14}\ S_{eff}\ T_8^{-2/3}\ \exp\left(-32.81\ T_8^{-1/3}\right)\ \text{ergs gm}^{-1}\ \text{sec}^{-1} \qquad (6)$$

where ρ is the density, x_{14} is the concentration by mass of N_{14}, $T_8 = T/10^8\ ^oK$ is the temperature in units of $10^8\ ^oK$ and S_{eff} is the effective cross section factor for the $N^{14}(p,\gamma)O^{15}$ reaction in keV-barns. The density is relatively low at hydrogen burning in supermassive stars and the usual correction for electron screening can be neglected.

It is also found at equilibrium in the temperature range previously noted that $x_{14} \approx 0.93\ x_{CNO}$ where x_{CNO} is the fraction by mass of carbon, nitrogen and oxygen. Hebbard and Bailey[14] give the empirical values $S_0 = 2.75 \pm 0.50$ keV-barns and $\langle dS/dE \rangle = 0$ from an extrapolation to zero energy of their low energy measurements on $N^{14}(p,\gamma)$. These values yield $S_{eff} \approx 2.79$ keV-barns and thus

$$\varepsilon_{HCNO} \approx 3.19 \times 10^{26}\ \rho x_H\ x_{CNO}\ T_8^{-2/3}\ \exp\left(-32.81\ T_8^{-1/3}\right)\ \text{ergs gm}^{-1}\ \text{sec}^{-1} \qquad (7)$$

The supermassive stars will be treated as Population I stars and thus reasonable values for the composition are $x_{CNO} = 0.03$ and $x_H = 0.4$ after some exhaustion of the original hydrogen. In a massive star with polytropic index n = 3 the density can be expressed[15] in terms of the temperature as

$$\rho \approx 130 \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_8^{\ 3}\ \text{gm cm}^{-3} \qquad (8)$$

The substitution of $x_H x_{CNO} = 0.012$ and of Eq. (8) into Eq. (7) yields

$$\varepsilon_{HCNO} \approx 4.98 \times 10^{26} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_8^{7/3} \exp\left(-32.81\ T_8^{-1/3}\right) \text{ ergs gm}^{-1} \text{ sec}^{-1} \qquad (9)$$

For the purposes at hand it is necessary to express the average energy generation throughout the star in terms of the central temperature. When the energy generation can be written as proportional to $\rho^{u-1}\ T^{s}$ the ratio of the average energy generation to the central energy generation in a polytrope of index $n = 3$ is given by[14]

$$\frac{\bar{\varepsilon}}{\varepsilon_c} = \frac{3.2}{(3u + s)^{3/2}} = 0.042 \qquad (10)$$

The numerical value has been arrived at by using $u = 2$ from the ρ-dependence and $s = 12$ as the exponent in the best power law fit to the temperature dependent factors in Eq. (7) in the range $0.5 < T_8 < 0.8$. Thus

$$\bar{\varepsilon}_{HCNO} \approx 2.09 \times 10^{25} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_{8c}^{7/3} \exp\left(-32.81\ T_{8c}^{-1/3}\right) \text{ ergs gm}^{-1} \text{ sec}^{-1} \qquad (11)$$

$$\sim 2.72 \times 10^{8} \left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} (T_{8c}/0.65)^{15} \qquad 0.5 < T_{8c} < 0.8 \qquad (12)$$

The power law exponent in Eq. (12) includes the dependence of density on temperature. T_{8c} is the <u>central</u> temperature in $10^8\ {}^\circ K$.

The average energy generation given by Eq. (11) can now be equated to the energy required by Eq. (4), i.e.,

$$\bar{\varepsilon}_{HCNO} = \bar{\varepsilon}_{SMS} \qquad (13)$$

This gives the result

$$\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \approx 2.09 \times 10^{20}\ T_{8n}^{7/3} \exp\left(-32.81\ T_{8n}^{-1/3}\right) \qquad (14)$$

by which it is possible to calculate the central temperature, T_{8n}, at which nuclear energy generation through HCNO-burning yields 10^5 ergs gm^{-1} sec^{-1} on the average in a supermassive star of mass M. Equation (12) can be employed to give the rough approximation

$$T_{8n} = T_n \times 10^{-8} \sim 0.38 \left(\frac{M}{M_\odot}\right)^{1/30} \tag{15}$$

The temperature given by the more accurate Eq. (14) is plotted against mass in Figs. 2 and 3.

IV. STABILITY OF SUPERMASSIVE STARS DURING HYDROGEN BURNING

The stability of supermassive stars has been discussed by several authors.[8-13,16-20] In what follows the discussion will be limited to the post-Newtonian approximation to the general relativistic treatment of the problem for rotating stars. In this approximation the binding energy E_b of a star of mass M and radius R is given by[12]

$$\frac{E_b}{Mc^2} \approx \frac{3\bar{\beta}_n}{4(5-n)}\left(\frac{2GM}{Rc^2}\right) + \frac{1}{4}\,(K\alpha)_n^2\left(\frac{2GM}{Rc^2}\right) - \zeta_n\left(\frac{2GM}{Rc^2}\right)^2 \tag{16}$$

where $2GM/Rc^2$ is the characteristic expansion parameter in the general relativity of spherically symmetric systems with G the gravitational constant and c the velocity of light. Distortion from spherical symmetry under rotation has been neglected. In this approximation it is not necessary to distinguish between the rest mass of the constituents and the inertial or gravitational mass of the contracted system measured by an external observer. Quantities which depend on the polytropic structure are designated by the use of the polytropic index n as a subscript. Numerical values will be given for n = 3 on the grounds that this is approximately the minimum index for convective stability. The ratio of gas pressure to total pressure, gas plus radiation, averaged throughout the star, is designated by $\bar{\beta}_n$. For the polytrope of index $n = 3$, β is a constant throughout the star, and it has been shown[15] that

$$(\mu\bar{\beta})_3 = (\mu\beta)_3 = 4.28\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} \qquad n = 3 \qquad (17)$$

where μ is the mean molecular weight of the stellar material. With μ of order unity, note that β is small for large M.

The coefficient of the negative post-Newtonian, general relativistic term is of order unity and is designated by ζ_n. For $n = 3$, $\zeta_3 = 1.265$. K_n is given in terms of the Newtonian rotational energy Ψ_0 and the angular velocity at the periphery ω_R by the equation

$$K_n = (2\Psi_0/MR^2\omega_R^{\,2})^{\frac{1}{2}} \qquad (18)$$

K_n can be determined when the polytropic index is given and the angular velocity $\omega = \omega(r)$ is specified as a function of radius. A model for differential rotation in a polytrope of index $n = 3$ due to Stoeckly[21] has been used in the calculations discussed here. For this model $K_3^2 = 2.47$. For uniform rotation $\omega = \omega_R$ and $K_n = k_n$, the radius of gyration in units of R. For $n = 3$, $k_3^2 = 0.075$. The quantity α_n is a measure of the amount of rotation in terms of the critical angular velocity which occurs when centrifugal forces match gravitational forces at some point in the equatorial plane. In the case of uniform rotation this first occurs at the periphery and α_n is independent of n being given by

$$\alpha_n = \alpha = \omega_R/\omega_{CR} \qquad (19)$$

where

$$\omega_{CR} = (GM/R^3)^{\frac{1}{2}} \qquad (20)$$

For numerical calculations on uniform rotation $\alpha = 1$ will be taken as the limiting case in the sense that angular momentum loss or transfer to the external surroundings will limit the rotation to that which corresponds to this value. In the case of differential rotation, the limit on the angular velocity occurs at the center and numerical integrations on Stoeckly's model yield the limiting value $\alpha_3^2 = 0.456$ so that $(K\alpha)_3^2 = 1.125$. This is to be compared with the limit for uniform rotation $(K\alpha)_3^2 = (k\alpha)_3^2 = 0.075$. For no rotation $\alpha_n = 0$.

In the post-Newtonian approximation the angular frequency σ_R of first order periodic variations in the fundamental mode of radial oscillations is closely related to the binding energy and is given by

$$\sigma_R^2 \approx \frac{1}{3}\left(\frac{c^3}{2GMk_n}\right)^2 \left[\frac{3\bar{\beta}_n}{2(5-n)}\left(\frac{2GM}{Rc^2}\right)^3 + (K\alpha)_n^2\left(\frac{2GM}{Rc^2}\right)^3 - 4\zeta_n\left(\frac{2GM}{Rc^2}\right)^4\right] \sec^{-2} \tag{21}$$

The period of the fundamental mode is given by

$$\begin{aligned} \Pi_R &= 2\pi/\sigma_R \quad \sec \\ &= 2\pi/86400\,\sigma_R \ \text{day} \end{aligned} \tag{22}$$

For zero rotation

$$\sigma_R^2 \approx -\frac{2}{3k_n^2 MR}\frac{dE_b}{dR} \qquad \alpha_n = 0 \tag{23}$$

When the term in $\bar{\beta}_n$ is small enough to be neglected

$$\sigma_R^2 \approx \frac{4E_b}{3k_n^2 MR^2} \qquad \bar{\beta}_n = 0 \tag{24}$$

The star becomes dynamically unstable when σ_R becomes imaginary or σ_R^2 becomes negative. The role of the negative post-Newtonian term is thus apparent from Eq. (21). When the dimensionless parameter $2GM/Rc^2$ becomes large enough during contraction, σ_R^2 will become negative and small perturbations will become exponentially large rather than periodic in nature. Since $\bar{\beta}_n$ is small for supermassive stars this instability sets in at large radii early in contraction when there is no rotation. It will be clear that rotation will postpone the onset of this instability until much smaller radii are reached and that differential rotation ($K_n^2\alpha_n^2 = 1.125$) will be much more effective in this regard than uniform rotation ($k_n^2\alpha_n^2 = 0.075$).

In order to determine the stability and binding energy during hydrogen burning and nuclear energy generation, it is necessary to express E_b and σ_R in terms of the central temperature. This can be done by using the Newtonian relation between radius and central temperature of a polytrope which is

$$\frac{2GM}{Rc^2} = 2(n+1)\frac{M_n}{R_n c^2}\left(\frac{\Re T}{\mu\beta}\right)_c \qquad (25)$$

where M_n and R_n are constants of integration corresponding to mass and radius respectively for the Lane-Emden second order differential equation governing the structure of a polytrope of index n. For the polytrope of index n = 3, $M_3 = 2.018$ and $R_3 = 6.987$ so that Eqs. (17) and (25) yield

$$\begin{aligned}\frac{2GM}{Rc^2} &= 2.60\times10^{-21}\left(\frac{\Re T}{\mu\beta}\right)_c \\ &= 5.05\times10^{-6}\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} T_{8c}\end{aligned} \qquad n = 3 \qquad (26)$$

In Eqs. (25) and (26), $\Re$ is the gas constant.

When Eq. (26) is substituted into Eqs. (16) and (21) the results in terms of the central temperature, T_{8c}, in units of 10^8 °K are

$$\frac{E_b}{Mc^2} \approx 1.112\times10^{-5}\,T_{8c} + 1.266\times10^{-6}\,(K\alpha)_3^2\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} T_{8c} - 3.236\times10^{-11}\left(\frac{M}{M_\odot}\right)T_{8c}^2 \qquad n=3 \qquad (27)$$

$$\sigma_R^2 \approx 2.614\times10^{-5}\left(\frac{M_\odot}{M}\right)T_{8c}^3 + 5.951\times10^{-6}(K\alpha)_3^2\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} T_{8c}^3 - 1.521\times10^{-10}\,T_{8c}^4 \qquad n=3 \qquad (28)$$

During the contraction E_b reaches a maximum value at the temperature given by

$$T_{8c}(E_b^{max}) = T_c(E_b^{max}) \times 10^{-8}$$

$$= 1.956 \times 10^4 \ (K\alpha)_3^2 \ (M_\odot/M)^{\frac{1}{2}} + 1.718 \times 10^5 \ (M_\odot/M) \qquad n = 3 \qquad (29)$$

In general this temperature is somewhat less than that at which dynamic instability sets in. Setting $\sigma_R = 0$ in Eq. (28) the result is

$$T_{8c}(\sigma_R = 0) = T_c(\sigma_R = 0) \times 10^{-8}$$

$$= 3.913 \times 10^4 (K\alpha)_3^2 (M_\odot/M)^{\frac{1}{2}} + 1.718 \times 10^5 (M_\odot/M) \quad n = 3 \quad (30)$$

For no rotation, Eqs. (29) and (30) show that instability sets in when E_b reaches its maximum value. When rotational effects are large so that the first terms on the right-hand sides of Eqs. (27) and (28) can be neglected, instability sets in at twice the temperature at which E_b reaches its maximum value. At this temperature $E_b \approx 0$. This means that beyond $T_c(E_b^{max})$ a large amount of internal energy must be supplied if hydrostatic equilibrium is to be maintained. If this energy is to be made available from the nuclear resources of the star, it is necessary that T_n be less than $T_c(E_b^{max})$. If this is not the case, rapid adiabatic collapse occurs until T_n is reached at which point rapid nuclear burning takes place in such a way as to lead to reversal of the collapse and the setting up of large amplitude, non-linear relaxation oscillations which have been discussed in detail by Fowler.[11] Thus in the case of no rotation quasi-stable hydrogen burning through the CNO bi-cycle can occur at temperatures somewhat in excess of $T_c(E_b^{max}) = T_c(\sigma_R = 0)$ but when rotational effects are large T_n must not exceed a value intermediate between $T_c(E_b^{max})$ and $T_c(\sigma_R = 0) = 2\, T_c(E_b^{max})$.

These points are illustrated in Figs. 2 and 3 where $T_c(E_b^{max})$ and

$T_c(\sigma_R = 0)$ are respectively plotted as functions of $M/M_\odot$ with T_n also plotted for comparison. Three cases are plotted

(a) No rotation, $\alpha_3 = 0$

(b) Maximum uniform rotation, $(K\alpha)_3^2 = (k\alpha)_3^2 = 0.075$

(c) Maximum differential rotation, $(K\alpha)_3^2 = 1.125$.

It is also of interest to know E_b/Mc^2 and Π_R as a function of stellar mass at the temperature at which hydrogen burning occurs. These quantities can be obtained by substituting T_{8n} from Eq. (14) into Eqs. (27), (28) and (22). The results are illustrated in Figs. 4 and 5.

V. CONCLUSIONS

Figures 2 and 3 show that nuclear energy generation through HCNO-burning occurs under conditions of stability up to the following mass limits: for no rotation, $M \lesssim 2 \times 10^5\ M_\odot$; for maximum uniform rotation, $M \lesssim 10^7\ M_\odot$; for maximum differential rotation, $M \lesssim 10^9\ M_\odot$. In the case of no rotation it has been shown[11] that relaxation oscillations under quasi-stable conditions extend the limiting mass to $M \lesssim 10^6\ M_\odot$.

Figure 4 shows that the binding energy at the onset of nuclear energy generation is quite small except for $M > 10^8\ M_\odot$ in the case of differential rotation. In this case the binding energy reaches slightly over one per cent of the rest mass energy and is thus greater than the total nuclear energy of the star. For all masses the binding energy reaches a maximum which can be determined by the substitution of Eq. (29) into Eq. (27). This maximum is usually reached upon further contraction after the termination of nuclear burning and is given by

$$\frac{E_b^{\max}}{Mc^2} \approx \frac{[(K\alpha)_n^2 + 3\bar{\beta}_n/(5-n)]^2}{64\zeta_n} \tag{31}$$

In the case of maximum differential rotation for n = 3 this yields

$$\frac{E_b^{\max}}{Mc^2} \approx 0.016 \qquad (K\alpha)^2 = 1.125,\ n = 3 \tag{32}$$

Thus in the case of maximum differential rotation the binding energy eventually becomes 1.6 per cent of the rest mass energy. Since this energy must be lost by the star, it represents a substantial contribution to the energy available for light and radio emission.

Figure 5 indicates that the period of small radial oscillations during hydrogen burning falls in the range from 1 to 20 days. It must be borne in mind, however, that the pulsations set up by the onset of nuclear burning will be very non-linear. The periods required for the transfer of energy to the surface and for the emission therefrom will be the order of years rather than days. The observed light curves should show variations characteristic of a wide range of frequencies corresponding to periods from a few days to tens of years. This may already have been observed[22] in the quasi-stellar radio source, 3C 273.

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Figure Captions

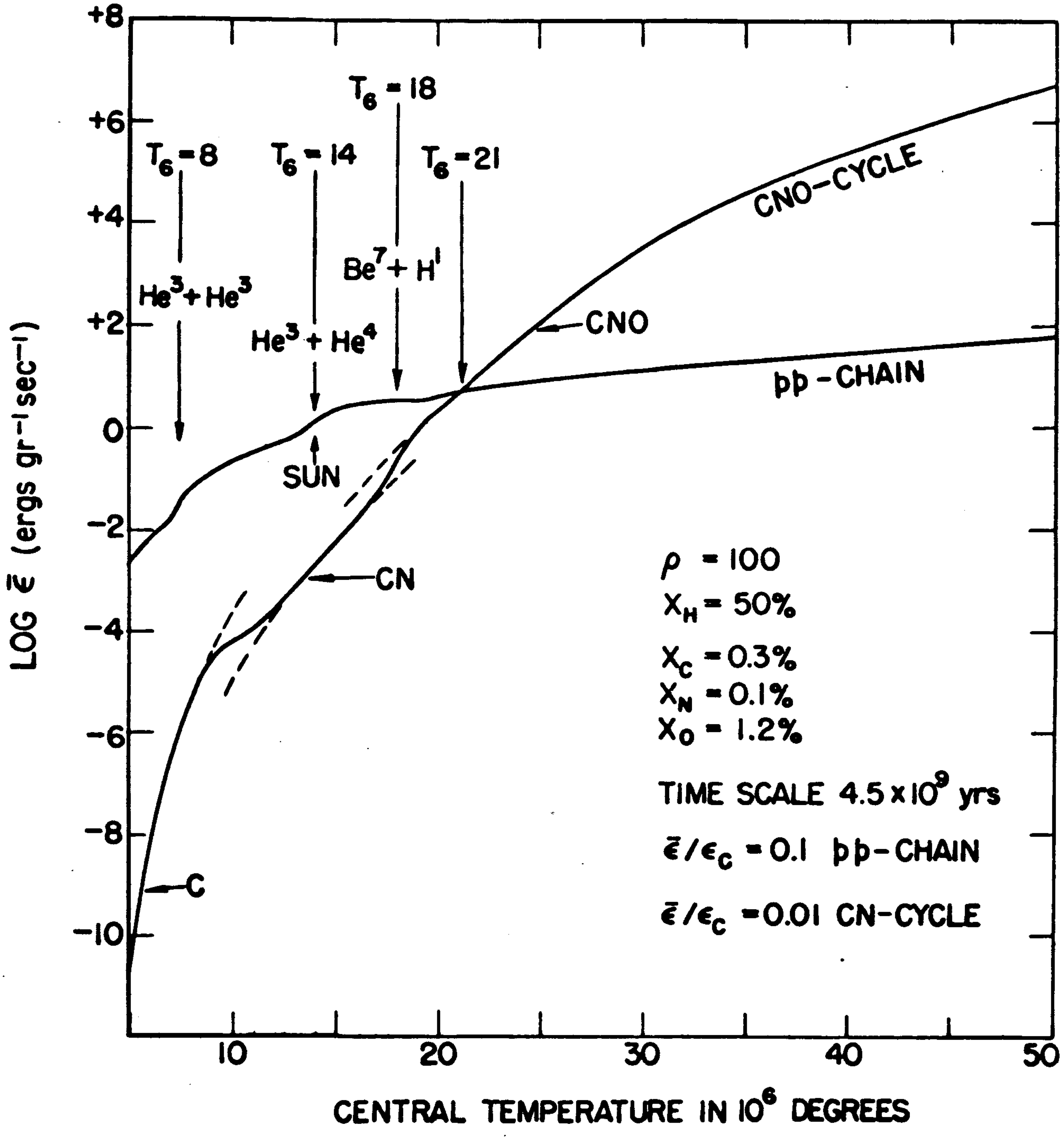
Fig. 1. Average energy generation throughout a star in ergs per gram-second as a function of central temperature for the p-p chain and the CNO bi-cycle. The central density is taken as $\rho = 100\ g/cm^3$, and the hydrogen concentration by weight as $x_H = 0.50$. Concentrations of C, N, and O by weight as given are those for a typical Population I star. The age of the star is taken to be 4.5×10^9 years. The points of inflection in the p-p chain arise from the onset of the indicated interactions. Similarly C, N, and O are successively involved in the CNO cycle. Note that the sun and the cool stars operate on the p-p chain; hot stars operate on the CNO cycle.

Fig. 2. The temperature, $T_c(E_b^{max})$, at which the binding energy reaches its maximum value plotted as a function of stellar mass for three cases: no rotation, maximum uniform rotation, and maximum differential rotation. The temperature, T_n, required for the operation of the CNO bi-cycle is shown for comparison.

Fig. 3. The temperature, $T_c(\sigma_R = 0)$, at which dynamic instability sets in plotted as a function of stellar mass for three cases: no rotation, maximum uniform rotation, and maximum differential rotation. The temperature, T_n, required for the operation of the CNO bi-cycle is shown for comparison.

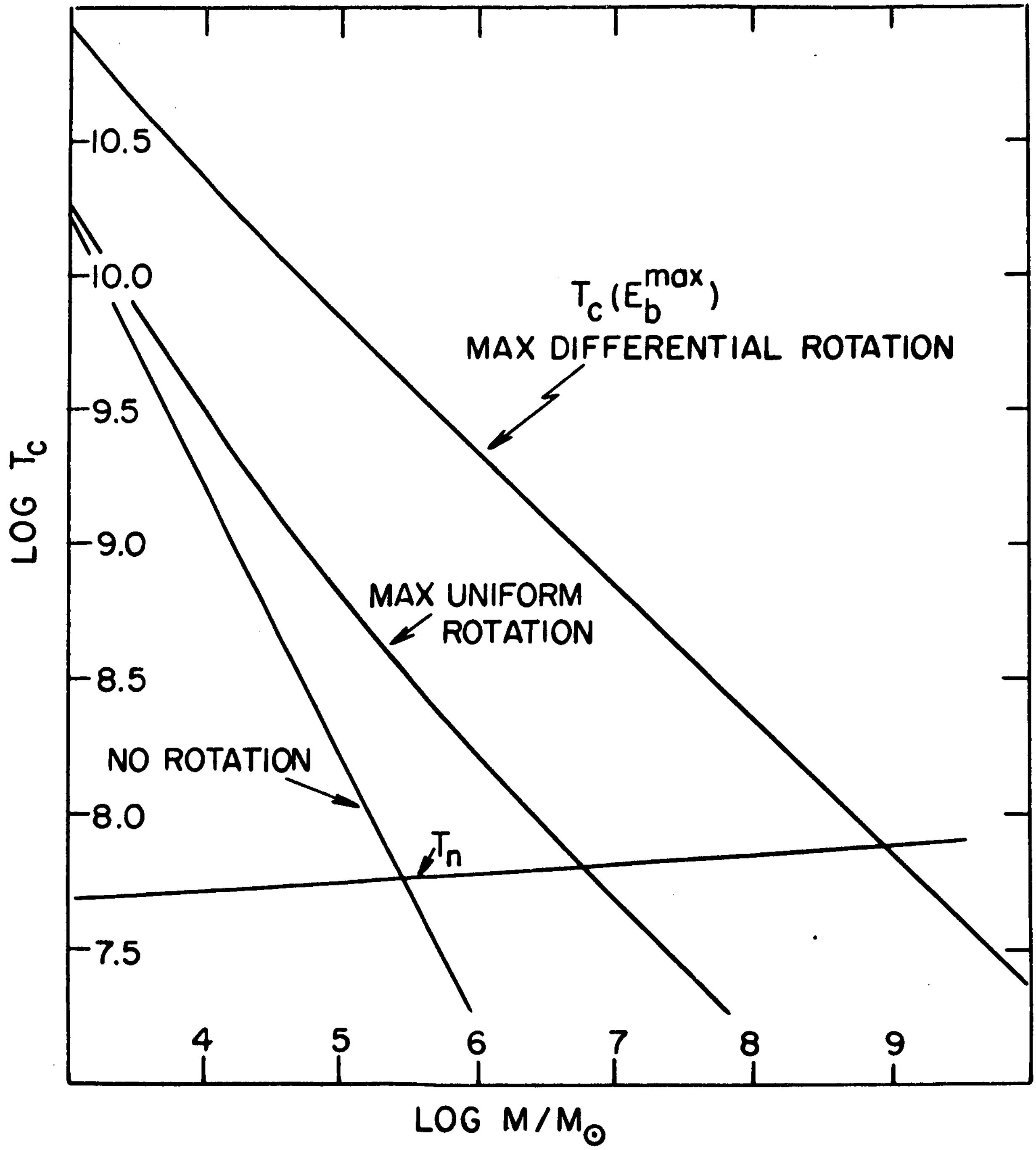
Fig. 4. The binding energy at the onset of nuclear energy generation through HCNO-burning plotted as a function of stellar mass for three cases: no rotation, maximum uniform rotation, and maximum differential rotation.

Fig. 5. The period in days of small, linear oscillations during nuclear energy generation through HCNO-burning plotted as a function of stellar mass for three cases: no rotation, maximum uniform rotation, and maximum differential rotation. The periods indicated in this plot do not hold for the large amplitude, non-linear relaxation oscillations discussed in the text. For such oscillations the periods are a few years rather than a few days.



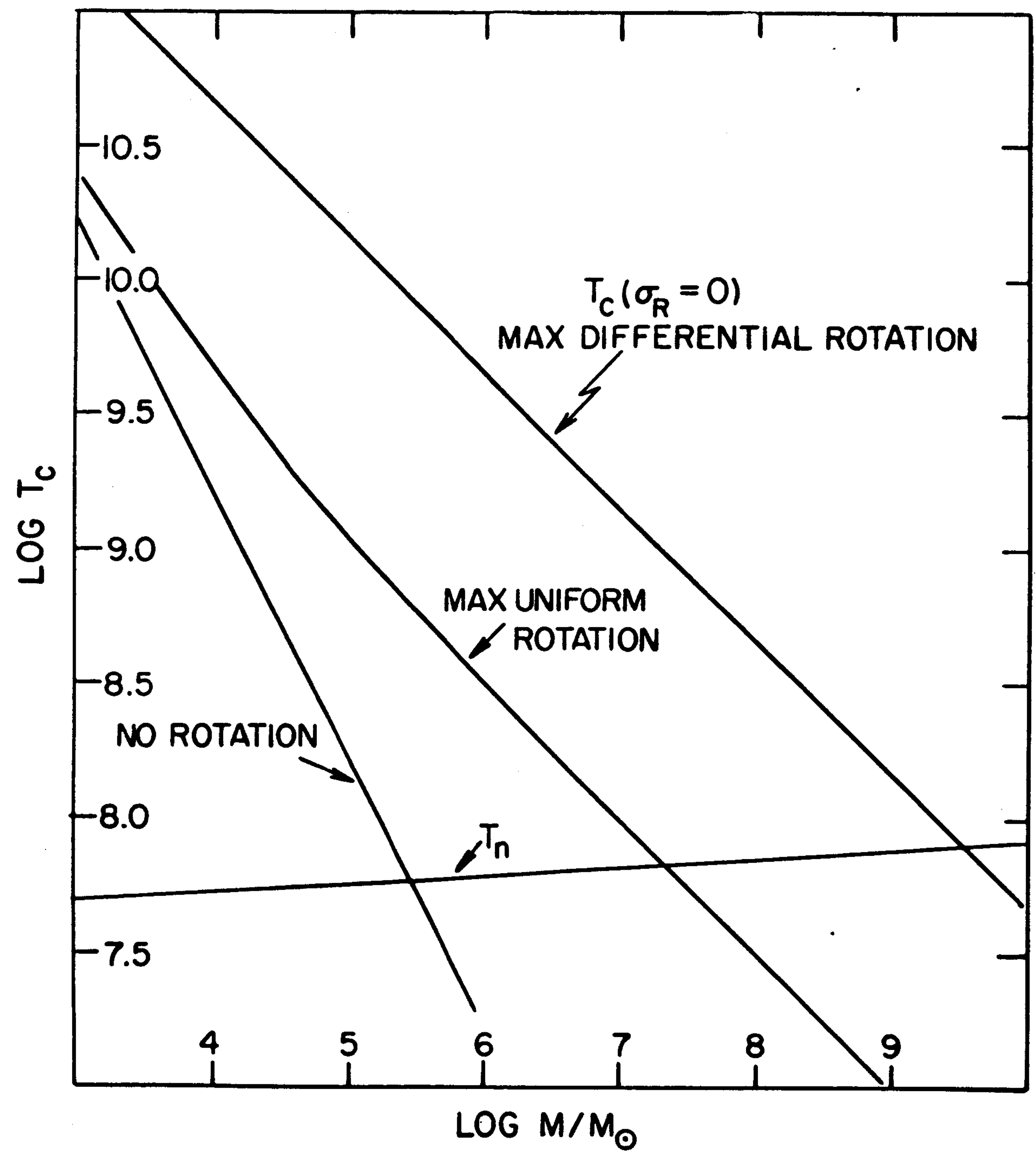
+8
+6
+4
+2
0
−2
−4
−6
−8
−10
10
20
30
40
50
LOG $\bar{\epsilon}$ (ergs gr^{-1}sec^{-1})
CENTRAL TEMPERATURE IN 10^6 DEGREES
$T_6 = 8$
$T_6 = 14$
$T_6 = 18$
$T_6 = 21$
$He^3 + He^3$
$He^3 + He^4$
$Be^7 + H^1$
SUN
CNO-CYCLE
CNO
pp-CHAIN
CN
C
ρ = 100
X_H = 50%
X_C = 0.3%
X_N = 0.1%
X_O = 1.2%
TIME SCALE 4.5×10^9 yrs
$\bar{\epsilon}/\epsilon_c$ = 0.1 pp-CHAIN
$\bar{\epsilon}/\epsilon_c$ = 0.01 CN-CYCLE

Fig. 1



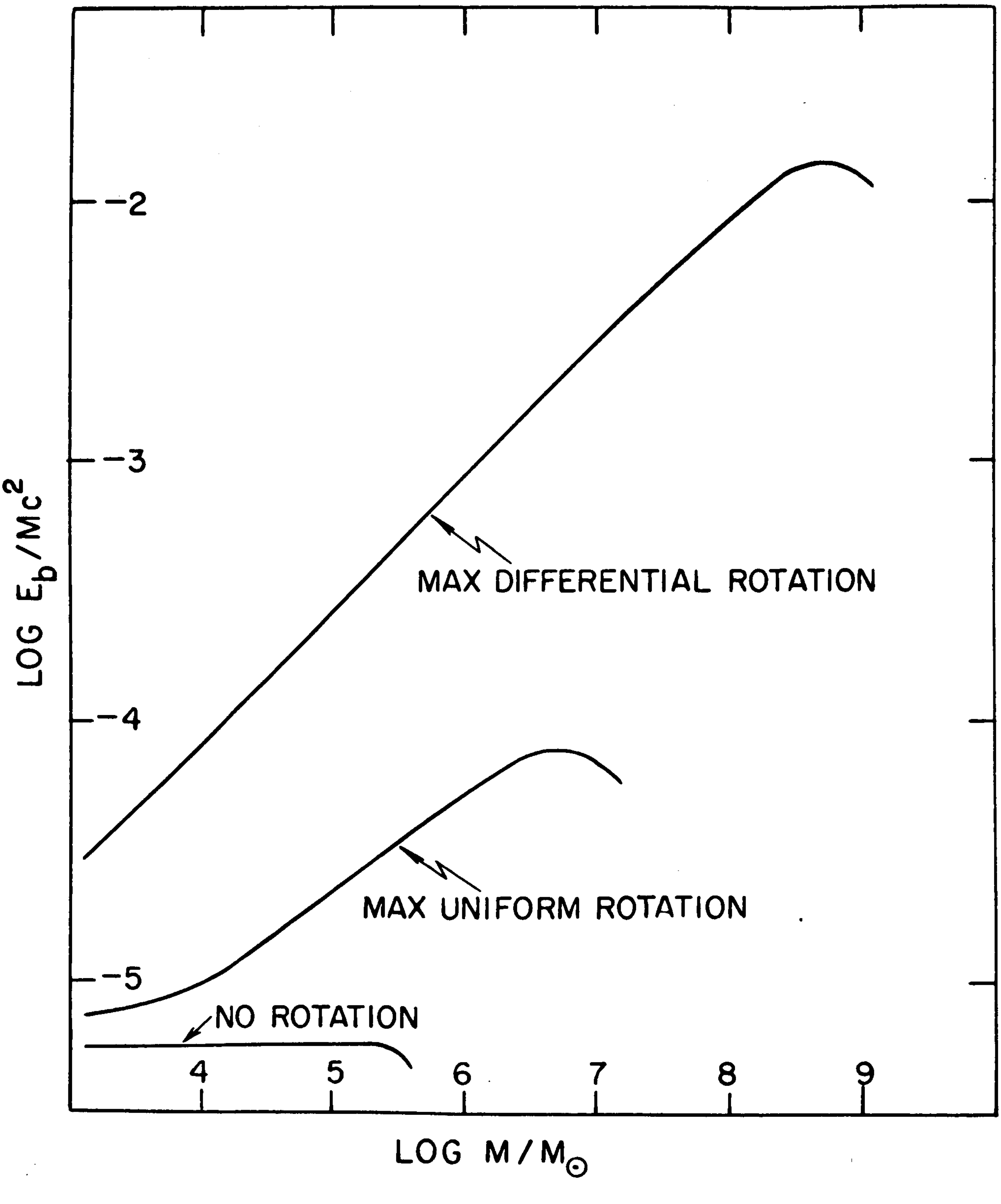
10.5
10.0
9.5
9.0
8.5
8.0
7.5
LOG T_c
$T_c(E_b^{max})$
MAX DIFFERENTIAL ROTATION
MAX UNIFORM ROTATION
NO ROTATION
T_n
4
5
6
7
8
9
LOG $M/M_\odot$

Fig. 2



LOG T_c
10.5
10.0
9.5
9.0
8.5
8.0
7.5
4
5
6
7
8
9
LOG $M/M_\odot$
$T_c(\sigma_R = 0)$
MAX DIFFERENTIAL ROTATION
MAX UNIFORM
ROTATION
NO ROTATION
T_n

Fig. 3



-2
-3
-4
-5
LOG E_b/Mc^2
MAX DIFFERENTIAL ROTATION
MAX UNIFORM ROTATION
NO ROTATION
4
5
6
7
8
9
LOG $M/M_\odot$

Fig. 4

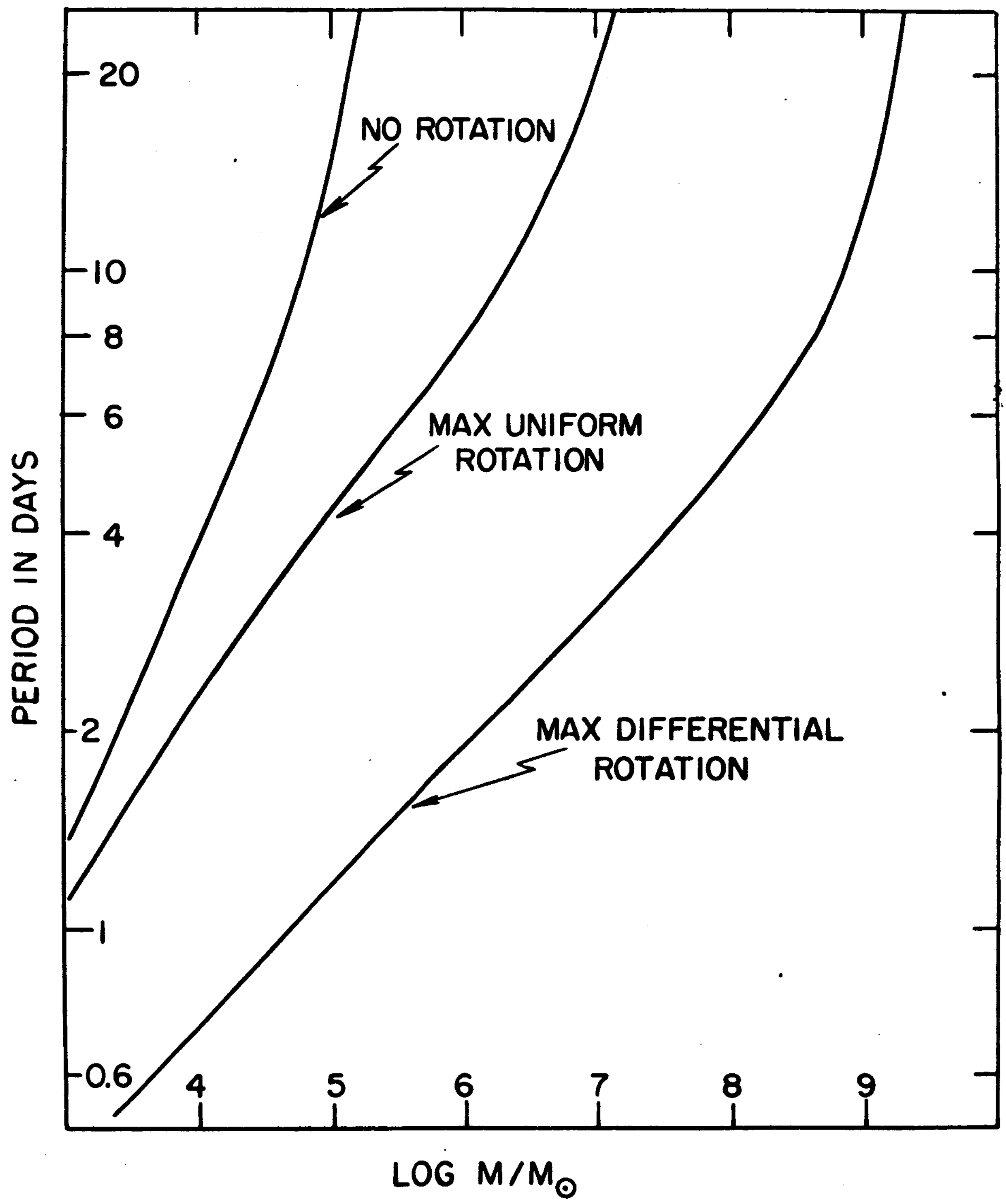

NO ROTATION
MAX UNIFORM ROTATION
MAX DIFFERENTIAL ROTATION
PERIOD IN DAYS
20
10
8
6
4
2
1
0.6
4
5
6
7
8
9
LOG $M/M_\odot$

Fig. 5