Final Report

on

Research Contract NAS 8-2619

APPLICATIONS OF CALCULUS OF VARIATIONS TO TRAJECTORY ANALYSIS

Department of Mathemati
Vanderbilt University
Nashville, Tennessee

March, 1966

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Submitted to

National Aeronautics and Space Administration

Marshall Space Flight Center

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by

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TABLE OF CONTENTS

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APPLICATIONS OF CALCULUS OF VARIATIONS TO TRAJECTORY ANALYSIS

by M. G. Boyce and J. L. Linnstaedter

Other participants on parts of the project were
G. E. Tyler, Richard K. Williams,
Florian Hardy, Donald F. Bailey

SUMMARY

This report describes in the introduction the general nature of the work done on Contract NAS 8-2619, and the numbered sections include in shortened form the principal contributions that were made.

Section I extends the classical calculus of variations theory to include control variables. Section II is a treatment of a special multistage fuel minimization trajectory problem in which the lengths of the time intervals of the several stages are known. Section III is a simplified example of such a multistage problem. Section IV extends the Denbow multistage theory to allow discontinuities in variables and functions at stage boundaries, and in Section V further extensions are made to include control variables and inequality and finite equation constraints. Section VI gives an application of the theory of Section V to a three stage re-entry problem, and Section VII is an application to a six stage earth-moon problem, for which partial results are obtained.

INTRODUCTION

The principal field of study and research on this contract has been the optimization of multistage rocket trajectories. A part of the work has been on needed extensions of basic calculus of variations theory and a part on applications of the theory. Some of the results obtained have been published in the Progress Reports of the Aero-Astrodynamics Laboratory under the following titles:

> "An Application of Calculus of Variations to the Optimization of Multistage Trajectories," by M. G. Boyce, Progress Report No. 3 on Studies in the Fields of Space Flight and Guidance Theory, MTP-AERO-63-12, Feb. 6, 1963.

> "Necessary Conditions for a Multistage Bolza-Mayer Problem Involving Control Variables and Having Inequality and Finite Equation Constraints," by M. G. Boyce and J. L. Linnstaedter, Progress Report No. 7 on Studies in the Fields of Space Flight and Guidance Theory, NASA TM X-53292, July 12, 1965.

The following reports were made to contractor conferences of the Aero-Astrodynamics Laboratory:

> "Transversality Conditions in the Optimization of Multistage Trajectories," by M. G. Boyce, July 18, 1962.

> "A Simple Multistage Problem Having Discontinuities in its Lagrange Multipliers," by M. G. Boyce, Dec. 19, 1962.

> "Extensions of the Denbow Multistage Calculus of Variations Problem to Include Control Variables and Inequality Constraints," by J. L. Linnstaedter, Oct. 22, 1964.

> "The Multistage Weierstrass and Clebsch Conditions with Some Applications to Trajectory Optimization," by M. G. Boyce, Feb. 4, 1965.

> "Applications of Multistage Calculus of Variations Theory to Two and Three Stage Rocket Trajectory Problems," by G. E. Tyler, Aug. 4, 1965.

In addition to the foregoing reports, informal oral and written reports were made from time to time to William E. Miner, former chief of the

Astrodynamics and Guidance Theory Division, and to Clyde D. Baker, present chief of the division.

Informal consultations of one to three days each, some in Nashville and some in Huntsville, were held during the time of the contract with one or more of the following: William E. Miner, Robert Silber, Robert W. Hunt, Grady Harmon, R. M. Chapman, Richard Hardy, W. A. Shaw, D. Lynn, C. C. Dearman, Ben Lisle, J. A. Lovingood, and Clyde Baker. Among the subjects treated were transformations of the Lagrange multipliers, series expansion methods for the solution of systems of differential equations, series methods for in-flight corrections of trajectories, extensions of the Denbow multistage theory, and applications of calculus of variations to re-entry problems.

In this report the first section is a summary of necessary conditions for one stage calculus of variations problems in the Mayer form which involve control variables.

The second section concerns rocket trajectories with a specified time interval for each stage except the last. The necessary conditions of Section I can be applied to each stage in succession, the transversality conditions at the end of a stage giving initial conditions for the next stage.

In Section III a multistage extension of Zermelo's navigation problem is given as an example to illustrate some features of multistage problems.

In Section IV a summary of the general multistage theory of C. H. Denbow is given, with modifications to allow discontinuities in functions and variables at stage boundaries.

Section V extends the multistage theory to problems involving control variables and having inequality and finite equation constraints. The Mayer formulation is used, and the system of differential equation constraints is taken in normal form since in trajectory problems the equations of motion are in such form. Proof of theorems are omitted in this report but are given in the paper by Boyce and Linnstaedter in Progress Report No. 7.

A three stage re-entry rocket optimization problem is treated in Section VI as an example of the theory in Section V. To avoid computational complexity, simple intermediate point constraints are assumed and a first order approximation to gravitational attraction is used.

Section VII is an application of the theory of Section V to an earth-moon problem in which six stages are determined by intermediate point conditions and by specified thrust magnitudes. Euler-Lagrange equations are obtained and some vector relations deduced from them. The Weierstrass condition yields a maximum principle. Transversality conditions are given in matrix form.

SECTION I. NECESSARY CONDITIONS FOR ONE STAGE CALCULUS OF VARIATIONS PROBLEMS INVOLVING CONTROL VARIABLES

Adaptations of classical calculus of variations theory to one stage Bolza problems containing control variables have been made by Hestenes and others (References 6, 7, 8). The resulting principal necessary conditions are stated in this section for the Mayer form of the problem.

NOTATION

t	independent variable
$\underline{x} = (x_1,\cdots,x_n)$	state variables, functions of t
$\underline{y} = (y_1,\cdots,y_m)$	control variables, functions of t
$\underline{b} = (b_1,\cdots,b_r)$	parameters occurring in end conditions
$T_1,\underline{X}_1 = (X_{11},\cdots,X_{1n})$	functions of $\underline{b}$ defining first end point
$T_2,\underline{X}_2 = (X_{21},\cdots,X_{2n})$	functions of $\underline{b}$ defining second end point
$\underline{g} = (g_1, \cdots,g_n)$	functions of (t,x,y) defining derivative constraints
$\underline{L} = (L_1,\cdots,L_n)$	Lagrange multipliers, functions of t
$H = \underline{L} \cdot \underline{g}$	generalized Hamiltonian function
$h(\underline{b})$	function to be minimized

Variables occurring as subscripts will denote partial derivatives, and a superimposed dot will indicate differentiation with respect to t. A set $t,\underline{x},\underline{y},\underline{b}$ will be called <u>admissible</u> if it belongs to a given open set R, and a set $\underline{x}(t),\underline{y}(t),\underline{b}$ will be an <u>admissible arc</u> if its elements are all admissible and if $\underline{x}(t)$ is continuous and $\dot{\underline{x}}(t),\underline{y}(t)$ are piece-wise continuous. The functions occurring in T, $\underline{X}$, $\underline{g}$, and h are assumed to have continuous partial derivatives of at least the second order.

STATEMENT OF A PROBLEM

In a given class of admissible functions and parameters $\underline{x}(t),\underline{y}(t),\underline{b}$

it is required to find a set which satisfies the differential equations and end conditions

$$\dot{\underline{x}} = \underline{g}(t,\underline{x},\underline{y}) \ , \ t_1 \leqq t \leqq t_2$$

$$t_1 = T_1(\underline{b}) \ , \ t_2 = T_2(\underline{b}) \ , \ \underline{x}(t_1) = \underline{X}_1(\underline{b}) \ , \ \underline{x}\ (t_2) = \underline{X}_2(\underline{b})$$

and which minimizes the given function $h(\underline{b})$.

Let C be an admissible arc $\underline{x}(t)$, $\underline{y}(t)$, $\underline{b}$ which is a solution of the problem. Also let C be assumed normal (Ref. 6, p. 15) and to have $\dot{\underline{x}}(t)$ and $\dot{\underline{y}}(t)$ continuous. Then C must satisfy the following four conditions.

NECESSARY CONDITIONS

I. First Necessary Condition. For every minimizing arc C there exist unique multipliers $L_i(t)$, having continuous first derivatives, such that the equations (Euler-Lagrange)

$$\dot{x}_i = H_{L_i} \ , \ \dot{L}_i = -H_{x_i} \ , \ H_{y_j} = 0, \ i = 1,\cdots,n, \quad j = 1,\cdots,m,$$

hold along C . Also the end values of C satisfy the transversality conditions

$$H_1 T_{1b_k} - \underline{L}_1 \cdot \underline{X}_{1b_k} - H_2 T_{2b_k} + \underline{L}_2 \cdot \underline{X}_{2b_k} + h_{b_k} = 0, \quad k = 1,\cdots,r,$$

where subscripts 1 and 2 on H and $\underline{L}$ indicate evaluation for $t = t_1$ and $t = t_2$, respectively.

As a consequence of the above Euler-Lagrange equations it follows that also along a minimizing arc C

$$dH/dt = H_t \ ,$$

and hence that, if H does not involve t explicitly, then H is constant along C .

II. Weierstrass Condition. Along a minimizing arc C the inequality

$$H(t,\underline{x},\underline{Y},\underline{L}) \leqq H(t,\underline{x},\underline{y},\underline{L})$$

must hold for every admissible element $(t,\underline{x},\underline{Y})$.

III. Clebsch (Legendre) Condition. At each element $(t,\underline{x},\underline{y},\underline{L})$ of a minimizing arc C the inequality

$$\sum_{i,j=1}^{m} H_{y_i y_j} Y_i Y_j \leqq 0$$

must hold for every set $(Y_1,\cdots,Y_m)$.

IV. Second Order Condition. The second variation of h along a minimizing arc C is non-negative for every variation of C satisfying the equations of variation.
(Cf. Ref. 6, p. 16.) No use of this condition is made in this paper.

SECTION II. THE OPTIMIZATION OF MULTISTAGE TRAJECTORIES WHOSE STAGES HAVE SPECIFIED DURATIONS

INTRODUCTION

The problem is to determine the fuel minimizing trajectory of a rocket whose flight consists of several stages caused by engine shut-offs at specified times. Initial position and velocity are assumed given and target conditions specified. In each stage the analytic formulation is similar to that of Cox and Shaw (Ref. 1), and we make their basic assumptions that the earth can be considered spherical, the inverse square gravity law holds, the only forces acting on the rocket are thrust and gravity, the direction of thrust is the axial direction of the rocket, rotation effects can be ignored, in each stage the magnitude of thrust and the fuel burning rate are constant, and the center of mass of the rocket is fixed with respect to the rocket.

The general procedure is roughly as follows. Using the fixed initial conditions for the first stage, determine as solutions of the Euler-Lagrange equations the family of minimizing trajectories satisfying those conditions. The given time t_1 for the end of the first stage will fix on each minimizing trajectory ~~of~~ a definite point. The totality of these points will constitute a subspace S_1 , which will be the locus of initial points for the second stage. New values of mass, thrust, and fuel burning rate determine new Euler-Lagrange equations. Minimizing trajectories must satisfy these new equations in this stage and also must satisfy transversality conditions for initial points in subspace S_1 . Through each point of S_1 these conditions determine a unique trajectory, and on each of these trajectories the given time t_2 for the end of the second stage will fix a definite point. The totality of these points will be subspace S_2 , which in turn will be the locus of initial points for the third stage, and transversality conditions will again determine a family of minimizing trajectories, one issuing from each point of S_2 . This procedure is repeated until in the final stage the mission objectives will impose criteria for selecting a pieced trajectory satisfying the given initial conditions and extending through the several stages. Closed form solutions are not attainable in most cases. However, it would seem possible to extend the single stage adaptive guidance mode computational procedures through several successive stages.

FORMULATION OF THE PROBLEM

A plumbline coordinate system is used (Ref. 1, p. 108; Ref. 2, p. 11), with the center of mass of the rocket designated by $\underline{x} = (x_1, x_2, x_3)$ and its velocity by $\underline{u} = (u_1, u_2, u_3)$. The time t is taken as indepent variable, and $\underline{u} = d\underline{x}/dt$. The thrust vector $\underline{F} = (0, F, 0)$, having its magnitude F constant for each stage, is assumed to be directed along the axis of the rocket. The orientation of the rocket axis relative to the plumbline system is designated by $\angle = (\chi_1, \chi_2, \chi_3)$, where χ_1, χ_2, χ_3 are the pitch, roll and yaw angles, respectively.

If $\ddot{\underline{x}}_g$ denotes the gravitational acceleration and $[A]$ the matrix for transformation of vectors from the missile to the plumbline coordinate system, then Newton's second law gives as equations of motion of the rocket

$$\dot{\underline{u}} = m^{-1}\, \underline{F}\, [A] + \ddot{\underline{x}}_g \;, \quad \dot{\underline{x}} = \underline{u} \;. \tag{1}$$

In terms of pitch, roll, and yaw, the matrix A has the following form (Ref. 1, p. 108; Ref. 2, p. 26):

$$A = \begin{bmatrix} CPCR & SPCR & SR \\ -SPCY - CPSRSY & CPCY - SPSRSY & CRSY \\ SPSY - CPSRCY & -CPSY - SPSRCY & CRCY \end{bmatrix} \qquad \begin{array}{c} CP = \cos\chi_1 \\ SP = \sin\chi_1 \\ \text{etc.} \end{array}$$

Since roll effects are to be ignored, the roll χ_2 will be assumed identically zero. Hence $CR = 1$, $SR = 0$, and the variable χ_2 may be dropped. Since fuel consumption is monotonically increasing with time, minimization of time of flight is equivalent to minimizing fuel consumption. It is more convenient to treat the problem from the minimum time standpoint.

In the terminology of the general theory of Section I we now have state variables $u_1, u_2, u_3, x_1, x_2, x_3$, control variables χ_1 and χ_3, and independent variable t. The function to be minimized, the function $h(\underline{b})$, is simply the final time t_f. Hence t_f is one of the parameters in $\underline{b}$; other parameters may occur in the initial and end conditions and in stage boundary conditions. The mass m is assumed a known function of t in each stage so is not included in the state variables.

Thus the problem is to find in a class of admissible sets of functions $\underline{u}(t)$, $\underline{x}(t)$, $\underline{\chi}(t)$ and parameters $\underline{b}$ a set that will satisfy the differential equations (1) and the given end conditions and that will minimize the final time t_f.

FIRST STAGE

Let the time interval for the first stage be $t_o \leqq t \leqq t_1$, and the initial conditions, $\underline{u}(t_o) = \underline{u}_o$, $\underline{x}(t_o) = \underline{x}_o$. On putting $\chi_2 = 0$ in A and using $\mu r^{-3}\underline{x}$ for $\underline{x}_g$, where μ is the gravitational constant times the mass of the earth, we get equations (1) in the form

$$\begin{aligned}
\dot{u}_1 &= -Fm^{-1}SPCY - \mu r^{-3}x_1 \\
\dot{u}_2 &= Fm^{-1}CPCY - \mu r^{-3}x_2 \\
\dot{u}_3 &= Fm^{-1}SY - \mu r^{-3}x_3 \\
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_3
\end{aligned} \tag{2}$$

In order to apply the necessary conditions of Section I, we now define a generalized Hamiltonian

$$\begin{aligned}
H = {} & L_1(-Fm^{-1}SPCY - \mu r^{-3}x_1) + L_2(Fm^{-1}CPCY - \mu r^{-3}x_2) \\
& + L_3(Fm^{-1}SY - \mu r^{-3}x_3) + L_4u_1 + L_5u_2 + L_6u_3 .
\end{aligned}$$

By condition I, Section I, the Euler-Lagrange equations are

$$\dot{u} = H_{L_i}, \ \dot{x}_i = H_{L_{i+3}}, \ \dot{L}_i = -H_{u_i}, \ \dot{L}_{i+3} = -H_{x_i}, \ H_{\chi_j} = 0, \ i = 1,2,3, \ j = 1,3.$$

These formulas give the six equations (2) plus the following eight:

$$
\begin{aligned}
L_1 &= -L_4 \\
L_2 &= -L_5 \\
L_3 &= -L_6 \\
L_4 &= \mu r^{-3} L_1 - 3\mu r^{-5} x_1 (L_1 x_1 + L_2 x_2 + L_3 x_3) \\
L_5 &= \mu r^{-3} L_2 - 3\mu r^{-5} x_2 (L_1 x_1 + L_2 x_2 + L_3 x_3) \\
L_6 &= \mu r^{-3} L_3 - 3\mu r^{-5} x_3 (L_1 x_1 + L_2 x_2 + L_3 x_3)
\end{aligned}
\qquad (3)
$$

$$
\begin{aligned}
&L_1 CPCY + L_2 SPCY = 0 \\
&L_1 SPSY - L_2 CPSY + L_3 CY = 0
\end{aligned}
\qquad (4)
$$

Assuming $CY \neq 0$ and letting $D^2 = L_1^2 + L_2^2$, $E^2 = L_1^2 + L_2^2 + L_3^2$, $D > 0,\ E > 0$ we get from equations (4) that

$$
\begin{aligned}
&\tan \chi_1 = -L_1/L_2 \ , \ SP = -L_1/D \ , \ CP = L_2/D \ , \\
&\tan \chi_3 = L_3/D \ , \ SY = L_3/E \ , \ CY = D/E \ ,
\end{aligned}
\qquad (5)
$$

the choice of signs in SP,CP,SY,CY being a consequence of the Weierstrass and Clebsch conditions, as will be shown in the next section. From (5) it follows that the thrust vector in the plumbline system can be expressed as

$$\underline{F}\,[A] = F(-SPCY, CPCY, SY) = F(L_1/E, L_2/E, L_3/E) \ .$$

Equations (5) may be used to eliminate the control variables from equations (2), thus giving, together with equations (3), a system of 12 differential equations of the first order in 12 dependent variables. This system may be written as six equations of second order, which in vector notation are

$$
\begin{aligned}
\ddot{\underline{x}} &= F\underline{E}/mE - \mu \underline{x}/r^3 \ , \\
\ddot{\underline{E}} &= -\mu \underline{E}/r^3 + 3\mu (\underline{x}\cdot\underline{E})\underline{E}/r^5 \ ,
\end{aligned}
\qquad (6)
$$

where $\underline{E}$ denotes the vector (L_1, L_2, L_3) .

Although the result is not utilized in this paper, it is of interest to note that three first integrals of the system (6) can be readily obtained by the following device. Cross multiply the first of equations (6) by $\underline{E}$ and the second by $\underline{x}$ and add the resulting equations to get

$$\underline{E} \times \ddot{\underline{x}} + \underline{x} \times \ddot{\underline{E}} = 0 \ . \tag{7}$$

This now yields

$$\underline{E} \times \dot{\underline{x}} + x \times \dot{\underline{E}} = \underline{M} \ , \tag{8}$$

where $\underline{M}$ is a constant vector, since the derivative with respect to t of the left member of (8) is the left member of (7).

The equations (2) and (3), after elimination of the control variables, or, equivalently, system (6), will have a six-parameter family of solutions satisfying the given initial conditions $\underline{u}(t_o) = \underline{u}_o, \underline{x}(t_o) = \underline{x}_o$. However, since the equations are homogeneous in the L's, if $\underline{u}(t), \underline{x}(t), \underline{L}(t)$ is a solution, then so is $\underline{u}(t), \underline{x}(t), c\underline{L}(t)$ for any non-zero constant c . Thus, if initial values of the L's are taken as parameters, only their ratios are significant in determining $\underline{u}(t), \underline{x}(t)$. Hence the value of one L may be fixed, or some function of the L's may be assigned a value at $t = t_o$, say $L_1^2(t_o) + L_2^2(t_o) + L_3^2(t_o) = 1$. Thus there is a five-parameter family of trajectories satisfying the Euler-Lagrange equations and having the given initial values. If $b_1, \cdots, b_5$ denote the parameters, the equations of the family may be written

$$\begin{aligned} \underline{u} &= \underline{u}(t, b_1, b_2, b_3, b_4, b_5) \ , \\ \underline{x} &= \underline{x}(t, b_1, b_2, b_3, b_4, b_5) \ . \end{aligned} \tag{9}$$

Each of these curves is the path of least time from the initial point to any other point on it, assuming that a minimum exists and that only one of the curves joins the two points. (The geometrical terminology refers to the seven dimensional space $t, \underline{u}, \underline{x}$ and not to three dimensional physical space.) Putting $t = t_1$ gives a point on each curve, and the totality of such points constitutes a subspace S_1 . If S_1 is considered as a given locus of variable end-points for the first stage, then, since t has constant value t_1 on S_1 , each trajectory is a time minimizing trajectory from the initial point to S_1 , and hence must satisfy the transversality conditions at S_1 . This property will be utilized in the discussion of continuity properties of the Lagrange multipliers.

THE WEIERSTRASS AND CLEBSCH CONDITIONS

We now show that, with the choice of signs adopted in (5), the necessary conditions II and III of Section I are satisfied by solutions of equations (2), (3), (4). For the Weierstrass test let circumflexes denote arbitrary values of the control variables. Then

$$H(t,\underline{u},\underline{x},\underline{\chi},\underline{L}) - H(t,u,x,\hat{\chi},L)$$
$$= Fm^{-1}(-L_1SPCY + L_2CPCY + L_3SY + L_1\hat{S}\hat{P}\hat{C}\hat{Y} - L_2\hat{C}\hat{P}\hat{C}\hat{Y} - L_3\hat{S}\hat{Y})$$
$$= Fm^{-1}(E + L_1\hat{S}\hat{P}\hat{C}\hat{Y} - L_2\hat{C}\hat{P}\hat{C}\hat{Y} - L_3\hat{S}\hat{Y}) > 0 ,$$

as is implied by the general inequality

$$(a^2 + b^2 + c^2)^{1/2} \geqq (a \sin A + b \cos A) \cos B + c \sin B ,$$

which holds for all real values of a, b, c, A, B .

For the Clebsch test, the matrix of the quadratic form involved is

$$\begin{bmatrix} L_1SPCY - L_2CPCY & L_1CPSY + L_2SPSY \\ L_1CPSY + L_2SPSY & L_1SPCY - L_2CPCY - L_3SY \end{bmatrix} .$$

By virtue of equations (5) this becomes

$$\begin{bmatrix} -D^2/E & 0 \\ 0 & -E \end{bmatrix} ,$$

which implies that the quadratic form is negative definite.

There are in all four sets of values of SP, CP, SY, CY in terms of the L's that will satisfy equations (4). Two of them reverse the inequality signs in conditions II and III, but there is one other set besides that given in (5) that satisfies conditions II and III. It can be got from (5) by replacing D by -D . This amounts to changing χ_1 to $\chi_1 + \pi$

and χ_3 to $\pi - \chi_3$, and it is found that this actually produces the same direction of thrust as before.

SECOND AND SUBSEQUENT STAGES

For the second stage the range of t is $t_1 \leqq t \leqq t_2$. The initial point is required to be in S_1 , the equations of which are obtained by putting $t = t_1$ in (9):

$$\left.\begin{aligned} \underline{u} &= \underline{u}(t_1,b_1,b_2,b_3,b_4,b_5) \\ \underline{x} &= x(t_1,b_1,b_2,b_3,b_4,b_5) \end{aligned}\right\} \equiv \underline{X}_1(\underline{b}) , \tag{10}$$

the six functions in the right members being denoted by $\underline{X}_1(\underline{b})$ to conform with the notation in Section I. The function $T_1(b)$ is the constant t_1 .

The differential equations of motion are of the same form as for the first stage, although F and m have different values. To allow for possible discontinuities in the L's , we denote their right hand limits at t_1 by $L(t_1 +)$. There are five transversality conditions (Condition I, Section I) which must be satisfied at $t = t_1$:

$$L(t_1+) \cdot \underline{X}_{1b_k} = 0 , \qquad k = 1,2,3,4,5 . \tag{11}$$

Since these equations are homogeneous in the L's , and so are the equations analogous to (2), (3), and (4), it follows that for the determination of $\underline{u}(t)$ and $\underline{x}(t)$ again only the ratios of the L's are significant. Thus again there will be an eleven parameter family of minimizing trajectories. When values are given to the b's to fix a point in S_1 , there will be six values $\underline{u}(t_1)$, $\underline{x}(t_1)$ and five transversality conditions to determine the eleven parameters. This in general will fix a unique minimizing trajectory issuing from each point of S_1 . Let the equations of these trajectories be expressed by the same equations (9) as for the first stage except that now the range for t is from t_1 to t_2 . Putting $t = t_2$ will determine a definite point on each trajectory, and the locus of these points will be a subspace S_2 with equations

$$\left.\begin{aligned} \underline{u} &= \underline{u}(t_2,b_1,b_2,b_3,b_4,b_5) \\ \underline{x} &= \underline{x}(t_2,b_1,b_2,b_3,b_4,b_5) \end{aligned}\right\} \equiv \underline{X}_2(\underline{b}) .$$

Note that again the transversality and other conditions involving the end point need not be used to determine the five parameter family of trajectories but only the conditions at the initial point.

For subsequent stages the procedure is like that for the second stage. The initial point for the third stage would be restricted to subspace S_2 and transversality conditions involving $\underline{X}_2(\underline{b})$ and $\underline{L}(t_2+)$ would be used.

The computational procedure given by Cox and Shaw (Ref. 1, p.118) could be used in the first stage. Modifications would be needed in the other stages to approximate the partial derivatives of the $\underline{X}(\underline{b})$ functions and to solve the transversality equations.

In the final stage the mission objective must be fulfilled at the end point. Since there is little hope for closed form solutions, the proposed procedure is to estimate initial conditions and use them to extend a solution by approximate integration methods through the several stages. If the objectives are not attained, make new estimates of the initial conditions and new computations of a minimizing trajectory, continuing thus until a trajectory is obtained that achieves the desired objectives with sufficient accuracy.

CONTINUITY PROPERTIES OF THE LAGRANGE MULTIPLIERS

In each stage the trajectories which are without corners and which satisfy the Euler-Lagrange equations will have Lagrange multipliers that are continuous and differentiable (Ref. 3, pp.202-204; Ref. 6,p. 12). However, on passing from one stage to the next, there are discontinuities in the functions defining $\dot{\underline{u}}$. From equations (2) it follows that there will be corners for the functions $\underline{u}$, and hence discontinuities might be expected in the L's . But the functions defining $\dot{\underline{x}}$ and $\dot{\underline{L}}$ are continuous in t, $\underline{u}$, $\underline{x}$ and have continuous partial derivatives. Thus continuous solutions for the L's can be obtained by taking $\underline{u}$ continuous across boundaries, provided the transversality conditions can be satisfied.

In obtaining the family of solutions of the Euler-Lagrange equations in each stage the homogeneity of the equations in the L's was utilized to decrease the number of parameters by one, say by assigning an initial value to one of the L's . As remarked in the discussion of the first stage, the five transversality conditions for parameters $b_1, \cdots, b_5$, namely,

$$\underline{L}(t_1-) \cdot \underline{X}_{1b_k}(\underline{b}) = 0 \ , \qquad k = 1,2,3,4,5,$$

are satisfied on S_1 . These conditions are the same as conditions (11) in $\underline{L}(t\ +)$ which hold for S_1 as locus of initial points in stage two. Hence $L_1(t_1-), \cdots, L_6(t_1-)$ and $L_1(t_1+), \cdots, L_6(t_1+)$ are proportional. By assigning equal values to one pair from the two sets, all can be made continuous at t_1 .

The transversality condition involving the final time as parameter in each stage is not homogeneous in the L's because of the term h_{b_k} . This condition would make the set of L's unique and not necessarily continuous across the boundary; however, it is not essential to use this condition for the determination of the trajectory equations. Hence it is possible to obtain Lagrange multipliers that are continuous through the several stages and to use their ratios at the initial point $t = t_o$ as parameters $b_1, \cdots, b_5$ for a five parameter family extending through all the stages.

SECTION III. A MULTISTAGE NAVIGATION PROBLEM

A simple form of Zermelo's navigation problem (Ref. 4), extended to multiple stages, serves to illustrate some features of trajectory problems. Zermelo stated his problem for air flight in a plane, but we follow Cicala's formulation (Ref. 5, p.19) and consider a motor boat on a plane water surface. A rectangular coordinate system is associated with the plane surface, and the boat is considered a point (x,y). The water current is assumed to have known velocity components u and v as functions of x and y and the time t. Let the velocity vector of the boat relative to the water make an angle θ with the positive x-axis and assume that the magnitude of the velocity vector is a known constant in each stage. The path of the boat is determined by the control variable θ, and the problem is to find θ as a function of t so as to minimize the time t_f for the boat to go from the origin to a specified point (x_f, y_f) that is assumed remote enough to require three stages. In order to get a problem that will have an easily obtained closed form solution, we take the water velocity components to be constants and choose the coordinate system so that $u = 0$, $v = a$.

The problem then is to find functions $x(t)$, $y(t)$, $\theta(t)$ such that

$$\dot{x} = v \cos \theta \ , \ \dot{y} = a + v \sin \theta \ ; \tag{1}$$

$$v = v_1 \text{ for } 0 \leqq t < t_1; \ v = v_2 \text{ for } t_1 \leqq t < t_2; \ v = v_3 \text{ for } t_2 \leqq t:$$

$$x(0) = y(0) = 0 \ ; \ x(t_f) = x_f \ , \ y(t_f) = y_f;$$

and such that t_f is a minimum.

FIRST STAGE

As in Section I, define the generalized Hamiltonian

$$H = L_1 v_1 \cos \theta + L_2(a + v_1 \sin \theta) \ .$$

From this H the Euler-Lagrange equations are found to be

$$\dot{x} = v_1 \cos \theta \ , \quad \dot{y} = a + v_1 \sin \theta \ ,$$

$$\dot{L}_1 = 0 \ , \quad \dot{L}_2 = 0 \ , \quad -L_1 \sin \theta + L_2 \cos \theta = 0 \ . \tag{2}$$

Hence L_1 and L_2 are constants, say $L_1 = L_{11}$, $L_2 = L_{21}$. It then follows that θ is constant, and integration of the first two of the above equations gives

$$x = (v_1 \cos \theta)t \ , \quad y = (a + v_1 \sin \theta)t \ , \tag{3}$$

on using initial conditions $x = y = 0$ when $t = 0$. Thus paths of minimum time are straight lines.

If our problem were a one-stage problem with end point (x_1, y_1) and time t_1 to be a minimum, we would have for the determination of θ, t_1, L_{11} and L_{21} the following equations

$$x_1 = (v_1 \cos \theta)t_1 \ , \quad y_1 = (a + v_1 \sin \theta)t_1 \ , \tag{4}$$

$$-L_{11} \sin \theta + L_{21} \cos \theta = 0 \ , \tag{5}$$

plus the transversality condition

$$L_{11} v_1 \cos \theta + L_{21}(a + v_1 \sin \theta) = 1 \ . \tag{6}$$

Equation (6) is found from the transversality equation in Section I by putting

$$k = 1, \ b_1 = t_1, \ T_1 = 0, \ T_2 = t_1, \ X_{11} = 0, \ X_{21} = 0, \ X_{12} = x_1, \ X_{22} = y_1, \ h = t_1.$$

Equations (4) determine θ and t_1, while (5) and (6) give unique multipliers

$$L_{11} = \cos \theta/(v_1 + a \sin \theta), \ L_{21} = \sin \theta/(v_1 + a \sin \theta) \ . \tag{7}$$

Now if we consider (x_1, y_1) variable and inquire as to the locus of such points each of which is reached in a minimum time equal to t_1, we get from (4), with θ variable, that the locus of (x_1, y_1) is the circle with center $(0, at_1)$ and radius $v_1 t_1$.

SECOND STAGE

The locus of initial points for the second stage is the circle mentioned in the preceding sentence. We write it as

$$x_1 = (v_1 \cos \alpha)t_1, \quad y_1 = (a + v_1 \sin \alpha)t_1 , \tag{8}$$

with the parameter α replacing the θ of equations (4), since we shall continue to use θ as the control variable. The differential equations of constraint for this stage are the same as for the first stage except that v_2 replaces v_1.

The Euler-Lagrange equations are as before, with v_2 replacing v_1, and hence L_1 and L_2 are constant, say $L_1 = L_{12}$, $L_2 = L_{22}$. It follows that θ is constant.

If the end point for the second stage is considered fixed at (x_2,y_2), then transversality conditions for parameters α and t_2 are

$$\begin{aligned} &L_{12}v_1t_1 \sin \alpha - L_{22}v_1t_1 \cos \alpha = 0 , \\ &L_{12}v_2 \cos \theta + L_{22}(a + v_2 \sin \theta) = 1 . \end{aligned} \tag{9}$$

The first of these equations, together with the last of the Euler-Lagrange equations, implies that $\theta = \alpha$. Then, from the pair of equations (9), it follows that

$$L_{12} = \cos \theta/(v_2 + a \sin \theta), \quad L_{22} = \sin \theta/(v_2 + a \sin \theta) . \tag{10}$$

Thus L_{12} and L_{22} are not equal to L_{11} and L_{21}, indicating discontinuities in the multipliers at stage boundaries. However, the control variable θ is continuous, being in fact the same constant in the two stages.

On integrating the Euler-Lagrange equations for x and y and using (8) as initial conditions, one finds that

$$\begin{aligned} x &= (v_2 \cos \theta)t + (v_1 - v_2)t_1 \cos \theta , \\ y &= (a + v_2 \sin \theta)t + (v_1 - v_2)t_1 \sin \theta . \end{aligned} \tag{11}$$

For each constant θ , the path is a straight line.

Now consider the locus of end points (x_2,y_2) that will each be reached in minimum time t_2 . Fixing $t = t_2$ in (11) and considering θ variable shows the locus to be the circle with center $(0,at_2)$ and radius $v_1t_1 + v_2(t_2 - t_1)$.

THIRD STAGE

With the circle of the preceding sentence as locus of initial points, the end point is required to be (x_f,y_f) and time t_f is to be a minimum. In the same way as before the path is shown to be a straight line with the control variable constant and equal to its value in the preceding stages. The new equations for x and y are

$$\begin{aligned} x &= (v_3 \cos \theta)t + \left[(v_1 - v_2)t_1 + (v_2 - v_3)t_2 \right] \cos \theta , \\ y &= (a + v_3 \sin \theta)t + \left[(v_1 - v_2)t_1 + (v_2 - v_3)t_2 \right] \sin \theta . \end{aligned} \qquad (12)$$

By putting the given values x_f,y_f in equations (12), one can solve for the minimum time $t = t_f$ and for the constant control angle θ . Then equations (11) with $t = t_2$, $x = x_2$, $y = y_2$ and equations (8) determine the corner points (x_1,y_1) and (x_2,y_2) .

CONCLUSIONS

This problem illustrates the extension of a trajectory across stage boundaries where the differential equations of constraint are discontinuous. The effect of the homogeneity in the Lagrange multipliers is similar to that in the more general problem.

The unique Lagrange multipliers that satisfy the Euler-Lagrange equations and the transversality conditions of I, Section I, are discontinuous at stage boundaries. However, the ratio $L_2/L_1 = \tan \theta$ is the same for each stage. The equations containing L's are homogeneous in the L's , except that the transversality condition computed for the final time as parameter in each stage is not homogeneous. But this transversality condition is not needed to determine the family of minimizing trajectories which satisfy initial conditions in each stage. That is, in order to obtain a pieced trajectory extending through the several stages, only the ratio of the L's is needed, and, since the ratio is preserved, the L's may be chosen continuous.

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SECTION IV. ON MULTISTAGE PROBLEMS HAVING DISCONTINUITIES AT STAGE BOUNDARIES

Discontinuities will be allowed in the functions appearing in the differential equation constraints and in the dependent variable coordinates defining admissible paths. Let t be the independent variable. For fixed p, define a set of variables $(t_o, t_1, \ldots, t_p)$ to be a <u>partition set</u> if and only if $t_o < t_1 < \ldots < t_p$. Let I denote the interval $t_o \leq t \leq t_p$ and I_a the subinterval $t_{a-1} \leq t < t_a$ for $a = 1, \ldots, p - 1$ and $t_{a-1} \leq t \leq t_a$ for $a = p$. Let $z(t)$ denote the set of functions $(z_1(t), \ldots, z_N(t))$, where each $z_\alpha(t)$, $\alpha = 1, \ldots, N$, is continuous on I except possibly at partition points $t_1, \ldots t_{p-1}$. At these points right and left limits $z_\alpha(t_1^-)$, $z_\alpha(t_1^+)$, $\ldots z_\alpha(t_{p-1}^+)$ are assumed to exist and we let $z_\alpha(t_b) = z_\alpha(t_b^+)$, $b = 1, \ldots, p - 1$.

The problem will be to find in a class of admissible arcs

$$z(t), \qquad (t_o,\ldots,t_p), \qquad t_o \leq t \leq t_p,$$

satisfying differential equations

$$(1) \qquad \phi^a_\beta\,(t,z,\dot z) = 0, \qquad t \text{ in } I_a, \qquad \beta = 1,\ldots,M < N,$$

and end and intermediate point conditions

$$(2) \qquad f_\gamma(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p)) = 0,$$
$$\gamma = 1, \ldots, K \leq (N+1)(p+1),$$

$$(3) \qquad z_\alpha(t_b^+) - z_\alpha(t_b^-) - d_{\alpha b} = 0$$

one that will minimize

$$f_o(t_o,\ldots,t_p,\ z(t_o),\ z(t_1^-),\ z(t_1^+),\ \ldots,\ z(t_p)).$$

Let R_a' be an open connected set in the 2N+1 dimmensional $(t,z,\dot z)$ space whose projection on the t-axis contains I_a. The functions ϕ^a_β are required to have continuous third partial derivatives in R_a' and each matrix $\|\phi^a_{\beta\dot z_\alpha}\|$ is assumed of rank M in R_a'.

Let S' denote an open connected set in the 2Np+p+1 dimmensional space of points $(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p))$ in which the functions f_ρ, $\rho = 0, 1, \ldots, K$ have continuous third partial derivatives and the matrix

$$(4) \qquad \left\| f_{\rho t_o} \quad f_{\rho t_b} \quad f_{\rho t_p} \quad f_{\rho z_\alpha(t_o)} \quad f_{\rho z_\alpha(t_b^-)} \quad f_{\rho z_\alpha(t_b^+)} \quad f_{\rho z_\alpha(t_p)} \right\|$$

is of rank K+1.

An admissible set is a set $(t,z,\dot{z})$ in R_a' for some a=1,...,p. An admissible subarc C_a' is a set of functions $z(t)$, t on I_a, with each $(t,z,\dot{z})$ an admissible set and such that $z(t)$ is continuous and $\dot{z}(t)$ is piecewise continuous on I_a. An admissible arc E' is a partition set $(t_o,\ldots,t_p)$ together with a set of admissible subarcs C_a', $a = 1, \ldots, p$, such that the set $(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p))$ is in S'.

Multiplier Rule. An admissible arc E' that satisfies equations (1), (2), (3) is said to satisfy the multiplier rule if there exist constants e_ρ not all zero and a function

$$F(t,z,\dot{z},\lambda) = \lambda_\beta \phi_\beta^a(t,z,\dot{z}), \ t \ \text{in} \ I_a,$$

with multipliers $\lambda_\beta(t)$ continuous except possibly at corners or discontinuities of E', where left and right limits exist, such that the following equations hold:

$$(5) \qquad F_{\dot{z}_\alpha} = \int_{t_{a-1}}^{t} F_{z_\alpha} dt + c_\alpha^a, \quad t \ \text{in} \ I_a,$$

$$e_\rho f_{\rho t_o} + \Big[\dot{z}_\alpha F_{\dot{z}_\alpha}\Big]^{t_o} = 0,$$

$$e_\rho f_{\rho t_b} + \Big[\dot{z}_\alpha F_{\dot{z}_\alpha}\Big]_{t_b^-}^{t_b^+} = 0,$$

$$e_\rho f_{\rho t_p} + \Big[\dot{z}_\alpha F_{\dot{z}_\alpha}\Big]_{t_p} = 0,$$

$$e_\rho f_{\rho z_\alpha(t_o)} - \Big[F_{\dot{z}_\alpha}\Big]^{t_o} = 0,$$

$$e_\rho (f_{\rho z_\alpha}(t_b^+) + f_{\rho z_\alpha}(t_b^-) - \left[F_{\dot z_\alpha}\right]_{t_b^-}^{t_b^+} = 0,$$

$$e_\rho f_{\rho z_\alpha}(t_p) - \left[F_{\dot z_\alpha}\right]_{t_p} = 0.$$

Every minimizing arc must satisfy the multiplier rule.

An extremal is defined to be an admissible arc and set of multipliers

$$z_\alpha(t),\ (t_o,\ldots,t_p),\ \ \lambda_\beta(t),\ \ t_o \leq t \leq t_p,$$

satisfying equations (1) and (5) and such that the functions $\dot z_\alpha(t)$, $\lambda_\beta(t)$ have continuous first derivatives except possibly at partition points, where finite left and right limits exist. An extremal is non-singular in case the determinant

$$\begin{Vmatrix} F_{\dot z_\alpha \dot z_\eta} & \phi_{\delta \dot z_\alpha} \\ \phi_{\beta \dot z_\eta} & 0 \end{Vmatrix} \qquad \begin{matrix} \alpha, \eta = 1, \ldots, N \\ \beta, \delta = 1, \ldots, M \end{matrix}$$

is different from zero along it. An admissible arc with a set of multipliers satisfying the multiplier rule is called normal if $e_o = 1$. With this value of e_o the set of multipliers is unique.

Weierstrass Condition. An admissible arc E' with a set of multipliers $\lambda_\beta(t)$ is said to satisfy the Weierstrass condition if

$$(t,z,\dot z,\lambda,\dot Z) = F(t,z,\dot Z,\lambda) - F(t,z,\dot z,\lambda) - (\dot Z'_\alpha - \dot z_\alpha)F_{\dot z_\alpha}(t,z,\dot z,\lambda) \geq 0$$

holds at every element $(t,z,\dot z,\lambda)$ of E' for all admissible sets $(t,z,\dot Z)$ satisfying the equations $\phi^a_\beta = 0$. Every normal minimizing arc must satisfy the Weierstrass condition.

Clebsch Condition. An admissible arc E' with a set of multipliers $\lambda_\beta(t)$ is said to satisfy the Clebsch condition if

$$F_{\dot z_\alpha \dot z_\eta}(t,z,\dot z,\lambda)\ \pi_\alpha \pi_\eta \geq 0$$

holds at every element $(t,z,\dot{z},\lambda)$ of E' for all sets $(\pi_1,\ldots,\pi_N)$ satisfying the equations

$$\phi^a_{\beta\dot{z}_\alpha}(t,z,\dot{z})\,\pi_\alpha = 0.$$

Every normal minimizing arc must satisfy the Clebsch condition.

SECTION V. ON MULTISTAGE PROBLEMS INVOLVING CONTROL VARIABLES AND HAVING INEQUALITY AND FINITE EQUATION CONSTRAINTS

By the introduction of new variables and by notational transformations the theory of Section I/can be utilized to establish necessary conditions for the more general formulation of this section. As before, let t be the independent variable and define a set of variables $(t_o, \ldots, t_p)$ contained in the range of t to be a partition set if and only if $t_o < t_1 < \ldots < t_p$. Let I denote the interval $t_o \leq t \leq t_p$, and let I_a denote the sub-interval $t_{a-1} \leq t < t_a$ for $a = 1, \ldots, p - 1$ and $t_{a-1} \leq t \leq t_a$ for $a = p$.

Let $x(t)$ denote the set of functions $(x_1 (t), \ldots, x_n (t))$. For each $i, i = 1, \ldots, n$, assume $x_i(t)$ to be continuous on I except possibly at partition points t_b, $b = 1, \ldots, p - 1$, where finite left and right limits exist; denote these limits by $x_i(t_b^-)$ and $x_i(t_b^+)$, respectively. The amount of discontinuity of each member of $x(t)$ at each partition point will be assumed known, and we write

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0,$$

with each d_{ib} a known constant. Also let $x_i(t_b) = x_i(t_b^+)$. Thus $x_i(t)$ is continuous at t_b if and only if $d_{ib} = 0$.

Let $y(t)$ denote the set $(y_1(t), \ldots, y_m(t))$, where $y_j(t)$ is piecewise continuous on I, $j = 1, \ldots, m$, finite discontinuities being allowed between, as well as at, partition points. In the formulation of the problem the $y_j(t)$ will occur only as undifferentiated variables and will not occur in the function to be minimized nor in the end and intermediate point constraints. Such variables are called control variables, while the $x_i(t)$ are called state variables.

The problem is to find in a class of admissible arcs

$$x(t), \quad y(t), \quad (t_o, \ldots, t_p), \quad t_o \leq t \leq t_p,$$

which satisfy differential equations

$$\dot{x}_i = L_i^a(t,x,y,), \quad t \text{ in } I_a, \quad a = 1, \ldots, p, \quad i = 1, \ldots, n,$$

finite equations

$$M^a_g(t,x,y) = 0, \qquad g = 1, \ldots, q,$$

inequalities

$$N^a_h(t,x,y) \geq 0, \qquad h = 1, \ldots, r, \qquad q + r \leq m,$$

and end and intermediate point conditions

$$J_k(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(_1^+), \ldots, x(t_p)) = 0,$$

$$k = 1, \ldots, s < (n + 1) \ (p + 1),$$

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0, \quad b = 1, \ldots, p - 1,$$

one that will minimize

$$J_o(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)).$$

In order to state precisely the properties of the functions involved in the problem, let R_a be an open connected set in the $m + n + 1$ dimensional (t,x,y) space whose projection on the t-axis contains the interval I_a, and let S be an open connected set in the $2np + p + 1$ dimmensional space of points

$$(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)).$$

The functions L^a_i, M^a_g, N^a_h are assumed continuous with continuous partial derivatives through those of third order in R_a, and J_o, J_k are to have such continuity properties in S. For each a, the matrix

$$\left\| \begin{matrix} M^a_{gy_j} & 0 \\ N^a_{hy_j} & D^a_1 \end{matrix} \right\|$$

is assumed of rank $q + r$ in R_a, where D^a_1 is an r by r diagonal matrix with $N^a_1, \ldots, N^a_r$ as diagonal elements. The matrix

$$\left\| J_{ct_o} \quad J_{ct_b} \quad J_{ct_p} \quad J_{cx_i(t_o)} \quad J_{cx_i(t_b^-)} \quad J_{cx_i(t_b^+)} \quad J_{cx_i(t_p)} \right\| \quad , c = 0, \ldots, s,$$

is assumed of rank $s + 1$ in S.

An admissible set is a set (t,x,y) in R_a for some $a = 1, \ldots, p$. An admissible sub-arc C_a is a set of functions $x(t), y(t), t$ on I_a, with each (t,x,y) admissible, and such that $x(t)$ is continuous and $\dot{x}(t), y(t)$ are piecewise continuous on I_a. An admissible arc is a partition set $(t_o, \ldots, t_p)$ together with a set of admissible sub-arcs C_a, $a = 1, \ldots, p$, such that the set $(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p))$ is in S.

On introducing a generalized Hamiltonian function H as defined below and utilizing the normal form of the differential equation constraints, one can now apply the theory of Section I to obtain the following multiplier rule.

The Multiplier Rule

An admissible arc E for which

$$J_k(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)) = 0,$$

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0,$$

is said to satisfy the multiplier rule if there exists a function

$$H(t,x,y,\lambda,\mu,\nu) = \lambda_i L_i^a - \mu_g M_g^a + \nu_h N_h^a,$$

with multipliers $\lambda_i(t), \mu_g(t), \nu_h(t)$ continuous except possibly at partition points or corners of E, where finite left and right limits exist, such that for each t in I_a, $a = 1, \ldots, p$,

$$(1) \quad \lambda_i = -\int_{t_{a-1}}^{t} H_{x_i} dt + c_i^a, \; H_{y_j} = 0, \; \dot{x}_i = L_i^a, \; M_g^a = 0, \; N_h^a \geq 0,$$

and such that the transversality matrix

$$(2) \quad \left\| \begin{matrix} H(t_o) & H(t_b^+) - H(t_b^-) & -H(t_p) & -\lambda_i(t_o) & -\lambda_i(t_b^+) + \lambda_i(t_b^-) & \lambda_i(t_p) \\ J_{ct_o} & J_{ct_b} & J_{ct_p} & J_{cx_i(t_o)} & J_{cx_i(t_b^+)} + J_{cx_i(t_b^-)} & J_{cx_i(t_p)} \end{matrix} \right\|$$

is of rank $s + 1$. The multipliers ν_h are zero when $N_h > 0$. Every minimizing arc E must satisfy the multiplier rule.

Between corners of a minimizing arc E *the equations*

$$\dot{x}_i = H_{\lambda_i}, \quad \dot{\lambda}_i = -H_{x_i}, \quad H_{y_j} = 0, \quad \nu_h H_{\nu_h} = 0 \quad \text{(not summed)}$$

hold and hence also

$$\frac{dH}{dt} = H_t.$$

Transversality Conditions for Normal Arcs

Under the usual normality assumptions, the transversality matrix can be put into a form having one fewer rows. This leads to the following statement of transversality conditions.

For a normal minimizing arc the transversality matrix

$$\left\| \begin{matrix} H(t_o)+J_{ot_o} & H(t_b^+)-H(t_b^-)+J_{ot_b} & -H(t_p)+J_{ot_p} & -\lambda_i(t_o)+J_{ox_i}(t_o) & -\lambda_i(t_b^+)+\lambda_i(t_b^-)+J_{ox_i}(t_b^+)+J_{ox_i}(t_b^-) & \lambda_i(t_p)+J_{ox_i}(t_p) \\ J_{kt_o} & J_{kt_b} & J_{kt_p} & J_{kx_i}(t_o) & J_{kx_i}(t_b^+)+J_{kx_i}(t_b^-) & J_{kx_i}(t_p) \end{matrix} \right\|$$

is of rank s.

Since the matrix is of order $s + 1$ by $(n+1)(p+1)$, the requirement that the rank be s imposes $(n+1)(p+1) - s$ conditions. This is one more condition than was imposed by (2), which was sufficient to determine the multipliers up to an arbitrary proportionality factor.

Weierstrass Condition

For a normal minimizing arc E *the inequality*

$$\lambda_i L_i(t,x,y) \geq \lambda_i L_i(t,x,Y)$$

must hold at each element (t,x,y,λ,μ,ν) *of* E *for all admissible sets* (t,x,Y) *satisfying* $M_g(t,x,Y) = 0$ *and* $N_h(t,x,Y) \geq 0$.

Clebsch Condition

For a normal minimizing arc E *the inequality*

$$H_{y_j y_e} \pi_j \pi_e \leq 0$$

must hold at each element (t,x,y,λ,μ,ν) *of* E *for all sets* $\pi_1, \ldots, \pi_m$ *satisfying* $M_{gy_j}(t,x,y)\pi_j = 0$ *and* $N_{hy_j}(t,x,y)\pi_j = 0$, *where in the last equation* h *ranges only over the subset of* $1, \ldots, r$ *for which* $N_h(t,x,y) = 0$.

For a normal minimizing arc the multipliers ν_h *are all non-negative*.

SECTION VI. A THREE STAGE RE-ENTRY OPTIMIZATION PROBLEM

In this section the theory of Section II is applied to a three stage re-entry problem. Since it is primarily an illustrative example, certain simplifying assumptions are made. In particular, the vehicle is assumed to be a particle of variable mass, with thrust magnitude proportional to mass flow rate and thrust direction subject to instantaneous change. Moreover, external forces are required to be functions of position only, while the earth is assumed spherically symmetrical and nonrotating with respect to the coordinate system of the vehicle. Finally, motion is restricted to two dimensions, gravitational acceleration is approximated by first order terms, and air resistance is neglected.

The foregoing conditions allow the motion of the vehicle to be described by the following equations:

$$\dot{u} = \begin{cases} -a^2x + cB_1m^{-1}\cos\theta, & t_o \leq t < t_1, \\ -a^2x, & t_1 \leq t < t_2, \\ -a^2x + cB_3m^{-1}\cos\theta, & t_2 \leq t \leq t_3, \end{cases}$$

$$\dot{v} = \begin{cases} -g_o + 2a^2y + cB_1m^{-1}\sin\theta, & t_o \leq t < t_1, \\ -g_o + 2a^2y, & t_1 \leq t < t_2, \\ -g_o + 2a^2y + cB_3m^{-1}\sin\theta, & t_2 \leq t \leq t_3, \end{cases}$$

$$\dot{x} = u, \quad t_o \leq t \leq t_3,$$

$$\dot{y} = v, \quad t_o \leq t \leq t_3,$$

$$\dot{m} = \begin{cases} -B_1, & t_o \leq t < t_1, \\ 0, & t_1 \leq t < t_2, \\ -B_3, & t_2 \leq t \leq t_3, \end{cases}$$

where t_o is initial time, t_3 is final time, and t_1, t_2 are intermediate staging times. The symbols a, g_o represent gravitation constants, and B_1, B_3 denote constant mass flow rates. This description implies a burning arc, a coast arc, and finally a burning arc, with B_3 not necessarily different from B_1.

The following end and intermediate point conditions will be imposed.

$$J_1 \equiv t_o = 0,$$
$$J_2 \equiv u\ (t_o) - u_o = 0,$$
$$J_3 \equiv v\ (t_o) = 0,$$
$$J_4 \equiv x\ (t_o) = 0,$$
$$J_5 \equiv y\ (t_o) - y_o = 0,$$
$$J_6 \equiv x\ (t_1) - x_1 = 0,$$
$$J_7 \equiv y\ (t_2) - y_2 = 0,$$
$$J_8 \equiv x\ (t_3) - x_3 = 0,$$
$$J_9 \equiv y\ (t_3) - y_3 = 0,$$
$$J_{10} \equiv m\ (t_3) - m_3 = 0,$$

and

$$m\ (t_1^-) - m\ (t_1^+) = d_1,$$

with u_o, y_o, x_1, y_2, x_3, y_3, m_3, d_1, B_1, and B_3 known constants.

The function to be minimized is taken to be the sum of the times of the powered stages, that is,

$$J_o \equiv t_1 - t_o + t_3 - t_2.$$

If $B_1 = B_3$, this is equivalent to requiring that the fuel used be minimized, or $J_o \cong m\ (t_o)$. The conditions J_6 and J_7 insure the existence of three stages.

The Multiplier Rule of Section IV allows the following Hamiltonian to be written:

$$
H = \begin{cases}
\lambda_1 \ (-a^2x + cB_1m^{-1}\cos\ \theta) + \lambda_2 \ (-g_o + 2a^2y + cB_1m^{-1}\sin\ \theta) \\
\qquad + \lambda_3 u + \lambda_4 v + \lambda_5 \ (-B_1), \ t_o \leq t < t_1, \\
\lambda_1 \ (-a^2x) + \lambda_2 \ (-g + 2a^2y) + \lambda_3 \ u + \lambda_4 v, \quad t_1 \leq t < t_2, \\
\lambda_1 \ (-a^2x + cB_3m^{-1}\cos\ \theta) + \lambda_2 \ (-g_o + 2a^2y + cB_3m^{-1}\sin\ \theta) \\
\qquad + \lambda_3 u + \lambda_4 v + \lambda_5 \ (-B_3), \quad t_2 \leq t \leq t_3.
\end{cases}
$$

The Euler equations for this Hamiltonian are:

$$\dot\lambda_1 + \lambda_3 = 0,$$
$$\dot\lambda_2 + \lambda_4 = 0,$$
$$\dot\lambda_3 - a^2\lambda_1 = 0,$$
$$\dot\lambda_4 + 2a^2\lambda_2 = 0,$$
$$\dot\lambda_5 - cB_1m^{-2} \ (\lambda_1 \cos\ \theta + \lambda_2 \sin\ \theta) = 0,$$
$$cB_1m^{-1} \ (\lambda_1 \sin\ \theta - \lambda_2 \cos\ \theta) = 0,$$

for t in $[t_o, t_1)$;

$$\dot\lambda_1 + \lambda_3 = 0,$$
$$\dot\lambda_2 + \lambda_4 = 0,$$
$$\dot\lambda_3 - a^2\lambda_1 = 0,$$
$$\dot\lambda_4 + 2a^2\lambda_2 = 0,$$

for t in $[t_1, t_2)$; and

$$\dot\lambda_1 + \lambda_3 = 0,$$
$$\dot\lambda_2 + \lambda_4 = 0,$$
$$\dot\lambda_3 - a^2\lambda_1 = 0,$$
$$\dot\lambda_4 + 2a^2\lambda_2 = 0,$$
$$\dot\lambda_5 - cB_3m^{-2} \ (\lambda_1 \cos\ \theta + \lambda_2 \sin\ \theta) = 0,$$
$$cB_3m^{-1} \ (\lambda_1 \sin\ \theta - \lambda_2 \cos\ \theta) = 0,$$

for t in $[t_2, t_3]$.

Simple techniques for integration allow these equations to be expressed in integrated form as follows:

$$\lambda_1 = A_1 \sin a(t + C_1),$$
$$\lambda_2 = A_2 \sinh a\sqrt{2}(t + C_2),$$
$$\lambda_3 = -aA_1 \cos a(t + C_1) ,$$
$$\lambda_4 = -aA_2 \sqrt{2} \cosh a\sqrt{2} (t + C_2),$$

for $t_o \leq t < t_1$;

$$\lambda_1 = A_1^{'} \sin a(t + C_1^{'}),$$
$$\lambda_2 = A_2^{'} \sinh a\sqrt{2}(t + C_2^{'}),$$
$$\lambda_3 = -aA_1^{'} \cos a(t + C_1^{'}),$$
$$\lambda_4 = -aA_2^{'}\sqrt{2} \cosh a\sqrt{2} (t + C_2^{'}),$$

for $t \leq t < t_2$; and

$$\lambda_1 = A_1^{''} \sin a(t + C_1^{''}),$$
$$\lambda_2 = A_1^{''} \sinh a\sqrt{2}(t + C_2^{''}),$$
$$\lambda_3 = -aA_1^{''}\cos a(t + C_1^{''}),$$
$$\lambda_4 = -aA_2^{''} \sqrt{2} \cosh a\sqrt{2}(t + C_2^{''}),$$

for $t_2 \leq t \leq t_3$. It is clear in expressing λ_1, λ_2, λ_3, λ_4 as functions of time with two constants of integration, that the last two Euler equations in stage 1 and stage 3 have been ignored. These equations together with the Weierstrass condition will be used to expressed the control angle as a function of the multipliers λ_1 and λ_2. From the last Euler equation of stage 1 and stage 3 we have (for $\lambda_1 \neq 0$, $\cos\theta \neq 0$)

$$\tan\theta = \lambda_2 / \lambda_1$$

and hence

$$\sin\theta = \pm \lambda_2 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}$$

and

$$\cos\theta = \pm \lambda_1 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2} .$$

From the Weierstrass condition of section VI,

$$cB_1 m^{-1}(\lambda_1 \cos\theta + \lambda_2 \sin\theta - \lambda_1 \cos\alpha - \lambda_2 \sin\alpha) \geq 0$$

for $t_o \leq t < t$. Here θ is the control angle that actually optimizes, and α ranges over all possible control angles for which the original equations of motion are satisfied. This expression being non-negative is equivalent to maximizing the following function (with respect to α):

$$W = \lambda_1 \cos\alpha - \lambda_2 \sin\alpha.$$

Thus $\frac{\partial W}{\partial \alpha} = 0$ and $\frac{\partial^2 W}{\partial \alpha^2} \leq 0$ which gives

$$-\lambda_1 \sin\alpha + \lambda_2 \cos\alpha = 0$$

and $-\lambda_1 \cos\alpha - \lambda_2 \sin\alpha \leq 0.$

thus $\tan\alpha = \lambda_2 / \lambda_1$ and

$$\lambda_1 \left(\pm \lambda_1 / \sqrt{\lambda_1^2 + \lambda_2^2}\right) + \lambda_2 \left(\pm \lambda_2 / \sqrt{\lambda_1^2 + \lambda_2^2}\right) \geq 0$$

which implies that $\cos\alpha = \lambda_1 / \sqrt{\lambda_1^2 + \lambda_2^2}$ and similarly that $\sin\alpha = \lambda_2 / \sqrt{\lambda_1^2 + \lambda_2^2}$. Hence the control angle θ is expressed as follows:

$$\tan\theta = \lambda_2 / \lambda_1, \quad \lambda_1 \neq 0, \ \cos\theta \neq 0,$$

$$\cos\theta = \lambda_1 / \sqrt{\lambda_1^2 + \lambda_2^2},$$

$$\sin\theta = \lambda_2 / \sqrt{\lambda_1^2 + \lambda_2^2}$$

for stage 1. The same expressions for control angle θ hold for stage 3. The fifth Euler equation on stage 1 and stage 3 becomes

$$\dot{\lambda}_5 = \begin{cases} cB_1 m^{-2}\sqrt{\lambda_1^2 + \lambda_2^2}, & t_o \leq t < t_1, \\ cB_3 m^{-2}\sqrt{\lambda_1^2 + \lambda_2^2}, & t_2 \leq t \leq t_3. \end{cases}$$

The transversality matrix which is given at the end of this section has eleven rows and twenty-four columns and is of rank ten. From this matrix fourteen end and intermediate conditions are found. These conditions imply that all multipliers, except possibly λ_3 at t_1 and λ_4 at t_2, are continuous across staging times.

Also the following condition holds at t_1:

$$cB_1 m^{-1} (\lambda_1 \cos\theta + \lambda_2 \sin\theta) + \lambda_5 (B_1) + (\lambda_3^+ - \lambda_3^-) u + 1 = 0$$

where $\lambda_3^+ = \lambda_3 (t_1^+)$.

A similar condition that holds at t_2 is:

$$cB_3 m^{-1} (\lambda_1 \cos\theta + \lambda_2 \sin\theta) - \lambda_5 (B_3) + (\lambda_4^+ - \lambda_4^-) v - 1 = 0.$$

The other four conditions implied by the transversality condition are:

$$\lambda_5(t_o) = 0,$$

$$\lambda_1(t_3) = 0,$$

$$\lambda_2(t_3) = 0,$$

$$- H(t_3) + 1 = 0$$

An optimal trajectory for this problem requires the finding of fifteen constants of integration from the equations of motion, a like number from the Euler equations, and the four times t_o, t_1, t_2, and t_3. Fourteen transversality conditions, ten end and intermediate conditions, and ten requirements on state variables at staging points provide the necessary number of conditions for the determination of these constants.

It is possible to start at the last stage to determine the integration constants for the Euler equations in terms of multiplier values. The constants for the third stage are:

$$A_1'' = -\lambda_{33}/a, \quad (\lambda_{33} \text{ is the final value of } \lambda_3),$$

$$A_2'' = -\lambda_{43}/a\sqrt{2},$$

$$C_1'' = C_2'' = -t_3.$$

Because of the continuity of λ_1 and λ_3 at t_2,

$$A_1'' = A_1',$$

$$C_1'' = C_1'.$$

The values for A_2'' and C_2'' only hold for the third stage. To proceed from the third stage back into the second stage we need the value of the difference

$\lambda_4(t_2^+) - \lambda_4(t_2^-)$. This can be found from the transversality condition above which holds at t_2. Supposing this equation solved, the determination of constants A_2', C_2' for the second stage can proceed, and these values also hold for the first stage for λ_2 and λ_4. An analogous procedure is applied to λ_1 and λ_3 for the first stage.

TRANSVERSALITY MATRIX FOR THREE STAGE PROBLEM

$H(t_o)-1$	$H(t_1)\Big\vert_{-}^{+}+1$	$H(t_2)\Big\vert_{-}^{+}-1$	$-H(t_3)+1$	$-\lambda_1(t_o)$	$-\lambda_2(t_o)$
1	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$-\lambda_3(t_o)$	$-\lambda_4(t_o)$	$-\lambda_5(t_o)$	$\lambda_1(t_1)\Big\vert_{+}^{-}$	$\lambda_2(t_1)\Big\vert_{+}^{-}$	$\lambda_3(t_1)\Big\vert_{+}^{-}$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$\lambda_4(t_1)\Big\vert_{+}^{-}$	$\lambda_5(t_1)\Big\vert_{+}^{-}$	$\lambda_1(t_2)\Big\vert_{+}^{-}$	$\lambda_2(t_2)\Big\vert_{+}^{-}$	$\lambda_3(t_2)\Big\vert_{+}^{-}$	$\lambda_4(t_2)\Big\vert_{+}^{-}$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$\lambda_5(t_2)\Big\vert_{+}^{-}$	$\lambda_1(t_3)$	$\lambda_2(t_3)$	$\lambda_3(t_3)$	$\lambda_4(t_3)$	$\lambda_5(t_3)$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

SECTION VII. NECESSARY CONDITIONS FOR A SIX STAGE EARTH-MOON TRAJECTORY OPTIMIZATION PROBLEM

INTRODUCTION

The problem is to determine a fuel minimizing trajectory for a earth-moon rocket for which six definite stages are defined by specified thrust magnitudes and by intermediate and end point constraints. The procedure will be to apply the Denbow multistage calculus of variations theory as modified by R. W. Hunt (unpublished paper presented to contractor conference Oct. 9, 1963, on "A Generalized Bolza-Mayer Problem with Discontinuous Solutions and Variable Intermediate Points") and by Boyce and Linnstaedter (Progress Report No. 7). The problem studied here is similar to one treated by Dr. Jan Andrus (in an unpublished paper also presented to the Oct. 9, 1963 conference, entitled "A Variational Formulation of Earth to Moon Trajectories"), but our approach is somewhat different.

In this section the Euler-Lagrange equations are obtained from the multiplier rule of Section V, simplified vector forms of the equations are developed, the Weierstrass condition is used to deduce a maximum principle, and the transversality matrix is given.

ASSUMPTIONS

1. The first part of the rocket flight, from blast off through the atmosphere, is not included in this study. Initial values of position, velocity, and mass are supposed given at sufficient altitude to make atmospheric resistance negligible.

2. The only forces acting on the rocket are the motor thrust and the gravitational forces of the earth, moon, and sun.

3. The fuel burning rate and the thrust magnitude are assumed to be known constants in each stage.

4. The direction of thrust is along the axial direction of the rocket and the center of mass of the rocket is fixed with respect to the rocket.

5. Roll effects on the rocket are ignored.

FORMULATION OF THE PROBLEM

The independent variable is the time t, and the state variables are the position coordinates x,y,z in an ephemeris coordinate system, the velocity components u,v,w, and the mass m. The control variables are the pitch angle χ_P and the yaw angle χ_Y determining the direction of thrust. The burning rate $\dot{m}$ is constant in each stage and is denoted by $\beta_a, a = 1,\cdots,6$. Thrust magnitude F_a is also constant in each stage. Staging intervals are denoted by

$$I_a : t_{a-1} \leq t < t_a \quad \text{for} \quad a = 1,2,3,4,5 \quad \text{and} \quad t_{a-1} \leq t \leq t_a \text{ for } a = 6.$$

Gravitational forces are functions of position coordinates only and have components represented by $X_a(x,y,z)$, $Y_a(x,y,z)$, $Z_a(x,y,z)$.

Let underlined symbols denote vectors as follows:

$$\underline{x} = (x,y,z), \quad \underline{u} = (u,v,w), \quad \underline{\lambda} = (\lambda_1,\lambda_2,\lambda_3), \quad \underline{\mu} = (\lambda_4,\lambda_5,\lambda_6),$$

$$\underline{A} = (-\sin\chi_P \cos\chi_Y, \cos\chi_P \cos\chi_Y, \sin\chi_Y), \quad \underline{X}_a = (X_a, Y_a, Z_a),$$

and let

$$M_a = \left\| \begin{matrix} X_{ax} & X_{ay} & X_{az} \\ Y_{ax} & Y_{ay} & Y_{az} \\ Z_{ax} & Z_{ay} & Z_{az} \end{matrix} \right\|$$

where subscripts x,y,z indicate partial derivatives.

The equations of motion of the rocket then are

$$\begin{aligned} &\dot{\underline{x}} = \underline{u}, \\ (1) \qquad &\dot{\underline{u}} = F_a m^{-1} \underline{A} + \underline{X}_a, \\ &\dot{m} = -\beta_a . \end{aligned}$$

The end and intermediate conditions, in the notation of the paper by Boyce and Linnstaedter, are assumed to be

$J_o = m_o - m(t_6)$, the function to be minimized,

$$
(2)\quad
\begin{aligned}
&J_k = 0,\ k = 1,\cdots,18, \text{ where}\\
&J_1 = t_o,\ J_2 = x(t_o) - x_o,\ J_3 = y(t_o) - y_o,\ J_4 = z(t_o) - z_o\\
&J_5 = u(t_o) - u_o,\ J_6 = v(t_o) - v_o,\ J_7 = w(t_o) - w_o,\ J_8 = m(t_o) - m_o,\\
&J_9 = t_1 - c_1,\ J_{10} = \underline{x}(t_2)\cdot x(t_2) - c_2,\ J_{11} = \underline{u}(t_2)\cdot\underline{u}(t_2) - c_3\\
&J_{12} = \underline{x}(t_2)\cdot u(t_2) - c_4,\ J_{13} = t_6 - t_3 - c_5\ ,\\
&J_{13+i} = \psi_i(t_6,\underline{x}(t_6),\ \underline{u}(t_6)),\ i = 1,\cdots,5.
\end{aligned}
$$

The ψ_i are functions defining the mission orbit about the moon. The following constraints are also assumed:

$$
(3)\quad m(t_1^+) - m(t_1^-) - c_6 = 0,\ m(t_4^+) - m(t_4^-) - c_7 = 0,\ \beta_3 = F_3 = \beta_5 = F_5 = 0.
$$

and the numbers β_a, F_a for $a = 1,2,4,6$ are known positive constants.

MULTIPLIER RULE AND EULER-LAGRANGE EQUATIONS

Lagrange multipliers $\lambda_1,\cdots,\lambda_7$ will be introduced through a generalized Hamiltonian H defined as follows:

$$
H(t,\underline{x},\underline{u},m,\underline{\lambda},\underline{\mu},\chi_P,\chi_Y) = F_a m^{-1}\,\underline{\lambda}\cdot\underline{A} + \underline{\lambda}\cdot\underline{X}_a + \underline{\mu}\cdot\underline{u} - \lambda_7\beta_a
$$

for t in I_a . We apply the corollary to the multiplier rule in Section IV to obtain the following Euler-Lagrange equations:

$$
(4)\quad
\begin{aligned}
&\dot{\underline{\lambda}} = -\underline{\mu}\\
&\dot{\underline{\mu}} = -\underline{\lambda}\, M_a\\
&\dot{\lambda}_7 = F_a\, m^{-2}\,\underline{\lambda}\cdot\underline{A}\\
&0 = F_a\, m^{-1}\,\underline{\lambda}\cdot\partial\underline{A}/\partial\chi_P\\
&0 = F_a\, m^{-1}\,\lambda\cdot\partial\underline{A}/\partial\chi_Y
\end{aligned}
$$

For stages in which $F_a = 0$ (or $\beta_a = 0$) the optimizing arc is singular in the sense defined in Section V in that the determinant

$$(5) \qquad \begin{vmatrix} H_{\chi_P \chi_P} & H_{\chi_P \chi_Y} \\ H_{\chi_Y \chi_P} & H_{\chi_Y \chi_Y} \end{vmatrix}$$

is zero.

If F_a and $\cos \chi_Y$ are not zero, the next to the last of equations (4) implied that

$$\lambda_1 \cos \chi_P + \lambda_2 \sin \chi_P = 0,$$

and hence that

$$(6) \qquad \tan \chi_P = -\lambda_1 / \lambda_2 , \; \sin \chi_P = -\lambda_1 / \sqrt{\lambda_1^2 + \lambda_2^2} , \; \cos \chi_P = \lambda_2 / \sqrt{\lambda_1^2 + \lambda_2^2} \; .$$

The sign of the radical would be ambiguous but is later shown to be positive. From the last of equations (4), assuming $\cos \chi_P \neq 0$, it follows that

$$\sqrt{\lambda_1^2 + \lambda_2^2} \; \sin \chi_Y - \lambda_3 \cos \chi_Y = 0 \; .$$

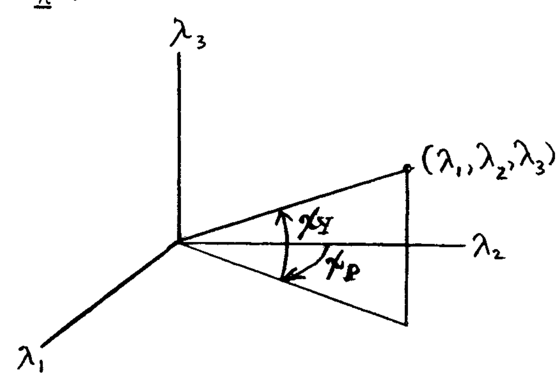
Therefore

$$(7) \qquad \tan \chi_Y = \lambda_3 / \sqrt{\lambda_1^2 + \lambda_2^2} , \; \sin \chi_Y = \lambda_3 / \lambda , \; \cos \chi_Y = \sqrt{\lambda_1^2 + \lambda_2^2} \; / \lambda ,$$

where $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$, the magnitude of $\underline{\lambda}$.

From these results it follows that

$$(8) \qquad \begin{aligned} \lambda_1 &= -\lambda \sin \chi_P \cos \chi_Y \\ \lambda_2 &= \lambda \cos \chi_P \cos \chi_Y \\ \lambda_3 &= \lambda \sin \chi_Y \end{aligned}$$



Thus $\lambda_1, \lambda_2, \lambda_3$ are rectangular coordinates of a point having distance λ and angles χ_P measured from the λ_2 axis and χ_Y from the $\lambda_1 \lambda_2$ plane as shown in the figure.

By use of equations (8), one can now write the equations of motion (1) and the Euler-Lagrange equations in the following form, which is free of the control variables χ_P, χ_Y .

$$
(9) \quad
\begin{aligned}
\dot{\underline{x}} &= \underline{u} \\
\dot{\underline{u}} &= F_a\, m^{-1}\, \lambda^{-1}\, \underline{\lambda} + \underline{X}_a \\
\dot{m} &= -\beta_a \\
\dot{\underline{\lambda}} &= -\underline{\mu} \\
\dot{\underline{\mu}} &= -\underline{\lambda}\, M_a \\
\dot{\lambda}_7 &= F_a\, m^{-2}\, \lambda
\end{aligned}
$$

Also, along an extremal, the Hamiltonian can be written

$$
(10) \quad
\begin{aligned}
H &= F_a\, m^{-1}\, \lambda + \underline{\lambda} \cdot \underline{X}_a + \underline{\mu} \cdot \underline{u} - \lambda_7 \beta_a , \\
&= \underline{\lambda} \cdot \underline{X}_a + \underline{\mu} \cdot \underline{u} + d(\lambda_7 m)/dt ,
\end{aligned}
$$

since $F_a\, m^{-1}\, \lambda - \lambda_7 \beta_a = m\dot{\lambda}_7 + \dot{m}\lambda_7$.

The system (9) can be written as a system of six second order differential equations in the six dependent variables $x, y, z, \lambda_1, \lambda_2, \lambda_3$:

$$
(11) \quad
\begin{aligned}
\ddot{\underline{x}} &= F_a\, \lambda^{-1} (m_o - \beta_a t)^{-1}\, \underline{\lambda} + \underline{X}_a \\
\ddot{\underline{\lambda}} &= -\underline{\lambda}\, M_a
\end{aligned}
$$

By taking the vector cross product of $\underline{\lambda}$ with the first of equations (11) and of $\underline{x}$ with the second we get

$$
(12) \quad
\begin{aligned}
\underline{\lambda} \times \ddot{\underline{x}} &= \underline{\lambda} \times \underline{X}_a \\
\ddot{\underline{\lambda}} \times \underline{x} &= -\underline{\lambda}\, M_a \times \underline{x}
\end{aligned}
$$

If the rocket is near enough the earth that other gravitational forces can be neglected, then the gravitational force vector $\underline{X}$ can be written

$$
\underline{X}_a = -g_a\, r^{-3}\, \underline{x},
$$

where g_a is a constant. It follows that

$$
X_{ax} = g_a (x^2 r^{-5} - r^{-3}) \; ,
$$

with similar expressions for the other elements in the matrix M_a. Hence

$$\underline{\lambda}\, M_a = g_a(r^{-5}(x + y + z)\underline{x} - r^{-3}\, \underline{\lambda}),$$

and equations (13) become

$$(13) \qquad \begin{aligned} \underline{\lambda} \times \ddot{\underline{x}} &= -g_a r^{-3}\, \underline{\lambda} \times \underline{x} \\ \ddot{\underline{\lambda}} \times \underline{x} &= g_a\, r^{-3}\, \underline{\lambda} \times \underline{x} \end{aligned}$$

On adding equations (14), we get

$$(14) \qquad \underline{\lambda} \times \ddot{\underline{x}} + \ddot{\underline{\lambda}} \times \underline{x} = 0 \;,$$

which holds along an extremal.

THE WEIERSTRASS CONDITION AND MAXIMUM PRINCIPLE

From the Weierstrass condition of Section V, it follows that, in case $F_a \neq 0$, the inequality

$$-\lambda_1 \sin \chi_P \cos \chi_Y + \lambda_2 \cos \chi_P \cos \chi_Y + \lambda_3 \sin \chi_Y \geqq -\lambda_1 \sin X_P \cos X_Y + \lambda_2 \cos X_P \cos X_Y + \lambda_3 \sin X_Y$$

must hold for all admissible X_P, X_Y . This implies the Maximum Principle that an optimum trajectory must have control variables χ_P, χ_Y maximizing the function

$$(15) \qquad L = -\lambda_1 \sin \chi_P \cos \chi_Y + \lambda_2 \cos \chi_P \cos \chi_Y + \lambda_3 \sin \chi_Y \;.$$

The first partial derivatives of L must therefore be zero:

$$L_{\chi_P} = -\lambda_1 \cos \chi_P \cos \chi_Y - \lambda_2 \sin \chi_P \cos \chi_Y = 0$$

$$L_{\chi_Y} = \lambda_1 \sin \chi_P \sin \chi_Y - \lambda_2 \cos \chi_P \sin \chi_Y + \lambda_3 \cos \chi_Y = 0$$

These are the same as the last two of the Euler equations (4) with the factor $F_a\, m^{-1}$ removed. Substitution of the solutions (6), (7) of (4) into (15) gives

$$(16) \qquad L = s\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = s\lambda,$$

where s is a sign factor ± 1 arising from the ambiguity of the radicals in (6) and (7). However, it is clear that s must be $+1$, since if $s = -1$ some choice of χ_P, χ_Y would give L a greater value than $-\lambda$.

Thus

(17) $$L = \lambda$$

on an optimum trajectory.

TRANSVERSALITY CONDITIONS

The transversality matrix in the form given for normal arcs in Section V will have 19 rows and 56 columns and must be of rank 18. Formal calculation of the matrix will show that certain columns have zero elements in all but the first row. It will then follow that the element in that row must also be zero. Since such an element is of the type $f(c+) - f(c-)$, its vanishing implies the continuity of the function at the point. Thus some of the transversality conditions simplify to the requirements that the following functions be continuous at the points specified:

H at t_2, t_4, t_5;

λ_i at t_1, t_3, t_4, t_5 for $i = 1, \cdots, 6$;

λ_7 at t_1, t_2, t_3, t_4, t_5 .

Also $\lambda_7(t_6) = 1$.

Elementary row and column transformations now make it possible to express the remaining transversality conditions as the requirement that the following matrices be of ranks 3 and 5, respectively:

$$\left\| \begin{matrix} \lambda_{12}^- - \lambda_{12}^+ & \lambda_{22}^- - \lambda_{22}^+ & \lambda_{32}^- - \lambda_{32}^+ & \lambda_{42}^- - \lambda_{42}^+ & \lambda_{52}^- - \lambda_{52}^+ & \lambda_{62}^- - \lambda_{62}^+ \\ x_2 & y_2 & z_2 & 0 & 0 & 0 \\ u_2 & v_2 & w_2 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & u_2 & v_2 & w_2 \end{matrix} \right\|$$

$$\begin{Vmatrix} G & \lambda_{16} & \lambda_{26} & \lambda_{36} & \lambda_{46} & \lambda_{56} & \lambda_{66} \\ \psi_{1t} & \psi_{1x} & \psi_{1y} & \psi_{1z} & \psi_{1u} & \psi_{1v} & \psi_{1w} \\ \psi_{2t} & \psi_{2x} & \psi_{2y} & \psi_{2z} & \psi_{2u} & \psi_{2v} & \psi_{2w} \\ \psi_{3t} & \psi_{3x} & \psi_{3y} & \psi_{3z} & \psi_{3u} & \psi_{3v} & \psi_{3w} \\ \psi_{4t} & \psi_{4x} & \psi_{4y} & \psi_{4z} & \psi_{4u} & \psi_{4v} & \psi_{4w} \\ \psi_{5t} & \psi_{5x} & \psi_{5y} & \psi_{5z} & \psi_{5u} & \psi_{5v} & \psi_{5w} \end{Vmatrix}$$

In the first of the above matrices subscripts 2 have been used to indicate evaluation at t_2. In the second matrix all of the ψ's are evaluated at t_6, and subscript 6 on the λ's indicates such evaluation. The symbol G denotes $H(t_3^+) - H(t_3^-) - H(t_6)$.