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YSIS OF THE EFFECTS OF
..-ZERO BANDWIDTHS ON THE
SECOND ORDER
NONLINEAR OPTICAL INTERACTIONS

by

Joseph T. Mayhan

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Department of ELECTRICAL ENGINEERING

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REPORT 1093-24

REPORT
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Submitted by	Joseph T. Mayhan Antenna Laboratory Department of Electrical Engineering
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The material contained in this report is also used as a thesis submitted to the Department of Electrical Engineering, The Ohio State University, as partial fulfillment for the degree of Master of Science.

ABSTRACT

The purpose of this report is to analyze the effects of small non-zero bandwidth on the second order optical interactions. In particular we are interested in possible applications in the sub-millimeter wavelength. A classical, deterministic approach is considered throughout and Maxwell's equations govern all phenomena. Both mixing and parametric effects are considered, the former somewhat in more detail than the latter. The basic equations derived are quite general; to present specific mixing results a gaussian frequency distribution is employed to crudely represent the laser radiation. The resulting equations for the mixing phenomena are complicated and must be integrated numerically. This results in the fact that the bandwidth has more predominant effects in generating the difference frequency rather than for the sum frequency. In particular the peak magnitude of the sum frequency remains approximately the same as that obtained in the monochromatic analysis whereas the difference frequency signal is more critically affected. Our model predicts no change in the phase matching conditions obtained with the monochromatic analysis. The phenomena of parametric amplication is seen to be more difficult to deal with. A set of equations are derived

which reduce to the proper results when mixing effects are considered. The equations are not solved, but are used to point out some possibly unexpected results.

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CHAPTER I
INTRODUCTION

The availibility of intense laser signals has led to a means of studying the various effects in nonlinear media which heretofore have been investigated very little. In particular, mixing of laser light or the parametric amplification using the laser radiation as the pump could lead to interesting generations and amplifications of the submillimeter waves. Since then much analysis has been carried out[1,3,4] in which the laser source was treated as a monochromatic plane wave to obtain an analytical description. This is, to a certain extent, a good approximation, since the laser does produce a signal with a very narrow bandwidth. However, it seems plausible that even a narrow bandwidth could lead to a significant change in nonlinear effects. This change may be particularly significant in achieving the difference frequency mixing (or parametirc amplification) of the two optical signals, since a small percentage bandwidth at the optical frequency may result in a large percentage bandwidth at the difference frequency, especially in view of the fact that very few difference signals have been observed experimentally. This thesis reports an investigation of the effects of this narrow bandwidth on nonlinear interactions.

A perturbation of the polarization vector[1] is used to derive an equation governing the second-order interactions. The equations governing mixing effects are then obtained from a further perturbation of the electric field, and the parametric amplification is treated by a generalization of a method correct for the monochromatic analysis. The above formulation is independent of any particular model used to represent the laser radiation, as are the basic governing equations.

As a specific example of the mixing effect we then present results based on a particular model of the incident laser radiation. As our model we choose one which has a gaussian frequency spectrum of bandwidth $\Delta\omega$. This implies the time function of the signal is also a gaussian pulse with reciprocal bandwidth $(1/\Delta\omega)$. This is not a realistic representation of either a Q-switched laser or a c.w. laser; although it is a fairly good approximation to the spiking solid state laser. Our model serves as the first step toward the analysis of the nonlinear effects of the Q-switched or c.w. laser when a statistical distribution function is considered. This point will be discussed more in detail in a later report.

In addition to the special deterministic gaussian model used to describe the laser signal, several assumptions must be made concerning the nonlinear media for purposes of calculation. We use for the dispersion function the Cauchy dispersion formula

$$n(\omega) = \alpha + \beta\omega^2 ,$$

where we take $\partial n/\partial\omega$ small at optical frequencies. The second order susceptibility, $\underline{\underline{X}}^{(2)}$, will be taken constant with respect to the sharp peak of the gaussian variation. These are properties satisfied by crystals such as potassium dihydrogen phospate. All effects arising at the boundary of the nonlinear media will be neglected.

The nonlinear media, the generated fields, and the input gaussian signals are assumed to be governed completely by Maxwell's field equations. The radiation terms at the second harmonic, sum, and difference frequencies in the solution of Maxwell's equations then constitute the desired predictions of nonlinear interactions.

The polarization vector $\underline{P}(\underline{r}, t)$ is introduced into the field equations following the presentation of Butcher [1]. This consists of a perturbation series in the various orders describing $\underline{P}(\underline{r}, t)$, assumed convergent due to the small values of the higher order susceptibility tensors. An equation is then obtained giverning all second-order phenomena. There are two methods of approximation used to solve this equation. The perturbation method assumes the electric field is expandable in a perturbation series similar to that used for the polarization vector. The series is shown to be convergent because of the powers of $\underline{\underline{X}}^{(2)}$ involved in each succeeding perturbation. In the specific example of the gaussian model the signal itself is assumed perfectly parallel or non-divergent, and thus there is only variation along the direction of propagation. Numerical results are then obtained for the mixing process based on this model. The other method of solution of the governing equation is by direct substitution. This is used to obtain a set of approximate equation for the parametric amplification process. These equations are shown to reduce to the results obtained by the perturbation method when the gaussian model is employed. We do not attempt here to solve the parametric problem but only point out some effects based on these equations due to the non-zero bandwidth. The numerical results for the mixing process are then given in Chapter IV.

The conclusions of our analysis are summarized in Chapter VI. In particular we emphasize that our gaussian model predicts no change in the monochromatic phase matching condition and the output signal is more critically affected when treating the difference frequency signal.

CHAPTER II
GENERAL FORMULATION

Solutions to the homogeneous field equations for anisotropic media are generally well known[1,2]. We wish to present here a restrictive method for treating the non-monochromatic, nonlinear problem based on the methods of Butcher[1]; Armstrong, Bloembergen, Ducuing and Pershan[3]; Kleinman[4]; Tien[5]; and others. Results include the phenomena of mixing, second harmonic generation, parametric amplification and the like, which arise as a result of the constituitive relations between the polarization vector and the electric field. We use gaussian units throughout the analysis.

We consider the higher order processes to be governed by the field equations and derive all our results based on their applicability. In the absence of sources, the field equations appear as

$$(1) \qquad \nabla \times \underline{H} - \frac{1}{c}\dot{\underline{D}} = 0$$

and

$$\nabla \times \underline{E} + \frac{1}{c}\dot{\underline{B}} = 0 ,$$

where the $(\cdot)$ denotes $\partial/\partial t$. Then with the constituitive relations

$$(2) \qquad \underline{B} = \underline{H} \quad \text{and} \quad \underline{D} = \underline{E} + 4\pi\underline{P}$$

we can eliminate $\underline{H}$, $\underline{B}$ and $\underline{D}$ from Eq. (1) and thus obtain

$$(3) \qquad \nabla \times \nabla \times \underline{E}(\underline{r}, t) + \frac{1}{c^2} \ddot{\underline{E}}(\underline{r}, t) = - \frac{4\pi}{c^2} \ddot{\underline{P}}(\underline{r}, t) \quad .$$

Particular solutions to the nonhomogeneous wave Eq. (3) are governed by the forcing function, $\underline{P}(\underline{r}, t)$. Complete specification of this function, plus initial and boundary conditions, uniquely determine all solutions. The nonlinear media itself must be specified via the function $\underline{P}(\underline{r}, t)$. This $\underline{P}(\underline{r}, t)$ is describable (P. N. Butcher [1]) by a sequence of tensor functions, $\underline{\underline{X}}^{(1)}(\omega)$, $\underline{\underline{\underline{X}}}^{(2)}(\omega_1, \omega_2)$, $\underline{\underline{\underline{\underline{X}}}}^{(3)}(\omega_1, \omega_2, \omega_3)$, . . . , such that, expanding $\underline{P}(\underline{r}, t)$ in the perturbation series,

$$(4) \qquad \underline{P}(\underline{r}, t) = \underline{P}^{(1)}(\underline{r}, t) + \lambda \underline{P}^{(2)}(\underline{r}, t) \ldots + \lambda^{(N-1)} \underline{P}^{(N)}(\underline{r}, t) + \ldots \quad ,$$

the defining equations of $\underline{P}^{(k)}(\underline{r}, t)$ are

$$(5) \qquad \underline{P}^{(1)}(\underline{r}, t) = \int_{-\infty}^{\infty} \underline{\underline{X}}^{(1)}(\omega_1) \cdot \underline{E}(\underline{r}, \omega_1)\, e^{i\omega_1 t}\, d\omega_1 \quad ,$$

$$\underline{P}^{(2)}(\underline{r}, t) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} \underline{\underline{\underline{X}}}^{(2)}(\omega_1, \omega_2) : \underline{E}(\underline{r}, \omega_1) \underline{E}(\underline{r}, \omega_2)\, e^{i(\omega_1 + \omega_2)t}\, d\omega_1\, d\omega_2 \quad ,$$

. . . .

$$\underline{P}^{(k)}(\underline{r}, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{\underline{X}}^{(k)}(\omega_1, \ldots, \omega_k) \,\vdots\, \underline{E}(\underline{r}, \omega_1) \ldots \underline{E}(\underline{r}, \omega_k)\, e^{i(\omega_1 + \ldots + \omega_k)t}\, d\omega_1 \ldots d\omega_k \quad ,$$

. . . .

where the k^{th} partial sum of the series with $\lambda = 1$ yields the k^{th} order approximation to the polarization $\underline{P}(\underline{r}, t)$. The series is assumed convergent to $\underline{P}(\underline{r}, t)$ as $k \to \infty$.

Substituting Eqs. (4) and (5) into Eq. (3) and carrying over to the frequency domain by means of the Fourier transform, where we define

$$\underline{E}(\underline{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{E}(\underline{r}, t)\, e^{-i\omega t}\, dt$$

(6)

$$\underline{P}(\underline{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{P}(\underline{r}, \omega)\, e^{-i\omega t}\, dt,$$

we obtain, after integrating twice,

$$\nabla \times \nabla \times \underline{E}(\underline{r}, \omega) - \frac{\omega^2}{c^2}\, \underline{\underline{\epsilon}} \cdot \underline{E}(\underline{r}, \omega) = \tag{7}$$

$$\lambda \frac{4\pi}{c^2} \omega^2 \int_{-\infty}^{\infty} \underline{\underline{\underline{X}}}^{(2)} : \underline{E}(\underline{r}, \omega-\omega_1)\, \underline{E}(\underline{r}, \omega_1)\, d\omega_1 \quad +$$

$$\lambda^2 \frac{4\pi}{c^2} \omega^2 \int_{\infty}^{\infty} \int_{-\infty}^{\infty} \underline{\underline{\underline{\underline{X}}}}^{(3)}(\omega-\omega_2-\omega_3, \omega_2, \omega_3) \vdots\, \underline{E}(\underline{r}, \omega-\omega_2-\omega_3)$$

$$\underline{E}(\underline{r}, \omega_2)\, \underline{E}(\underline{r}, \omega_3) d\omega_2\, d\omega_3$$

$$+ \ldots \quad ,$$

where the linear dielectric tensor $\underline{\underline{\epsilon}}(\omega)$ is given by

$$\underline{\underline{\epsilon}}(\omega) = \frac{\omega^2}{c^2}\left(1 + 4\pi\, \underline{\underline{X}}^{(1)}(\omega)\right) . \tag{8}$$

The various phenomena occuring in nonlinear processes can all be obtained as solutions to Eq. (7). In particular, the processes of mixing and parametric amplification can be obtained in media for which the tensor functions $\underline{\underline{X}}^{(3)}, \ldots$, are insignificant with respect to $\underline{\underline{X}}^{(1)}$ and $\underline{\underline{X}}^{(2)}$. Other phenomena are treated by considering $\underline{\underline{X}}^{(3)}$ and so forth. Since this thesis is concerned with the second-order processes, we set $\lambda = 1$, retain only $\underline{\underline{X}}^{(2)}$ in Eq. (7), and obtain solutions to the differential-integral equation

$$\nabla \times \nabla \times \underline{E}(\underline{r}, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}(\underline{r}, \omega) = \frac{4\pi}{c^2} \omega^2 \int_{-\infty}^{\infty} \underline{\underline{X}}^{(2)}(\omega-\omega'', \omega'') : \underline{E}(\underline{r}, \omega-\omega'') \underline{E}(\underline{r}, \omega'') d\omega''. \tag{9}$$

Equation (9) is, in general, complicated and we must resort to approximate methods of solution. Since $\underline{E}(\underline{r}, t)$ is specified, the only remaining approximation in its solution must involve $\underline{E}(\underline{r}, \omega)$. To be specific, for the mixing process (including harmonic generation) a perturbation series in $\underline{E}(\underline{r}, \omega)$ is employed, but does not hold for parametric amplification. The difference arises since for mixing we assume that the two input signals present are both large in magnitude and that the crystal length is small enough to neglect any depletion of the wave. In other words, we assume that the energy given the sum or difference frequency signal generated is small with respect to the energy present in the input signals. The input signals can then be introduced into the first order solutions. However in the case of parametric amplification we assume a large signal input and one of second order in magnitude. Then the second-order signal changes appreciably over the crystal length, growing exponentially under the proper conditions, taking energy from the large signal input. In this case we solve Eq. (9) by direct substitution. This method of solution is more general and is shown to reduce to the results obtained by the perturbation method. The correlation of results for the two methods reinforces the validity of the parametric solution.

CHAPTER III
MIXING OF NON-MONOCHROMATIC WAVES

The mixing process involves an input of two signals, relatively large in magnitude, which mix via the nonlinear term in Eq. (9). This nonlinear term, being a product of two frequency spectra, yields many frequencies not present in the input spectrum. For example, in the case of two monochromatic waves of frequencies ω_1 and ω_2[1], and frequency spectra $A\delta(\omega-\omega_1) + A^*\delta(\omega+\omega_1)$ and $B\delta(\omega-\omega_2) + B^*\delta(\omega+\omega_2)$, frequencies of the type $2\omega_1$, $2\omega_2$, $\omega_2 + \omega_1$, and $\omega_2 - \omega_1$ will be generated. The generation of $2\omega_1$ or $2\omega_2$ is commonly denoted as the second harmonic generation. Complications arise with respect to propagation of these generated waves. A necessary, but not sufficient, criterion for any propagation of significant amplitude is propagation in an anisotropic medium.

In the rigorous non-monochromatic analysis, the exact solution of Eq. (9) appears to be complicated. However, for the mixing process, approximate solutions are easily obtainable using a perturbation approach. It is essential here that the input signals are of the same order of magnitude so that they can be introduced in the first-order approximation.

We assume, then, a perturbation series expansion of $\underline{E}(\underline{r}, \omega)$, similar to the approach used by Butcher[1]; Armstrong, Bloembergen, Ducuing and Pershan[3]; Tien[5]; and others.

$$\underline{E}(\underline{r}, \omega) = \underline{E}^{(1)}(\underline{r}, \omega) + \lambda \underline{E}^{(2)}(\underline{r}, \omega) + \lambda^2 \underline{E}^{(3)}(\underline{r}, \omega) + \dots \,, \tag{10}$$

where λ is the perturbation parameter. Assuming for the present that the series converges, we substitute Eq. (10) into Eq. (9). Rearranging and equating coefficients of λ^k for $k = 0, 1, \dots$, we obtain an equation for each of the orders of perturbation:

$$\nabla \times \nabla \times \underline{E}^{(1)}(\underline{r}, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}^{(1)}(\underline{r}, \omega) = 0 \,, \tag{11}$$

$$\nabla \times \nabla \times \underline{E}^{(2)}(\underline{r}, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}^{(2)}(\underline{r}, \omega) = \frac{4\pi}{c^2} \omega^2 \int_{-\infty}^{\infty} \underline{\underline{\underline{X}}}^{(2)}(\omega-\omega'', \omega''): \underline{E}^{(1)}(\underline{r}, \omega-\omega'') \underline{E}^{(1)}(\underline{r}, \omega'') \, d\omega'', \tag{12}$$

$$\nabla \times \nabla \times \underline{E}^{(3)}(\underline{r}, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}^{(3)}(\underline{r}, \omega) = \frac{4\pi}{c^2} \omega^2 \int_{-\infty}^{\infty} \underline{\underline{\underline{X}}}^{(2)}(\omega-\omega'', \omega''): [\underline{E}^{(1)}(\underline{r}, \omega-\omega'') \underline{E}^{(2)}(\underline{r}, \omega'') + \underline{E}^{(2)}(\underline{r}, \omega-\omega'') \underline{E}^{(1)}(\underline{r}, \omega'')] \, d\omega'' \,, \tag{13}$$

. . . .

Once the first-order solution is specified, all higher order solutions are determined. This emphasizes the importance of the functional form of the frequency spectrum of the input signal.

It is now easy to see that the solution (Eq. (10)) converges quite rapidly. Assuming that the input $\underline{E}^{(1)}$ is of magnitude $E^{(1)}$, from Eq. (11) the order of magnitude of $\underline{E}^{(2)}$ is $X^{(2)} E^{(1)} E^{(1)}$. Since $X^{(2)}$ is quite small, we can say $E^{(2)} \sim \epsilon$, where $\epsilon << 1$. Carrying on we find $E^{(3)} \sim \epsilon X^{(2)}$, ... , $E^{(N)} \sim \epsilon (X^{(2)})^{N-1}$, ... However, $\epsilon X^{(2)} << X^{(2)}$ and consequently even the second-order solution gives a very good approximation to the exact solution.

Our first-order solutions must satisfy the homogeneous wave equation for an anisotropic medium given by Eq. (11). These solutions are treated in the references[1,2] and will only be summarized here. Allowable solutions are of the form

$$(14) \qquad \underline{E}^{(1)}(\underline{r}, \omega) = \underline{a}_o E_o(\omega) e^{ik_o(\omega)z} + \underline{a}_o E_o^*(-\omega) e^{-ik_o(-\omega)z}$$

$$+ \underline{a}_e E_e(\omega) e^{ik_e(\omega)z} + \underline{a}_e E_e^*(-\omega) e^{-ik_e(-\omega)z},$$

where the subscripts o and e represent the ordinary and extraordinary rays, repsectively, and the $\underline{a}_j (j = 0, e)$ are the unit polarization vectors of the ordinary and extraordinary rays. We have assumed z variation only, which considerably simplifies the problem. This assumption implies that the input laser signal is spatially non-divergent, or perfectly parallel in the z-direction. The $k_j(\omega)$ designate the spatial variations along the z-direction and allow for dispersive media,

$$k_j(\omega) = \frac{\omega}{c}\, n_j(\omega)\,, \tag{15}$$

where the $n_j(\omega)$ are the indices of refraction for the media. Thus far $E_j(\omega)$ represents any general frequency spectrum. $\underline{E}^{(1)}(z,\omega)$ represents a real signal, as is shown in the Appendix, a necessary condition being the symmetry in ω about $\omega = 0$ axis.

The index of refraction varies depending on the medium considered. For the functional form we use here the Cauchy dispersion formula

$$n_j(\omega) = \alpha_j + \beta_j\omega^2 \quad . \tag{16}$$

Flynn and Hsu[6] have taken data collected by Dennis[7] for potassium dihydrogen phospate and evaluated constants for the functions

$$n_o(\omega) = \alpha_o + \beta_o\omega^2 \tag{17}$$

and

$$n_e'(\omega) = \alpha_1 + \beta_1\omega^2 \;,$$

where $n_e'(\omega)$ is the index of refraction of the extraordinary ray along the optic axis of the medium. $n_o(\omega)$ and $n_e'(\omega)$ are related to the index of refraction, $n_e(\omega)$, for the extraordinary ray propagating at an angle θ to the optic axis by the equation

(18) $$\frac{1}{n_e^2(\omega)} = \frac{\cos^2\theta}{n_o^2(\omega)} + \frac{\sin^2\theta}{n_e'^2(\omega)} \quad .$$

We can put $n_e(\omega)$ in the approximate form[8]

(19) $$n_e(\omega) = \alpha_e(\theta) + \beta_e(\theta)\omega^2$$

by setting

(20) $$\alpha_e(\theta) = \alpha_1 \{\gamma \cos^2\theta + \sin^2\theta\}^{-\frac{1}{2}}$$

and

$$\beta_e(\theta) = \beta_1 \{\gamma \cos^2\theta + \sin^2\theta\}^{-\frac{1}{2}},$$

where

(21) $$\gamma = \left(\frac{n_e'(\omega)}{n_o(\omega)}\right)^2 \Bigg|_{\omega_o}$$

and ω_o is the frequency at which we are primarily interested in $n_e(\omega)$. The approximation arises in assuming γ is constant since rigorously it does change with frequency. However the approximation is good for a limited region about ω_o. Thus $n_o(\omega)$ and $n_e(\omega)$ are of the same functional form and α_o, β_o, α_1, and β_1 are the constants determined by Flynn and Hsu.[6] α_e and β_e are functions of θ and phase matching will occur for a given θ. The region of validity of Eq. (19), for our purposes of calculations in Chapter IV, is for wavelengths between 0.3 and 1.0 microns. Presumably by changing α_j and β_j the expressions for $n_j(\omega)$ can be made to fit other portions of the experimental curves. The constants used in Chapter IV are chosen so as to render $n_o(\omega)$ and $n_e'(\omega)$ exact at the ruby fundamental and second harmonic frequencies. For greater accuracy when dealing with other frequencies in the region 0.3 to 1.0 microns, these constants should be re-evaluated to fit the experimental curves at the respective frequencies. This is not done in this thesis however, and consequently when dealing with frequencies other than the ruby fundamental and its second harmonic, some of the metching angles are slightly in error with those published.

The reference made to input signals implies satisfying boundary conditions at the surface of a nonlinear medium. This has been treated in detail by Bloembergen and Pershan.[9] This thesis is not concerned with these effects and we shall assume that the input signals are present in the media in the form of extraordinary or ordinary rays, or both, depending on the case considered.

Taking $E_o(t)$ to be quasi monochromatic, we assume, for the present (other possibilities will be considered later), a frequency spectrum gaussian in form given by

$$(22) \qquad E_o(\omega) = \frac{E_1}{\sqrt{2\pi}\,\Delta\omega_1} e^{-(\omega-\omega_1)^2/2\Delta\omega_1^2} + \frac{E_2}{\sqrt{2\pi}\,\Delta\omega_2} e^{-(\omega-\omega_2)^2/2\Delta\omega_2^2} \quad ;$$

$$E_e(\omega) = 0 \,.$$

Allowance has been made that the two signals have different bandwidths, $\Delta\omega_1$ and $\Delta\omega_2$ respectively, and E_1 and E_2 may be complex to represent phase differences between the two signals. A gaussian frequency input in free space arises from a gaussian time pulse of reciprocal bandwidth, i.e., the gaussian frequency wave packet in free space (bandwidth $\Delta\omega_o$),

$$E(z, \omega) = \frac{E_o}{\sqrt{2\pi}\,\Delta\omega_o}\, e^{-(\omega-\omega_o)^2/2\Delta\omega_o^2}\, e^{i\frac{\omega}{c}z} \tag{23}$$

has the associated time function

$$E(z, t) = E_o\, e^{-(t-z/c)^2/2\left(\frac{1}{\Delta\omega_o}\right)^2}\, e^{i\omega_o(t+z/c)} \tag{24}$$

of a gaussian time pulse (bandwidth $1/\Delta\omega_o$).

We use the identity

$$\lim_{\Delta\omega_o \to 0} \frac{1}{\sqrt{2\pi}\,\Delta\omega_o}\, e^{-(\omega-\omega_o)^2/2\Delta\omega_o^2} = \delta(\omega-\omega_o) \tag{25}$$

to reduce our results to the monochromatic analysis given in Ref. 1.

Given our first-order solutions, $\underline{E}^{(1)}(z, \omega)$, we proceed to solve Eq. (12):

(12) $$\nabla \times \nabla \times \underline{E}^{(2)}(z,\omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}^{(2)}(z,\omega)$$

$$= \frac{4\pi}{c^2}\omega^2 \int_{-\infty}^{\infty} \underline{\underline{X}}^{(2)}(\omega-\omega'', \omega''): \underline{E}^{(1)}(z, \omega-\omega'') \underline{E}^{(1)}(z,\omega'')\, d\omega''.$$

For the form assumed in Eq. (22) the nonlinear factor in the integrand yields 16 terms. Each term will generate a contribution to the net solution by way of the superposition principle. In particular, each term yields a gaussian function peaked at either $\omega = \pm(\omega_1 + \omega_2)$, $\omega = 0$, $\omega = \pm(\omega_2 - \omega_1)$, $\omega = \pm 2\omega$, or $\omega = \pm 2\omega_2$. Hence there are possible any of five output frequencies, each of some linear combination of ω_1 and ω_2. Which frequency we obtain depends primarily on the technique of phase matching. The basis of this technique can be described as follows: assume we have a system governed by the differential equation

(26) $$\frac{d^2E}{dz^2} + k^2E = A\, e^{ik'z}$$

with homogeneous solutions of the form $B_1\, e^{ikz}$. When $k' = k$, the forcing function in Eq. (26) is exciting the system at its resonant frequency. It is known that this gives a large output of the form $B_2 z\, e^{ikz}$. In turn, if $k' = k + \Delta k$, Δk small, we have a solution of the form

(27) $$B_2(z)\, e^{ikz},$$

where $B_2(z) \to B_2 z$ as $\Delta k \to 0$. Note that for $\Delta k = 0$, $d^2(B_2 z)/dz^2 = 0$; hence for Δk small we can approximate solutions by setting $d^2 B_2(z)/dz^2 \approx 0$. In effect this is what we encounter in the following analysis, except that the equations will be in tensor form and k and k' are functions of ω. k and k' are different for each ω_j output. The technique of phase matching, then, is to construct the medium and physical experiment so that the $\Delta k_j(\omega_j)$ is zero for the frequency we want as output. We intrinsically assume that the other $\Delta k_i(\omega_i)$ are not close to zero so that the corresponding signals propagate with insignificant amplitude and are negligible. This phase matching effect is exhibited more clearly in the actual solution for the sum frequency radiation in the following paragraphs.

Consider first the term in $\underline{E}^{(o)}(z, \omega-\omega'')\ \underline{E}^{(o)}(z, \omega'')$ contributing to the sum frequency radiation given by

$$(28) \qquad \frac{E_1 E_2}{2\pi\Delta\omega_1\Delta\omega_2}\ \underline{a}_o \underline{a}_o\, e^{-(\omega-\omega''-\omega_1)^2/2\Delta\omega_1^2}\, e^{-(\omega''-\omega_2)^2/2\Delta\omega_2^2} \times e^{i[k_o(\omega-\omega'')+k_o(\omega'')]z} .$$

We manipulate the exponentials in this term into the form

$$(29) \qquad e^{-\frac{\eta}{1+\eta}[\omega-(\omega_1+\omega_2)]^2/2\Delta\omega_1^2}\, e^{-(1+\eta)\left[\omega''-\left(\frac{\omega-\omega_1+\eta\omega_2}{1+\eta}\right)\right]^2/2\Delta\omega_1^2}\, e^{i[k_o(\omega-\omega'')+k_o(\omega'')]z} ,$$

where we have defined $\eta = \Delta\omega_1^2/\Delta\omega_2^2$. Furthermore we have, because of the functional form assumed for $k_o(\omega)$,

$$(30) \qquad k_o(\omega-\omega'') + k_o(\omega'') = k_o(\omega) - \frac{3\beta_o}{4c}\omega^3 + \frac{3\beta_o\omega}{c}(\omega''-\omega/2)^2 .$$

If we assume that $\underline{\underline{\underline{X}}}^{(2)}(\omega-\omega'', \omega'')$ is a slowly varying function with respect to the sharp peak of the exponential involving ω'' in expression (29) (for $\Delta\omega_1$ small enough), we can remove $\underline{\underline{\underline{X}}}^{(2)}$ from under the integral. Equation (12) then becomes

$$(31) \qquad \nabla \times \nabla \times \underline{E}_s^{(2)}(z,\omega) - \frac{\omega^2}{c^2}\underline{\underline{\epsilon}}(\omega)\cdot\underline{E}_s^{(2)}(z,\omega) = \frac{4\pi}{c^2}\omega^2\,\underline{\underline{\underline{X}}}^{(2)}(\omega_1,\omega_2):\underline{a}_o\,\underline{a}_o$$

$$\frac{E_1 E_2}{2\pi\Delta\omega_1\Delta\omega_2}\, e^{-\frac{\eta}{1+\eta}[\omega-(\omega_2+\omega_1)]^2/2\Delta\omega_1^2}\, e^{i\left[k_o(\omega)-\frac{3\beta_o}{4c}\omega^3\right]z}$$

$$\times \int_{-\infty}^{\infty} e^{-(1+\eta)\left[\omega''-\left(\frac{\omega-\omega_1+\eta\omega_2}{1+\eta}\right)\right]^2/2\Delta\omega_1^2}$$

$$e^{i\frac{3\beta_o\omega}{c}(\omega''-\omega/2)^2 z}\, d\omega'',$$

where we have defined $\underline{E}_S(z, \omega)$ as that part of $\underline{E}^{(2)}(z, \omega)$ contributing to the sum frequency. The integral can be put into the form* (for crystals with similar properties as KDP)

$$e^{i \frac{3\beta_0 \omega}{c} z \left[\frac{\omega}{2} \left(\frac{\eta-1}{\eta+1}\right) + \frac{(\omega_1 - \eta\omega_2)}{1+\eta}\right]^2} \tag{32}$$

$$\int_{-\infty}^{\infty} e^{-\left(\frac{1+\eta}{2\Delta\omega_1^2}\right)[\omega'' - \delta]^2} d\omega'' ,$$

where δ is a complex constant not affecting the integration. Integrating we obtain the important form of Eq. (31)

$$\nabla \times \nabla \times \underline{E}_S^{(2)}(z, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_S^{(2)}(z, \omega) = \frac{2\sqrt{2\pi}}{c^2} \underline{\underline{X}}^{(2)}(\omega_1, \omega_2) : \underline{a}_o \underline{a}_o \tag{33}$$

$$\frac{E_1 E_2}{\sqrt{2\pi} \Delta\omega_2} \left(\frac{1}{1+\eta}\right)^{\frac{1}{2}} \omega^2 \, e^{-[\omega - (\omega_2 + \omega_1)]^2 / 2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2} \, e^{ik_s'(\omega) z} ,$$

* Consider the identity

$$\alpha[(x-a)^2 + \gamma(x-b)^2] = \frac{\alpha\gamma}{1+\gamma}(a-b)^2 + \alpha(1+\gamma)\left[x - \left(\frac{a+\gamma b}{1+\gamma}\right)\right]^2$$

in the integrand by setting

$\alpha = (1+\eta)/2\Delta\omega_1^2$, $\gamma = 6\beta_o \omega \Delta\omega_1^2 z / ic(1+\eta)$,

$a = (\omega - \omega_1 + \eta\omega_2)/(1+\eta)$, $b = \omega/2$, and $\delta = (a+\gamma b)/(1+\gamma)$.

Use the fact that for crystals with similar properties as KDP, $|\gamma|$ is small with respect to unity, i.e., for KDP

$\partial n/\partial\omega \sim 10^{-17}$, $\Delta\omega_1 \sim 10^{11}$, $c \sim 10^{10}$, $|z| \sim 1$

we get

$|\gamma| \sim 10^{-5}$.

Consequently

$\alpha\gamma/(1+\gamma) \sim \alpha\gamma$, $\alpha(1+\gamma) \sim \alpha$

and the expression (32) follows.

where we define the phase variation $k_s'(\omega)$ as

$$(34) \qquad k_s'(\omega) = k_o(\omega) - \frac{3\beta_o\omega}{c}\left\{\frac{\omega^2}{4}\left[1-\left(\frac{\eta-1}{\eta+1}\right)^2\right] - \omega\left[\frac{\eta-1}{(\eta+1)^2}\right](\omega_1-\eta\omega_2) - \left(\frac{1}{1+\eta}\right)^2(\omega_1-\eta\omega_2)^2\right\}.$$

Equation (33) is the tensor form of the analagous Eq. (26) discussed previously. The output radiation $\underline{E}_s^{(2)}(z,\omega)$ could obviously take the form of either the extraordinary or the ordinary wave. Assuming the special case that $\underline{E}_s^{(2)}(z,\omega)$ is an extraordinary wave, and following the example of the monochromatic analysis,[1] we write

$$(35) \qquad \underline{E}_s^{(2)}(z,\omega) = \underline{a}_e\, E_s'(z,\omega)\, e^{ik_e(\omega)z},$$

where we neglect $\partial^2 E_s'/\partial z^2$, anticipating that $k_e(\omega)$ is close to $k'(\omega)$ for near resonance solutions. The operators in Eq. (33) are written relative to the crystal axes as the coordinate system. Denoting $\underline{k}$ as the unit vector parallel to the direction of propagation, we substitute Eq. (33) into Eq. (31), apply Fresnel's equation,[1,2] take the scalar product of the resulting equation with $\underline{a}_e$, and finally obtain

$$(36) \qquad \frac{dE_s'(z)}{dz} = +i\frac{2\pi}{c^2}\,\underline{a}_e \cdot \underline{\underline{X}}^{(2)}(\omega_1,\omega_2) : \underline{a}_o\underline{a}_o\left(\frac{1}{\underline{k}\times\underline{a}_e}\right)^2 E_1E_2\,\frac{\omega^2}{k_e(\omega)}$$

$$\frac{1}{\sqrt{2\pi}\left(\frac{1+\eta}{\eta}\right)^{\frac{1}{2}}\Delta\omega_1}\; e^{-[\omega-(\omega_2+\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2}\; e^{i\Delta k_s(\omega)z},$$

where we define

(37) $$\Delta k_s(\omega) = k_s'(\omega) - k_e(\omega) .$$

We can put $\Delta k_s(\omega)$ into the form

(38) $$\Delta k_s(\omega) = \frac{\omega}{c} \Bigg\{ \omega^2 \left[\frac{\beta_o}{4} \left(1 + 3 \left(\frac{\eta - 1}{\eta + 1} \right)^2 \right) - \beta_e \right] + \omega \left[3\beta_o \frac{\eta - 1}{(\eta+1)^2} (\omega_1 - \eta\, \omega_2) \right] + \alpha_e - \alpha_o + 3\beta_o \left(\frac{1}{1+\eta} \right)^2 (\omega_1 - \eta\, \omega_2)^2 \Bigg\} .$$

Since we desire a mixing process leading to a radiation field fixed at $\omega_1 + \omega_2$, we require

(39) $$\Delta k_s(\omega_2 + \omega_1) \approx 0.$$

If we desire, say, the difference frequency, we would ignore Eq. (39) and require $\Delta k_d(\omega_2 - \omega_1) \approx 0$ for the $\Delta k_d(\omega)$ encountered there.

Integrating Eq. (36) and applying the condition that $E_s'(0) = 0$, we obtain

(40) $$E_s'(z) = + i \frac{2\pi}{c^2} \underline{a}_e \cdot \underline{\underline{X}}^{(2)}(\omega_1, \omega_2) : \underline{a}_o \underline{a}_o \left(\frac{1}{\underline{k} \times \underline{a}_e} \right)^2 E_1 E_2 \frac{\omega^2}{k_e(\omega)} \frac{1}{\sqrt{2\pi} \left(\frac{1+\eta}{\eta} \right)^{\frac{1}{2}} \Delta\omega_1} e^{-[\omega - (\omega_2 + \omega_1)]^2 / 2\left(\frac{1+\eta}{\eta} \right) \Delta\omega_1} e^{i \frac{\Delta k_s(\omega)}{2} z} \frac{\sin \frac{\Delta k_s(\omega)}{2} z}{(\Delta k_s(\omega)/2)} ,$$

and solutions for the term (29) are given by Eq. (35).

Equation (40) specifies only a partial solution to Eq. (12) for the output frequency $\omega_2 + \omega_1$. The product term in Eq. (12) yields three other terms which contribute to the output at this frequency. One of these can be shown to give equivalent results, as above, by a simple change of variable. The two remaining contributions complete the negative side of the frequency spectrum to yield a real signal output. Results for these can be obtained by assuming solution of the form $\underline{a}_e\ E'(\omega)\ e^{-ik_e(-\omega)}$, letting $\omega_1 \to -\omega_1$, $\omega_2 \to -\omega_2$ and $E_i \to E_i^*$, in Eq. (40) and assuming $X^{(2)}(\omega_1, \omega_2) = X^{(2)}(-\omega_1, -\omega_2)$ (for $X^{(2)}$ constant this is obviously true). We can collect our results and obtain the complete solution for the sum frequency, $\underline{E}_s^{(2)}(z, t)$, in the form of the real integral:

$$\underline{E}_s^{(2)}(z,t) = -\,\underline{a}_e \frac{2\pi}{c}\left(\underline{a}_e \cdot \underline{\underline{X}}^{(2)} : \underline{a}_o \underline{a}_o\right)\left(\frac{1}{\underline{k}\times\underline{a}_e}\right)^2 \frac{|E_1 E_2|}{\sqrt{2\pi}\left(\frac{1+\eta}{\eta}\right)^{\frac{1}{2}}\omega_1}$$
$$\int_{-\infty}^{\infty} \frac{\omega}{n_e(\omega)} \frac{\sin\left(\frac{\Delta k_s(\omega)}{2}\right)z}{(\Delta k_s(\omega)/2)}\, e^{-[\omega-(\omega_2+\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2}$$
$$\sin\left[\omega t + \left(k_e(\omega) + \frac{\Delta k_s(\omega)}{2}\right)z\right] d\omega \,. \tag{41}$$

We note immediately the effect of η on the weighting function in the output integral. The weighting function is now peaked at $\omega_2 + \omega_1$ with bandwidth

(42) $$\Delta\omega_s^2 = \left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2 = \Delta\omega_1^2 + \Delta\omega_2^2 \ .$$

The time function of a gaussian frequency pulse in free space with this bandwidth, $\Delta\omega_s$, would have bandwidth $\left(\frac{1}{\Delta\omega_s}\right)$. We expect the time function of Eq. (41) to have a bandwidth close to this, since if $\Delta\omega_s$ is small enough Eq. (41) yields a plane wave modulated by a gaussian envelope. The integral, as it stands, is complicated; Chapter IV gives numerical results with η as a parameter.

The output in the case of second harmonic generation is now quite simple to obtain. We merely let $\omega_2 \to \omega_1$ and $\eta \to 1$. Our output signal radiated at $2\omega_1$ becomes

(43) $$\underline{E_2}^{(2)}(z,t) = -\underline{a}_e \frac{2\pi}{c} \underline{a}_e \cdot \underline{\underline{X}}^{(2)} : \underline{a}_o \underline{a}_o \left(\frac{1}{\underline{k} \times \underline{a}_e}\right)^2 \frac{|E_1|^2}{2\sqrt{\pi}\Delta\omega_1}$$

$$\int_{\infty}^{\infty} \frac{\omega}{n_e(\omega)} \frac{\sin \frac{\Delta k_z(\omega)}{2} z}{(\Delta k_z(\omega)/2)} e^{-(\omega-2\omega_1)^2/4\Delta\omega_1^2}$$

$$\sin\left[\omega t + \left(k_e(\omega) + \frac{\Delta k_z(\omega)}{2}\right) z\right] d\omega,$$

where now

(44) $$\Delta k_z(\omega) = \frac{\omega}{c}\left\{\omega^2 \left[\frac{\beta_o}{4} - \beta_e\right] + \alpha_o - \alpha_e\right\} .$$

The output for the difference frequency corresponding to the input given in Eq. (22) is obtained in a manner analogous to that of the sum frequency. The results are

(45) $$\underline{E}_d^{(2)}(z,t) = -\,\underline{a}_e \frac{2\pi}{c}\left(\underline{a}_e \cdot \underline{\underline{X}}^{(2)}(\omega_1,\omega_2):\underline{a}_o\underline{a}_o\right)\left(\frac{1}{\underline{k}\times\underline{a}_e}\right)^2$$

$$\cdot \frac{|E_1 E_2|}{\sqrt{2\pi}\left(\frac{1+\eta}{\eta}\right)^{\frac{1}{2}}\Delta\omega_1} \int_{-\infty}^{\infty} \frac{\omega}{n_e(\omega)} \frac{\sin\frac{\Delta k_d(\omega)}{2} z}{(\Delta k_d(\omega)/2)}\, e^{-[\omega-(\omega_2-\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2}$$

$$\times \sin\left[\omega t + \left(k_e(\omega) + \frac{\Delta k_d(\omega)}{2}\right) z\right] d\omega\,,$$

where

(46) $$\Delta k_d(\omega) = \frac{\omega}{c}\left\{\omega^2\left[\frac{\beta_o}{4}\left(1 + 3\left(\frac{1-\eta}{1+\eta}\right)^2\right) - \beta_e\right]\right.$$

$$\left. + \omega\left[\frac{1-\eta}{(1+\eta)^2}\right](\omega_1 + \eta\omega_2) + \alpha_o - \alpha_e + \left(\frac{1}{1+\eta}\right)^2(\omega_1 + \eta\,\omega_2)^2\right\}\,.$$

As with the sum frequency radiation, Eqs. (43) and (45) are subject to the constraints

(47) $$\Delta k_2(2\omega_1) = 0 \quad \text{and} \quad \Delta k_d(\omega_2 - \omega_1) = 0\,,$$

respectively.

The results given thus far are based on the fact that the signals given by Eq. (22) are present in the media. They are in the form of two ordinary rays phase-matched to an extraordinary ray. However this is not the only case which may occur; we can have other combinations of ordinary and extraordinary rays which satisfy the matching and the symmetry condition.[1] As an example consider the case of an ordinary and extraordinary ray matched to an ordinary ray at the difference frequency, i.e.,

(48) $$k_e(\omega_2) - k_o(\omega_1) = k_o(\omega_2 - \omega_1) \ .$$

This arises from input signals of the form (rather than Eq. (22))

(49) $$E_o(\omega) = \frac{E_1}{\sqrt{2\pi}\,\Delta\omega_1} e^{-(\omega - \omega_1)^2/2\Delta\omega_1^2}$$

and

$$E_e(\omega) = \frac{E_2}{\sqrt{2\pi}\,\Delta\omega_2} e^{-(\omega-\omega_2)^2/2\Delta\omega_2^2}$$

being present. We consider the term contributing to the difference frequency from Eq. (12),

(50) $$\int_{-\infty}^{\infty} e^{-(\omega-\omega''+\omega_1)^2/2\Delta\omega_1^2}\, e^{-(\omega''-\omega_2)^2 2\Delta\omega_2^2}\, e^{i[k_o(\omega-\omega'')+k_e(\omega'')]z},$$

which combines to give

(51) $$e^{-[\omega-(\omega_2-\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2} \int_{\infty}^{\infty} e^{-(1+\eta)\left[\omega''-\left(\frac{\omega+\omega_1+\eta\omega_2}{1+\eta}\right)\right]^2/2\Delta\omega_1^2}$$

$$\times\, e^{i[k_o(\omega-\omega'') + k_e(\omega'')]z}\, d\omega'' \ .$$

This integral is complicated since it involves a cubic in ω''. Hence we resort to approximate methods of evaluation.

We assume $\Delta\omega_1^2$ and $\Delta\omega_2^2$ are small enough so that ω is close to $\omega_2 - \omega_1$, and consequently ω'' is close to ω_2. We then expand $n_o(\omega-\omega'')$ and $n_e(\omega'')$ each in a Taylor series about $\omega = -\omega_1$ and $\omega = \omega_2$, respectively;

$$n_e(\omega'') \approx n_e(\omega_2) + \left.\frac{\partial n_e}{\partial \omega}\right|_{\omega_2} (\omega''-\omega_2)$$

(52)

$$n_o(\omega-\omega'') \approx n_o(-\omega_1) + \left.\frac{\partial n_o}{\partial \omega}\right|_{-\omega_1} (\omega-\omega''+\omega_1) \quad .$$

Taking $n_o(\omega) = \alpha_o + \beta_o\omega^2$ and $n_e = \alpha_e + \beta_e\omega^2$, we have

$$(53) \qquad \left.\frac{\partial n_e}{\partial \omega}\right|_{\omega_2} = 2\beta_e\omega_2 \quad \text{and} \quad \left.\frac{\partial n_o}{\partial \omega}\right|_{-\omega_1} = -2\beta_o\omega_1 \quad .$$

Then by substituting expressions (51) and (52) into Eq. (12) and integrating (subject to a condition similar to that of footnote on page 20), we obtain, for crystals with properties similar to KDP,

$$(54) \qquad \nabla \times \nabla \times \underline{E}_{dp}^{(2)}(z,\omega) - \frac{\omega^2}{c^2}\, \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_{dp}^{(2)}(z,\omega) = \frac{4\pi}{c^2}\, \underline{\underline{X}}^{(2)}(\omega_1,\omega_2) : \underline{a}_o\, \underline{a}_e$$

$$\frac{E_1 E_2}{\sqrt{2\pi}\left(\frac{1+\eta}{\eta}\right)^{\frac{1}{2}} \Delta\omega_1}\; e^{-[\omega-(\omega_2-\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2}\; e^{ik_{dp}'(\omega)\, z} ,$$

where we define

(55) $$k_{dp}'(\omega) = \frac{\omega}{c}\left[n_o(\omega_1) - 2\beta_o\omega_1(\omega+\omega_1)\right] + 2\left(\frac{\beta_e\omega_2 - \beta_o\omega_1}{c}\right)\left(\frac{\omega+\omega_1+\eta\omega_2}{1+\eta}\right)^2$$

$$- \left(\frac{\omega+\omega_1+\eta\omega_2}{c(1+\eta)}\right)\left[n_o(\omega_1) - n_e(\omega_2) - 2\beta_o\omega_1(2\omega+\omega_1) + 2\beta_e\omega_2^2\right] .$$

Solutions to Eq. (54) are similar to those previous. Collecting the remaining terms contributing to the frequency $\omega_2 - \omega_1$, we have

(56) $$E_{dp}^{(2)}(z,\omega) = -\underline{a}_o \frac{2\pi}{c}\left(\underline{a}_o \cdot \underline{\underline{X}}^{(2)}(-\omega_1,\omega_2):\ \underline{a}_o\underline{a}_e\right)\left(\frac{1}{\underline{k}\times\underline{a}_o}\right)^2$$

$$\frac{|E_1 E_2|}{\sqrt{2\pi}\left(\frac{1+\eta}{\eta}\right)^{\frac{1}{2}}\Delta\omega_1}\int_{-\infty}^{\infty}\frac{\omega}{n_o(\omega)}\ \frac{\sin\frac{\Delta k_{dp}(\omega)}{2}z}{(\Delta k_{dp}(\omega)/2)}\ e^{-[\omega-(\omega_2-\omega_1)]^2/2\left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2}$$

$$\sin\left[\omega t + \left(k_o(\omega) + \frac{\Delta k_{dp}(\omega)}{2}\right)z\right]d\omega ,$$

where

(57) $$\Delta k_{dp}(\omega) = k_{dp}'(\omega) - k_o(\omega) .$$

Equation (57) is subject to the constraint

(58) $$\Delta k_{dp}(\omega_2 - \omega_1) \simeq 0 .$$

Results for the dc output, where we are interested in $\omega = 0$, can be obtained by letting $\omega_2 \rightarrow \omega_1$ in the difference calculation.

It is especially interesting to compare the results of phase matching for the non-monochromatic signal with the monochromatic plane wave. The monochromatic phase matching condition for the sum frequency is

$$(59) \qquad k_o(\omega_1) + k_o(\omega_2) = k_e(\omega_1 + \omega_2)$$

or

$$\left(\frac{\alpha_e - \alpha_o}{c}\right)(\omega_1 + \omega_2) + \omega_1^3 \left(\frac{\beta_e - \beta_o}{c}\right) + \omega_2^3 \left(\frac{\beta_e - \beta_o}{c}\right)$$

$$+ \frac{3\beta_e}{c}\,\omega_1 \omega_2^2 + \frac{3\beta_e}{c}\,\omega_2 \omega_1^2 = 0 .$$

Factoring out $\left(\frac{\omega_1 + \omega_2}{c}\right)$ we conclude that

$$(61) \qquad \alpha_o - \alpha_e + (\omega_1 + \omega_2)^2 \left(\frac{\beta_o}{4} - \beta_e\right) + \frac{3\beta_o}{4}\,(\omega_1 - \omega_2)^2 = 0 .$$

But taking $\Delta k_s(\omega_1 + \omega_2) = 0$ for the non-monochromatic signal we obtain Eq. (59). Hence, for our gaussian model presented here, the phase matching condition does not change. One can easily show that $\Delta k_{dp}(\omega)$ obtained by approximation reduces to the corresponding matching condition also.

CHAPTER IV
NUMERICAL RESULTS

In this chapter we investigate the dependence of the integrated output, $\underline{E}(z,t)$, on the bandwidths $\Delta\omega_1$ and $\Delta\omega_2$. Each of the integrals, Eqs. (41), (43) and (56), is convergent because of the strong gaussian variation. We note that if $\Delta\omega_1 = 0$ and $\Delta\omega_2 = 0$, our integrals simply involve the delta function. As $\Delta\omega_1$ and $\Delta\omega_2$ change, however, the integrals become more complicated due to the functional forms of $\Delta k(\omega)$ and $k(\omega)$. This leads to numerical calculation of the integrals. We must keep in mind that the integrals are meaningful only as long as

$$(62) \qquad |\gamma| = \frac{6\beta_0\omega\Delta\omega_1^2 z}{c(1+\eta)} << 1 .$$

For a $\Delta\omega_1 \sim 10^{13}$ in KDP at the ruby frequency we have $|\gamma| \sim .1$. This determines the limits of the validity of our results. Our numerical calculations take $\Delta\omega_1 = 10^{11}$ so we are well within the range of our approximation.

We shall use Eq. (41) to illustrate the method for presenting our results. The integrand is a very fast oscillating function of time because of the large optical frequencies; hence it is necessary to change the variable such that the integral consists mainly of the envelope of the optical radiation as follows: we change variables in the form

(63) $$\omega = \omega_1 + \omega_2 + \xi x ,$$

where we set

(64) $$\xi = \sqrt{2 \frac{1+\eta}{\eta}} \Delta\omega_1 .$$

If we omit the factor $\frac{2\sqrt{2\pi}}{c} \underline{a}_e \cdot \underline{\underline{X}}^{(2)}(\omega_1, \omega_2) : \underline{a}_o \underline{a}_o \left(\frac{1}{\underline{k} \times \underline{a}_e}\right)^2 |E_1 E_2|$, then $|E_s^{(2)}(z_o, t)|$ varies as

(65) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(\omega_1 + \omega_2 + \xi x)}{n_e(\omega_1 + \omega_2 + \xi x)} \frac{\sin\left[\frac{\Delta k(\omega_1 + \omega_2 + \xi x)}{2} z_o\right]}{(\Delta k(\omega_1 + \omega_2 + \xi x)/2)}$$

$$\times e^{-x^2} \sin\left[(\omega_1 + \omega_2)t + \xi \times t + k_e(\omega_2 + \omega_1 + \xi x) z_o + \frac{\Delta k(\omega_1 + \omega_2 + \xi x)}{2}\right] dx .$$

We expand $k_e(\omega_1 + \omega_2 + \xi x)$ and remove the fast varying terms from the integrand to obtain

(66) $$|E^{(2)}(z_o, t)| \sim A(z_o, t) \sin\left[(\omega_1 + \omega_2)t + k_e(\omega_1 + \omega_2) z_o\right]$$

$$+ B(z_o, t) \cos\left[(\omega_1 + \omega_2)t + k_e(\omega_1 + \omega_2) z_o\right] ,$$

where

(67) $$A(z_O, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(\omega_1 + \omega_2 + \xi x)}{n_e(\omega_1 + \omega_2 + \xi x)} \frac{\sin\left(\frac{\Delta k(\omega_1 + \omega_2 + \xi x)}{2} z\right)}{(\Delta k(\omega_1 + \omega_2 + \xi x)/2}$$

$$\times e^{-x^2} \cos \xi \times \left[t + \frac{\alpha_2 z_O}{c} + \frac{3\beta_2}{c} (\omega_1 + \omega_2)^2 z_o \right.$$

$$\left. + \frac{3\beta_1}{c} (\omega + \omega_2) \xi x \, z_o + \frac{\beta_1}{c} x^2 \xi^2 z_o + \frac{\Delta k(\omega_1 + \omega_2 + \xi x)}{2\xi x} z_O \right] dx .$$

$B(z_O, t)$ is identical, with the cosine factor replaced by a sine factor in the integrand. We simplify Eq. (66) to obtain

(68) $$|E^{(2)}(z_o, t)| \sim [A(z_O, t)^2 + B(z_o, t)^2]^{\frac{1}{2}} \sin [(\omega_1 + \omega_2) t$$

$$+ k_e(\omega_1 + \omega_2) z_o + \psi(z_O, t)],$$

where we define

(69) $$\psi(z_O, t) = \tan^{-1} \left[\frac{A(z_O, t)}{B(z_O, t)} \right] .$$

We plot the envelope and the phase of the plane wave $\sin[(\omega_1 + \omega_2)t + k_e(\omega_1 + \omega_2)z_O]$. An analagous procedure gives similar results for the second harmonic and difference frequencies, with respective changes in ω and $\Delta k(\omega)$.

The calculations for the envelopes for various ω_1 and ω_2 are shown in Figs. 1, 3, 4, and 5. Results are given for $z_o = -1$ cm and for η as the parameter for each set of ω_1 and ω_2, with $\Delta\omega_1 = 10^{11}$ rad/sec. Since $A(z_o, t) \gg B(z_o, t)$ the envelope varies as $A(z_o, t)$ and the phase is constant at 90^o. There is an appreciable difference in the form of the output between the sum and difference results.

It is interesting to plot approximation curves for the envelope which are obtained by using the fact that $\Delta\omega_1$ is small with respect to $\omega_1 + \omega_2$. We then treat the factors of the gaussian function in the expression for $A(z_o, t)$ as constant with respect to the sharp variation about the peak, and neglect the x^2 and x^3 terms contributing to the argument of the cosine factor. We obtain

$$(70) \qquad A(z_o, t) \simeq \frac{1}{\sqrt{2}} \frac{(\omega_1 + \omega_2)}{n_e(\omega_1 + \omega_2)} \; e^{-(t-t_{max})^2/2\left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{\Delta\omega_1}\right)^2} ,$$

where t_{max} is a constant. The analagous approximation for the difference frequency is

$$(71) \qquad A(z_o, t) \simeq \frac{1}{\sqrt{2}} \frac{(\omega_2 - \omega_1)}{n_o(\omega_2 - \omega_1)} \; e^{-(t-t_{1max})^2/2\left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{\Delta\omega_1}\right)^2} .$$

The peak value of these approximation curves represents the monochromatic calculated magnitude ($\Delta\omega_1 \to 0 \quad \Delta\omega_2 \to 0$).

The approximation curves for the sum frequency are shown in Fig. 2 for values of η to compare with Fig. 1. The value of t_{max}

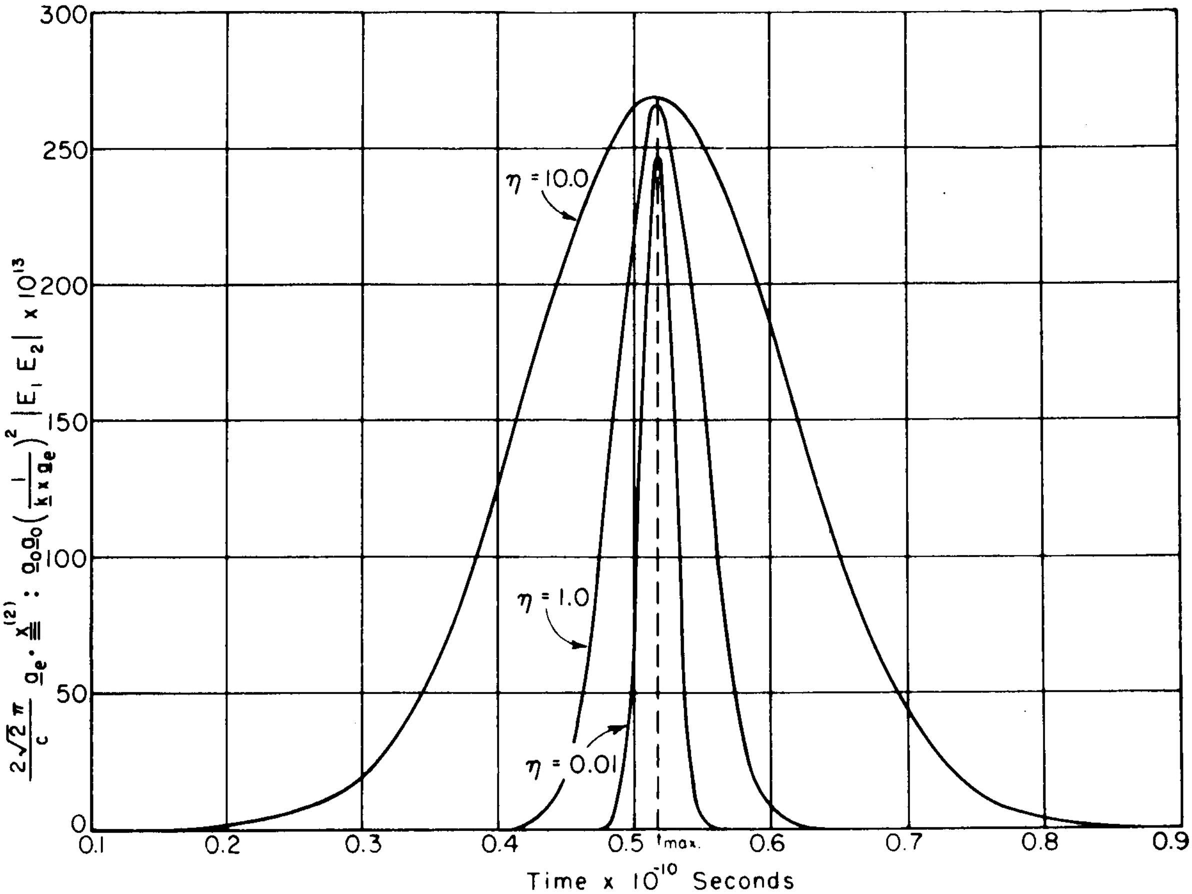

Fig. 1. Sum frequency calculations for the mixing of ruby and HeNe. Curves approximately symmetric about t_{max} = 0.518 $\times$ 10^{-10} seconds. The calculated standard deviations are (a) η = 10.0, S. D. 0.680 $\times$ 10^{-11}; (b) η = 0.10, S. D. = 219 $\times$ 10^{-11}; (c) η = 0.01, S. D. = 0.074 $\times$ 10^{-11}. ω_1 = 2.7164 $\times 10^{15}$ rad/sec; ω_2 = 3.0195 $\times$ 10^{15} rad/sec; z_o = - 1 cm.

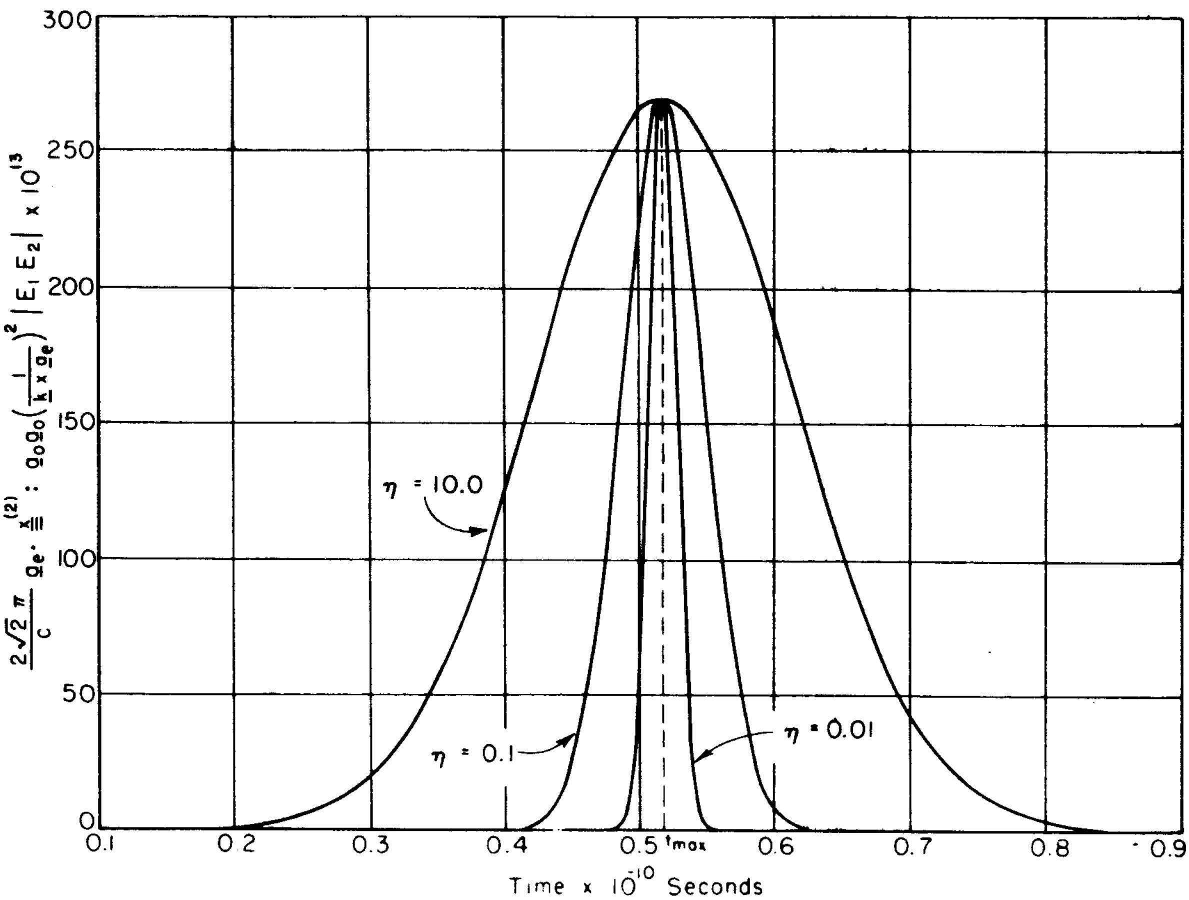

Fig. 2. Approximation functions for the mixing of ruby and HeNe. Compare to Figure 1.

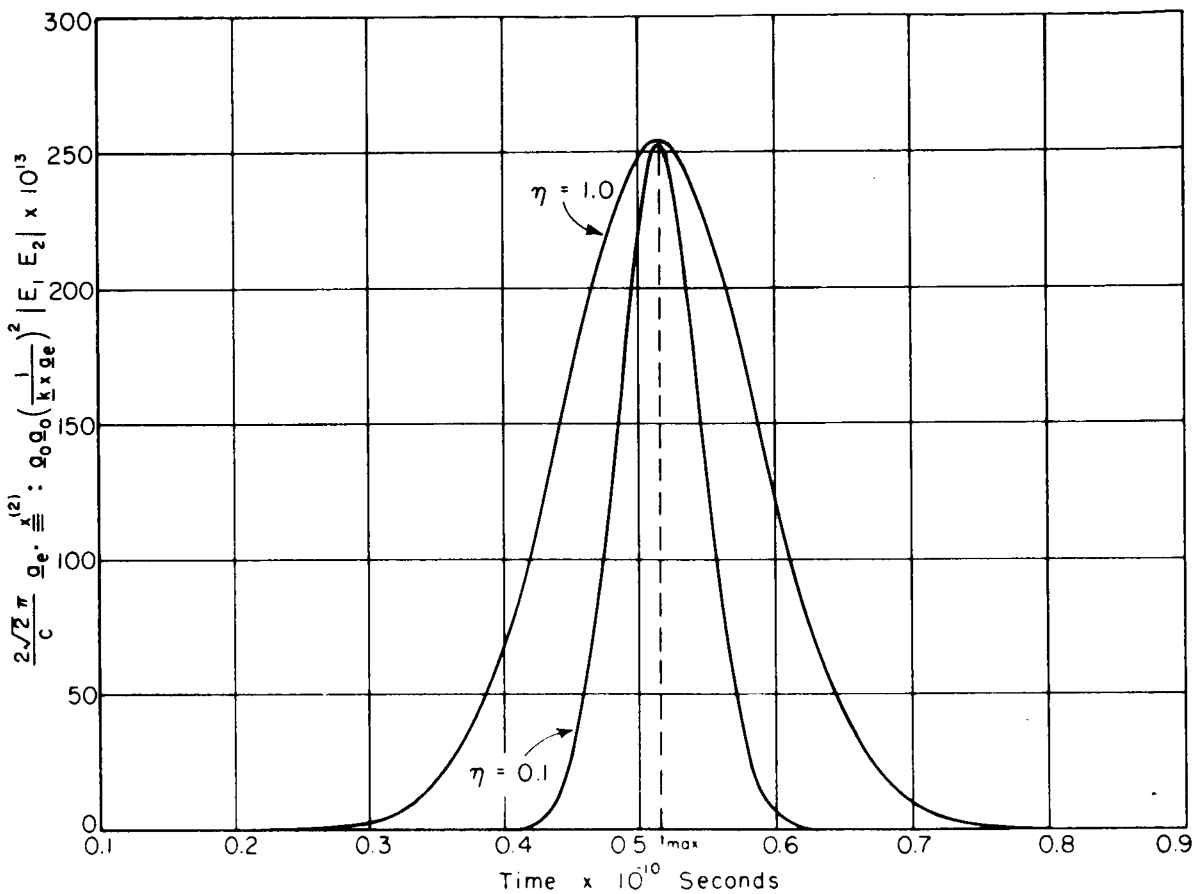

Fig. 3. Sum frequency calculations for the mixing of cooled ruby with ruby. Curves approximately symmetric about $t_{max} = 0.516 \times 10^{-10}$ seconds. The calculated standard deviations are (a) $\eta = 1.0$, S. D. $= 0.054 \times 10^{-11}$; (b) $\eta = 0.10$, 0.217×10^{-11}. $\omega_1 = 2.7164 \times 10^{15}$ rad/sec; $\omega_2 = 2.7121 \times 10^{15}$ rad/sec; $z_o = -1$ cm.

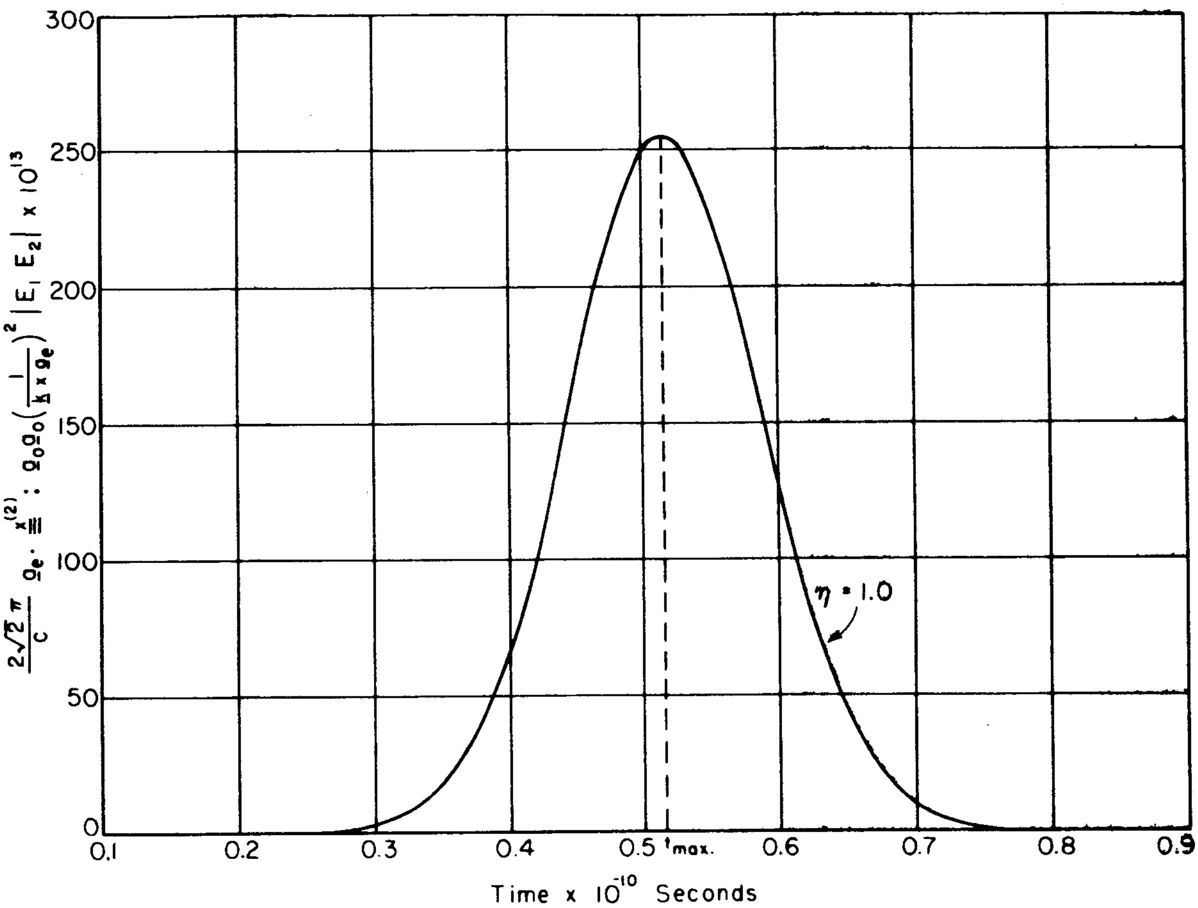

Fig. 4. Calculations for ruby second harmonic curve approximately symmetric about $t_{max} = 0.516 \times 10^{-10}$ seconds. The calculated standard deviation is 9.504×10^{-11}. $\omega = 2.7164 \times 10^{15}$ rad/sec; $z_o = -1$ cm.

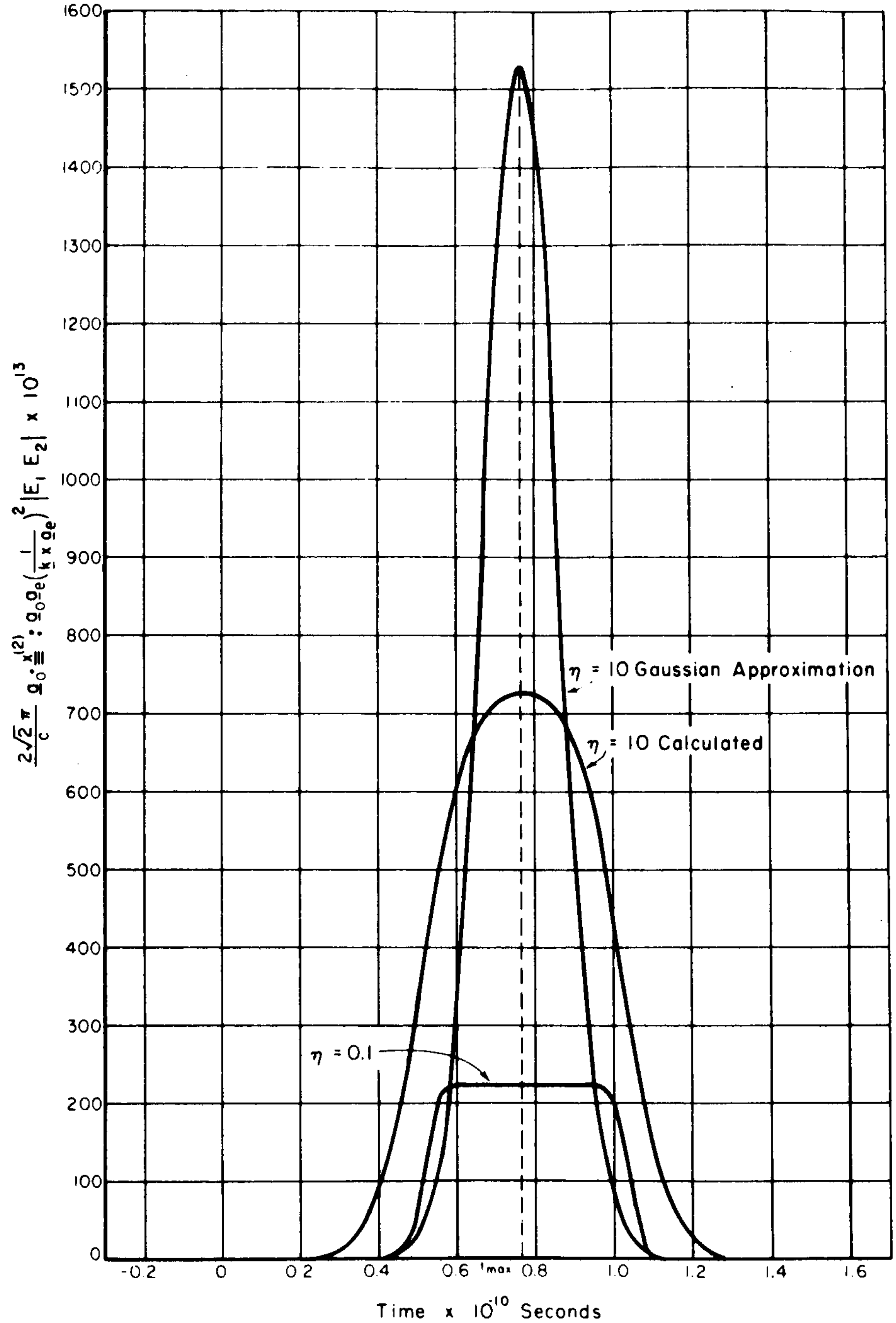

Fig. 5. Difference frequency calculations for the mixing of ruby with a mercury arc. Curves approximately symmetric about $t_{max} = 0.766 \times 10^{-10}$ seconds. $\omega_2 = 6.0467 \times 10^{15}$ rad/sec; $\omega_1 = 2.7121 \times 10^{15}$ rad/sec; $z_0 = -1$ cm. Approximation curve given for $\eta = 10$ where peak represents the monochromatic calculation.

must be altered to fit the numerical results since the calculated t_{max} does not coincide. The approximation is almost exact for η large $\Delta\omega_2 << \Delta\omega_1$), but the magnitude decreases as η decreases ($\Delta\omega_2 >> \Delta\omega_1$). It should be recalled that we are fixing $\Delta\omega_1$ and varying $\Delta\omega_2$ and that

$$(72) \qquad \left(\frac{1+\eta}{\eta}\right)\Delta\omega_1^2 = \Delta\omega_1^2 + \Delta\omega_2^2 .$$

The bandwidths of the numerical results and the approximate results compare remarkably well for this case.

Much different results are obtained for the approximation to the difference frequency output. The numerical result for $\eta = 10$, which is of the order of the best obtainable for $\Delta\omega_1 = 10''$ is much lower in magnitude than the approximation function (whose peak denotes the monochromatic calculation). In particular the bandwidth $\Delta\omega_1 = 10''$ predicts a much lower output (of the order of $\frac{1}{2}$) than the monochromatic result, and this decrease becomes greater as the effect of $\Delta\omega_2$ is considered (η decreases).

One of the main considerations governing the numerical results are the $\Delta k_j(\omega)$ functions, which change with each problem. Since the $\Delta k_{dp}(\omega)$ was obtained by approximation, we would like a check on the validity of our method. The check is based on the fact that we know the exact expression for $\Delta k_s(\omega)$. Hence we calculate a $\Delta k_{sp}(\omega)$ via the same approximation used to obtain $\Delta k_{dp}(\omega)$. We then compare the function

$$(73) \qquad \frac{\sin\left(\frac{\Delta k_s(\omega)}{2} z\right)}{(\Delta k_s(\omega)/2)} \quad \text{and} \quad \frac{\sin\left(\frac{\Delta k_{sp}(\omega)}{2} z\right)}{(\Delta k_{sp}(\omega)/2)}$$

contributing to the integrand over the range of integration. Numerical results show that these are of the same sign and magnitude ($\pm .003$) for all practical contributions to the integral (± 4 standard deviations from the peak). This reinforces the validity of the method used to obtain $\Delta k_{dp}(\omega)$.

CHAPTER V
PARAMETRIC FORMULATION

The previous chapters have presented the various mixing effects as solutions to Eq. (9) by means of a perturbation calculation. This method must be abandoned when trying to obtain results for parametric amplification since the perturbation series would not converge because of amplification of the higher order terms. However Eq. (9) still governs all such solutions and only our method of approximation must change. In this chapter we shall present a set of equations with contraints, which, if solutions exist, govern the parametric output. We do not propose to solve these equations, but only point out some useful facts caused by the narrow bandwidth spread. The method of approximation is basically a generalization of that used for the monochromatic calculations given in the references.

We assume solutions of Eq. (9),

$$(9) \qquad \nabla \times \nabla \times \underline{E}(z,\omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}(z,\omega) = \frac{4\pi}{c^2} \omega^2 \underline{\underline{\underline{\chi}}}^{(2)} : \int_{-\infty}^{\infty} \underline{E}(z,\omega-\omega'') \, \underline{E}(z,\omega'') \, d\omega'' ,$$

of the form

(74) $$\underline{E}(z, \omega) = E_1(z,\omega) + E_2(z,\omega) + E_3(z,\omega) +$$

$$+ E_1^*(z, -\omega) + E_2^*(z, -\omega) + E_3^*(z, -\omega),$$

where

$E_1(z, \omega)$ is peaked at ω_1 with bandwidth $\Delta\omega_1$,

$E_2(z, \omega)$ is peaked at ω_2 with bandwidth $\Delta\omega_2$,

and

$E_3(z, \omega)$ is peaked at ω_3 with bandwidth $\Delta\omega_3$.

Further we shall assume that all bandwidths are narrow. Note that Eq. (9) contains a convolution integral in ω as its forcing function. The convolution of $\underline{E}(z, \omega)$ with itself involves nine terms of the form

(75) $$\int_{-\infty}^{\infty} \underline{E}_i(z, \omega-\omega'') \underline{E}_j(z, \omega'') d\omega'' .$$

We assume this convolution yields terms peaked about $\pm \omega_i \pm \omega_j$, with sufficiently narrow bandwidths (for the monochromatic case with delta functions for E_i, E_j, this is clearly satisfied). Further we assume that

$E_1(z, \omega)$ convolved with $E_3(z, \omega)$ has bandwidth $f(\Delta\omega_1, \Delta\omega_3)$,

$E_2(z, \omega)$ convolved with $E_3(z, \omega)$ has bandwidth $g(\Delta\omega_2, \Delta\omega_3)$,

and

$E_3(z, \omega)$ convolved with $E_1(z, \omega)$ has bandwidth $h(\Delta\omega_2, \Delta\omega_1)$, where f, g, and h are general functions of two variables. We assume that the medium acts as a filter (accomplished by phase matching conditions) to all frequencies except ω_1, ω_2, and ω_3, where we pick

$$\omega_3 = \omega_1 + \omega_2 \, . \tag{76}$$

We substitute Eq. (74) into Eq. (9) and collect terms peaked about ω_1, ω_2, and ω_3, respectively. If we further require the constraint

$$\Delta\omega_2 = f(\Delta\omega_1, \Delta\omega_3), \tag{77}$$

$$\Delta\omega_1 = g(\Delta\omega_2, \Delta\omega_3),$$

and

$$\Delta\omega_3 = h(\Delta\omega_2, \Delta\omega_1),$$

each of the terms peaked about ω_j has bandwidth $\Delta\omega_j$ ($j = 1, 2, 3$). Since $\Delta\omega_j$ is sufficiently narrow, the sum of terms peaked about each ω_j is approximately zero. (In the monochromatic case the sum is exactly zero as the linear independence of the delta functions is employed). We then obtain the equations

$$\nabla \times \nabla \times \underline{E}_1(z,\omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_1(z,\omega) = \frac{4\pi}{c^2} \omega^2 \underline{\underline{\underline{X}}}^{(2)} \vdots \tag{78}$$

$$\int_{-\infty}^{\infty} \underline{E}_2^*(z, \omega-\omega'') \, \underline{E}_3(z, \omega'') d\omega'',$$

$$\nabla \times \nabla \times \underline{E}_2(z, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_2(z, \omega) = \frac{4\pi}{c^2} \omega^2 \underline{\underline{\underline{X}}}^{(2)} :$$

$$\int_{-\infty}^{\infty} \underline{E}_1^*(z, \omega-\omega'') \, \underline{E}_3(z, \omega'') d\omega'' ,$$

$$\nabla \times \nabla \times \underline{E}_3(z, \omega) - \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_3(z, \omega) = \frac{4\pi}{c^2} \omega^2 \underline{\underline{\underline{X}}}^{(2)} :$$

$$\int_{-\infty}^{\infty} \underline{E}_1(z, \omega-\omega'') \, \underline{E}_2(z, \omega'') d\omega'' .$$

Thus far we have made no assumptions regarding initial input functions and hence Eqs. (78) are applicable for the mixing process also, subject to the constraint Eq. (77). We can easily show that for two input signals large in magnitude, Eqs. (78) yield results identical to those for the mixing effects. Since the crystal length and $|\underline{\underline{\underline{X}}}^{(2)}|$ are small, the forcing function governing the solutions $\underline{E}_1(z, \omega)$ and $\underline{E}_2(z, \omega)$ are small with respect to $|E_1|$ and $|E_2|$. Hence we can neglect depletion of these waves similar to Butcher's[1] monochromatic analysis. We then take $\underline{E}_1(z, \omega)$ and $\underline{E}_2(z, \omega)$ of the form Eqs. (14) and (22). Substituting into the equation governing $E_3(z, \omega)$, we obtain the previous results of Chapter III. This gives us a fairly firm basis as to the validity of Eqs. (78).

In the parametric amplification problem only one of the input signals is large enough to neglect depletion. For this choose $E_3(z, \omega)$ and denote it as the pump wave. We take

$$(79) \qquad \underline{E}_3(z,\omega) = \underline{a}_j\, E_p(\omega)\, e^{ik_j(\omega)z} \;,$$

where the polarization of the wave remains general. Substituting this pump signal into Eq. (78) yield two coupled equations in $E_1(z, \omega)$ and $E_2(z, \omega)$;

$$(80) \qquad \nabla \times \nabla \times \underline{E}_1(z,\omega) - \frac{\omega^2}{c^2}\,\underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_1(z,\omega) = \frac{4\pi}{c^2}\,\omega^2\, \underline{\underline{X}}^{(2)} : \underline{a}_j$$

$$\int_{-\infty}^{\infty} \underline{E}_2^*(z,\omega-\omega'')\, E_p(\omega'')\, e^{ik_j(\omega'')z}\, d\omega''$$

(78)

$$\nabla \times \nabla \times \underline{E}_2(z,\omega) - \frac{\omega^2}{c^2}\,\underline{\underline{\epsilon}}(\omega) \cdot \underline{E}_2(z,\omega) = \frac{4\pi}{c^2}\,\omega^2\, X^{(2)} : \underline{a}_j \;\cdot$$

$$\int_{-\infty}^{\infty} \underline{E}_1^*(z,\omega-\omega'')\, E_p(\omega'')\, e^{ik_j(\omega'')z}\, d\omega'' ,$$

subject to the constraints Eq. (77).

This thesis will not attempt to solve Eqs. (80) since it is not certain as to how to obtain solutions subject to the proper constraints. However a few effects of bandwidth on a parametric amplification can be pointed out, assuming solutions exist. These are enumerated below.

(1) For a given non-monochromatic $E_p(\omega)$ and a purely monochromatic input for $E_1(0,\omega)$ (or $E_2(0,\omega)$), the solutions for $E_2(z,\omega)$ and $E_1(z,\omega)$ are necessarily not monochromatic. This can be shown by assuming

(81) $$E_2(z,\omega) = E_2(z)\,\delta(\omega-\omega_2)$$

and

$$E_3(z,\omega) = E_3(z)\,\delta(\omega-\omega_3)\ ,$$

and substituting into Eqs. (80). It becomes obvious they cannot be satisfied.

(2) For a given monochromatic $E_p(\omega)$ and a monochromatic input there exists solutions such that $E_2(z,\omega)$ and $E_1(z,\omega)$ are monochromatic. However, one cannot conclude from these equations that these are the only solutions. The monochromatic solutions are those given in the references.[1]

(3) If we pick $E_p(\omega)$ to be gaussian as with the mixing case, then the solutions for $E_1(z,\omega)$ and $E_2(z,\omega)$ are necessarily not gaussian. This can be seen by assuming solutions which are gaussian in form, and by substituting in Eqs. (80).

CHAPTER VI
DISCUSSION OF RESULTS

The basic objective of this thesis has been to shed some light on the effects of non-zero bandwidth on the second-order processes. The monochromatic signal is generally used because of its simplicity and fairly accurate phase matching predictions. However, even intuitively one can see that the interaction of non-monochromatic signals should in some way alter the output signal. One of the basic problems involved is the analytical description of the laser signal, to say nothing of the differential-integral equations then encountered. It is generally extremely difficult to find a model which can take into account all the experimentally observed results and we must resort to the model which describes the effects in which we are interested, but is still feasible to work with analytically. Since the limit of the gaussian function is a very good representation of the delta function encountered in the monochromatic analysis, and since there is a general lack of understanding of the statistical description of the laser radiation, it is not difficult to accept it as a limited physical model for investigation of the effects of small bandwidth. Once we accept this model the conclusions are very interesting.

One of the most interesting results is that the effects of bandwidth are more predominant in treating the difference frequency. The peak magnitude of the sum frequency output hardly changes from that obtained in the monochromatic analysis except when η becomes very small ($\Delta\omega_2^2 >> \Delta\omega_1^2$). However the magnitude of the difference frequency radiation is far less than the monochromatic prediction for all η and decreases even more when the effects of $\Delta\omega_2$ become predominant. This is consistent with the experimental fact that the difference frequency is much harder to observe (though this is also due to the measuring techniques required) and has a lower conversion efficiency than for the sum frequency signal.

We have also shown that the sum frequency radiated can be approximated satisfactorily by a gaussian function (for the input functions assumed) for most reasonable η (as could the difference frequency for a smaller $\Delta\omega$ and reasonable η). The main deviation of the calculations from the gaussian curve is near the peak where the magnitude decreases for $\underline{E}_S(z,t)$ as η decreases. The approximations used to obtain the approximation functions essentially consist of treating the sharp gaussian variation of the weighting function as a delta function. Since the approximation curves are much better for the sum-generated fields than for the difference fields, we would expect the effects of bandwidth to be much more predominant for the difference-generated fields.

Our model predicts no change in the phase matching conditions obtained in the monochromatic analysis. This is consistent with the fact that most monochromatic calculated matching angles are correct to within ± 0.2 degrees, as observed from experimental results.

The phenomenon involving parametric amplification was seen to be more difficult to deal with because of the coupled equations encountered. The equations derived appear to be valid to the extent that they reduce to the mixing results obtained by perturbation analysis. The conclusions listed in Chapter V are particularly interesting since they are somewhat unexpected.

APPENDIX
REAL SIGNAL SPECTRUM

The short analysis which follows is presented to verify certain results obtained on the assumption of real signals. In particular we shall derive the frequency spectrum for a general signal satisfying Maxwell's field equations.

Any real signal which is Fourier transformable can be written in the form

$$(82) \qquad \underline{E}(z, t) = \int_{-\infty}^{\infty} \underline{E}(\omega) \cos\,[\omega t + k(\omega)z + \alpha]\, d\omega,$$

where α is a phase constant. We can write the cosine function in exponential form as

$$(83) \qquad \cos\,[\omega t + k(\omega)z + \alpha] = \frac{1}{2}\; e^{i\omega t}\, e^{ik(\omega)z}\, e^{i\alpha} + \frac{1}{2}\; e^{-i\omega t}\, e^{-ik(\omega)z}\, e^{-i\alpha} \; .$$

Substituting Eq. (83) into Eq. (82) results in

$$(84) \qquad \underline{E}(z, t) = \frac{1}{2} \int_{-\infty}^{\infty} \underline{E}(\omega)\, e^{ik(\omega)z}\, e^{i\alpha}\, e^{i\omega t}\, d\omega + \frac{1}{2} \int_{-\infty}^{\infty} \underline{E}(\omega)\, e^{-ik(\omega)z}\, e^{-i\alpha}\, e^{-i\omega t}\, d\omega \, .$$

Using the Fourier transform relation

$$(85) \qquad f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \quad ,$$

we put our integral $\underline{E}(z, t)$ into this form by changing variables $\omega \to -\omega$ in the latter integral and setting

$$(86) \qquad A = e^{i\alpha} \; .$$

We than obtain

$$(87) \qquad \underline{E}(z, t) = \frac{1}{2} \int_{-\infty}^{\infty} \{A\, \underline{E}(\omega)\, e^{ik(\omega)z} + A^{*}\, \underline{E}(-\omega)\, e^{-ik(-\omega)z}\} \times e^{i\omega t}\, d\omega \; .$$

Thus we have shown that

$$(88) \qquad \underline{E}(z, \omega) = A\, \underline{E}(\omega)\, e^{ik(\omega)z} + A^{*}\, \underline{E}(-\omega)\, e^{-ik(-\omega)z}$$

represents a real frequency signal. Note in particular the symmetry about the $\omega = 0$ axis.

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