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BOUNDED PHASE COORDINATE CONTROL

Prepared under Contract No. NASw-986 *by*

HONEYWELL, INC.
St. Paul, Minn.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1966

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ABSTRACT

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Two methods based on different concepts are presented for solving bounded phase-coordinate time-optimal control problems. One method originates from Pontryagin's maximum principle and jump conditions in the modified adjoint solution. The computational scheme for this method is derived from sufficiency conditions. The other method is based on the introduction of a measure of excursion of phase-trajectories outside their restraint sets and gives an approximate solution. The computational procedure for this method is developed from the necessary and sufficient conditions which are relatively easy to apply. The method is extended to the solution of optimal problems with integral cost. An on-line analog computer program is developed which proves satisfactory for time-optimal problems with no phase-coordinate bounds, but not directly applicable for problems with bounds. The cause of the difficulty is revealed by an analysis. Areas for further investigation on the bounded phase-coordinate optimal control problems are also outlined.

Author

FOREWORD

This report was prepared by Honeywell Inc., Systems and Research Division, Research Department staff members, for the National Aeronautics and Space Administration under NASA Contract No. NAS w-986.

The studies began on 15 June 1964, and ended on 14 June 1965. Mr. C. R. Stone was the project supervisor. Dr. E. R. Rang oversaw the final report preparation.

This report was edited by Dr. J. Y. S. Luh who also prepared Chapters 1, 6, and 8. The main contributors to Chapter 2 were Dr. David L. Russel* and Mr. H. E. Gollwitzer. Dr. Russell also prepared Chapter 3. Dr. E. Bruce Lee* contributed to Chapters 4 and 7 and Mr. D. D. Fairchild to Chapter 5. Dr. C. A. Harvey provided consulting assistance.

This is the final report and concludes the work on Contract No. NAS w-986.

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CHAPTER 1
INTRODUCTION

1.1 HISTORICAL DEVELOPMENT

Bounded phase-coordinate problems arise naturally in many practical applications. In many flight vehicles, engine deflection, angle of attack and bending moment contribute to the phase-coordinate constraints. As an example, if the controller input is engine gimbal rate, the engine displacement may be considered as a phase-coordinate of the dynamical system. Normally, the allowable engine displacements are small; an efficient use of the available control input often demands operating on the engine displacement limit. Unfortunately, such intuition is not always correct. The efficient use of the control input implies not only operating at the displacement limit, but also considering the displacement limit explicitly in the over-all design of the controller. The term "efficient use" can be specifically defined by a given minimization criterion, and problems of this type are called optimal control problems with phase-coordinate inequality constraints.

As indicated by Bolza [1, pp. 125-126], Weierstrass formulated the analogous problems of calculus of variations with phase-coordinate inequality constraints in 1882, and developed the "corner" conditions for the two-dimensional Lagrange problems. The "corner" conditions deal with the discontinuities of the solutions of the analogous problems. According to Bliss [2, p. 43], the necessary and sufficient conditions for a minimum solution were studied subsequently by Carathéodory, Bolza, Dresden, Graves, Reid, Smiley, Bliss, and Underhill. Most of the studies were completed between 1904 and 1937.

In 1961, Berkovitz [3] reduced the general control problem with constraints to a problem of calculus of variations. In his discussion, a translation of necessary conditions for the problem of calculus of variations into the necessary conditions for the optimal control was established, including the application of Pontryagin's maximum principle [4]. His results, however, are not applicable to control problems with phase-coordinate inequality constraints that do not explicitly involve the control variable. In an independent study, Gamkrelidze [5, also Chapter VI of 4] treated the latter problem entirely based on the maximum principle. Berkovitz [6] then showed that Gamkrelidze's results can be achieved by solving the relevant problem of calculus of variations. Dreyfus [7] studied the same problem by means of the dynamic programming formulation. His results are in agreement with that of Berkovitz [8]. Among all the studies, sufficiency conditions were virtually ignored. For the practical applications, even when solutions do exist, the necessary conditions derived by various authors are difficult to apply.

During 1961-1962, Chang derived a simpler necessary condition for a more restricted class of problems [9], and existence theorems based on the extension of Ascoli's Theorem [10]. For linear time-optimal control systems with convex restraint set, the necessary condition is also the sufficient condition. The result, however, was not adequately proved. An elegant proof of the necessity of the condition can be deduced from Neustadt's recent work [27], while a rigorous proof of the sufficiency will be shown in Chapter 3. This condition is an improvement on Gamkrelidze's result. It establishes the fact that the normal vector appearing in the modified adjoint differential equation is always outward with respect to the set of attainability, and hence the necessary and sufficient condition is relatively easy to apply.

As to the computational aspects of the problem, there are essentially two classes of methods. One class is the direct method which includes the method of the gradient, steepest-descent or their equivalent. This method was studied by Dreyfus [7], Denham [11, 12] and Bryson [13], using the necessary conditions of the optimal control, and by Paiewonsky, et. al. [14], using conditions both of the optimal control and from the calculus of variations. The other class is the indirect method which was discussed by Kahne [15], Ho and Brentani [16], and Nagata, et. al. [17]. Each method has its advantages and disadvantages, and, in general, neither one is the best method.

1.2 PROBLEM STATEMENT

The general problem of interest is stated as follows. Given a linear control process as described by the differential system

$$\dot{x} = A(t)\, x + B(t)\, u(t) \tag{1}$$

where x and u(t) are n-dimensional state vector and m-dimensional control vector, respectively, A(t) and B(t) are n by n and n by m matrices of measurable functions for t in some interval $[t_o, t_1]$. Let G be a closed convex subset of E^n and Ω be a compact convex subset of E^m. Let the cost functional of control be

$$C(u) = g\left[x(t_1)\right] + \int_{t_o}^{t_1} \left[f^\circ (x, t) + h^\circ (u, t)\right] dt \tag{2}$$

where $f^\circ (x, t)$ and $h^\circ (u, t)$ are real-valued, non-negative, convex and continously differentiable functions with respect to t, while g is convex and differentiable. The general problem of optimal control of bounded phase-coordinate systems is to choose an admissible control $u(t) \in \Omega$ on the time interval $[t_o, t_1]$ which steers the system (1) from its given initial state $x(t_o) = x_o$ at time t_o to a closed target set $\hat{G} \subset G$ at time t_1, such that the response $x(t) \subset G$ for all $t \in [t_o, t_1]$ and the cost functional is a minimum.

The general problem described above is difficult to solve. Instead, solutions of more restricted classes of problems were attempted which yield relatively simple results. Approximate solutions were also considered. These are outlined in the following section.

1.3 METHOD OF SOLVING THE PROBLEM

The first question that needs to be answered is "Under what conditions does such an admissible control exist?" Chang [10] discussed the existence theorems thoroughly. However, the fundamental requirement of the compactness of the set of allowed "control-and-path" pairs (u, x) was not clearly presented. The subject is re-examined rigorously in Chapter 2. With the value of $g(x)$ in Equation (2) chosen as constant and $f^{o} + h^{o}$ as unity, the problem is reduced to a problem of time-optimal control. This subject is specifically discussed in Chapter 3, in which a sufficiency theorem for the optimum control is established. Using this theorem, a computational scheme by means of "backing out" procedure is derived and illustrated by a numerical example. Uniqueness theorems are also given for various restraint sets in the phase-coordinate system.

A method of approximate solution based on an entirely different concept is given in Chapter 4. The solution obtained by this method allows excursions of the phase-coordinate trajectories outside their restraint sets. The excursions can be adjusted to be as small as pleased and thereby the trajectory so determined approximates the solution obtained by the method given in Chapter 3. The necessary and sufficient conditions upon which the computational scheme is proposed are given and illustrated by a numerical example. A brief discussion is also given for the optimum problem when $g(x)$ and $f^{o} + h^{o}$ are not constants. This problem is later discussed in detail in Chapter 7.

Chapter 5 describes an analog computational scheme which mechanizes Neustadt's algorithm [18] for an on-line computation. The program worked well for a time-optimal problem with no constraints in phase-coordinates but failed when applied to the problem of bounded phase-coordinate as proposed in Chapter 4. The implementation of the approximate solution on the analog computer for an on-line operation is then discussed in Chapter 6. The main difficulty lies in the existence of singular arcs in the adjoint solution which is also illustrated in the example given in Chapter 4.

Chapter 8 summarizes the results of the studies on bounded phase-coordinate optimal control and recommends areas of further investigation.

CHAPTER 2

FUNDAMENTAL REQUIREMENT FOR THE EXISTENCE OF OPTIMAL CONTROLS IN BOUNDED PHASE-COORDINATE SYSTEMS

2.1 INTRODUCTION

The problem of the existence of optimal controls in bounded phase-coordinate systems was discussed by Chang [10, pp. 3-37]. His results are constructed on the basis of the compactness of the set of all allowed "control-and-path" pairs. This fundamental proposition was given as Theorem 3 in his report [10, pp. 15-18]. Although his sufficiency conditions for the resulting existence theorems are correct, the proof of Theorem 3 is not clear.

In this chapter, the proposition of the compactness of the set of all allowed "control-and-path" pairs is re-examined. A rigorous treatment within the framework of currently used mathematics [19] of this topic is presented. After necessary preliminaries, an extension of Ascoli's Theorem is proved and is then applied in the proof of a theorem which is analogous to Chang's Theorem 3.

2.2 AN EXTENSION OF ASCOLI'S THEOREM

The following definitions and lemmas [19] are needed in the extension of Ascoli's Theorem:

<u>Definition 1.</u>

A family of n-vector functions, $\{w(t)\}$, on an interval, T, is said to be piecewise equicontinuous if:

(i) Each $w(t)$ is piecewise continuous on T, i.e., continuous except at finitely many points in T, where it may or may not be defined.

(ii) Given $\epsilon > 0$, there exists $\delta > 0$ such that for any $w(t) \epsilon \{w(t)\}$, if t_1 and t_2 lie in an open interval in which $w(t)$ is continuous and $|t_1 - t_2| < \delta$, then $\| w(t_1) - w(t_2) \| \leq \epsilon$. Here, $\|w(t)\|$ denotes the Euclidian norm of the n-vector, $w(t)$.

Definition 2.

A sequence, $\{w_k(t)\}$, of n-vector functions is said to converge almost uniformly on T to the n-vector function, $w(t)$, if for each $\delta > 0$ it is possible to select a measurable set, N_δ, whose measure is less than δ, such that $\{w_k(t)\}$ converges uniformly to $w(t)$ on $T-N_\delta$.

Definition 3.

Let X denote the set of all n-vector functions defined on T. For a given member, $w(t)$, of X, define the (ϵ, δ) neighborhood, $N\left[\epsilon, \delta, w(t)\right]$, to be the class of all members, $\hat{w}(t)$, of X, such that $\|w(t) - \hat{w}(t)\| < \epsilon$, except on a measurable set of measure less than δ. The resulting topology will be called the A. U. (almost uniform) topology of X.

Lemma 1.

If a sequence, $\{w_k(t)\}$, converges to $w(t)$ in the A. U. topology, then it converges to $w(t)$ almost uniformly and conversely.

Proof: Choose $\delta > 0$. Since $\{w_k(t)\}$ converges to $w(t)$ in the A. U. topology, it is possible to choose for each integer, $n > 0$, another integer, $\beta(n) > 0$, such that for $K \geq \beta(n)$, we have $w_k(t) \in N\left[\frac{1}{n}, \frac{\delta}{2^n}, w(t)\right]$. Let $N = \bigcup_{n=1}^{\infty} N\left[\frac{1}{n}, \frac{\delta}{2^n}, w(t)\right]$.

Then, the measure of N is, at most, δ, and for $t \in T - N$ and $K \geq \beta(n)$, $\| w_k(t) - w(t) \| < \frac{1}{n}$. Consequently, $w_k(t)$ converges to $w(t)$ in the A. U. topology since δ was arbitrary. The converse is obvious from the definition of almost uniform convergence.

The following theorem gives results which are essential in the extension of Ascoli's theorem (Theorem 2):

Theorem 1.

Let B denote a closed, uniformly bounded set of piecewise equicontinuous functions on the interval, T. Assume that:

(i) There is an integer, $N \geq 0$, such that $w(t) \in B \implies w(t)$ has, at most, N discontinuities in T.

(ii) T is bounded, i. e., there is a positive real number, R, such that $t \in T \implies |t| \leq R$.

Then, B is sequentially compact in the A. U. topology.

Proof:

Let $\{w(t)\}$ be any infinite sequence of members of B. For each k assume that $N(k) \leq N$ is the number of discontinuities of $w_k(t)$ in T. Now, define $\tau_k = (t_{k_1}, t_{k_2}, \ldots, t_{k_{N(k)}}, t_N, t_N, \ldots, t_N)$ to be a vector such that the first N(K) components are, in order, the points of discontinuity of $w_k(t)$. The remaining components, if any, are equal to the right hand endpoint of T. The set, $\{\tau_k\}$, is a uniformly bounded set of points of R^N. In fact, $\|\tau_k\| \leq \sqrt{N}\, R$ for each k. By the Bolzano-Weierstrass Theorem there exists a subsequence, $\tau_k^{(1)}$, which converges to a vector, $\tau = (\tau_1, \ldots, \tau_N)$. Let $w_k^{(1)}(t)$ be the corresponding subsequence of vector functions.

Now, choose $\delta > 0$ and let I_δ be the closed subset of T such that if $t \epsilon I_\delta$, then $|t - T_i| \geq \delta$, where T_i denote the endpoints of the interval, T. Then, there is a $K > 0$ such that for $k \geq K$, the functions $w_k^{(1)}(t)$ are continuous on I_δ. This result is a consequence of the fact that for $k \geq K$, the points of discontinuity lie within a distance, δ, of the limit, $\tau = (\tau_1, \ldots, \tau_N)$. The interval, I_δ, is a compact subset of the real numbers, and the functions, $w_k^{(1)}(t)$, are a uniformly bounded and equicontinuous family defined on I_δ. Then, from the theorem of Ascoli, there is a subsequence, $\{w_k^{(2)}(t)\}$, of $\{w_k^{(1)}(t)\}$, which converges uniformly to a vector function, $w^{(1)}(t)$, on I_δ. Moreover, $w^{(1)}(t)$ satisfies the same bounds and continuity hypothesis as do the $w_k(t)$ on I_δ.

Let δ_ℓ be a sequence of positive numbers such that $\delta_1 = \delta$ and $\lim_{\ell \to \infty} \delta_\ell = 0$, and let I_{δ_ℓ} be defined as above, replacing δ by δ_ℓ. Now, for each $\ell > 1$, let $\{w_k^{(\ell+1)}(t)\}$ be a subsequence of $\{w_k^{(\ell)}(t)\}$ such that:

(i) $\{w_k^{(\ell+1)}(t)\}$ converges uniformly to some $w^{(\ell)}(t)$ on I_{δ_ℓ}.

(ii) $\{w_k^{(\ell+1)}(t)\}$ excludes $w_1^{(\ell)}(t)$.

Let $w(t)$ denote the function whose value on $t \in \bigcup_{\ell=1}^{\infty} I_{\delta_\ell}$ is the common value of all $w^{\ell}(t)$ which are defined at t. Then, let $\{w^*_\ell(t)\}$ be the subsequence of $\{w_k(t)\}$ such that $w_\ell(t) = w_1^{(\ell+1)}(t)$. Since $\delta_\ell \longrightarrow 0$, this clearly implies that $w^*_\ell(t)$ converges almost uniformly to the function, $w(t)$. The limit, $w(t)$, is clearly in B and the proof is completed.

Definition 4.

Let B be a uniformly bounded set of n-vector functions defined on a finite interval, T. For each integer, $M > 0$, let S_M be a family of n-vector functions such that $S_M \subseteq B$ and S_M satisfies the hypothesis applied to the family B of Theorem 1. The set of families, S_M, is uniformly dense in the A. U. topology of B if, given any $\epsilon > 0$ and $\delta > 0$, it is possible to find $M(\epsilon, \delta)$ such that every $w(t) \in B$ lies in a (ϵ, δ) neighborhood of some member of S_M.

Theorem 2. (An extension of Ascoli's Theorem)

Let B be a uniformly bounded set of n-vector functions which is closed in its A. U. topology and contains a uniformly dense set of families, S_M. Then B is sequentially compact in its A. U. topology.

Proof:

Let $\{w_\ell(t)\}$ be an infinite sequence in B, and let $\eta > 0$ be a small number. Corresponding to the positive number, $\frac{\eta}{2^k}$, let M_k be chosen so that $M_{k+1} > M_k$ and each number of B lies in a $\left(\frac{\eta}{2^k}, \frac{\eta}{2^k}\right)$ neighborhood of some member of S_{M_k}. For each ℓ, let $S_{M_k,\ell}(t)$ be such a function in S_{M_k}

corresponding to $w_\ell(t)$. Since S_{M_k} satisfies the hypothesis of Theorem 1 for each k, let $\{\ell_1\}$ be a subsequence of the natural numbers such that $S_{M_1, \ell_1}(t)$ converges almost uniformly to a function, $S_{M_1}(t)$. In general, let ℓ_{k+1} be a subsequence of $\{\ell_k\}$ such that:

(i) $\{\ell_{k+1}\}$ does not include the first member of ℓ_k.

(ii) $\{S_{M_{k+1}, \ell}(t)\}$ converges almost uniformly to $S_{M_{k+1}}(t)$.

Let ℓ^* denote the subsequence of the natural numbers which consists of the first members of the $\{\ell_k\}$. Then, for each k, $\{S_{M_{k,\ell}}(t)\}$ converges almost uniformly to $S_{M_k}(t)$.

Let ℓ^*_k be chosen successively so that $\ell^*_{k+1} > \ell^*_k$ and $\ell^* \geq \ell^*_k \Longrightarrow S_{M_k \ell^*}(t)$ lies in the $\frac{\eta}{2^k}, \frac{\eta}{2^k}$ neighborhood of $S_{M_k}(t)$. Now, assume that $\ell^* \geq \ell^*_k$ and $\bar{\ell}^* \geq \ell^*_k$. Then the following inequalities hold:

(i) $\| w_{\ell^*}(t) - S_{M_{k,\ell^*}}(t) \| < \frac{\eta}{2^k}$ except on a set of measure at most $\frac{\eta}{2^k}$.

(ii) $\| S_{M_{k,\ell^*}}(t) - S_{M_{k,\bar{\ell}^*}}(t) \| < \frac{\eta}{2^{k-1}}$ except on a set of measure at most $\frac{\eta}{2^{k-1}}$.

(iii) $\| S_{M_{k,\bar{\ell}^*}}(t) - w_{\bar{\ell}^*}(t) \| < \frac{\eta}{2^k}$ except on a set of measure at most $\frac{\eta}{2^k}$.

Consequently, from the triangle inequality, $\| w_{\ell^*}(t) - w_{\bar{\ell}^*}(t) \| < \frac{\eta}{2^{k-2}}$ except on a set of measure at most $\frac{\eta}{2^{k-2}}$.

Let E denote the union of these sets for all k. Then, from the countable sub-additivity of lebesque measure, the measure of E is at most 4η. Clearly, the sequence $\{w_{\ell^*}(t)\}$ converges uniformly on T-E to $w^*(t)$. Now, take a sequence, η, such that $\lim_{i \to \infty} \eta_i = 0$.

By a now familiar diagonalization process, a subsequence, $\{w_\ell(t)\}$, can be found which converges almost uniformly on T to a function, w(t), which, since B is closed, belongs to B. This completes the proof.

2.3 COMPACTNESS OF "CONTROL-AND-PATH" PAIRS

In many control problems, one is given a system of first-order differential equations of the type

$$\dot{x}^i = f^i(t, x, u), \quad i = 1, 2, \ldots, n \tag{1}$$

where x is an n-vector and u is an m-vector. The functions, $f^i(t, x, u)$, together with their partial derivatives, $\frac{\partial f^i}{\partial x^j}$, i, j = 1, ..., n, are bounded, single-valued, and continuous functions in the vector arguments, x, u, and the scalar argument, t, on a product region, $X_1 \times U_1 \times T$, in $R^n \times R^m \times R^1$. X_1 and U_1 are closed regions in R^n and R^m, respectively, and T is a compact interval in R^1. A control function, $u(t) = [u^1(t), \ldots, u^m(t)]$, is a vector-valued measurable function defined on T with a graph in U_1. It follows from standard existence theorems that if x(t) is specified initially and u(t) is a given control function, then there exists a unique, absolutely continuous solution, x(t), of Equation (1), which passes through the given initial conditions [20]. The following definitions are introduced for the convenience of discussion:

Definition 5.

Let $f(t)$ be a function of a real variable defined on a compact interval, $T = [t_1, t_2]$. Let π_r: $t_1 = \xi_1 \leq \xi_2 \leq \cdots \xi_r = t_2$ be a partition of T. Then, $f(t)$ is said to be of bounded variation if

$$\sup_{\pi_r} \sum_{k=1}^{r} \left| f(\xi_k) - f(\xi_{k-1}) \right| < \infty .$$

A vector function, $g(t) = [g_1(t), \ldots g_r(t)]$, is of bounded variation if each of its components are of bounded variation.

Definition 6.

Let A be a family of vector functions and suppose every member of A is of bounded variation on an interval, $T = [t_1, t_2]$. Then, for each member, f_i, of A,

$$\sup_{\pi_r} \sum_{k=1}^{r} \left| f_i(\xi_k) - f_i(\xi_{k-1}) \right| = M(f_i) < \infty .$$

If there exists a number, $M > 0$, such that $M(f_i) \leq M$ for each $f_i \in A$, then A is said to be of uniform bounded variation.

The class of allowable controls is given in Definition 7.

Definition 7.

Let $\Omega \subseteq U_1$ be a compact subset of R^m and let A be a family of uniform bounded variation vector functions defined on T with range in Ω. Furthermore, let $X \subseteq X_1$ be a compact arc-wise connected subset of R^n. Then a control, $u(t) \epsilon A$, is allowable with respect to X if the solution, $x(t)$, of Equation (1) satisfies the condition $x(t) \epsilon X$ for $t \epsilon T$. Similarly, the solution of Equation (1) with an allowable control is called an allowable path.

The set of all allowable controls with respect to X, hereafter designated as allowable, is denoted by c, and the set of all allowed paths by p. The m + n vector, $[u(t), x(t)]$, denotes an allowable control-and-path pair where u(t) is allowable and x(t) the corresponding allowed path. The set of all control-and-path pairs is denoted by F.

A cost functional, c(u), is defined by the function, $x^o(t)$, $t \epsilon T$, and is given by

$$c(u) = x^o(t_2) - x^o(t_1) \tag{2}$$

Note that $T = [t_1, t_2]$.

An allowable control, $\hat{u}(t)$, is said to be optimal if

$$c(\hat{u}) = \sup_{u \epsilon A} c(u)$$

The types of terminating conditions considered are as follows:

(i) If t_2 is fixed but $x(t_2)$ is not specified, then the problem is referred to as a free endpoint type.

(ii) If $x(t_2)$ is specified and t_2 is contained in a compact interval, $\widetilde{T} = [t_1, t_3]$, the problem is referred to as a fixed endpoint problem.

Definition 8.

The set of optimal controls under the free endpoint condition is denoted as S_o, and the set of optimal controls under a fixed endpoint condition is denoted as $S_o(\widetilde{T})$.

The following lemma establishes the fact that A contains a uniformly dense family of step functions:

Lemma 2.

The family, A, of allowable controls contains a uniformly dense set of families of step functions.

Proof:

Let the bound on the variation of the members of A on T be $M > 0$. Let $f(t) \epsilon A$, and let $\epsilon, \delta > 0$ be given. Let N be the least integer for which

$$N > \frac{M(t_2 - t_1)}{\epsilon \delta} . \tag{3}$$

Now, divide the interval, T, into N subintervals of equal length,

τ_i, $i = 1, \ldots, N$.

Define

$$a_i = \sup_{t \epsilon \tau_i} f(t), \quad b_i = \inf_{t \epsilon \tau_i} f(t) \tag{4}$$

and let g(t) be a step function defined on T and given by

$$g(t) = \frac{1}{2}(a_i + b_i), \; t \; \epsilon \; \tau_i, \; i = 1, \ldots, N. \tag{5}$$

Since the maximum variation of f(t) is M, the number of intervals for which $a_i - b_i \geq \epsilon$ is, at most, $\frac{M}{\epsilon}$.

Consequently,

$$\| f(t) - g(t) \| = \| f(t) - \frac{1}{2} (a_i + b_i) \| \leq \frac{1}{2} \| f(t) - a_i \| + \frac{1}{2} \| f(t) - b_i \| \tag{6}$$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ with exception of, at most, $\frac{M}{\epsilon}$, intervals, whose measure is less than $\frac{M}{\epsilon} \frac{(t_2 - t_1)}{N}$. But, from (3), $\frac{M}{\epsilon} \frac{(t_2 - t_1)}{N} < \delta$, hence, A contains a uniformly dense set of families of step functions. This concludes the proof of Lemma 2.

Theorem 3.

The set, F, of all allowed control-and-path pairs in sequentially compact in the almost uniform topology.

Proof:

Let $[\,u(t), x(t)]$ be an allowed control-and-path pair. Define $n + m + 1$ dimensional vector function, $w(t)$, as follows; $w(t) = [\,u(t), x(t), x^{o}(t)]$. Let B be a set which contains infinitely many $w(t)$ and let $\{w_k(t)\}$ be a sequence such that $w_k(t) \in B$. From the hypothesis on the $f^i(t, x, u)$, and from Lemma 2, we have that B is closed. From Lemma 2, B contains a uniformly dense set of families, S_m. By Theorem 2, we can select a subsequence, $\{w_k(t)\}$, which converges to a limit function, $w(t) = [\,w^1(t), \ldots, w^m(t), w^{m+1}(t), \ldots, w^{m+n+1}(t)]$, in the almost uniform topology.

Let $u(t) = [\,w^1(t), \ldots, w^m(t)]$ and $x(t) = [\,w^{m+1}(t), \ldots, w^{n+m}(t)]$. Since X was closed, $[\,w^{m+1}, \ldots, w^{n+m}(t)] \in X$, and, from Theorem 2, we can conclude that $[\,w^1(t), \ldots, w^m(t)] \in \Omega$ almost everywhere on T. Let $\hat{u}(t)$ be defined as follows:

$$\hat{u}(t) = \begin{cases} u(t) & \text{where } u(t) \text{ is defined, or} \\ \tilde{u} & \text{where } \tilde{u} \text{ is any vector } \tilde{u} \in \Omega \\ & \text{on a subset of measure zero} \\ & \text{where } w(t) \text{ is not defined.} \end{cases}$$

As a result, $\hat{u} \in \Omega$ on T.

The next step is to establish that w(t) is an allowed control-and-path pair. Suppose for contradiction that there exists at least one t_3, $t_1 \leq t_3 \leq t_2$, for which

$$\left\| \hat{x}(t_3) - \int_{t_1}^{t_3} f(t, \hat{x}, \hat{u})\, dt - x(t_1) \right\| > a > 0 \tag{7}$$

for some a. The function, $\hat{x}$, is the allowed path corresponding to $\hat{u}$. The function, f (t, x, u), being continuous on a compact set, $X \times \Omega \times T$, is uniformly continuous, and hence, for any $\epsilon > 0$, it is possible to find $\delta(\epsilon) > 0$ such that

$$\| f(t, x, u) - f(t, \hat{x}, \hat{u}) \| < \epsilon$$

if

$$\begin{aligned} \| x - \hat{x} \| &< \delta(\epsilon), \\ \| u - \hat{u} \| &< \delta(\epsilon). \end{aligned} \tag{8}$$

Define $M = \max\limits_{X \times \Omega \times T} \| f(t, x, u) \|$,

$$\begin{aligned} \epsilon &= \frac{a}{3(t_2 - t_1)}, \\ \epsilon &= \min\left[\frac{a}{3}, \delta(\epsilon)\right], \\ \delta_1 &= \frac{a}{6M}. \end{aligned} \tag{9}$$

From Theorem 2, there is a (δ_1, ϵ_1) neighborhood in the almost uniform topology such that if $k \geq N_1(\delta_1, \epsilon_1)$, then

$$|| x_k(t) = \hat{x}(t) || < \frac{a}{3} \tag{10}$$

$$|| u_k(t) = \hat{u}_i(t) || < \frac{a}{3}$$

except on a set of measure less than ϵ. Here, $u_k(t)$ and $x_k(t)$ are an allowed control-and-path pair from $\left\{ w_k(t) \right\}$ which from definition satisfy

$$x_k(t) = x(t_1) + \int_{t_1}^{t} f \quad (t, x_k, u_k) \ dt.$$

Now,

$$\begin{aligned} 0 < a < || x(t_3) - \int_{t_1}^{t_3} f(t, \hat{x}, \hat{u}) \, dt - x(t_1) || \leq || \, & x(t_3) \\ - x_k(t_3) || + \int_{t_1}^{t_3} || f(t, \hat{x}, \hat{u}) - f(t, x_k, u_k) || \ & dt, \end{aligned} \tag{11}$$

Now, if k is chosen sufficiently large, one gets from (10) that

$$|| x(t_3) - x_k(t_3) || < \frac{a}{3} \tag{12}$$

Furthermore, since $\hat{x}_k \to x$ and $u_k \to u$ except on a set of measure less than δ_1, call it E, one obtains from (11) the following inequality:

$$\int_{t_1}^{t_3} ||f(t, \hat{x}, \hat{u})\, dt - f(t, x_{k'}, u_{k'})||\, dt \leq \int_{t_1}^{t_2} || f(t, \hat{x}, \hat{u})$$

$$- f(t, x_k, u_k)||\, dt \leq \int_E || f(t, \hat{x}, \hat{u}) - f(t, x_k, u_k)||\, dt \qquad (13)$$

$$+ \int_{T-E} || f(t, \hat{x}, \hat{u}) - f(t, x_k, u_k)||\, dt \leq 2 M \delta_1$$

$$+ \epsilon (t_2 - t_1) = \frac{2a}{3} .$$

Hence, from (11), (12), (13), one gets the inequality

$$0 < a < || \hat{x} (t_3) - \int_{t_1}^{t_3} f(t, \hat{x}, \hat{u})\, dt - x(t_1)|| \leq \frac{a}{3} + \frac{2a}{3} = a.$$

This is obviously a contradiction and

$$\hat{x} (t_3) = \int_{t_1}^{t_3} f(t, \hat{x}, \hat{u})\, dt + x(t_1)$$

for all $t_3 \epsilon$ T. From the Lebesgue differentiation theorem, $\hat{x}(t)$ is a solution of the system and hence is an allowed control-and-path pair. This completes the proof.

2.4 CONCLUSION

Theorem 3 given in the preceding section established the compactness of the "control-and-path" pairs. The theorem is analogous to Chang's Theorem 3

[10, p. 15]. This result is a fundamental requirement for the proof of existance theorems for optimal control with bounded phase-coordinates as shown in Reference 10.

CHAPTER 3

TIME-OPTIMAL BOUNDED PHASE-COORDINATE CONTROL OF LINEAR SYSTEMS

3.1 INTRODUCTION

This chapter discusses time-optimal bounded phase-coordinate control of linear systems of the form

$$\dot{x} = A(t)x + B(t)u.$$

The basic ideas have been gleaned from the sources listed in the references; very little is entirely new. These results have been obtained from a study of two papers by Chang, [9], [10]. Chang's ideas are very fruitful but his mathematical proofs are apparently incorrect.

It is shown that the results presented here place the problem of time-optimal control of linear systems with convex phase constraints in a position where calculation of trajectories by the "backing out" procedure is feasible.

3.2 PROBLEM STATEMENT

In this section, the problem treated herein will be precisely described.

The control system has the form

$$\dot{x} = A(t)x + B(t)u = f(x, u, t). \tag{14}$$

Vectors x and f (x, u, t) are n-dimensional while the vector, u, is m-dimensional. $A(t)$ is a measurable n by n matrix function for t in some interval, $[T_0, T_1]$ of R^1, while $B(t)$ is a measurable n by m matrix function on the same domain. It is assumed that $m \le n$.

Let G be a closed convex subset of E^n with non-empty interior. G is not necessarily compact. x_o and x_1 are points in Int(G) such that $x_o \neq x_1$. Let Ω be a compact convex subset of E^m with non-empty interior. A measurable m-vector function, u(t), defined on a subinterval, $[t_o, t_u]$, of $[T_o, T_1]$, is an admissible control relative to t_o and the point $x_o \in G$ if:

(i) $u(t) \in \Omega$ for each t in $[t_o, t_u]$;

(ii) The solution x(t) of Equation (14) with $x(t_o) = x_o$ and u replaced by u(t) lies in G on the interval, $[t_o, t_u]$.

The cost of an admissible control, u(t), is denoted C(u) and is equal to $t_u - t_o$.

The problem is as follows: Let U denote the class of all admissible controls, u(t), defined on intervals $[t_o, t_u]$ of $[T_o, T_1]$ relative to t_o and the point x_o, with the additional property that t_u is the <u>first</u> time at which the corresponding solution, x(t), of Equation (14), with $x(t_o) = x_o$, is equal to x_1. Assuming U is non-empty, find a member $\hat{u}(t)$ of U such that for all $u(t) \in U$, $C(\hat{u}) \leq C(u)$, i.e., $t_{\hat{u}} \leq t_u$. This problem will, for brevity, be called the problem, P, and $\hat{u}(t)$, if it exists, will be called an optimal control for the problem, P.

3.3 PRELIMINARY CONCEPTS AND NOTATION

Throughout this chapter, t_o and x_o will remain fixed. Therefore, instead of saying that a control, u(t), is admissible relative to t_o and x_o, it will simply be said that u(t) is admissible. Whenever a symbol, such as $\hat{u}(t)$ or $u^*(t)$, is used for an admissible control, the corresponding solution of Equation (14), with $x(t_o) = x_o$, will have the same type of symbol, i.e., $\hat{x}(t)$ or $x^*(t)$ respectively.

It is useful to augment the system, i.e., Equation (14), by first setting

$$x^{\circ}(t_o) = 0, \; \dot{x}^{\circ}(t) \equiv 1 \tag{15}$$

and then letting

$$\xi(t) = \begin{bmatrix} x^{\circ} \\ x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{bmatrix}, \tag{16}$$

where

$$x = \begin{bmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{bmatrix}.$$

Thus, for each admissible control, $u(t)$, the value of the zero-th component of $\xi(t_u)$ gives the cost of control. $\xi(t)$ satisfies the differential equation

$$\dot{\xi}(t) = \phi(\xi, u, t) = \begin{bmatrix} 1 \\ f^1(x, u, t) \\ \cdot \\ \cdot \\ \cdot \\ f^n(x, u, t) \end{bmatrix}, \tag{17}$$

where

$$f(x, u, t) = \begin{bmatrix} f^1(x, u, t) \\ \cdot \\ \cdot \\ \cdot \\ f^n(x, u, t) \end{bmatrix}.$$

The same notational conventions will apply to $\xi(t)$ as to $x(t)$; i.e., $\hat{u}(t)$ or $u^*(t)$, etc., will correspond to $\hat{\xi}(t)$ or $\xi^*(t)$, etc., respectively. The constraints set, G, will be replaced by $\Gamma = R^1 \otimes G$.

Given a point, $\xi \in \partial\Gamma$ (boundary of Γ), let $\eta(\xi)$ denote a unit vector pointing into Ext (Γ) such that:

(i) If Γ has a unique exterior normal at ξ, then $\eta(\xi)$ is that normal;

(ii) If Γ does not possess a unique exterior normal at ξ, then $\eta(\xi)$ is normal to some supporting hyperplane to Γ at ξ, which exists, since Γ is convex.

Given an admissible control, $u(t) \in U$, it will be assumed that the interval $[t_o, t_u]$ can be broken up into a finite set of closed subintervals, $I_k = [t_{k-1}, t_k]$, $k = 1, 2, \ldots, r$, where r is necessarily odd, such that $t_o \leq t_1 \leq t_2 \leq \ldots \leq t_r = t_u$ and :

(i) If k is odd, the $\xi(t)$ corresponding to $u(t)$ lies entirely in Int(Γ) for $t \in \mathrm{Int}\left(\left[t_{k-1}, t_k\right]\right)$;

(ii) If k is even, $\xi(t)$ lies on $\partial\Gamma$ for $t \in \left[t_{k-1}, t_k\right]$.

Note that if k is even, one may have $t_{k-1} = t_k$; but, this does not happen if k is odd. The times, t_k, $k \neq 0, r$, are called junction times for the solution, $\xi(t)$, and the points, $\xi(t_k)$, are called junction points.

Let us suppose the matrix, A(t), in Equation (14) has entries $a_{ij}(t)$, $i, j = 1, 2, \ldots, n$, and B(t) has entries $b_{ij}(t)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. The matrices, $A^*(t)$ and $B^*(t)$, have entries $a^*_{ij}(t)$, $i, j = 0, 1 \ldots, n$, and $b^*_{ij}(t)$, $i = 0, 1 \ldots, n$, $j = 0, 1 \ldots, m$, respectively, which are defined as follows:

$$a^*_{ij}(t) = \begin{cases} 0 \text{ if } i = 0 \text{ or } j = 0, \\ a_{ij}(t) \text{ otherwise;} \end{cases}$$

$$b^*_{ij}(t) = \begin{cases} 0 \text{ if } i = 0, \\ b_{ij}(t) \text{ otherwise.} \end{cases}$$

Consequently, Equation (17) may be written as

$$\dot{\xi} = A^*(t)\,\xi + B^*(t)u + e_o, \tag{18}$$

where

$$e_o = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Let $\psi(t)$ be a covariant (n+1)-dimensional vector function which is a non-trivial solution of the so-called "adjoint system"

$$\dot{\psi} = -\psi\, A^*(t). \tag{19}$$

If, for all such $\psi(t)$, the equation

$$\psi(t)B^*(t)u(t) = \max_{u \in \Omega} \left\{ \psi(t)B^*(t)u \right\}$$

has a unique solution, $u(t) \in \Omega$, for almost all t in the domain of $\psi(t)$, the systems, represented by Equations (14) and (18), will be called normal systems.

3.4 A SUFFICIENT CONDITION FOR TIME-OPTIMAL BOUNDED PHASE-COORDINATE CONTROL

Theorem 1.

Let $\hat{u}(t)$ be a member of U defined on $[t_o, t_u]$. Let $\psi(t)$ be a covariant vector function defined and continuous on $[t_o, t_{\hat{u}}]$, with the possible exception of the points $t_1, t_2, \ldots, t_{r-1}$. Let $\nu_1, \nu_2, \ldots, \nu_{r-1}$ be non-negative real numbers.

Let $\xi(t)$ be a non-negative measurable function defined on $[t_o, t_{\hat{u}}]$. Assume that

(i) $\left\| \psi(t_{\hat{u}}) \right\| = 1;$

(ii) $\psi_o(t)$ is a constant, $\psi_o < 0$,
where $\psi(t) = [\psi_o(t), \psi_1(t), \ldots, \psi_n(t)]$; (The form of $A^*(t)$ guarantees that $\psi_o(t)$ is constant.)

(iii) If k is odd, then for almost all $t \in (t_{k-1}, t_k)$,

$$\dot{\psi} = -\psi A^*(t); \tag{20}$$

(iv) If k is even, then for almost all $t \in (t_{k-1}, t_k)$,

$$\dot{\psi} = -\psi A^*(t) + \zeta(t)\eta[\hat{\xi}(t)]; \tag{21}$$

(v) For $k = 1, 2, \ldots, r-1$,

$$\psi(t_k + 0) - \psi(t_k - 0) = \nu_k \eta[\hat{\xi}(t_k)] . \tag{22}$$

Let $H(u, t)$ be defined for $u \in \Omega$ and each t in $[t_o, t_{\hat{u}}]$, except possibly for t_1, $t_2, \ldots, t_{r-1}$ by

$$H(u, t) = \psi(t) B^*(t) u. \tag{23}$$

Suppose that for almost all t in $[t_o, t_{\hat{u}}]$,

$$H[\hat{u}(t), t] = \max_{u \in \Omega} H(u, t).$$

Assume that for almost all $t \in [t_o, t_{\hat{u}}]$, there exists a $u^+(t) \in \Omega$ such that

$$A(t)x_1 + B(t)u^+(t) = 0 \tag{25}$$

and $u^+(t)$ is measurable on $[t_o, t_{\hat{u}}]$.

Then, $\hat{u}(t)$ is an optimal controller for the problem, P.

Proof:

Suppose for the sake of contradiction there exists another controller, $u^*(t) \in U$, such that $t_{u^*} < t_{\hat{u}}$. One may extend the definition of $u^*(t)$ by letting $u^*(t)$ equal $u^+(t)$ almost everywhere on the interval $\left[t_{u^*}, t_{\hat{u}}\right]$. This choice of $u^*(t)$ maintains $x^*(t)$ at x_1 during this interval because of Equation (25).

Consider the expression $\psi(t_{\hat{u}}) \left[\hat{\xi}(t_{\hat{u}}) - \xi^*(t_{\hat{u}})\right] - \psi(t_o) \left[\hat{\xi}(t_o) - \xi^*(t_o)\right]$. Since $\hat{x}(t_o) = x^*(t_o) = x_o$, and $\hat{x}(t_{\hat{u}}) = x^*(t_{\hat{u}}) = x_1$, and the component $\psi_o(t)$ of $\psi(t)$ is a constant,

$$\psi(t_{\hat{u}})\left[\hat{\xi}(t_{\hat{u}}) - \xi^*(t_{\hat{u}})\right] - \psi(t_o)\left[\hat{\xi}(t_o) - \xi^*(t_o)\right] = \psi_0(t_{\hat{u}} - 0) - \psi_o(t_{\hat{u}} - 0) = 0. \tag{26}$$

With the possible exceptions of the times, t_k, $k = 1, 2, \ldots, r-1$, the function $\psi(t)\left[\hat{\xi}(t) - \xi^*(t)\right]$ is absolutely continuous on $\left[t_o, t_{\hat{u}}\right]$.
Therefore,

$$\psi(t_{\hat{u}})\left[\hat{\xi}(t_{\hat{u}}) - \xi^*(t_{\hat{u}})\right] - \psi(t_o)\left[\hat{\xi}(t_o) - \xi^*(t_o)\right] = \sum_{k=1}^{r-1}\left[\psi(t_k+0) - \psi(t_k - 0)\right]\left[\hat{\xi}(t_k) - \xi^*(t_k)\right] + \sum_{k=1}^{r}\int_{t_{k-1}}^{t_k} \frac{d}{dt}\left\{\psi(t)\left[\hat{\xi}(t) - \xi^*(t)\right]\right\} dt = 0. \tag{27}$$

Taking note of Equations (20), (21) and (22), Equation (27) yields:

$$
\begin{aligned}
&\sum_{k=1}^{r-1} \nu_k \, \eta \left[\hat{\xi}(t_k)\right]\left[\hat{\xi}(t_k) - \xi^*(t_k)\right] \\
&+ \sum_{\ell=1}^{\frac{r-1}{2}} \int_{t_{2\ell-1}}^{t_{2\ell}} \left[\left\{-\psi(t)A^*(t) + \zeta(t)\eta\left[\hat{\xi}(t)\right]\right\} \left[\hat{\xi}(t) - \xi^*(t)\right]\right. \\
&+ \left.\psi(t)\left\{A^*(t)\left[\hat{\xi}(t) - \xi^*(t)\right] + B^*(t)\left[\hat{u}(t) - u^*(t)\right]\right\}\right] dt \qquad (28) \\
&+ \sum_{\ell=0}^{\frac{r-1}{2}} \int_{t_{2\ell}}^{t_{2\ell+1}} \left[-\psi(t)A^*(t)\left[\hat{\xi}(t) - \xi^*(t)\right]\right. \\
&+ \left.\psi(t)\left\{A^*(t)\left[\hat{\xi}(t) - \xi^*(t)\right] + B^*(t)\left[\hat{u}(t) - u^*(t)\right]\right\}\right] dt \\
&= 0,
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
&\sum_{k=1}^{r-1} \nu_k \, \eta \left[\hat{\xi}(t_k)\right]\left[\hat{\xi}(t_k) - \xi^*(t_k)\right] \\
&+ \sum_{\ell=1}^{\frac{r-1}{2}} \int_{t_{2\ell-1}}^{t_{2\ell}} \left\{\zeta(t)\eta\left[\hat{\xi}(t)\right]\left[\hat{\xi}(t) - \xi^*(t)\right] + \psi(t)B^*(t)\left[\hat{u}(t)-u^*(t)\right]\right\} dt \qquad (29) \\
&+ \sum_{\ell=0}^{\frac{r-1}{2}} \int_{t_{2\ell}}^{t_{2\ell+1}} \psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] dt = 0.
\end{aligned}
$$

Now, the fact that Γ is convex, together with the non-negativeness of ν_k, shows that for $k = 1, 2, \ldots, r-1$,

$$\nu_k \; \eta\left[\hat{\xi}(t)\right]\left[\hat{\xi}(t) - \xi^*(t)\right] \geq 0. \tag{30}$$

The fact that Γ is convex, together with the non-negativeness of $\xi(t)$, shows that for $t \in \left[t_{2\ell-1}, t_{2\ell}\right]$, $\ell = 1, 2, \ldots, \frac{r-1}{2}$,

$$\zeta(t)\, \eta\left[\hat{\xi}(t)\right]\left[\hat{\xi}(t) - \xi^*(t)\right] \geq 0. \tag{31}$$

By hypothesis on $\hat{u}(t)$,

$$\psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] \geq 0, \; t \in \left[t_o, t_{\hat{u}}\right] \tag{32}$$

Therefore, in order that (29) should hold, it is necessary that expressions (30), (31), and (32) should be identically zero wherever defined. Let $\left[t^+, t_{\hat{u}}\right]$ denote the intersection of $I_r = \left[t_{r-1}, t_r\right]$ with $\left[t_{u^*}, t_{\hat{u}}\right]$. The interval, $\left[t^+, t_{\hat{u}}\right]$, is easily seen to have non-zero length. On this interval,

$$\psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] = 0 \tag{33}$$

or

$$\psi(t)B^*(t)\; u^*(t) = \psi(t)B^*(t)\;\hat{u}(t) = \max_{u \in \Omega} H(u, t) \tag{34}$$

almost everywhere.

But, then, the normality of the system implies that, almost everywhere on $\left[t^+, t_{\hat{u}}\right]$,

$$\hat{u}(t) = u^*(t). \tag{35}$$

But then, since $\hat{x}(t_{\hat{u}}) = x_1$, and Equation (25) is true, it must be true that

$$\hat{x}(t^+) = x_1 \tag{36}$$

and this contradicts the definition of $t_{\hat{u}}$, since $t^+ < t_{\hat{u}}$.

Thus, $\hat{u}(t)$ must, indeed, be an optimal controller for the problem, P. This completes the proof of Theorem 1.

It will be shown next that, under certain assumptions which amount to a modified normality conditions, a time-optimal bounded phase trajectory, $\hat{x}(t)$ joining x_o to x_1, and its associated control function, $\hat{u}(t)$, are unique, provided they also satisfy the hypotheses of Theorem 1. Some of the results are for time varying systems and some for autonomous systems.

3.5 FURTHER ASSUMPTIONS AND NOTATION

It will be assumed now that an interval, $[t_{k-1}, t_k]$, (k necessarily even), during which $\hat{x}(t)$ lies on the boundary of G, is the union of finitely many subintervals which may be divided into two classes: subintervals during which $\zeta(t) > 0$, and subintervals during which $\zeta(t) \equiv 0$. On a subinterval during which $\zeta(t) \equiv 0$, Equation (21) and Equation (20) are identical, so the covariant vector, $\psi(t)$, will obey Equation (20) - the same equation that $\psi(t)$ satisfies on intervals of time during which $\hat{x}(t)$ lies in the interior of G.

The intervals, $I_k = [t_{k-1}, t_k]$, were defined for a given interval, $[t_o, t_{\hat{u}}]$, and path, $\hat{x}(t)$. It will now be assumed that there is a refinement of this subdivision such that $[t_o, t_{\hat{u}}]$ is thereby divided into subintervals, $J_k = [\tau_{k-1}, \tau_k]$, $k = 1, 2, \ldots, s$, where s is necessarily odd, such that:

(i) If $t \epsilon (\tau_{k-1}, \tau_k)$ and k is odd, then $\psi(t)$ obeys Equation (20);

(ii) If $t \epsilon (\tau_{k-1}, \tau_k)$ and k is even, then $\psi(t)$ obeys Equation (21) and $\zeta(t) > 0$.

It should be noted that each interval, I_k, (k odd), is an interval, $J_{k'}$, (k' odd), while each interval, J_k, (k even), is a subinterval of some $I_{k'}$, (k' even).

3.6 UNIQUENESS OF TIME-OPTIMAL BOUNDED PHASE-COORDINATE CONTROL FOR STRICTLY CONVEX CONSTRAINT SETS

Theorem 2.

Let the control, $\hat{u}(t) \epsilon U$, the augmented path variable, $\hat{\xi}(t)$, and the covariant vector, $\psi(t)$, satisfy all of the hypotheses of Theorem 1. Let it be assumed in addition that the constraint set, G, is strictly convex and that the rank of the n by m matrix, B(t), is almost everywhere equal to m. Then, $\hat{u}(t)$ and $\hat{x}(t)$ are the unique time-optimal bounded phase-coordinate control-and-path joining x_o to x_1, with $x(t_o) = x_o$.

Proof:

Let $u^*(t)$ and $x^*(t)$ be any bounded phase-coordinate control and corresponding path such that $x^*(t_o) = x_o$, $x^*(t_{\hat{u}}) = x_1$. From Equation (29) in the proof of Theorem 1, and from the assumptions introduced above, it is seen that

$$\sum_{k=1}^{r-1} \nu_k \eta\left[\hat{\xi}(t_k)\right]\left[\hat{\xi}(t_k) - \xi^*(t_k)\right] + \sum_{\ell=1}^{\frac{s-1}{2}} \int_{\tau_{2\ell-1}}^{\tau_{2\ell}}$$

$$\left\{ \zeta(t)\eta\left[\hat{\xi}(t)\right]\left[\hat{\xi}(t) - \xi^*(t)\right] + \psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] \right\} dt$$

$$+ \sum_{\ell=0}^{\frac{s-1}{2}} \int_{\tau_{2\ell}}^{\tau_{2\ell+1}} \psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] dt = 0. \tag{37}$$

As in the proof of Theorem 1, (30), (31), and (32) may be derived, with (31) holding for $t \in \left[\tau_{2\ell-1}, \tau_{2\ell}\right]$, $\ell = 1, 2, \ldots, \frac{s-1}{2}$. Also, as in the proof of Theorem 1, the equality sign must hold for (30), (31), and (32), with the new domain for (31). From the normality hypothesis, the equation, $\psi(t)B^*(t)\hat{u}(t) = \max_{u \in \Omega} \left\{ \psi(t)B^*(t)u \right\}$, has $\hat{u}(t)$ as its unique solution almost everywhere on an interval, (τ_{k-1}, τ_k), (k odd), $u^*(t) = \hat{u}(t)$ almost everywhere on such intervals. It should be clear then that the theorem will have been proved when the following result has been obtained: if k is even, and $x^*(\tau_{k-1}) = \hat{x}(\tau_{k-1})$, then $x^*(t) = \hat{x}(t)$ and $u^*(t) = \hat{u}(t)$ almost everywhere for t in the interval $\left[\tau_{k-1}, \tau_k\right]$.

To prove this, note that at any time instant, $t \in (\tau_{k-1}, \tau_k)$, $\eta\left[\hat{\xi}(t)\right]$ is the (n + 1)-dimensional exterior normal (or normal to a support plane) to Γ at $\hat{\xi}(t)$. Let $\eta\left[\hat{x}(t)\right]$ denote the n-dimensional exterior normal (or normal to a support plane) to G at the corresponding point, $\hat{x}(t)$. Then, since the zero-th component of $\eta\left[\hat{\xi}(t)\right]$ is zero,

$$\zeta(t)\eta\left[\hat{\xi}(t)\right]\left[\hat{\xi}(t) - \xi^*(t)\right] = \zeta(t)\eta\left[\hat{x}(t)\right]\left[\hat{x}(t) - x^*(t)\right]. \tag{38}$$

If Equation (37) is to be true, then

$$\zeta(t)\eta\left[\hat{x}(t)\right]\left[\hat{x}(t) - x^*(t)\right] = 0, \; t \in \left[\tau_{k-1}, \tau_k\right], \tag{39}$$

for K even. The strict convexity of G then implies $x^*(t) \equiv \hat{x}(t)$, $t \in \left[\tau_{k-1}, \tau_k\right]$. Since both $x^*(t)$ and $\hat{x}(t)$ satisfy Equation (14), it must be true that

$$B(t)u^*(t) = B(t)\hat{u}(t) \tag{40}$$

for almost all $t \in \left[\tau_{k-1}, \tau_k\right]$. Then, since the rank of B(t) is almost everywhere equal to m, $u^*(t) = \hat{u}(t)$ almost everywhere in $\left[\tau_{k-1}, \tau_k\right]$. Thus the proof of Theorem 2 is complete.

In the above theorem it would clearly be adequate to assume that $\zeta(t) > 0$ almost everywhere in $\left[\tau_{k-1}, \tau_k\right]$ when k is even.

This result has rather restricted application because, unless for some reason it is spherical or elliptical, the constraint set is usually a region bounded by hyperplanes and hence will not be strictly convex. Thus, another result is needed which will give uniqueness in such cases. A general result for measurable A(t) and B(t) and (not strictly) convex constraint sets appears very difficult at present. The results presented below will be valid for a more restricted class of problems.

3.7 MORE ASSUMPTIONS AND NOTATION

It will henceforth be assumed that A(t) is a bounded measurable matrix function of t, while $B(t) \equiv B$ is a constant matrix of rank, m. The constraint set, G, will be a closed convex set bounded by finitely many (n-1)-dimensional hyperplanes, $H_1, H_2, \ldots, H_p$. In other words, letting v_k, $k = 1, 2, \ldots, p$, be a unit vector perpendicular to H_k, there are real numbers, c_k, $k = 1, 2, \ldots, p$, such that

$$G = \left\{ x \;\middle|\; x \cdot v_k \le c_k,\ k = 1, 2, \dots, p \right\}. \tag{41}$$

Thus, for points, $x \in \partial G$, which lie in the interior of an $(n-1)$-dimensional face of G, which is part of the hyperplane, H_k, $\eta(x) = v_k'$. Finally, it will be assumed that the control restraint set, Ω, is a polyhedron in E^m with non-empty interior. Here the prime denotes the transpose.

3.8 UNIQUENESS OF TIME-OPTIMAL BOUNDED PHASE-COORDINATE CONTROL FOR CONSTRAINT SETS WHICH ARE POLYHEDRA

It is evident from the proof of Theorem 2 that it will be sufficient to consider the following problem: Assume that $\left[\tau_{k-1}, \tau_k\right]$, (k even), is an interval during which $\hat{x}(t)$ lies on the boundary of G, and that $\zeta(t) > 0$ for $t \in (\tau_{k-1}, \tau_k)$. Show that if $x^*(\tau_{k-1}) = \hat{x}(\tau_{k-1})$, then $x^*(t) \equiv \hat{x}(t)$ and $u^*(t) = \hat{u}(t)$ almost everywhere on $\left[\tau_{k-1}, \tau_k\right]$.

Equation (39) is still valid but, since G is not strictly convex, it cannot immediately be concluded that $x^*(t) \equiv \hat{x}(t)$ for $t \in \left[\tau_{k-1}, \tau_k\right]$. Let us consider some fixed time, $t \in \left[\tau_{k-1}, \tau_k\right]$. Then, $\hat{x}(t) \in \partial G$. If $\hat{x}(t)$ lies in the interior of an $(n-1)$-dimensional face of G, which is part of some hyperplane, H_k, then certainly, $\eta\left[\hat{x}(t)\right] = v_k'$. Equation (39) then implies that $x^*(t)$ also lies in H_k, and, consequently, on the same face of G. Now suppose, instead, that $\hat{x}(t)$ lies in an $(n-r)$-dimensional face of G, which is part of $H_{k_1} \cap H_{k_2} \cap \dots \cap H_{k_r}$. Then, $\eta\left[\hat{x}(t)\right] = \alpha_1 v_{k_1} + \alpha_2 v_{k_2} + \dots + \alpha_r v_{k_r}$, where $\alpha_1, \alpha_2, \dots \alpha_r$ are non-negative real numbers. Let us suppose the ordering is such that $\alpha_1, \alpha_2, \dots \alpha_s$, $s \le r$, are the non-zero coefficients. Then it is easy to see that Equation (39) implies $x^*(t) \in H_{k_1} \cap H_{k_2} \cap \dots \cap H_{k_s}$. Since both $\hat{x}(t)$ and $x^*(t)$ are continuous on $\left[\tau_{k-1}, \tau_k\right]$, the interval, $\left[\tau_{k-1}, \tau_k\right]$, is composed of subintervals on each of which both $\hat{x}(t)$ and $x^*(t)$ lie in some intersection, $H_{k_1} \cap H_{k_2} \cap \dots \cap H_{k_s}$.

Instead of presenting a complete analysis of this situation, we will consider a simplified situation in which $[\tau_{k-1}, \tau_k]$ is composed of finitely many subintervals in each of which only one of the coefficients, α_i, is non-zero and, hence, $\hat{x}(t)$ and $x^*(t)$ both belong to some hyperplane, H_{k_i}. But then, it is clearly sufficient to consider the case where $\hat{x}(t)$ and $x^*(t)$ both lie in a fixed hyperplane, H_1, on $[\tau_{k-1}, \tau_k]$, and $\eta[\hat{x}(t)] \equiv v_1'$ on $[\tau_{k-1}, \tau_k]$. We remark that the results in the general case referred to above are not essentially different from those we present here.

Using, if necessary, a non-singular change of coordinates, it may be assumed that v_1 is given by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \\ \vdots \\ 0 \end{bmatrix} = e_1 \tag{42}$$

and $c_1 = 1$. Thus, for all $t \in [\tau_{k-1}, \tau_k]$,

$$\eta[\hat{x}(t)]' = e_1, \tag{43}$$

where the prime denotes the transpose.

(This choice of $\eta[\hat{x}(t)]$ will be made even if $\hat{x}(t)$ lies on a face of G of dimension less than $n-1$.)

Conditions must be found under which the equation,

$$\psi(t)B^*(t)\left[\hat{u}(t) - u^*(t)\right] = 0, \tag{44}$$

already known to be valid almost everywhere in $[\tau_{k-1}, \tau_k]$, implies that $u^*(t) = \hat{u}(t)$ almost everywhere there, and hence $x^*(t) \equiv \hat{x}(t)$ there.

Let x be a point of G lying in H_1 and let $x(t)$ be, for some $u(t)$, a solution of Equation (14) passing through $x = x(\tau)$ at time, τ, such that $x(t)$ lies in H_1 on an interval, $[\tau, \tau + \delta]$. Then, it must be true that

$$\frac{d}{dt}\left[x(t) \cdot e_1\right]\Big|_{t=\tau} = 0. \tag{45}$$

If $\dot{x}(t)$ exists, then

$$\frac{d}{dt}\left[x(t) \cdot e_1\right]\Big|_{t=\tau} = \dot{x}(\tau) \cdot e_1 = 0. \tag{46}$$

But then,

$$\dot{x}(\tau) \cdot e_1 = \sum_{k=1}^{n} a_{1k}(\tau)\, x^k(\tau) + \sum_{k=1}^{m} b_{1k} u^k(\tau) \equiv$$

$$\alpha_1(\tau) \cdot x(\tau) + \beta_1 \cdot u(\tau) = 0. \tag{47}$$

For a given $x(\tau)$, therefore, the set of admissible values of $u(\tau)$ consists of the set

$$\Omega\left[x(\tau), \tau\right] = \left\{ u \mid u \in \Omega \text{ and } \beta_1 \cdot u = -\alpha_1(\tau) \cdot x(\tau) \right\} \tag{48}$$

which is the intersection with Ω of an (n-1)-dimensional hyperplane perpendicular to the fixed vector, β_1. Thus, $\Omega\left[x(\tau), \tau\right]$ is a convex set of dimension $\leq m-1$ which, in general, varies with $x(\tau)$ and τ.

Definition 1.

$\Omega(x, \tau) = \left\{ u \in \Omega \mid \beta_1 \cdot u = -\alpha_1(\tau) \cdot x \right\}$; $\Omega_c = \left\{ u \in \Omega \mid \beta_1 \cdot u = c, \text{ c real} \right\}$. Obviously, each $\Omega(x, \tau)$ corresponds to one and only one Ω_c, but a fixed Ω_c may correspond to none, one, or more than one $\Omega(x, \tau)$.

The following statement is easy to prove: There exist a set, W, of vectors of unit norm containing but finitely many members such that for all Ω_c the one-dimensional faces of Ω_c are parallel to members w of W.

Definition 2 (The Modified Normality Condition).

There are no subsets of $[\tau_{k-1}, \tau_k]$ with positive measure during which $\psi(t)B^*$ is perpendicular to any $w \in W$.

Assuming that the modified normality condition holds, the uniqueness of $\hat{x}(t)$ and $\hat{u}(t)$ can be shown; i.e., it can be shown that $u^*(t) = \hat{u}(t)$ almost everywhere on $[\tau_{k-1}, \tau_k]$.

From Equation (44) and the hypotheses of Theorem 1, it is known that both $u^*(t)$ and $\hat{u}(t)$ maximize $\psi(t)B^*u$ almost everywhere on $[\tau_{k-1}, \tau_k]$ for $u \in \Omega$. Therefore, if $u^*(t) \neq \hat{u}(t)$, the convexity of Ω implies that the line segment from $u^*(t)$ to $\hat{u}(t)$ almost always lies on the boundary of Ω, and the vector, $\psi(t)B^*$, is perpendicular to this line segment. Now, the line segment from $u^*(t)$ to $\hat{u}$ is either part of a one-dimensional face of Ω or else is such that its interior lies in the interior of a face of Ω of dimension higher than one. In the latter case, $\psi(t)B^*$ must be perpendicular to this entire higher dimensional face, for otherwise, $\psi(t)B^*\hat{u}$ would not be maximal. But this higher dimensional face of Ω contains a one-dimensional face of some Ω_c, and thus, $\psi(t)B^*$ is perpendicular to some $w \in W$.

Therefore, the modified normality condition makes it necessary that the line segment from $u^*(t)$ to $\hat{u}(t)$ lie in a one-dimensional face of Ω at almost all points of $[\tau_{k-1}, \tau_k]$. It will be assumed here that $[\tau_{k-1}, \tau_k]$ can be subdivided into finitely many subintervals during which this line segment lies in a fixed one-dimensional face of Ω. But then, for the proof, it may as well be assumed that this line segment lies in a fixed one-dimensional face of Ω on all of $[\tau_{k-1}, \tau_k]$.

This one-dimensional face of Ω cannot be perpendicular to the vector, β_1, for this would imply that it is a one-dimensional face of some Ω_c and hence parallel to some $w \in W$, and then one would have $\psi(t)B^* w \equiv 0$ on $\left[\tau_{k-1}, \tau_k\right]$. But, $\beta_1' = e_1' B = \eta\left[\hat{x}(t)\right]B$, and hence,

$$\left| \eta\left[\hat{x}(t)\right] B\left[\hat{u}(t) - u^*(t)\right] \right| \geq M_o \left\| \hat{u}(t) - u^*(t) \right\| \tag{49}$$

for $t \in \left[\tau_{k-1}, \tau_k\right]$, where M_o is a fixed positive number.

Now, for $t \in \left[\tau_{k-1}, \tau_k\right]$,

$$\eta\left[\hat{x}(t)\right]\left[x^*(t) - \hat{x}(t)\right] = 0$$

so that almost everywhere, by differentiation,

$$\eta\left[\hat{x}(t)\right]\left[\dot{x}^*(t) - \dot{\hat{x}}(t)\right] = 0$$

or, using Equation (14),

$$\eta\left[\hat{x}(t)\right]\left[A(t)X^*(t) + B\,u^*(t) - A(t)\,\hat{x}(t) - B\hat{u}(t)\right] = 0, \text{ i.e.,}$$

$$\eta\left[\hat{x}(t)\right] A(t)\left[x^*(t) - \hat{x}(t)\right] = \eta\left[\hat{x}(t)\right] B\left[u^*(t) - \hat{u}(t)\right]$$

which, using (49), implies

$$\left| \eta\left[\hat{x}(t)\right] A(t) \left[x^*(t) - \hat{x}(t)\right] \right| \geq M_o \left\| \hat{u}(t) - u^*(t) \right\| \tag{50}$$

for almost all $t \in \left[\tau_{k-1}, \tau\right]$.

Now, since the matrix, $A(t)$, is bounded,

$$\left| \eta\left[\hat{x}(t)\right] A(t) \left[x^*(t) - \hat{x}(t)\right] \right| \leq M_1 \left\| x^*(t) - \hat{x}(t) \right\|,$$

where M_1 is a fixed positive number. Thus,

$$||x^*(t) - \hat{x}(t)|| \geq M\, ||u^*(t) - \hat{u}(t)||, \tag{51}$$

where M is a positive number. Assume that

$$||u^*(t) - \hat{u}(t)|| < N(t - \tau_{k-1})^q \tag{52}$$

on the interval, $\left[\tau_{k-1}, \tau_k\right]$, where N is a positive real number and q is a non-negative integer. Since $||u^*(t) - \hat{u}(t)||$ is bounded (Ω is compact), this inequality certainly holds for some N when q= 0. Using the variation of parameters formula, and the fact that $x^*(\tau_{k-1}) = \hat{x}(\tau_{k-1})$,

$$\begin{aligned} x^*(t) - \hat{x}(t) &= \exp\left[\int_{\tau_{k-1}}^{t} A(s)ds\right] \int_{\tau_{k-1}}^{t} \exp\left[-\int_{\tau_{k-1}}^{s} A(s)ds\right] B\left[u^*(s)-\hat{u}(s)\right]ds \\ &\leq K \int_{\tau_{k-1}}^{t} ||u^*(s) - \hat{u}(s)||\, ds. \end{aligned} \tag{53}$$

But then, from (52), it is seen that

$$||x^*(t) - \hat{x}(t)|| \leq K\,N \int_{\tau_{k-1}}^{t} (s - \tau_{k-1})^q\, ds = K\,N \frac{(t - \tau_{k-1})^{q+1}}{q + 1}.$$

Then, (51) proves that

$$||u^*(t) - \hat{u}(t)|| \leq \frac{KN}{M} \frac{(t - \tau_{k-1})^{q+1}}{q + 1}.$$

Since this analysis is true for q = 1, mathematical induction shows that

$$||u^*(t) - \hat{u}(t)|| \leq \left[\frac{K}{M}\right]^q N \frac{(t - \tau_{k-1})^q}{q\,!} \tag{54}$$

for any positive integer, q, for each $t \in \left[\tau_{k-1}, \tau_k\right]$. But, the expression on the right hand side of (54) is the q-th term in the Taylor series for

$N \exp\left[\frac{K}{M}(t - \tau_{k-1})\right]$ and hence approaches zero as $q \to \infty$. Thus, the only possible conclusion is that

$$u^*(t) = \hat{u}(t)$$

for almost all $t \in \left[\tau_{k-1}, \tau_k\right]$. Therefore, the following theorem is proved.

Theorem 3.

Let it be required in the statement of the optimization problem that A(t) be bounded and measurable and $B(t) \equiv B$ is constant. Let $\hat{u}(t)$ be an optimal control for the problem, P, and $\hat{x}(t)$ the corresponding trajectory and assume that $\hat{u}(t)$ and $\hat{\xi}(t)$ satisfy the hypotheses of Theorem 1. Let the control restraint set, Ω, be a convex polyhedron in E^m and let the constraint set, G, be given by

$$G = \left\{ x \mid x \cdot v_k \le c_k, \cdot k = 1, 2, \ldots, p \right\}.$$

Let W_k denote the finite set of unit vectors parallel to one-dimensional faces of sets

$$\Omega_k(c) = \left\{ u \mid v_k' B u = c \right\}.$$

Suppose that there is no interval during which $\hat{x}(t) \cdot v_k = c_k$ and $\psi(t)B^* w_k = 0$ for some member w_k of W_k. Then, $\hat{u}(t)$ and $\hat{x}(t)$ are the unique solution of the bounded phase-coordinate optimization problem with initial point, x_o, initial time, t_o, and final point, x_1.

3.9 THE MODIFIED NORMALITY CONDITION FOR AUTONOMOUS SYSTEMS

It will now be shown that if both $A(t) \equiv A$ and $B(t) \equiv B$ are constant matrices, then the test for the validity of the modified normality condition is essentially the same as the test for the validity of the normality condition for the problem of time-optimal control without phase constraints.

It will be shown later that, in this case, the function, $\zeta(t)$, is piecewise analytic. Now, suppose there is an interval, (τ_0, τ_1), during which $\hat{x}(t)$ lies on the (n-1)-dimensional hyperplane, $x \cdot e_1 = 1$, and during which $\zeta(t)$ is analytic and for some $w \in W$, $\psi(t)B^* w = 0$, i.e., in the inner product notation

$$\left[\psi(t)', \; B^* w\right] \equiv 0, \tag{55}$$

where the prime denotes the transpose.

Differentiating yields

$$\left\{ -\psi(t)A^* + \zeta(t) \; \eta\left[\hat{\xi}(t)\right] \right\} B^* w \equiv 0. \tag{56}$$

But, as already known from the definition of W,

$$\eta\left[\hat{\xi}(t)\right] B^* \; w \equiv 0 \tag{57}$$

whence

$$\left[-\psi(t)A^*\right] B^* w \equiv 0 \Rightarrow \psi(t) \; A^* B^* \; w \equiv 0. \tag{58}$$

Continuing, it is seen that

$$\psi(t) \left[A^*\right]^k B^* w \equiv 0, \; k = 0, 1, 2, \ldots, n-1, \tag{59}$$

whence, letting $\hat{\psi}(t)$ denote the last n components of $\psi(t)$,

$$\hat{\psi}(t) \; A^k \; B \; w \equiv 0, \; k = 0, 1, 2, \ldots, n-1. \tag{60}$$

Then, if B w, A B w, ..., A^{n-1} Bw are linearly independent, and $\hat{\psi}(t) \not\equiv 0$, a contradiction is obtained. This should be compared with the corresponding test in Chapter III of Reference 4.

We will conclude this chapter with an example and some remarks regarding practical application of these results. All notational conventions so far introduced will continue to be used with the exception that the covariant vector, $\psi(t)$, will be an n-dimensional solution of

$$\dot{\psi}(t) = -\psi(t)A(t)$$

or

$$\dot{\psi}(t) = -\psi(t)A(t) + \zeta(t)\, \eta\left[\hat{x}(t)\right]$$

rather than a solution of the augmented equation used earlier.

3.10 EXAMPLE: THE LINEAR HARMONIC OSCILLATOR (LHO)

To illustrate the use of Theorem 1, consider the controlled LHO,

$$\ddot{x} + x = u,$$

where the variables, x and u, are scalars, with u subject to the control restraint

$$|u| \leq 1 \tag{61}$$

and with the phase constraint

$$\left|\frac{dx}{dt}\right| \leq \frac{1}{2}. \tag{62}$$

Letting $x(t) = \begin{bmatrix} x_1 & (t) \\ x_2 & (t) \end{bmatrix}$ now be a two-dimensional vector, the second-order system equivalent to LHO is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{63}$$

which obviously has the form $\dot{x} = Ax + Bu$. The phase constraint (62) then becomes

$$|x_2| \le \frac{1}{2} . \tag{64}$$

The adjoint system is

$$(\dot{\psi}_1, \dot{\psi}_2) = -(\psi_1, \psi_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ or } \dot{\psi} = -\psi A, \tag{65}$$

when $|x_2(t)| \le \frac{1}{2}$.

Time-optimal solutions of Equation (63) will be constructed in that portion of $G = \left\{ (x_1, x_2) \,\middle|\, |x_2| \le \frac{1}{2} \right\}$ (See Figure 1) which lies in the left half plane by a "backing out" procedure.

The general solution of Equation (65) is a vector $\psi(t)$ of the form

$$\psi(t) = (C_1 \cos t + C_2 \sin t, \ -C_1 \sin t + C_2 \cos t). \tag{66}$$

Thus, for the unconstrained time-optimal control problem, the Maximum Principle shows that

$$u(t) = \text{sign } (-C_1 \sin t + C_2 \cos t). \tag{67}$$

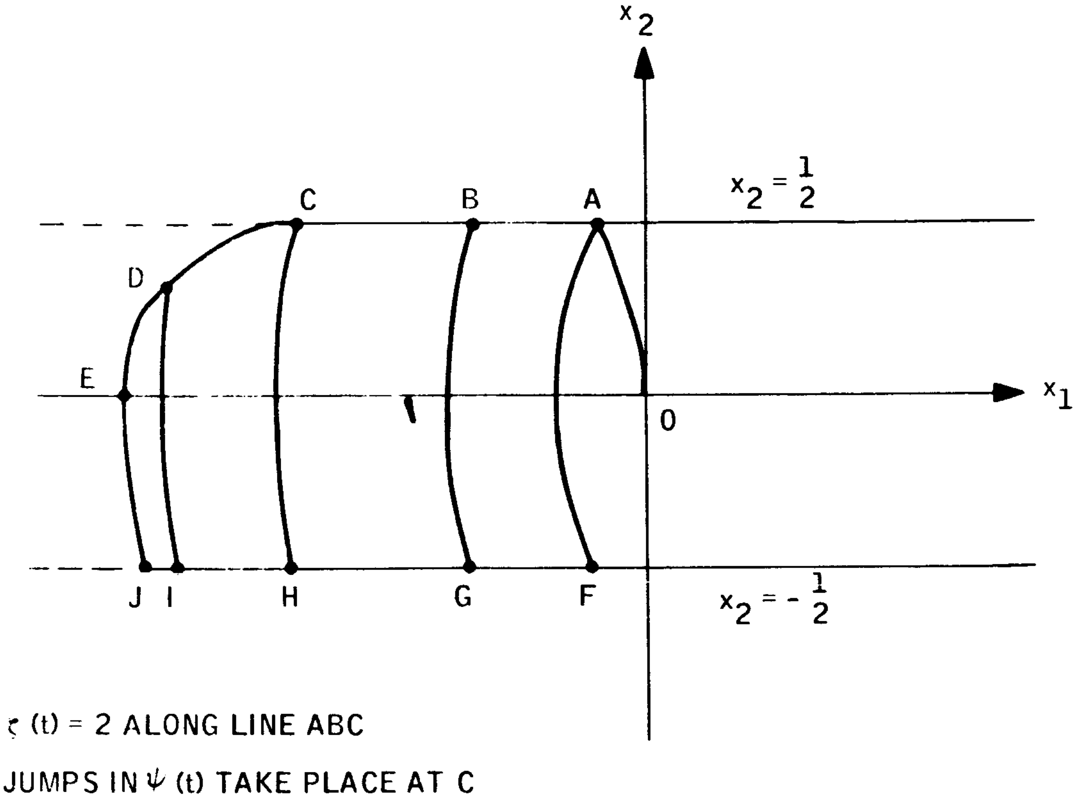

Figure 1. Optimal Solutions of the Linear Harmonic Oscillator with Bounded Phase-Coordinates

Let it be agreed that in every case the origin is reached at time $t = 0$ and thus the optimal trajectories are to be studied for $t < 0$.

From the study of the unconstrained problem, it is known that there is an optimal trajectory, $\hat{x}(t)$, with $\hat{u}(t) \equiv -1$ on the interval, $\left[-\frac{\pi}{6}, 0\right]$. In Figure 1, this is the arc AO. It is clear, then, that for $t \in \left(-\frac{\pi}{6}, 0\right)$,

$$\psi_2 = -C_1 \sin t + C_2 \cos t < 0 \tag{68}$$

because (66) and (67) describe the relationship between $\psi(t)$ and $\hat{u}(t)$.

At $t = -\frac{\pi}{6}$, $\hat{x}(t)$ encounters the phase boundary, $x_2 = \frac{1}{2}$. If, on an interval, $\left[-\frac{\pi}{6} - \delta, -\frac{\pi}{6}\right]$, $\hat{x}(t)$ is to lie on this boundary, one must have

$$\dot{\hat{x}}_2(t) = 0,$$

whence

$$\hat{u}(t) = \hat{x}_1(t), \tag{69}$$

and thus, the value of $\hat{u}(t)$ is not in general extremal within $\Omega = \left[-1, 1\right]$. But, the equation

$$\psi(t) B\hat{u}(t) = \max_{u \in \Omega} \psi(t)Bu \tag{70}$$

must hold on this interval if the hypotheses of Theorem 1 are to be satisfied. Hence, $\psi_2(t) \equiv 0$ on this interval.

To avoid a discontinuity in $\psi(t)$ at $t = -\frac{\pi}{6}$, set

$$\psi_2\left(-\frac{\pi}{6}\right) = -C_1 \sin\left(-\frac{\pi}{6}\right) + C_2 \cos\left(-\frac{\pi}{6}\right) = \frac{C_1}{2} + \frac{C_2\sqrt{3}}{2} = 0 \tag{71}$$

so that $C_1 = -\sqrt{3}\ C_2$. With Equation (68) in mind, the choices

$$C_2 = -1,\ C_1 = \sqrt{3} \tag{72}$$

are made, thus determining

$$\psi(t) = \left(\sqrt{3}\cos t - \sin t, -\sqrt{3}\sin t - \cos t\right) \tag{73}$$

on the interval, $\left[-\frac{\pi}{6}, 0\right]$.

If $\hat{x}(t)$ lies on the boundary, $x_2 = \frac{1}{2}$, on an interval, $\left[-\frac{\pi}{6} - \delta, -\frac{\pi}{6}\right]$, then $\psi_2(t) \equiv 0$ on this interval. But, on this interval,

$$\left[\dot{\psi}_1(t), \dot{\psi}_2(t)\right] = -\left[\psi_1(t), \psi_2(t)\right]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \zeta(t)\begin{bmatrix} 0 & 1 \end{bmatrix} \tag{74}$$

since $\eta\left[\hat{x}(t)\right] \equiv \begin{bmatrix} 0 & 1 \end{bmatrix}$. Therefore, $\dot{\psi}_2(t) \equiv 0$, which implies

$$\zeta(t) \equiv \psi_1(t). \tag{75}$$

But, since $\dot{\psi}_1(t) = \psi_2(t) \equiv 0$, this means

$$\zeta(t) \equiv \psi_1(t) \equiv \psi_1\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\sqrt{3} + \frac{1}{2} = 2. \tag{76}$$

The use of the function, $\zeta(t)$, may be discontinued at any time $t < -\frac{\pi}{6}$. If $\zeta(t)$ is not used at all, the optimal trajectory, FAO, results. If $\zeta(t)$ is used on an interval, $\left[-\frac{\pi}{6} - \delta, -\frac{\pi}{6}\right]$, and discontinued prior to $-\frac{\pi}{6} - \delta$, then, for $t < -\frac{\pi}{6} - \delta$, one has

$$\psi(t) = \left[2\cos\left(t + \frac{\pi}{6} + \delta\right), -2\sin\left(t + \frac{\pi}{6} + \delta\right)\right] \tag{77}$$

and a trajectory, GBAO, is the result. This analysis holds when δ is a positive number $\leq \sqrt{3}$. If $\delta > \sqrt{3}$, it does not apply because then $\hat{x}(t)$ would lie on $x_2 = \frac{1}{2}$ to the left of the point, $\begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$, and there is no control which can keep $\hat{x}(t)$ on $x_2 = \frac{1}{2}$ in this region.

Assuming now that $\zeta(t) = 2$ has been used on the interval, $\left[-\frac{\pi}{6}-\sqrt{3}, -\frac{\pi}{6}\right]$, the situation at $t = -\frac{\pi}{6} - \sqrt{3}$ will now be examined. It is clear that, under the circumstances, $\hat{x}\left(-\frac{\pi}{6} - \sqrt{3}\right) = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$. If $\psi(t)$ remains continuous at $t = -\frac{\pi}{6} - \sqrt{3}$, then for $t < -\frac{\pi}{6} - \sqrt{3}$ one has $\psi(t) = \left[2\cos\left(t + \frac{\pi}{6}+\sqrt{3}\right), -2\sin\left(t + \frac{\pi}{6} + \sqrt{3}\right)\right]$ and a trajectory, HCAO, is the result. Suppose, on the other hand, a jump discontinuity of the form

$$\psi(t + 0) - \psi(t - 0) = \nu\eta\left[\hat{x}(t)\right] = \begin{bmatrix} 0 & \nu \end{bmatrix} \tag{78}$$

is introduced at at $t = -\frac{\pi}{6} - \sqrt{3}$. Then

$$\psi\left(-\frac{\pi}{6}-\sqrt{3} - 0\right) = \begin{bmatrix} 2 & -\nu \end{bmatrix}. \tag{79}$$

The real number, ν, may be any positive number. Depending on the choice made, and this choice is entirely arbitrary, a number of trajectory types arise. With $\nu = 0$, the trajectory, HCAO, has already been mentioned. For $\nu > 0$, the trajectory takes the form indicated by IDCAO. The limiting case, $\nu = +\infty$, which actually corresponds to taking $\psi\left(-\frac{\pi}{6} - \sqrt{3} - 0\right) = \begin{bmatrix} 0 & 1 \end{bmatrix}$, leads to the trajectory, JECAO.

The trajectories described above and those obtained from the control given by Equation (67), with $C_1 = \cos\theta$ and $C_2 = \sin\theta$, where $-\frac{\pi}{6} < \theta < 0$, together with their symmetric counterparts, completely fill out the domain of controllability for this particular time-optimal bounded phase-coordinate control problem.

3.11 COMPUTATION OF TIME-OPTIMAL BOUNDED PHASE-TRAJECTORIES AND CONTROLS

In this section it will be shown that the method illustrated in the above example has rather general application. It will be shown that computation of optimal trajectories via the so-called "backing out" procedure is feasible in the sense that no difficult mathematical problems will normally arise in such computation.

The "backing out" method for problems without phase constraints involves the following operations for each trajectory: First, a value, ψ_0, for $\psi(t_{\hat{u}})$, the final value of $\psi(t)$, is selected. If the control process is linear, this leads immediately to the entire solution, $\psi(t)$, defined on any interval, $\left[t_0, t_{\hat{u}}\right]$, so that $\hat{u}(t)$ is chosen to maximize $\psi(t)B(t)u$; finally, $\hat{x}(t)$ is calculated as a solution of

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)\hat{u}(t)$$

with $\hat{x}(t_{\hat{u}}) = x_1$. For nonlinear problems, the basic idea is the same but $\psi(t)$ and $\hat{x}(t)$, in general, have to be calculated simultaneously.

The corresponding process for time-optimal bounded phase-coordinate control of linear systems of the form

$$\dot{x} = A(t)x + Bu \tag{80}$$

$\left[A(t) \text{ time varying, } B \text{ constant}\right]$ will be considered below. It is clear that the calculation of an optimal bounded phase-trajectory involves treatment of the following subsidiary problems:

(i) Determination of $\hat{x}(t)$ and $\hat{u}(t)$ when $\hat{x}(t)$ lies in the interior of the constraint set, G;

(ii) Analysis of the behavior of $\psi(t)$, $\hat{x}(t)$, and $\hat{u}(t)$ at times when $\hat{x}(t)$ leaves ∂G to enter Int(G);

(iii) Analysis of the behavior of $\psi(t)$, $\hat{x}(t)$, and $\hat{u}(t)$ during intervals for which $\hat{x}(t)$ lies on ∂G;

(iv) Analysis of the behavior of $\psi(t)$, $\hat{x}(t)$, and $\hat{u}(t)$ at times when $\hat{x}(t)$ leaves Int(G) to meet ∂G.

Each of these subsidiary problems will now be discussed.

Let (τ_0, τ_1) be a maximal interval such that $\hat{x}(t)$ lies in Int(G). If Theorem 1 is to be used, it is clear that the usual Maximum Principle must be employed on such intervals. Thus, the principle question that arises is how $\psi(\tau_1)$ is to be determined. For the unconstrained optimization problem, $\tau_1 = t_{\hat{u}}$, the choice of $\psi(t_{\hat{u}})$ is arbitrary, with each choice leading, in general, to a different optimal trajectory. The example above shows that this is no longer true in the present theory of bounded phase control. Every solution is such that $\psi_2(t) < 0$ on the interval, $\left(-\frac{\pi}{6}, 0\right)$, leads to the trajectory, AO, and the only extension backward in time occurs when, in addition, $\psi_2\left(-\frac{\pi}{6}\right) = 0$.

Let it be assumed that both Ω and G are convex polyhedrons, the former being compact. Assume that τ_0 is a time when $\hat{x}(t)$ leaves the boundary, $x \cdot e_1 = 1$ (without loss of generality), of G for Int(G). Assuming $\hat{u}(t)$ to be left continuous at time $t = \tau_0$, it must be true that

$$\psi(\tau_0 - 0)B\hat{u}(\tau_0) = \max_{u \in \Omega} \psi(\tau_0 - 0)Bu \tag{81}$$

and it also must be true that

$$\hat{u}(\tau_0) \in \Omega\left[\hat{x}(\tau_0), \tau_0\right] = \left\{ u \in \Omega \mid \beta_1 \cdot u = -\alpha_1(\tau_0) \cdot \hat{x}(\tau_0) \right\} \tag{82}$$

[See Definition 1 following Equation (48)]. In general, it is to be expected that $\hat{u}(t)$ will lie in the interior of a one-dimensional face of Ω on some interval, $\left[\tau_0 - \delta, \tau_0\right]$, as was the case in the above example. Letting F be this one-dimensional face of Ω, it is seen that $\psi(\tau_0 - 0)B$ must be perpendicular to F. Now, $\psi(\tau_0 - 0) - \left[\psi(\tau_0 - 0)e_1\right]e_1' = \psi(\tau_0 + 0) - \left[\psi(\tau_0 + 0)e_1\right]e_1'$.

Theorem 1 allows the possibilities, $\psi(\tau_0 + 0) = \nu e_1'$, where $\nu \geq 0$. Thus, the condition which must be imposed upon $\psi(\tau_0 + 0)$ in order that the trajectory be continuable for time $t < \tau_0$ is, in general, that there be a one-dimensional face, F, of Ω, whose interior intersects $\Omega\left[\hat{x}(\tau_0), \tau_0\right]$ in one point only and a real number, $\nu \geq 0$, such that with

$$\psi(\tau_0 + 0) - \psi(\tau_0 - 0) = \nu e_1', \tag{83}$$

$\psi(\tau_0 - 0)B$ is perpendicular to F.

In most cases, this is the rule which will enable one to determine $\psi(\tau_0 + 0)$. There are exceptional cases but these are not likely to arise very often in practice.

Now, attention will be turned to an interval, $[\tau_0, \tau_F]$, during which $\hat{x}(t) \in \partial G$. During such an interval, $\psi(t)$ satisfies

$$\dot{\psi}(t) = -\psi(t)A(t) + \zeta(t)\,\eta\left[\hat{x}(t)\right]. \tag{84}$$

It will be assumed that $[\tau_0, \tau_F]$ has a partition

$$[\tau_0, \tau_F] = \bigcup_{k=1}^{r} [\tau_{k-1}, \tau_k] \tag{85}$$

and that for $t \in [\tau_{k-1}, \tau_k]$, $\psi(t)$ is perpendicular to a one-dimensional face of Ω, namely, F_k. Without loss of generality, it will again be assumed that the portion of ∂G on which $\hat{x}(t)$ lies during the interval, $[\tau_0, \tau_1]$, is given by $x.e_1 = 1$. Letting z_k be a unit vector parallel to F_k, it is seen that on each $[\tau_{k-1}, \tau_k]$

$$\psi(t)Bz_k \equiv 0. \tag{86}$$

Differentiation of both sides of this identity yields

$$\left[-\psi(t)A(t) + \zeta(t)\,e_1'\right] Bz_k \equiv 0, \tag{87}$$

whence

$$\zeta(t) = \frac{\psi(t)A(t)Bz_k}{e_1' Bz_k}. \tag{88}$$

Thus, $\zeta(t)$ is determined explicitly. If A(t) is analytic in t, then $\psi(t)$ is piecewise analytic because

$$\dot{\psi}(t) = -\psi(t)A(t) + \frac{\psi(t)A(t)Bz_k e_1'}{e_1' Bz_k} \tag{89}$$

and hence, as claimed just prior to Equation (55), $\zeta(t)$ is piecewise analytic.

If the use of $\zeta(t)$ is discontinued arbitrarily at some time during such an interval, the fact that $\zeta(t) > 0$ will, in general, guarantee that the trajectory will re-enter the interior of G. (Compare with the example given above). In running of optimal trajectories in a "backing out" process, the use of $\zeta(t)$ must be discontinued at a reasonably large number of points so as to obtain these trajectories.

The next question to arise is that of determining the τ_k, i.e., when does $\zeta(t)$ cease trying to keep $\psi(t)B$ perpendicular to a given one-dimensional face of Ω and start keeping $\psi(t)B$ perpendicular to another ? This situation will usually be easy to detect; at such a time, τ_k, $\psi(t)B$ will be perpendicular to a face of Ω of dimension higher than one, of which the two one-dimensional faces already mentioned are bounding faces.

Finally, let it be assumed that τ_0 is a time when an optimal trajectory leaves Int(G) to meet ∂G. A large number of these cases have already been treated above when the discontinuance of the use of $\zeta(t)$ was discussed. The only additional remark necessary is that, if a number of discontinuities in $\psi(t)$ of the form

$$\psi(\tau_0 + 0) - \psi(\tau_0 - 0) = \nu\eta\left[\hat{x}(\tau_o)\right],$$

where $\nu > 0$, are possible (i.e., the resulting trajectories remain within G), then a representative number of these discontinuities should be made in order to obtain the desired trajectories. (Again note the situation at $t = -\frac{\pi}{6} - \sqrt{3}$ in the example.)

3.12 CONCLUSION

In this chapter it has been shown that if a control, $\hat{u}(t)$, and associated path, $\hat{x}(t)$, obey a certain maximum principle, stated earlier by Chang [9, 10], then they represent a solution to the time-optimal control problem. Also, conditions under which such an optimal solution is unique have been obtained. Finally, it has been shown that this theory can be used in a "backing out" procedure to obtain optimal trajectories.

CHAPTER 4

AN APPROXIMATION TO LINEAR BOUNDED PHASE-COORDINATE CONTROL PROBLEMS

4.1 INTRODUCTION

In many control problems both restraints on the magnitudes of the control variables and various system variables may occur. Certain results [4, 5, 9] are available for the determination of optimal controllers for some classes of linear and nonlinear systems involving such restraints. These results take the form of necessary or sufficient conditions for optimal control but not both, and are therefore only a partial solution to even the theoretical problem, leaving much to be desired in the way of a practical solution. To use the necessary or sufficient conditions for synthesizing an optimal controller it is necessary to solve a two-point boundary value problem in terms of a number of free parameters and multipliers, where the number of parameters is not even known, as well as certain jump conditions [4, 5]. A "backing out" procedure, given in Chapter 3, is also available if one is interested in flooding the domain of controllability with responses and then keeping track (storing) of the corresponding control magnitude for each such point.

In this chapter, we offer a procedure which has several advantages over the above schemes, but is only an approximate solution. Its main advantage is that no discontinuities will be encountered in the adjoint solution which determines the optimum controller, and therefore the resulting two-point boundary value problem may be more readily solved. The results provide both necessary and sufficient conditions, as well as existence, for the approximate problem.

The analysis is limited to linear control processes as described by the differential system

$$\mathcal{L}) \quad \dot{x} = A(t)\,x + B(t)\,u(t).$$

The coefficient matrices, A(t) and B(t), are composed of known continuous functions on the time interval, $[t_0, t_1]$. The controller, u(t), is to be chosen from a set, Ω: $|u^j| \leq 1$; $j = 1, 2, \ldots m$, so as to steer the response, $x_u(t)$, of $\mathcal{L}$) from an initial point, x_o, at time, t_o, to a prescribed compact target set, $\tilde{G} \subset R^n$, and it is required that $x_u(t)$ remain within a given constraint set, Λ, during its entire response. Here R^n is the n-dimensional real number space.

The problem of time-optimal control, as considered in Section 4.2, is to find a controller, u(t), which steers $x_u(t)$ from x_o to $\tilde{G} \subset \Lambda$ in minimum time, that is, which minimizes $C(u) = t_1 - t_o$, with $x(t_1) \in \tilde{G}$ and $x_u(t) \in \Lambda$, $t_o \leq t \leq t_1$. Later, in Section 4.4, other optimum control cost functionals are discussed.

Certain difficulties are involved when one directly solves for this optimum controller. We shall therefore be content with solving the following apparently simpler problem: Find that controller u(t) with graph in Ω which steers $x_u(t)$ from x_o at t_o to $\tilde{G}$ at t_1 with $x_u^\circ(t_1) \leq \beta$ and $t_1 - t_o$ a minimum. $x_u^\circ(t)$ is defined below.

It is assumed atat Λ is a closed convex set (for convenience we could even let $\Lambda = \{x \mid x' H x \leq c\}$, where H is a positive semi-definite matrix and c = constant > 0.) Let F(x) be a convex continuous differentiable function which is such that

$$\begin{aligned} F(x) &\neq 0 \quad \text{if } x \notin \Lambda, \\ &= 0 \quad \text{if } x \in \Lambda. \end{aligned}$$

Then define*

$$x_u^\circ(t_1) = \int_{t_o}^{t_1} F\left[x_u(t)\right] dt.$$

$x_u^\circ(t_1)$ essentially measures the excursions of the response, $x_u(t)$, to a controller, $u(t)$, outside of the region, Λ, during the time interval, $\left[t_o, t_1\right]$. By keeping $x_u^\circ(t_1)$ small, the response, $x_u(t)$, is restricted to stay close to or within Λ. The above minimum time optimal control problem is approximately solved by finding a controller which steers $\hat{x}_u(t) = \left[x_u^\circ(t), x_u(t)\right]$ from $(0, x_o)$ to $G = \left\{ x^\circ, x \mid x \in \tilde{G},\ 0 \le x^\circ \le \beta \right\}$ in the minimum time interval, $t_1 - t_o$, if $\beta > 0$ is sufficiently small.

In 4.2, necessary and sufficient conditions are given for this approximation problem using the time-optimal criterion. Section 4.3 contains an example and 4.4 is a discussion of the approximation problem for other cost functionals.

4.2 THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE APPROXIMATE LINEAR TIME-OPTIMAL PROBLEMS

We augment the system, $\mathcal{L}$), by considering the equation system,

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* There is, of course, some question as to whether such a function, $F(x)$, exists for an arbitrary convex set, Λ, contained in R^n. We now cite an example which shows that there are such functions in a number of interesting cases. Suppose $\Lambda = \left\{ x^1, x^2, \ldots x^n \,\middle|\, |x^2| \le 1 \right\}$. Then pick

$$F(x) = \begin{cases} (x^2 - 1)^2/_2 & \text{if } x^2 \geq 1 \;, \\ 0 & \text{if } |x^2| \le 1 \,, \\ (x^2 + 1)^2/_2 & \text{if } x^2 \le -1 \;. \end{cases}$$

Thus, if only one coordinate (or a linear combination) is restricted, the problem is easily handled, as in the example where $F(x)$ is continuous and has continuous partial derivatives. Other Λ's can be approximately handled as in the example.

$$\hat{\mathcal{L}}) \quad \dot{x}^\circ = F(x)$$

$$\dot{x} = A(t)x + B(t)\,u(t)$$

obtained from $\mathcal{L}$) by adding the equation for $\dot{x}^\circ$ with $x^\circ(t_o) = 0$. Here, $A(t)$, $B(t)$ are bounded and continuous on $[t_o, t_1]$ and $F(x)$ is a convex function, with $F(x) = 0$ for $x \in \Lambda$. $\frac{\partial F}{\partial x}(x)$ is assumed to exist and be continuous everywhere.

The set of attainability, $\hat{K}(t_1) \subset R^{n+1}$, is the collection of end points, $\hat{x}_u(t_1)$, of responses $\hat{x}_u(t) = [x_u^\circ(t), x_u(t)]$ of $\hat{\mathcal{L}}$) which initiate at $(0, x_o)$ at time, t_o, corresponding to all (Lebesgue) measurable controllers, $u(t)$, which are such that $|u^j(t)| \leq 1$ on $[t_o, t_1]$, for $j = 1, 2, \ldots, m$. (Such controllers are referred to as admissible controllers.)

In the following theorems various properties are established for $\hat{K}(t_1)$ and $\partial\hat{K}(t_1)$, as required in synthesizing optimal controllers.

Theorem 1.

Consider the above system $\hat{\mathcal{L}}$), with initial point $\hat{x}_o$, restraint set Ω, and set of attainability $\hat{K}(t_1)$. Then $\hat{K}(t_1)$ is a nonempty compact subset of R^{n+1} in variables (x°, x), with convex lower surface (as defined below) for each $t_o \leq t_1 < \infty$.

Proof:

$\hat{K}(t_1)$ is nonempty since any measurable controller, $u(t) \subset \Omega$, gives rise to an end point, $\hat{x}_u(t_1) \in \hat{K}(t_1)$. $\hat{K}(t_1)$ is compact because the system, $\mathcal{L}$), satisfies the hypothesis of the existence theorems of References 20 and 22.

Define a point, $\hat{x}_1$, to be in lower boundary (surface) of $\hat{K}(t_1)$ if its first component, $x_1^\circ = \inf. \{ x^\circ \}$, for all points, $\hat{x}$, of $\hat{K}(t_1)$, with $x = x_1$. The orthogonal projection of $\hat{K}(t_1)$ on the plane, $x^\circ = 0$, gives the compact convex set of attainability [21], $K(t_1)$, for the time-optimal problem (in the x-space). The lower boundary is convex if it defines a convex function on $K(t_1)$.

We now show that if $\hat{x}_1$ and $\hat{x}_2$ are points of $\hat{K}(t_1)$, then the point, $\hat{y} = \lambda \hat{x}_1 + (1-\lambda)\hat{x}_2 = (y^\circ, y)$, $0 \leq \lambda \leq 1$, is such that

$$y = x_{\bar{u}}(t_1)$$

and

$$y^\circ \geq x^\circ_{\bar{u}}(t_1).$$

where $\bar{u}(t) = \lambda\, u_1(t) + (1-\lambda)\, u_2(t)$ and $u_1(t)$ and $u_2(t)$ are such that $\hat{x}_{u_1}(t_1) = \hat{x}_1$ and $\hat{x}_{u_2}(t_1) = x_2$. The convexity of the lower surface of $\hat{K}(t_1)^1$ then follows because, in order for it to be nonconvex, it is necessary that there exist two points, $\hat{x}_1$, $\hat{x}_2$, on this lower boundary, with the property that the point, $\lambda\, \hat{x}_1 + (1-\lambda)\, \hat{x}_2$, is below the set, $\hat{K}(t_1)$, for some $0 < \lambda < 1$, which will then be impossible.

With $\bar{u}(t) = \lambda\, u_1(t) + (1-\lambda)\, u_2(t)$, we find that

$$x_{\bar{u}}(t_1) = \Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)\bar{u}(s)ds$$

$$= \lambda \left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_1(s)ds\right]$$

$$+ (1-\lambda)\left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_2(s)ds\right]$$

$$= \lambda\, x_{u_1}(t_1) + (1-\lambda)\, x_{u_2}(t_1)$$

$$= \lambda\, x_1 + (1-\lambda)\, x_2 = y$$

where $\Phi(t)$ is the fundamental solution matrix of $\mathcal{L}$), with $\Phi(t_o) = I$. We also calculate

$$x^o_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F\left[x_{\bar{u}}(t)\right] dt$$

and $\lambda\, x^o_{u_1}(t_1) + (1-\lambda)\, x^o_{u_2}(t_1)$ for comparison. Since $F(x)$ is a convex function of x, it follows that for $0 \le \lambda \le 1$,

$$F\left[x_{\bar{u}}(t)\right] = F\left[\lambda\, x_{u_1}(t) + (1-\lambda)x_{u_2}(t)\right] \le \lambda\, F\left[x_{u_1}(t)\right] + (1-\lambda)\, F\left[x_{u_2}(t)\right]$$

and so

$$x^o_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F\left[x_{\bar{u}}(t)\right] dt = \int_{t_o}^{t_1} F\left[\lambda\, x_{u_1}(t) + (1-\lambda)\, x_{u_2}(t)\right] dt$$

$$\le \lambda \int_{t_o}^{t_1} F\left[x_{u_1}(t)\right] dt + \int_{t_o}^{t_1} (1-\lambda)\, F\left[x_{u_2}(t)\right] dt = y^\bullet.$$

Q.E.D.

We will now consider those controllers, $u(t)$, on $[t_o, t_1]$ which steer $\hat{x}_u(t)$ from $\hat{x}_o$ at t_o to points $\hat{x}_1$ contained in the lower boundary of $\hat{K}(t_1)$ [written $\partial\hat{K}^-(t_1)$]. Such controllers will be called extremal and they will play a significant part in the selection of optimal controllers.

Let $u(t) \in \Omega$ on $t_o \leq t \leq t_1$ be an admissible controller for the convex control process

$$\hat{\mathcal{L}}) \quad \dot{x}^{\circ} = F(x)$$
$$\dot{x} = A(t)\,x + B(t)u(t)$$

with initial point, $\hat{x}_o = (0, x_o)$, at t_o. If the corresponding response, $\hat{x}_u(t)$, has an end point, $\hat{x}(t_1) \in \partial K^-(t_1)$, then $u(t)$ is called an extremal control and $\hat{x}_u(t)$ an extremal response on $[t_o, t_1]$.

The adjoint response, $\hat{\eta}(t) = [\eta_o(t), \eta(t)]$, corresponding to a controller, $u(t)$, is a row n+1 vector satisfying the differential system

$$\dot{\eta} = -\eta\, A(t) - \eta_o \frac{\partial F'}{\partial x}\,[x_u(t)]$$

$$\eta_o = \text{constant} \lesseqgtr 0.$$

where $x_u(t)$ is the response of $\mathcal{L}$) corresponding to the controller, $u(t)$. Define $u(t)$ on $[t_o, t_1]$ to be a maximal controller, in case there exists a nonvanishing adjoint response, $\hat{\eta}(t)$, $\eta_o \leq 0$, so that $\eta(t)B(t)u(t) = \underset{u \in \Omega}{\mathrm{Max}} \{\eta(t)B(t)u\}$ a.e. on $[t_o, t_1]$.

In Theorem 2, which follows, it is shown that extremal and maximal controllers are the same.

Theorem 2.

Consider the convex control process*

$\hat{\mathcal{L}}$) $\dot{x}^{\circ} = F(x)$

$\dot{x} = A(t)x + B(t)u(t)$

with initial point $\hat{x}_0 = (0, x_o)$ at time t_o. An admissible controller, $u(t) \subset \Omega$, on $[t_o, t_1]$ is extremal for $\hat{\mathcal{L}}$)if, and only if, it is a maximal controller, that is, if and only if there is a nonvanishing adjoint response, $\hat{\eta}(t)$, of

$$\dot{\eta} = -\eta\, A(t) - \eta_o \frac{\partial F'}{\partial x} [x_u(t)]$$

$$\eta_o = \text{constant} \leq 0$$

so that

$$\eta(t)B(t)u(t) = \underset{u \in \Omega}{\text{Max}} \left\{ \eta(t)B(t)u \right\} \text{ almost always on } [t_o, t_1].$$

Proof:

Assume $u(t)$ on $[t_o, t_1]$ is extremal and so steers $\hat{x}(t)$ from $(0, x_o)$ at t_o to $\hat{x}(t)$ from $(0, x_o)$ at t_o to $\hat{x}_1 \in \partial\hat{K}^-(t_1)$ at t_1. Choose $\hat{\eta}(t_1) = [\eta_o, \eta(t_1)]$ to be a nonzero vector normal to π directed into the halfspace defined by π which does not meet $\hat{K}(t_1)$. Note $\eta_o \bar{<} 0$. Then, let $\hat{\eta}(t)$, with $\hat{\eta}(t_1)$ as above, be the response of the adjoint equation corresponding to the controller, $u(t)$.

* The necessary portion of this theorem follows from L.S. Pontryagin's Maximum Principle [4]. For completeness, the simple arguments to establish the necessary part are presented.

The controller*, $\bar{u}(t) = \text{sgn}\{\eta(t)B(t)\}$, defined for $t \in [t_o, t_1]$ is admissible and

$$\eta(t)B(t)\bar{u}(t) = \underset{u \in \Omega}{\text{Max}} \{\eta(t)B(t)u\}$$

on $[t_o, t_1]$.

Let τ_ε be a closed subset of measure, $\varepsilon > 0$, contained in $\mathcal{J} = [t_o, t_1]$, whereon

$$\delta + \eta(t)B(t)u(t) < \underset{u \in \Omega}{\text{Max}} \{\eta(t)B(t)u\} \text{ for some } \delta > 0.$$

For given $\delta > 0$ consider the modified controller

$$\begin{aligned} u_\varepsilon(t) &= u(t) \text{ on } \mathcal{J} - \tau_\varepsilon \\ &= \bar{u}(t) \text{ on } \tau_\varepsilon, \end{aligned}$$

and calculate

$$\frac{d\hat{\eta}(t)\,\hat{x}_\varepsilon}{dt} = \dot{\hat{\eta}}\hat{x}_\varepsilon + \eta\dot{\hat{x}}_\varepsilon$$

and

$$\frac{d\hat{\eta}(t)\,\hat{x}}{dt} = \dot{\hat{\eta}}\hat{x} + \hat{\eta}\dot{\hat{x}},$$ where $\hat{x}_\varepsilon$ refers to a response of $\hat{\mathcal{L}}$)

corresponding to the modified controller, $u_\varepsilon(t)$.

$$\begin{aligned} * \text{ sgn}\{\,\} &= -1 \text{ if } \{\,\} < 0 \\ &= 0 \text{ if } \{\,\} = 0 \\ &= +1 \text{ if } \{\,\} > 0 \end{aligned}$$

Integration from t_o to t_1 yields

$$\hat{\eta}(t)\ \hat{x}_\varepsilon(t_1) - \hat{\eta}(t)\hat{x}_\varepsilon(t_o) = \int_{t_o}^{t_1}\left[\left\{-\eta\ A(t) + \frac{\partial F}{\partial x}[x(t)]\right\}x_\varepsilon(t)\right.$$

$$\left. + \quad \eta(t)\left[A(t)x_\varepsilon(t) + B(t)u(t)\right] - F[x_\varepsilon(t)]\right]dt$$

and

$$\hat{\eta}(t_1)\hat{x}(t_1) - \hat{\eta}(t_o)\hat{x}(t_o) = \int_{t_o}^{t_1}\left[\left\{-\eta\ A(t) + \frac{\partial F}{\partial x}[x(t)]\right\}x(t)\right.$$

$$\left. + \eta(t)\left\{A(t)x(t) + B(t)u(t)\right\} - F[x(t)]\right]dt \text{ for } \eta_o = -1.$$ The case when $\eta_o = 0$ is simpler and omitted.

Combining terms and using the assumed continuity for F and $\frac{\partial F}{\partial x}$, we easily find that

$\hat{\eta}(t_1)\hat{x}_\varepsilon(t_1) - \hat{\eta}(t_1)\hat{x}(t_1) \geq \delta\ \epsilon + 0(\epsilon)$ for ϵ sufficiently small, where $0(\epsilon)$ corresponds to terms of higher than first-order in ϵ, and therefore for ϵ sufficiently small

$\hat{\eta}(t)\ x_\varepsilon(t_1) - \hat{\eta}(t_1)\hat{x}(t_1) > 0$, contradicting the construction of $\hat{\eta}(t_1)$ as the outward normal to $\hat{K}(t_1)$ at $\hat{x}_1$.

Hence, there exists no such interval, τ_ε, so

$$\eta(t)B(t)u(t) = \operatorname*{Max}_{u\in\Omega}\eta(t)B(t)u \text{ almost everywhere on } \mathfrak{J}.$$

Conversely, assume that u(t) and corresponding response $\hat{\eta}(t) \neq 0$ are such that

$$\eta(t)B(t)u(t) = \operatorname*{Max}_{u\in\Omega}\eta(t)Bu$$

a.e. on $\mathcal{J}$ with $\eta_o \leq 0$. Let $\bar{u}$ (t) be any controller in Ω with corresponding response, $x_{\bar{u}}(t)$. If we calculate

$$\frac{d\hat{\eta}\hat{x}_u}{dt} \text{ and } \frac{d\hat{\eta}\hat{x}_{\bar{u}}}{dt} \text{ as above,}$$

and then integrate from t_o to t_1, using the assumed convesity of F(x), we find that

$$\hat{\eta}(t_1)\,\hat{x}_u(t_1) \geq \hat{\eta}(t_1)\,\hat{x}_{\bar{u}}(t_1) = \hat{\eta}(t_1)\,\hat{w}$$

where $\hat{w}$ is any point of $\hat{K}(t_1)$. Since $|\hat{\eta}(t_1)| \neq 0$, and $\eta_o \leq 0$, the above inequality implies that $\hat{x}_u(t_1)$ is contained in the lower boundary of the compact set, $\hat{K}(t_1)$, with convex lower boundary and, hence, u(t) is extremal.

Q.E.D.

Theorem 2 indicates that to stay at a lower boundary point we must continuously steer maximally in the direction of the vector, $\hat{\eta}(t)$. This remark is summarized as a corollary.

Corollary.

Let u(t) on $[t_o, t_1]$ be an extremal controller for $\hat{\mathcal{L}}$), with corresponding response, $\hat{x}_u(t)$, and adjoint response, $\hat{\eta}(t)$, so that,

$$\eta(t)\,B(t)u(t) = \max_{u\in\Omega} \eta(t)B(t)u$$

a.e. on $[t_o, t_1]$. Then on each subinterval, $[t_o\,\tau] \subset [t_o, t_1]$, u(t) is also an extremal controller with $\hat{x}_u(\tau) \in \partial\hat{K}(\tau)$. Moreover, $\hat{\eta}(\tau)$ is an exterior normal to $\hat{K}(\tau)$ at $\hat{x}(\tau)$.

Proof:

Replace t_1 by τ in the proof of Theorem 2 to obtain that

$$\hat{\eta}(\tau)\ \hat{x}_u(\tau) \geq \hat{\eta}(\tau)\ \hat{x}_{\bar{u}}(\tau) = \hat{\eta}(\tau)\ \hat{w}(\tau)$$

for all $\hat{w}(\tau)$ in $\hat{K}(\tau)$. From this inequality the conclusion of the corollary can be drawn.

We next show that the set of attainability, $\hat{K}(t_1)$, depends continuously on the parameter, t_1.

Define the distance between a point, p, and a compact set, $G_1\ \ R^n$, to be

$$d(p, G_1) = \min_{g \in G_1} |p-g|$$

and define the distance between two compact sets, G_1 and $G_2 \subset R^n$, to be

$$d(G_2, G_2) = \operatorname{Max}\left\{ \max_{p_1 \in G_1} d(p_1, G_2),\ \max_{p_2 \in G_2} d(p_2, G_1) \right\}.$$ Here

$$|p| = \sum_{i=1}^{n} |p^i|.$$

The set, $\hat{K}(t_2) \subset R^{n+1}$, varies continuously with t_2 if, given an $\epsilon > 0$, there exists a $\delta > 0$, so that for $|t_2 - t_1| < \delta$,

$$d\left[\hat{K}(t_1),\ \hat{K}(t_2)\right] < \epsilon.$$

Lemma 1.

Consider the system, $\hat{\mathcal{L}}$), as above with attainable set, $\hat{K}(t_1) \subset R^{n+1}$. Then, $\hat{K}(t_1)$ varies continuously with $t_1 < \infty$.

Proof:

We need only show that each point, $\hat{x}(t_1)$, of $\hat{K}(t_1)$ is close to some point, $\hat{x}(t_2)$, of $\hat{K}(t_2)$ and conversely. That is, we need to show that, given $\epsilon > 0$, there exists a $\delta > 0$, so that when $|t_1 - t_2| < \delta$, there exists $\hat{x}(t_1) \epsilon \hat{K}(t_1)$ such that $|x(t_1) - x(t_2)| < \epsilon$ for each $\hat{x}(t_2) \epsilon \hat{K}(t_2)$ and conversely.

Let $u_1(t)$ be an admissible controller on $[t_o, t_1+1]$ and $\hat{x}_1(t)$ the corresponding response. For $t_1 \leq t_2 \leq t_1+1$ calculate

$$x_1^\circ(t_2) - x_1^\circ(t_1) = \int_{t_o}^{t_2} F[x_1(t)]\, dt - \int_{t_o}^{t_1} F[x_1(t)]\, dt$$

and

$$x_1(t_2) - x_1(t_1) = \Phi(t_2) = \int_{t_o}^{t_2} \Phi(s)^{-1} B(s) u_1(s)\, ds$$

$$- \Phi(t_2) \int_{t_o}^{t_1} \Phi(s)^{-1} [B(s) u_1(s)]\, ds$$

$$+ [\Phi(t_2) - \Phi(t_1)] \int_{t_o}^{t_1} \Phi(s)^{-1} B(s) u_1(s)\, ds .$$

so

$$x_1^\circ(t_2) - x_1^\circ(t_1) = \int_{t_1}^{t_2} F[x_1(t)]\, dt$$

and

$$x_1(t_2) - x_1(t_1) = \Phi(t_2) \int_{t_1}^{t_2} \Phi(s)^{-1} u_1(s)\, ds$$

$$+ \left[\Phi(t_2) - \Phi(t_1) \right] \left[\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)\, u_1(s)\, ds \right]$$

Since A(t) is bounded and continuous on $\left[t_o, t_1 + 1\right]$, so is $\Phi(t)$, and therefore there exists a constant, C_1, so that

$$| \Phi(t)^{-1} | < C_1$$

and

$$| \Phi(t)^{-1} | < C_1 \text{ on } \left[t_o, t_1 + 1\right].$$

Also, since B(s) has bounded continuous elements, $b^i_j(t)$, and $u_1(t)$ is bounded and measurable, there exists the constant, C_2, so that

$$\left| \int_{t_o}^{t_1} \Phi(s)^{-1} B(s)\, u_1(s)\, ds \right| < C_2.$$

Integration is a continuous operation; therefore, given an $\epsilon > 0$, there exists a $\delta > 0$, so that

$$\left| \int_{t_1}^{t} F\left[x_1(t)\right] dt \right| < \frac{\epsilon}{3},$$

$$\left| \int_{t_1}^{t} \Phi(s)^{-1} B(s)\, u_1(s)\, ds \right| < \frac{\epsilon}{3C_2}$$

for $| t-t_1 | < \delta < 1.$

Hence,

$$|\hat{x}_1(t_2) - \hat{x}_1(t_1)| < \frac{\epsilon}{3} + C_1 \frac{\epsilon}{3C_1} + \frac{\epsilon}{3C_2} C_2 = \epsilon$$

for $|t_2 - t_1| < \delta < 1$.

The other way we consider $u_1(t) = u(t)$ on $[t_o, t_1]$, where $u(t)$ steers to $\hat{x}(t_1)$, and extend it to $[t_o, t_1 + 1]$ by letting $u_1(t) = u(t_1)$ for $t \in [t_1, t_1, + 1]$. The above calculation is then repeated to find $|\hat{x}(t_2) - \hat{x}(t_1)| < \epsilon$ for $|t_2 - t_1| < \delta < 1$ and so $\hat{K}(t_1)$ varies continuously with t_1.

Theorem 3.

Consider the system, $\hat{\mathcal{L}})$ as above, with initial data $\hat{x}_o = (0, x_o)$, compact restraint set Ω and set of attainability $\hat{K}(t_1)$. Let the target set, $G = [x^o, x \mid 0 < x^o \leq \beta, x \in \tilde{G}]$, where $\beta > 0$ is a constant and $\tilde{G}$ is a compact set of R^n. Suppose G meets the interior of $\hat{K}(t_1)$, then there is a $\delta > 0$ such that G meets $\hat{K}(t_1)$ for $|t - t_1| < \delta$.

Proof:

Since G meets the interior of $\hat{K}(t_1)$, there is a point, $\hat{p} \in [G \cap \text{Int. } \hat{K}(t_1)]$ and a ball neighborhood, $N(\hat{p})$, of radius $r > 0$ contained in $\hat{K}(t_1)$. Consider the hyperplane, $x^o = p^o - r/2$, of R^{n+1}, and in this plane pick $n + 1$ independent points, $\hat{x}_1, \hat{x}_2 \ldots \hat{x}_n, \hat{x}_{n+1}$, of the boundary of the ball, $N(\hat{p})$, all equally spaced. Let $\hat{x}_1(t), \hat{x}_2(t), \ldots \hat{x}_n(t), \hat{x}_{n+1}(t)$ be responses of $\hat{\mathcal{L}})$ with initial data, $\hat{x}_o = (0, x_o)$, and corresponding to controllers $u_1(t)$, $u_2(t), \ldots u_{n+1}(t)$, $t_o \leq t \leq t_1 + 1$, which are such that $\hat{x}_1(t_1) = \hat{x}_1, \ldots \hat{x}_{n+1}(t_1) = \hat{x}_{n+1}$. Pick $1 > \delta > 0$ so small that for $|t - t_1| \leq \delta$, the points, $\hat{x}_1(t)$, lie within spheres of radius, $r/10$, of the points, $\hat{x}_1 \ldots \hat{x}_{n+1}$, this being possible because of the previous Lemma 1.

Consider the convex combination of controllers, $u_\lambda(t) = \lambda_1 u_1(t) + \lambda_2 u_2(t) \ldots$ $\lambda_{n+1} u_{n+1}(t)$, $\lambda_i > 0$, $\Sigma \lambda_i = 1$ (Note: $|u_\lambda^i| \le 1$), and the corresponding responses, $\hat{x}_\lambda(t)$, of $\hat{\mathcal{L}}$ with initial data, $(0, x_o)$. For each fixed t, $|t-t_1| \le \delta$, these response end points, $x_\lambda(t)$ sweep out a surface section, $\widetilde{S}$, which lies below the plane, $x^\circ = p^\circ$, by convexity, above or on the plane, $x^\circ = 0$, because of the positive nature of F and intersect the line segment, $\left\{0 \le x^\circ \le p^\circ,\ x = p\right\}$ (see proof of Theorem 1). Hence, G meets K(t) for $|t-t_1| \le \delta < 1$.

We now consider the problem of existence of optimum controllers.

Theorem 4.

Consider the system, $\hat{\mathcal{L}}$ as above, with compact restraint set $\Omega = \left[u \,\middle|\, |u^1| \le 1,\ i = 1, 2 \ldots, m\right] \subset R^m$, initial point $(0, x_o) \in R^{n+1}$ at time t_o, and constant compact target set $G = \left[x^\circ, x \mid 0 \le x^\circ \le \beta,\ x \in \widetilde{G}\right]$ for $\beta > 0$. If there exists an admissible controller, $u(t) \subset \Omega$, steering $\hat{x}_o$ to G on $t_o \le t \le t_1$, then there exists on optimum controller (also admissible) steering $\hat{x}$ to G in minimum time duration $t^* - t_o$.

Proof:

If $(0, x_o) \in G$, then $t^* = t_o$ and optimum control is not required. So assume $(0, x_o) \notin G$ and consider the set of attainability, $\hat{K}(t_1)$, for $t_1 \ge t_o$. Since there is one controller which steers $(0, x_o)$ to G, the set, $\hat{K}(t_1)$, meets G for some $t_1 > \hat{t}_o$. Define t^* to be the greatest lower bound of all times, t_1, such that $\hat{K}(t_1)$ meets G. By the continuous dependence of $\hat{K}(t_1)$ on t_1, the set of times for which $\hat{K}(t_1)$ meets G is a closed set in R^1. Hence, t^* is the first time $\hat{K}(t_1)$ meets G and, therefore, pick as the optimum controller, $u^*(t)$, $t_o \le t \le t^*$, a controller which steers to $K(t^*) \cap G$.

The next theorem asserts that for optimum control we need only consider points of the lower boundary of the set of attainability and therefore, by Theorem 2, extremal controllers. A sufficiency condition is also included.

Theorem 5.

Consider the system, $\hat{\mathcal{L}}$ as above, with compact rectangular restraint set Ω, initial point $(0, x_o)$ at t_o and compact convex target set $G = \left[x^\circ, x \mid 0 \le x^\circ \le \beta;\ x \in \tilde{G};\ \beta > 0 \right]$. Let $u^*(t)$ be a minimal time-optimal controller steering $\hat{x}^*(t)$ from $\hat{x}_o$ to G. Then, $u^*(t)$ is extremal, that is, there exists a nonvanishing adjoint response, $\hat{\eta}(t) = \left[\eta_o, \eta(t)\right]$ with $\eta_o \le 0$ so that

$$\eta(t)\ B(t)\ u^*(t) = \operatorname*{Max}_{u \in \Omega} \left[\eta(t)\ B(t)\ u\right]$$

almost always on $\left[t_o, t^*\right]$, with $\hat{\eta}(t^*)$ an outward normal of $\hat{K}(t^*)$ at $x^*(t^*)$ on $\partial \hat{K}(t^*)$ and $\hat{\eta}(t^*)$ satisfies the transversality condition, namely, $\hat{\eta}(t^*)$ is normal to a supporting hyperplane, π, of G and the set of attainability, $\hat{K}(t^*)$, which separates $\hat{K}(t^*)$ from G.

Moreover, if for each point [23] $\bar{x} \in G$, there exists a nonmaximal controller, $\bar{u}(t) \subset \Omega$, so that on $\bar{t}_o \le t < \infty$, the response, $x_{\bar{u}}(t)$, initiating at $\bar{x} = x_{\bar{u}}(\bar{t}_o)$ is contained in G; then, when u(t) is an admissible extremal controller steering x_o to G by means of a response satisfying the transversality condition, it is an optimum controller.

Proof:

By assumption there exists a controller steering $\hat{x}_o$ to G, so G meets $\hat{K}(t^*)$. Suppose G meets the interior of $\hat{K}(t^*)$. This is impossible because then G meets the interior of $\hat{K}(t)$ for $|t-t^*|<\delta$, $\delta>0$, by Theorem 3, and this contradicts the optimality of the controller. Hence, ∂G meets $\partial \hat{K}(t^*)$ so that the optimum controller must steer to $\partial\hat{K}(t^*)$. We must show that it steers to a lower boundary point to conclude that it is extremal. This follows at once because $\hat{K}(t)$ always first makes contact with G at a lower boundary point as can be seen by considering how the compact set $\hat{K}(t_1)$ with convex lower surface moves with respect to the set, G. Thus, if $u^*(t)$ is optimal, it is extremal and, by Theorem 2, there exists the nonvanishing adjoint response $\hat{\eta}(t)$ so that

$$\eta(t)\, B(t)\, u^*(t) = \underset{u \in \Omega}{\operatorname{Max}}\ \eta(t)\, B(t)\, u$$

where $\hat{\eta}(t^*)$ satisfies the transversality condition; since G and the lower boundary of $\hat{K}(t^*)$ are convex, they can be separated by a supporting hyperplane, π, and we choose $\hat{\eta}(t^*)$ to be normal to π and directed into the halfspace containing G.

When u(t) is an admissible extremal controller steering $\hat{x}_o$ to G and satisfying the transversality condition, it must be an optimum controller if G has the property that through each point, $\bar{x} \in G$, there passes a nonmaximal response which remains forever in G. This follows because once G and $\hat{K}(t)$ come together, the interior of $\hat{K}(t)$ has a nonempty intersection with G, so that the transversality condition can only be satisfied once and therefore there is only one time, namely t^*, for which an extremal controller can steer to G and satisfy the transversality condition. Thus, any such extremal controller satisfying the transversality condition is an optimum controller.

Q. E. D.

We have, therefore, reduced the problem of finding an optimum controller for the approximation problem to that of finding a solution to the two-point boundary value problem as given by the 2n+2 equations:

$$\dot{x}^{\circ} = F(x)$$

$$\dot{x} = A(t)\, x + B(t) \underset{u \in \Omega}{\text{Max}} \left[\eta(t)\, B(t)\, u\right]$$

$$\dot{\eta} = -\eta\, A(t) - \eta_o \frac{\partial F'}{\partial x}(x)$$

$$\dot{\eta}_o = 0 \quad , \quad (\eta_o \leq 0)$$

with boundary conditions $\hat{x}(t_o) = \hat{x}_o$, $\hat{x}(t^*) \in \partial G$, with $\hat{\eta}(t^*)$ an interior normal to G at $\hat{x}(t^*)$.

4.3 AN EXAMPLE OF APPROXIMATE BOUNDED PHASE-COORDINATE TIME-OPTIMAL CONTROL

We shall consider a very simple example to illustrate some of the theory of the previous section. Consider a simple mechanism with position coordinate, x, and velocity coordinate, y. Suppose it is desired to bring the mechanism to rest by means of a thrust force, u(t), whose magnitude is bidirectional but limited to be less than one in magnitude, and suppose the velocity is not to exceed 0.6 in magnitude. That is, consider the linear system

$$\dot{x} = y$$

$$\dot{y} = u(t)$$

with $|u(t)| \leq 1$, $\Lambda = \left[x, y \,\middle|\, |y| \leq 0.6\right]$, $x(0) = 10$, and $y(0) = 0$.

Pick $$F(x,y) = \frac{1}{2}(y-\frac{1}{2})^2 \quad \text{for } y \geq \frac{1}{2}$$
$$= 0 \qquad \text{for } |y| \leq \frac{1}{2}$$
$$= +\frac{1}{2}(y+\frac{1}{2})^2 \quad \text{for } y \leq -\frac{1}{2} .$$

We shall later determine the parameter, $\beta > 0$, so that the strict bound on y is not exceeded. Problems in which the bound is soft are more easily handled since then we can generally pick β ahead of time and in a straightforward manner solve the two-point boundary value problem. Here, we have picked F(x, y) so that we are constraining the response, even before the boundary of Λ is exceeded, in hopes of maintaining the strict bound on y. To solve this approximate problem, it is merely required that we find a solution of the system:

$$\dot{x}^\circ = F(x,y)$$

$$\dot{x} = y$$

$$\dot{y} = \underset{u \in \Omega}{\text{Max}} \left[\eta_2 u\right]$$

$$\dot{\eta}_o = 0 \quad (\eta_o \leq 0)$$

$$\dot{\eta}_1 = 0$$

$$\dot{\eta}_2 = -\eta_1 - \eta_o \frac{\partial F}{\partial y}$$

with $x^\circ(0) = 0$, $x(0) = 10$, $y(0) = 0$, $x^\circ(t_1) \leq \beta$, $x(t_1) = 0$, $y(t_1) = 0$ for some $t_1 > 0$.

A simple calculation shows that picking $\beta = 0.08$, $\eta_o(0) = -10$, $\eta_1(0) = -1$, $\eta_2(0) \approx -0.55$ provides a time optimal solution for this problem. A plot of this response is shown in Figure 2. Note in this problem the exact optimum solution was obtained, but, in general, one would pick different F(x, y)'s to get better approximations.

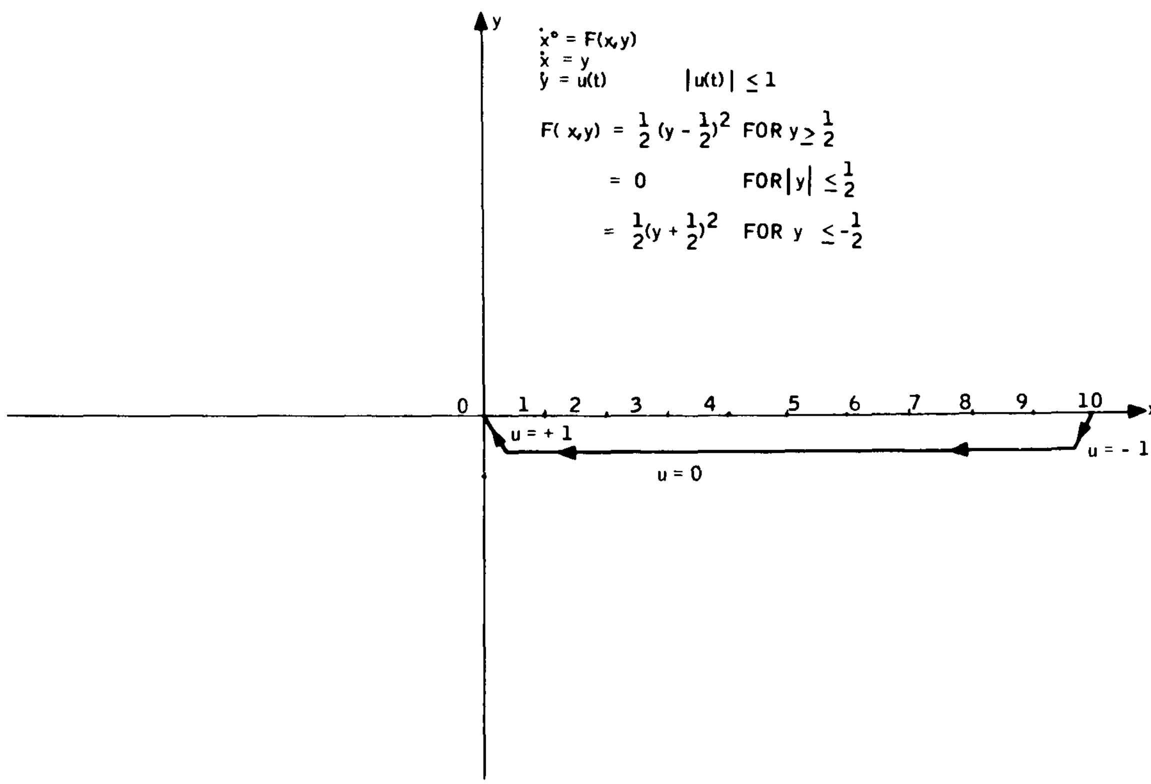

Figure 2. Optimal Solution of the Linear Harmonic Oscillator With Approximate Bounded Phase-Coordinates

4.4 REMARKS ON THE APPROXIMATE BOUNDED PHASE-COORDINATE PROBLEMS WITH INTEGRAL COST

As before, consider the linear control process

$$\mathcal{L})\ \dot{x} = A(t)\,x + B(t)\,u(t)$$

satisfying the conditions stated at the beginning of Section 4.1. As a cost functional of control, consider

$$C(u) = g\left[x(T)\right] + \int_{t_o}^{T} \left[f^{\circ}(x,t) + h^{\circ}(u,t)\right] dt$$

where T = fixed time $> t_o$ and the real functions, $f^{\circ}(x,t)$ and $h^{\circ}(u,t)$, are continuously differentiable and $f^{\circ}(x,t)$ is a convex function of x for each t.

The problem of optimal control is to pick an admissible controller, $u(t)$, on $\left[t_o, T\right]$ so that the response, $x_u(t)$, of $\mathcal{L}$ moves from x_o to a target set, $\tilde{G} \subset R^n$, at T, ($\tilde{G}$ may be whole space) and minimizes $C(u)$ with the entire response, $x_u(t)$, contained in the closed convex restraint set, Λ.

As before, we introduce the convex differentiable function, $F(x)$, satisfying the conditions:

$$\begin{aligned} F(x) &> 0 \text{ if } x \notin \Lambda\ , \\ &= 0 \text{ if } x \in \Lambda\ . \end{aligned}$$

The approximation problem is obtained by adding $F(x)$ to the integrand of the cost functional, $C(u)$, to obtain a new cost functional

$$\begin{aligned} C_{\lambda}(u) &= g\left[x(T)\right] + \int_{t_o}^{T} \left[f^{\circ}(x,t) + \lambda F(x) + h^{\circ}(u,t)\right] dt \\ &= \int_{t_o}^{T} \left[\tilde{f}^{\circ}(x,t) + h^{\circ}(u,t)\right] dt. \end{aligned}$$

Here, $\lambda \geq 0$. If λ is sufficiently large, then one would expect that the contribution from the term, $\lambda F(x)$, can be small only if the response stays near Λ or within it. The approximation problem is to find that controller, $u(t)$, which minimizes $C_\lambda(u)$ and steers to $\mathfrak{G} \subset R^n$.

We shall assume that $h^\circ(u,t)$ is convex in u for each t or that the controller is bounded and h is a positive function of u for each t. In either case, the previous theory can be applied after slight modification by noting that $\tilde{f}^\circ(x,t) = f^\circ(x,t) + \lambda F(x)$ is a convex function of x for each t; since both f° and F were convex functions, and by noting the contribution to $x^\circ(T)$ made by the terms , $h^\circ(u,t)$. That is, the problem has now been cast as one which is covered by the sufficiency results of Reference 24, which are also necessary [4] , and can be obtained as a slight modification of the results of section 4.2.

4.5 CONCLUSION

An approximate solution to linear bounded phase-coordinate control problems is presented in this chapter. The method relies on the introduction of a positive constant, β, which is a measure of the phase-trajectory lying outside the constraint set, Λ , in the phase-coordinate system. For this reason, the problem discussed in this chapter is commonly called the soft bounded phase-coordinate control problem.

CHAPTER 5

MECHANIZATION OF NEUSTADT'S ALGORITHM FOR TIME-OPTIMAL CONTROL ON AN ANALOG COMPUTER

5.1 INTRODUCTION

The technique described allows "on-line" simulation of the time-optimal regulator by adapting Neustadt's algorithm [18] to analog computation. The procedure is an out-growth of attempts to realize bounded-phase coordinate optimum controllers.

5.2 PROBLEM STATEMENT

Given the equations of the controlled system in the form of an n-th-order vector differential equation

$$\dot{x} = A(t)x + B(t)u(t) \tag{90}$$

we seek a control vector, $u^*(t)$, of m components, which steers the system state, $x(t)$, from an initial state, x_o, at time $t = 0$, to final state in which all components of, x, are zero, with the finite time of transition, T, a minimum. An additional provision is that

$$\left|u_j(t)\right| \leq 1; \quad j = 1, 2, \ldots m.$$

5.3 NEUSTADT'S STRATEGY [18]

The variation is parameters formula gives

$$x(t) = X(t)x_o + X(t)\int_o^t X^{-1}(s)B(s)u(s)ds \tag{91}$$

as a representation of the solution of Equation (90). The matrix, $X(t)$, is the matrix solution of the homogeneous part of (90) which becomes the identity matrix for $t = 0$. If $x(t_o) = 0$, multiplication of Equation (90) by $X^{-1}(t_o)$ yields the formula

$$-x_o = + \int_0^{t_o} X^{-1}(s)B(s)u(s)ds. \tag{92}$$

This may be interpreted as producing the set of initial conditions, x_o, from which the origin can be reached in t_o seconds by application of control function $u(t)$ on $[0, t_o]$. Taking the inner product of (92) with an n-vector η, yet to be determined, we have

$$-\eta \cdot x_o = \int_0^{t_o} \eta \cdot X^{-1}(s)B(s)u(s)ds. \tag{93}$$

By selecting

$$u(s) = \mathrm{Sgn}\,[\eta \cdot X^{-1}(s)B(s)] \tag{94}$$

the expression given in (93) is maximized for each η. The time-optimal regulator for normal systems is assured [21] for a particular value, $\eta = \eta_o$. To obtain Neustadt's relationship, define

$$Z(t, \eta) = \int_0^t X^{-1}(s)B(s)\,\mathrm{sgn}\,[\eta \cdot X^{-1}(s)B(s)]\;ds\,. \tag{95}$$

Making use of (95), Equation (93) may be written

$$-\eta \cdot x_o = \eta \cdot Z\,(t, \eta), \text{ when } t = t_o. \tag{96}$$

Expression (96) may be written as

$$0 = \eta \cdot [\, Z(t_o, \eta) + x_o] \,. \tag{97}$$

Equation (97) is satisfied by the η corresponding to the time-optimal regulator for the initial condition, x_o. Neustadt considered the function

$$f(t, \eta; x_o) = \eta \cdot [\, Z(t, \eta) + x_o] \,, \tag{98}$$

and proved the following properties:

(i) $f(t, \eta; x_o)$ is continuous in t and η .

(ii) $f(t, \eta; x_o)$ is strictly increasing with t for a fixed η.

Further insight to the significance of (98) can be gained by graphical arguments. Using (92), it is possible to construct a graph of the set of all initial states from which the origin can be reached in t seconds. Such a graph is shown in Figure 3 for $t_1 < t_2 < t_3 < t_o$. Selecting η_a arbitrarily, the corresponding $Z_a(t, \eta)$ is constructed in Figure 3. Examination of (98) reveals that $f(t, \eta; x_o)$ may be reduced to zero by either of two means:*

(i) Causing the vectors, η and $[\, Z(t, \eta) + x_o]$, to form a right angle, or

(ii) Reducing the vector $[\, Z(t, \eta) + x_o]$, to zero.

Returning to Figure 3, the vector, $[\, Z_a(t, \eta) + x_o]$, is constructed for the particular η_a shown. It is apparent that the angle between η_a and $[\, Z_a(t, \eta) + x_o]$ (i. e., angle θ) will be 90° for some time, $t_1 < t_a < t_3$. Thus, the first of

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*The discussion which follows assumes that $f(t, \eta; x_o)$ is initially negative, which is equivalent to saying that the angle between x_o and η is greater than 90°.

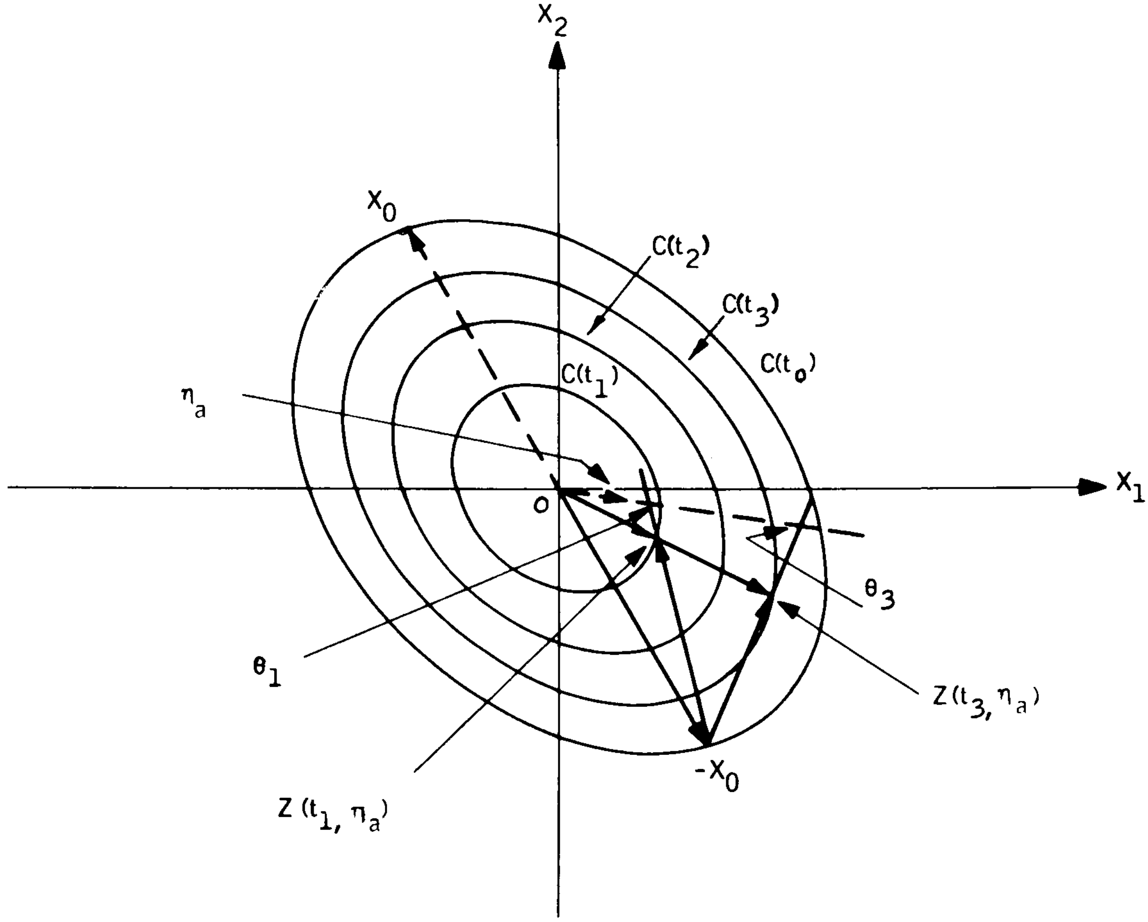

Figure 3. Set C(t) of Reachable Initial States

two conditions necessary for (98) to be zero has been illustrated. It can be further observed that the projection of $[Z_a(t, \eta) + x_o]$ on η_a will be negative for $t < t_a$, and will become positive for $t > t_a$ for a fixed x_o.

The second condition (i. e., reducing $[Z(t, \eta) + x_o]$ to zero) is only possible when $t = t_o$. In addition, $Z(t_o, \eta ; x_o)$ must be coincident with negative x_o, which fixes the corresponding η_o.

The vector, $[Z(t, \eta) + x_o]$, will be nonzero for all $t \neq t_o$, and the projection of η will be outward or positive for all $t > t_o$. Therefore, $t = t_o$ is the upper bound of the zero crossings of $f(t; \eta, x_o)$ considered as a function of t. In other words, $t = t_o$ is the upper bound of ω where $\omega = \{ T \mid f(T; \eta; x_o) = 0 \}$, and further $t_o \epsilon \omega$.

Plots of $f(t, \eta ; x_o)$ versus time for several values of η, as obtained from the preceeding graphical argument, are shown in Figure 4.

5.4 IMPLEMENTATION

Using a circuit which permits maximizing $T \epsilon \omega$, the optimal controller corresponding to a given initial condition can be obtained.

The Bang-Bang or Coulomb Friction Circuit driven by $f(t, \eta; x_o)$ can provide an output of the form

$$
\begin{aligned}
V(t, \eta; x_o) = e_{o(BB)} &= k, \qquad \text{for } f(t, \eta; x_o) < 0, \\
&= 0, \qquad \text{for } f(t, \eta; x_o) > 0.
\end{aligned}
\tag{99}
$$

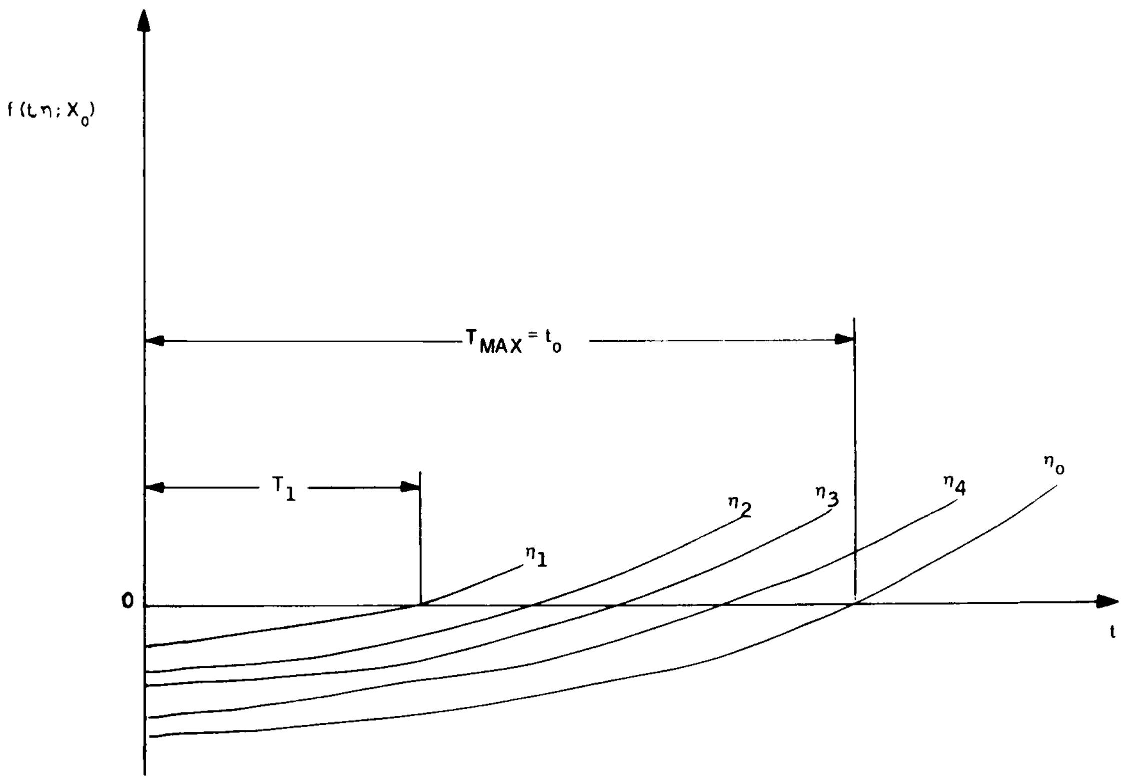

Figure 4. Neustadt's Function f(t, η; x_o) versus Time

By driving an integrator with the output of the Bang-Bang circuit, the output voltage of the integrator will be of the form

$$\begin{aligned} U(t, \eta; x_o) &= e_{o(int)} = kt, \text{ for } t \leq T \\ &= 0 + kT, \text{ for } t > T; \end{aligned} \tag{100}$$

where T is a particular time such that $f(T, \eta; x_o) = 0$. For a given initial condition, x_o, and a trial value, η, (100) is only a function of time. By changing η and repeating the solution, it is possible to obtain a set of curves, $U(t, \eta; x_o)$, which are a function of the time at which $f(t, \eta; x_o)$ is zero. A graph of the set of curves, $U(T, \eta; x_o)$ is shown in Figure 5. To generate $U(T, \eta; x_o)$ with the computer, the form of (98) was modified to

$$\bar{f}(t, \eta; x_o) = \eta(t) \cdot x(t) \tag{101}$$

which is shown equivalent to Neustadt's expression by the following argument.

If $\eta(t) = [X^T(t)]^{-1}\eta_o$, then

$$\eta(t) \cdot x(t) = [X^T(t)]^{-1}\eta_o \cdot [X(t)\, x_o + X(t)\int_o^t X^{-1}(s)B(s)u(s)\, ds] \tag{102}$$

LaSalle [21] proves the optimal steering function to be of the form

$$u(s) = \text{sgn}\, [\eta(t) \cdot X(t)\, X^{-1}(s)\, B(s)]\, , \quad 0 \leq s \leq t,$$

which is equivalent to

$$u(s) = \text{sgn}[\eta_o \cdot X^{-1}(s)B(s)]\, . \tag{103}$$

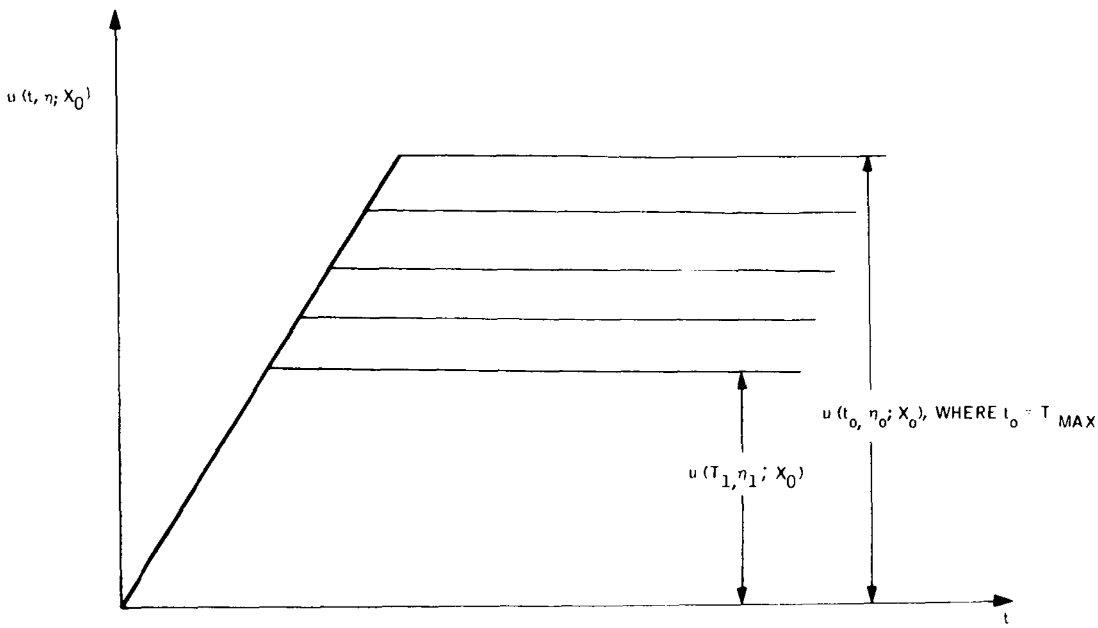

Figure 5. The Maximum Criteria for Analog Simulation

Substituting (103) into (102), we have

$$\eta(t) \cdot x(t) = \eta_o \cdot \left[x_o + \int_o^t X^{-1}(s)B(s)\operatorname{sgn}[\eta_o \cdot X^{-1}(s)B(s)]\, ds, \right. \tag{104}$$

but

$$\int_o^t X^{-1}(s)B(s)\operatorname{sgn}[\eta_o \cdot X^{-1}(s)B(s)]\,ds = Z(t, \eta; x_o).$$

Therefore, (104) can be expressed as

$$\eta(t) \cdot x(t) = \eta_o \cdot [x_o + Z(t, \eta; x_o)], \text{ which} \tag{105}$$

demonstrates that $\bar{f}(t, \eta; x_o) = f(t, \eta; x_o)$.

Advantage was taken of the repetitive solution capabilities of the REAC-C400 analog computer. In this mode, the circuitry functions such that during one-half of a square-wave cycle the integrator capacitor terminals are "shorted" to discharge the capacitor, and during the other half-cycle the integrator is placed in "Operate". Thus, the computer solves the program repeatedly at a frequency determined by an external square-wave generator. The repetitive solutions are then displayed on a large screen oscilloscope, such as the Electromec. By varying the initial values of $\eta_j(o)$; $j = 0, 1, 2, \ldots m$, a continuous display similar to that of Figure 5 is available. The effect of each new setting of the elements of η is immediately apparent and maximization of $\omega(T)$ is facilitated with a maximum of interpretation required.

5.5 APPLICATION

The intended application of Neustadt's algorithm was in conjunction with the "soft-bounded" phase-coordinate problem, more properly termed "the approximate linear time-optimal control process with bounded phase-coordinates" as discussed in Chapter 4.

The problem statement given in (90) is still applicable, with the additional requirement that $x_j(t)$, $j = 1, 2, \ldots m$ remain within a given constraint during its response.

The equations defining the specific case studied (i. e., an augmented harmonic oscillator) are

$$\dot{x}_o = F(x_2)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \operatorname{sgn} \eta_2(t)$$

$$\dot{\eta}_o = 0$$

$$\dot{\eta}_1 = \eta_2$$

$$\dot{\eta}_2 = -\eta_1 - \eta_o \frac{\partial F(x_2)}{\partial x_2},$$

where

$$\begin{aligned} F(x_2) &= \tfrac{1}{2}(x_2 - \tfrac{1}{2})^2, && \text{if } x_2 \geq \tfrac{1}{2} \\ &= 0, && \text{if } |x_2| < \tfrac{1}{2} \\ &= \tfrac{1}{2}(x_2 + \tfrac{1}{2})^2, && \text{if } x_2 \leq \tfrac{1}{2}. \end{aligned}$$

With reference to Chapter 4, the parameters subscripted with a zero result from augmenting the system to permit enforcing the "soft boundary". The augmented system was believed to be normal; however, this assumption later proved to be invalid. The systems non-normality was initially indicated by the potentiometer settings required to maximize the algorithm (i. e., η_o had to be zero, which corresponds to the unbounded case). The level of confidence in these first indications was improved when attempts to determine the required initial values of the elements of η using grid networks were also unsuccessful. Increments of $\frac{1}{100}$ over a range from 1 to 10 were used in the grid networks search . The phase-coordinates of the state vectors were plotted for each trial in the grid network. No trial combination within the grid network resulted in switching or tracking along the boundary. Further refinement of the grids was not considered worthwhile. These simulating results warranted further analytic studies which proved that the coordinate, $\eta(t)$, vanished for a finite time while tracking along the boundary, leaving the controller undefined and non-extremal. Having established that the controller is undefined over a portion of the switching boundary invalidates the mechanization scheme being used. It is interesting to note that the technique "recognized" this condition by only providing data for the unbounded case. Recall that normal systems are required to have no component of $[\eta \cdot X^{-1}(t)B(t)]$ vanish on any interval, with $\eta \neq 0$ [21] .

A detailed treatment of the soft-bounded problem will be discussed in Chapters 6 and 7, since the applicability of the technique under discussion is void for non-normal systems [18] .

In implementing the system of (99), it was observed that the solution obtained with $\eta_o = 0$ was the correct solution to the unbounded problem. This result was anticipated since the systems equations reduce to the unbounded case in that configuration. Several initial conditions of the state variables were investigated, and the required initial conditions of the adjoint vectors were obtained. It should be observed at this point that the selection of initial conditions of the state vectors was conditioned by the particular system of equations, and a desire to compare the results obtained with those obtained in Chapter 3, Section 3. 10, for the same system. The conditions investigated were in the range

$$1 \leq x_1(0) \leq 2, \text{ and}$$

$$x_2(0) = 0.$$

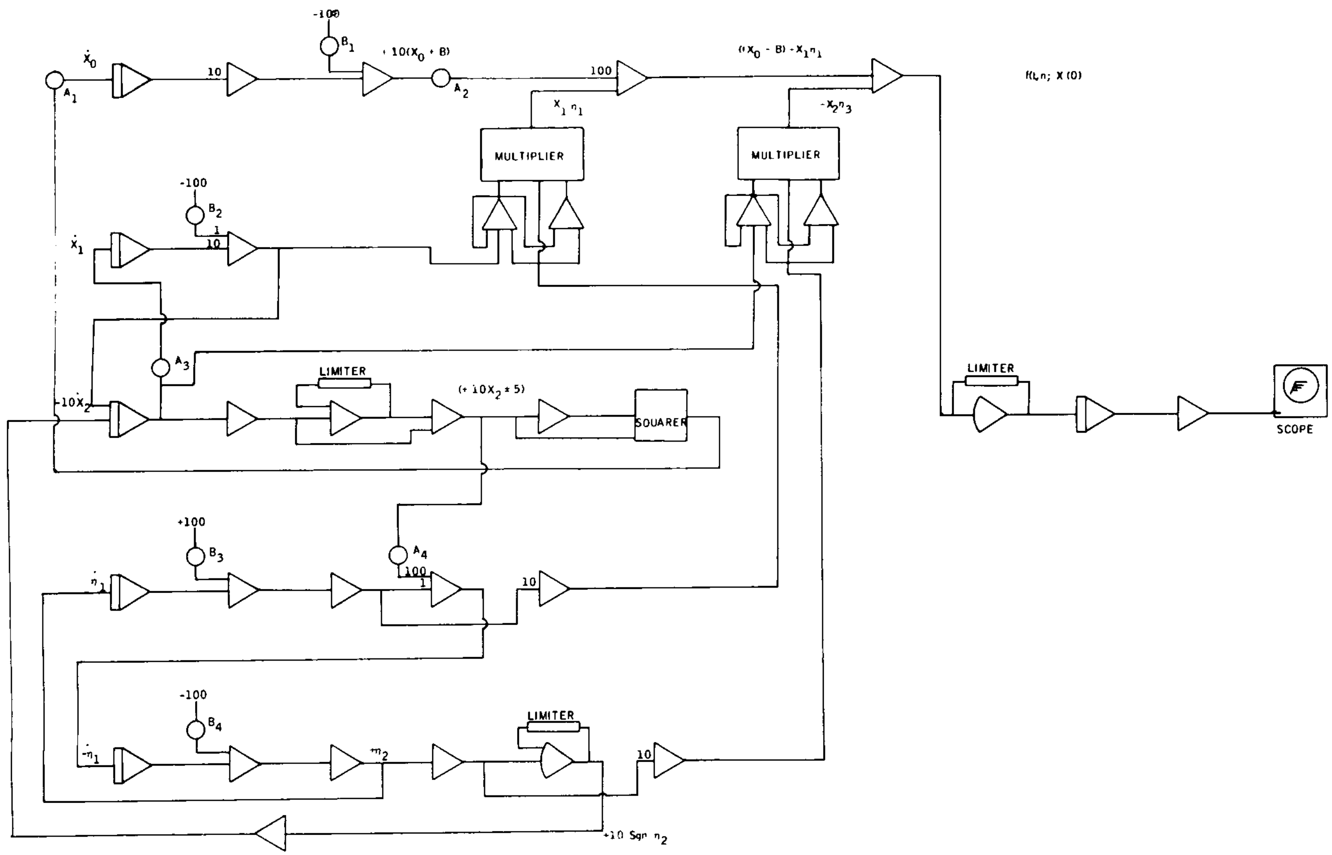

Figure 6. The Analog Computer Program

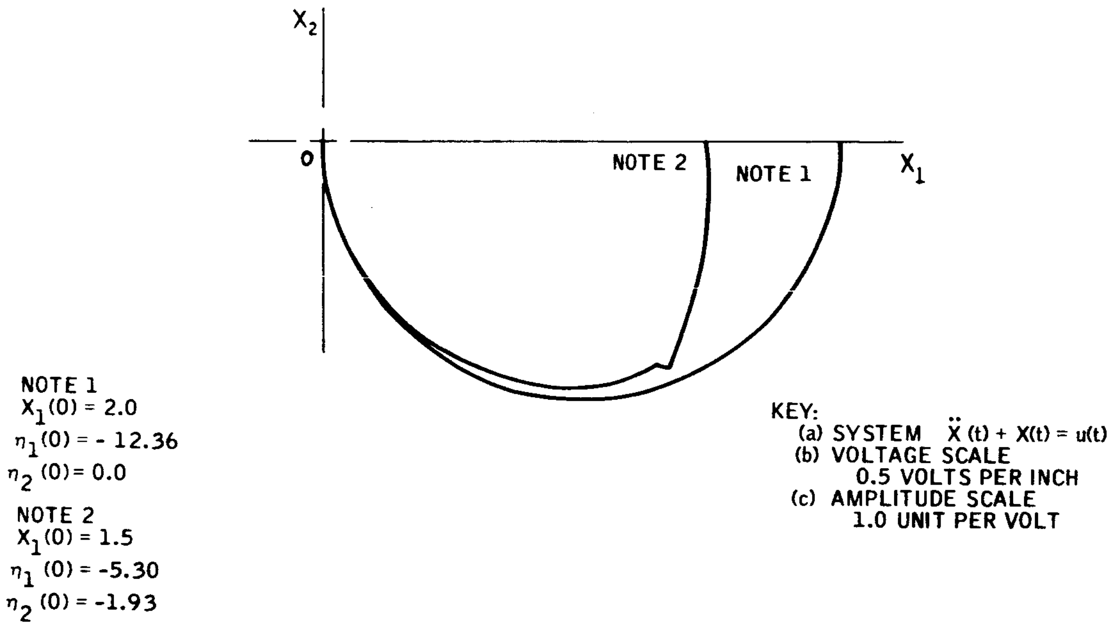

Figure 7. Phase-Plane Plot of the Linear Harmonic Oscillator

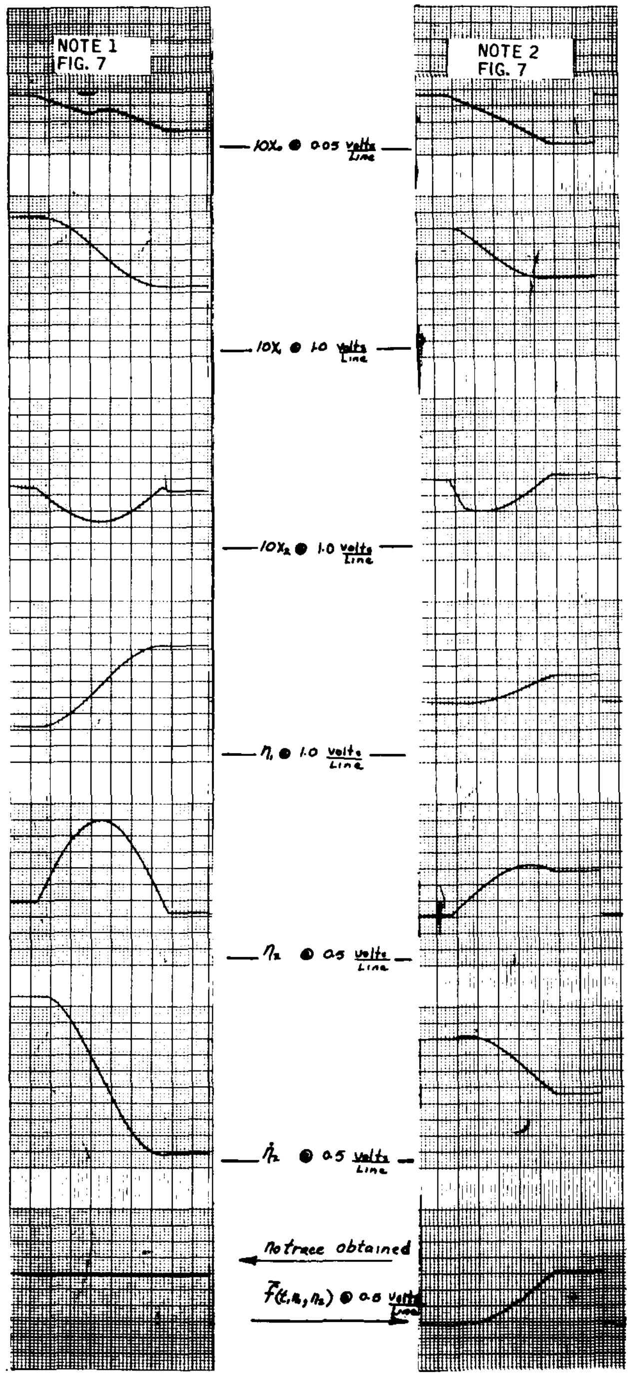

Figure 8. Analog Computer Results for the Linear Harmonic Oscillator

The analog computer program used to simulate the set of equations of (99) and implement Neustadt's algorithm is shown in Figure 6. Two examples of the phase-plane plots are given in Figure 7. For these examples, the time variations of individual parameters are shown in Figure 8.

5.6 CONCLUSION

It is concluded that:

1) The technical is not applicable to the bounded phase-coordinate problem as that problem is presently stated; this will be discussed further in Chapter 6.
2) On-line applications do exist for the unbounded phase-coordinate problem.
3) Where applications exist, complete mechanization of the search procedure (i. e., removing the operator from the loop) should be considered. Open-loop versus closed-loop augmentation also has interesting implications.

The results demonstrated that on-line time-optimal control could be obtained for possibly higher than second-order systems when the plant dynamics are slow, as in chemical reactor control problems where one or two minutes can be spent in obtaining a feasible solution. Also, in such applications a digital computer (of perhaps the Honeywell 200 class) could be programmed to seek the minimum, and thereby completely mechanize the search procedure.

Further, the method as it stands could be used to obtain feasible solutions as needed in training the feedback controller of the logic net mechanization [25] , or other such applications.

CHAPTER 6
ANALOG COMPUTATION OF TIME-OPTIMAL CONTROL FOR APPROXIMATE BOUNDED PHASE-COORDINATE SYSTEMS

6.1 INTRODUCTION

The synthesis method for approximate bounded phase-coordinate time optimal control discussed in Chapter 4 was studied on an analog computer. The results indicated that the method cannot be directly implemented. This chapter gives a detailed discussion on the subject.

As shown in Chapter 4, the subject problem can be stated as follows:

Consider a linear control process described by the system of differential equations

$$\dot{x} = A(t)x + B(t)u(t),$$

where the coefficient matrices, $A(t)$ and $B(t)$, are composed of known continuous functions on the time interval, $[t_o, t_1]$. Find an allowable controller, $u(t)$, which steers $x_u(t)$ from x_o at t_o to a prescribed compact target set, $\tilde{G}$, with $x_u^o(t_1) \leq \beta$ and $t_1 - t_o$ a minimum, where

$$x_u^o(t_1) = \int_{t_o}^{t_1} F[x_u(t)]\, dt,$$

$F(x)$ = convex continuous differentiable function such that

$$F(x) \begin{cases} = 0, & \text{if } x \text{ remains within a given constraint set, } \Lambda; \\ \neq 0, & \text{otherwise.} \end{cases}$$

To solve the problem, the proposed method in Chapter 4 considers an augmented 2n+2 system of equations

$$
S)\ \begin{cases}
\dot{x}^\circ = F(x) \\
\dot{x} = A(t)x + B(t)\ \mathrm{sgn}\ \eta B(t) \\
\dot{\eta}^\circ = 0 \\
\eta = -\eta A(t) - \eta^\circ \dfrac{\partial F'(x)}{\partial x}
\end{cases}
$$

with $x^\circ(t_o) = 0$, $x(t_o) = x_o$, and $\eta^\circ(t_o) < 0$. Let

$$\hat{x} = (x^\circ,\ x),$$

$$\hat{\eta} = (\eta^\circ,\ \eta),$$

and $$\hat{f} = \hat{\eta}\,[\,\hat{x} - \alpha]\,,$$

where

$$
\alpha = \begin{bmatrix} \beta \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}
$$

Then the system, S), could be solved on an analog computer by means of modified Neustadt's algorithm, as discussed in Chapter 5.

6.2 EXPERIMENTAL RESULTS

Two examples were studied on the analog computer:

(a) Pure inertia system $\ddot{x}_1 = u$, with $\Lambda = |\dot{x}_1| \le 0.5$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix},$$

$$F(x) = \begin{cases} 0.5(x_2 - 0.5)^2 & \text{for } x_2 \geq 0.5 \\ 0 & |x_2| \leq 0.5 \\ 0.5\,(x_2 + 0.5)^2 & x_2 \leq -0.5 \end{cases},$$

$$\text{Target} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Various initial conditions, $x(0)$, and different values of β were considered on the computation. A set of typical phase-coordinate trajectories, with $x(0) = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$, $\beta = 0.5$, and various trials of $\hat{\eta}(0)$ is shown in Figure 9. Note that the trajectory which supposedly corresponds to a time-optimal* solution does not pass through the origin, and the trajectory which ends at the origin does not correspond to a time-optimal solution. Moreover, there is no indication showing the effectiveness of the phase-coordinate constraint.

(b) Harmonic oscillator system $\ddot{x}_1 + x_1 = u$, with $\Lambda = |\dot{x}_1| \leq 0.5$

*A time-optimal solution, as discussed in Chapter 5, is determined as follows: Select $\eta^{\circ}(0)$ and $\eta(0)$ such that $\hat{f}(t)\ |(t = 0) > 0$. Integrate the system until $f(T) = 0$. Thus, for each set of $\eta^{\circ}(0)$ and $\eta(0)$, there is a corresponding T. Conditions that give maximum T correspond to a time-optimal solution.

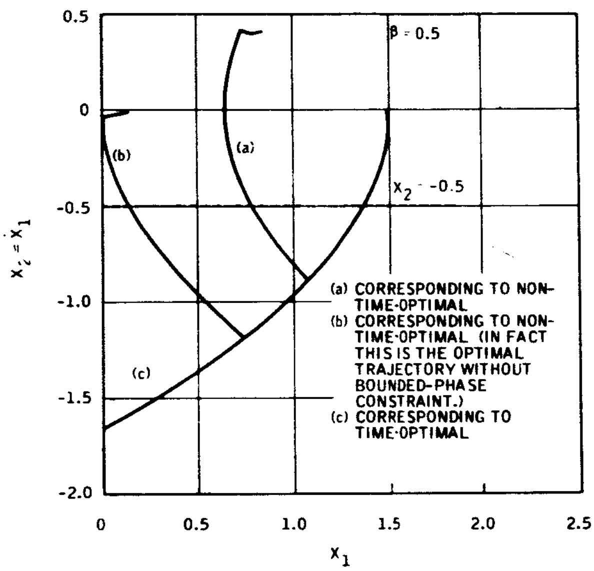

Figure 9. Phase-Coordinate Trajectories of the Pure Inertia System

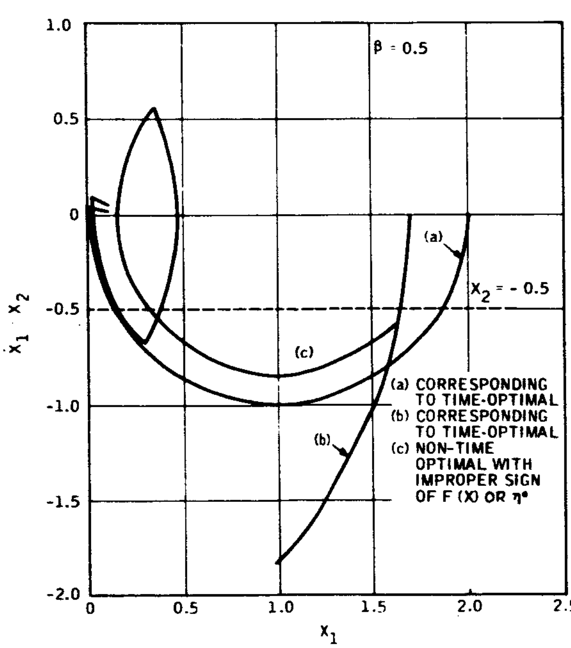

Figure 10. Phase-Coordinate Trajectories of the Linear Harmonic Oscillator

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}$$

F(x) and the target are defined in the same forms as shown in example (a). Figure 10 shows a set of typical phase-coordinate trajectories with $\beta = 0.5$. Note that for the case of $x(0) = \begin{bmatrix} 2.0 \\ 0 \end{bmatrix}$, the time-optimal trajectory with the bounded phase-coordinate constraint is identical to that without the constraint. For $x(0) = \begin{bmatrix} 1.7 \\ 0 \end{bmatrix}$, the trajectory which supposedly corresponds to a time-optimal solution does not pass through the origin. The trajectory which ends at the origin, however, is not only non-time-optimal, but also corresponds to an improper sign of $\eta^\circ \frac{\partial F'(x)}{\partial x}$. The latter violates either the requirement of convexity of F(x), in derivation of the computational method, or the negative value of constant η°, which is a result of the Maximum Principle.

6.3 EXPLANATION OF THE RESULTS

The fact that the computational method could not be implemented directly can be explained by an analysis of a numerical example. Consider the pure inertia system as given in the previous section. Since $x^\circ(0) = x_2(0) = 0$ and $\eta^\circ(0) < 0$, then

$$\hat{f}(0) = \hat{f}\,|_{t=0} = 0.5\,|\eta^\circ(0)| + 1.5\,\eta_1(0) < 0$$

implies $\eta_1(0) < -|\eta^\circ(0)|/3 < 0$. Since $\dot{\eta}_1 = 0$, hence $\eta_1(t)$ is a negative constant.

When $|x_2| \leq 0.5$, the function $F(x) = 0$ and $\frac{\partial F'(x)}{\partial x} = 0$ so that $\dot{\eta}_2 = -\eta_1 > 0$. Since $u(t) = \operatorname{sgn} \eta_2(t)$, hence $\eta_2(0) < 0$ for otherwise the resulting $u(0) = +1$ would steer the system away from the origin. Thus,

•

•

$\eta_2(0)$ = negative constant and

$\dot{\eta}_2(0)$ = positive constant.

Since $x_2(t) = -t$ for $u(t) = -1$, the trajectory will arrive at the phase-coordinate boundary $x_2 = -0.5$ at $t = 0.5$. This is illustrated in Figure 11. Note that $\eta_2(t)$ cannot change sign for $t < 0.5$. If it would, then the switch of the control occurred too early so that the trajectory would not be able to pass through the origin without crossing the other boundary, $x_2 = 0.5$, and followed by at least two more switches of control.

Let $\eta_2(t_1) = 0$ at some $t_1 > 0.5$. Then $\eta_2(t)$ is a parabola bending downward for $0.5 \le t \le t_1$, since $F\left[x(t)\right] = 0.5\left[x_2(t) + 0.5\right]^2 = 0.5\left[-t+0.5\right]^2$, $\int_{0.5}^{t} \frac{\partial F}{\partial x_2}\,dt = -0.5\left[-t + 0.5\right]^2$ and $\eta_2(t) = |\eta_1(0)| -0.5\,|\eta^\circ(0)| \cdot [-t + 0.5]^2$ for $0.5 \le t \le t_1$.

For $t = t_1 + \epsilon$, where ϵ is a small quantity, $\eta_2(t) > 0$, and hence $u(t) = +1$. Since $x_2(t) = +t - 2t_1$ for $u(t) = +1$,

$$\int_{t_1}^{t} \frac{\partial F}{\partial x_2}\,dt = +0.5\left[t - 2t_1 + 0.5\right]^2 - 0.5\left[-t_1 + 0.5\right]^2 \quad \text{and}$$

$$\eta_2(t) = |\eta_1(0)| + 0.5\,|\eta^\circ(0)| \cdot \left\{\left[t - 2t_1 + 0.5\right]^2 - 2\left[-t_1 + 0.5\right]^2\right\}$$ which indicates that $\eta_2(t)$ is now a parabola bending upwards for $t > t_1$. Since $\eta_2(t)$ cannot change sign thereafter, the trajectory will not pass through the origin unless the switch of the control occurs on the switching curve and, in which case, the trajectory with the bounded phase-coordinate constraint is identical to that without the constraint.

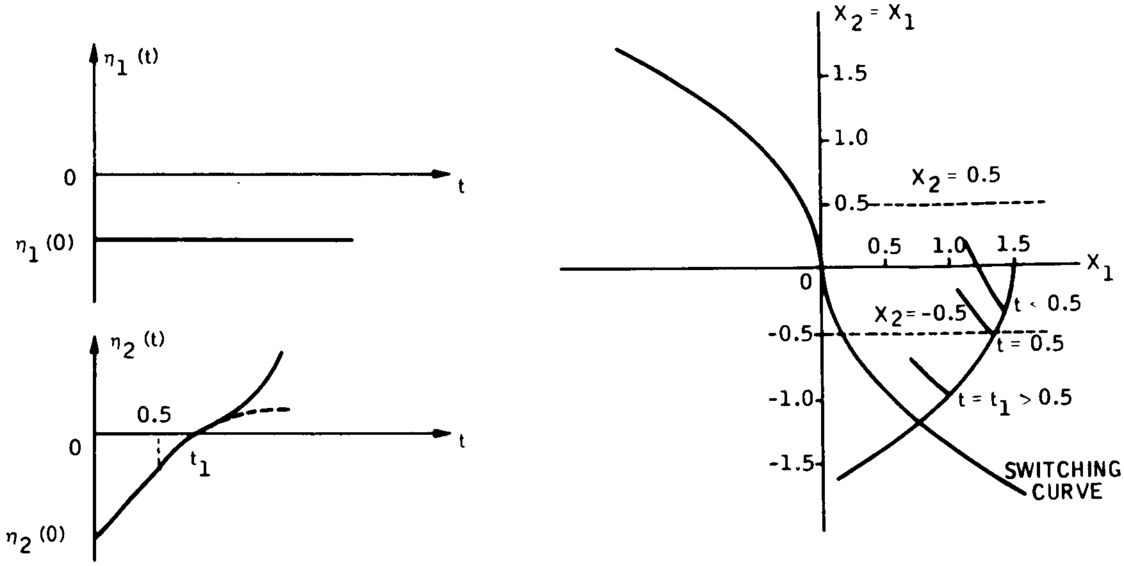

Figure 11. Illustration of the Computational Method

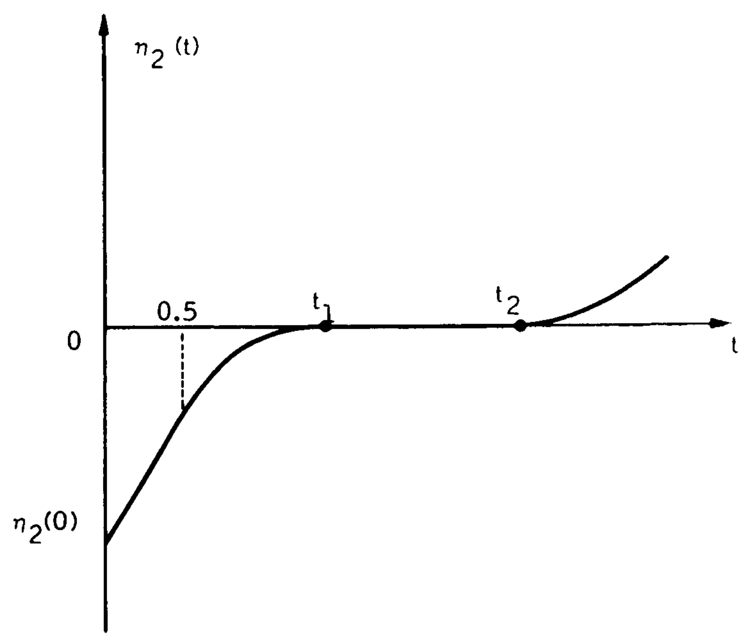

Figure 12. Adjoint Solution with a Singular Arc

6.4 CONCLUSION

From the preceding observation, it is concluded that the method proposed in Chapter 4 cannot be directly implemented. It is reasonable to conjecture, however, that the method is valid if the parabolic portion of η_2 is tangent to the horizontal axis (Figure 12) such that $\eta_2(t) = \dot{\eta}_2(t) = 0$ for $t_1 \leq t \leq t_2$ and $t_1 < t_2$. But this leaves $u(t) = \text{sgn}\ \eta_2(t)$ undefined on $t \in [t_1, t_2]$ which is equivalent to the introduction of a segment of singular arc. For this case, $\hat{f}$ cannot be readily computed, and hence, the modified Neustadt's algorithm given in Chapter 5 does not apply. Further study of the behavior of signular arc is therefore recommended.

CHAPTER 7

APPROXIMATION TO BOUNDED PHASE-COORDINATE CONTROL PROBLEMS WITH INTEGRAL COST

7.1 INTRODUCTION

In Chapter 4, a short discussion of bounded phase-coordinate problems was given. The motivation for this was the use of certain sufficiency conditions and the usual method of handling bounded phase-coordinate problems by soft constraints introduced through a penalty function. It was later realized that the method used to handle the bounded phase-coordinate time-optimal problem was a different method, involving the use of a transversality condition. The use of the transversality condition to obtain a soft constraint appears to be a new way of handling this problem and has not been completely developed and evaluated. It is the purpose of this chapter to indicate the extent to which the theory for the integral cost criterion can be developed along the lines of the previous theory for time-optimal control with the bounded phase-coordinate.

7.2 PROBLEM STATEMENT

As in Chapter 4, consider the linear control process

$$\mathcal{L})\ \dot{x} = A(t)x + B(t)u,\ x(t_o) = x_\bullet,\ u \epsilon \Omega$$

where A(t) and B(t) are n by n and n by m matrices, having bounded and continuous elements on each compact time interval. Ω, the controller restraint set, is assumed to be a compact convex set contained in R^m. The usual notions for compactness, convexity, and so on, for the real number space, R^n, will be used throughout (see Reference 26 for details).

The cost functional of control

$$C(u) = g\left[x(t_1)\right] + \int_{t_o}^{t_1}\left\{f^o\left[x(t), t\right] + h^o\left[u(t), t\right]\right\} dt,$$

where t_1 is a fixed time $> t_o$, and the real functions, $f^o(x, t)$ and $h^o(u, t)$, are continuously differentiable, with f^o and h^o being convex non-negative functions of x and u for each fixed t; and g is assumed to be convex in x and a C^1 function.

The problem of optimal control considered here is to choose an admissible controller, $u(t) \subset \Omega$, on the time interval, $[t_o, t_1]$, so that the response of $\mathcal{L}$ moves from the initial point, x_o, at time, t_o, to a closed target set, $G \subset R^n$, at time t_1, minimizes the cost functional, C(u), and the entire response, $x_u(t)$, is contained in the closed convex restraint set, $\Lambda \subset R^n$.

7.3 EXISTENCE THEOREM, NECESSARY AND SUFFICIENT CONDITIONS FOR THE OPTIMUM CONTROLLER

We introduce the convex differentiable function, $f^{n+1}(x)$, satisfying the conditions:

$$f^{n+1}(x) > 0, \text{ if } x \notin \Lambda,$$
$$= 0, \text{ if } x \in \Lambda.$$

It is at this point that we depart from the previous theory, which was to add $f^{n+1}(x)$ by means of a Lagrange Multiplier λ to the integrand of the integral part of the cost functional and then argue that if λ is sufficiently large the bound on the phase constraint is approximately enforced when C(u) is minimized. Instead, we prescribe a bound, $\beta > 0$, and require

$$\int_{t_o}^{t_1} f^{n+1}\left[x(t)\right] dt \leq \beta.$$

Of course, one way of handling this added inequality is to use the method of Lagrange multipliers, which leads back to the original formulation. We wish to prove existence, as well as give necessary and sufficient conditions, so we will not resort directly to such methods.

Let $\dot{x}^{n+1} = f^{n+1}(x)$,

and $\dot{x}^{o} = f^{o}(x,t) + h^{o}(u,t)$,

with $x^{o}(t_o) = 0 = x^{n+1}(t_o)$. We augment the system, $\mathcal{L}$), by adding these two equations, obtaining the system

$$\tilde{S}) \begin{cases} \dot{x}^{o} = f^{o}(x,t) + h^{o}(u,t) \\ \dot{x} = A(t)x + B(t)u \\ \dot{x}^{n+1} = f^{n+1}(x) \end{cases}$$

with initial data $\tilde{x}_o = \tilde{x}(t_o) = \left[x^{o}(t_o),\ x(t_o),\ x^{n+1}(t_o)\right] = (0,\ x_o,\ 0)$.
Here $\tilde{x} = (x^{o}, x,\ x^{n+1})$.

The soft bound problem (approximation problem) is to find an admissible steering function, $u(t) \subset \Omega$ on $\left[t_o,\ t_1\right]$, steering $\tilde{x}(t)$ from $\tilde{x}_o$ to the closed target set, $\tilde{G} = \{(x^{o},\ x,\ x^{n+1}) \mid 0 \le x^{o} < \infty,\ x \epsilon G,\ 0 \le x^{n+1} < \beta\}$ with minimum cost, $C(u) = x^{o}(t_1) + g\left[x(t_1)\right]$. We hereafter only consider the soft bound problem.

Define the set of attainability, $\tilde{K}(t_1)$, in variables, (x^{o}, x, x^{n+1}), to be the collection of end points, $\tilde{x}(t_1)$, of responses, $\tilde{x}(t)$, of $\tilde{S}$ corresponding to all admissible controllers, $u(t)$, on $\left[t_o, t_1\right]$, with $\tilde{x}(t_o) = \tilde{x}_o$. An admissible controller, $u(t)$, is any measurable controller belonging to the compact convex restraint set, $\Omega \subset R^{m}$.

We now establish that $\tilde{K}(t_1)$ is a compact subset of R^{n+2}, assuring us that optimum controllers exist. We also establish that the lower boundary, in the coordinates, x^{o} and x^{n+1}, is a convex surface, giving us a way of choosing optimum controllers.

We will not work directly with the set of attainability, but with a set, the vertical saturation of the set of attainability, which contains more points. Later, it will be shown that the optimum points for our problem in the vertical saturation set are also points of the set of attainability.

Define the vertical saturation, $\widetilde{K}_v$, of $\widetilde{K}$(assume t_1 is fixed so we drop the dependence on it) to be the set of all points, (x^o, x, x^{n+1}), of R^{n+2} for which there exists a point, (y^o, x, y^{n+1}), in $\widetilde{K}$, with $y^o \le x^o$, $y^{n+1} \le x^{n+1}$. Because of the positive nature of f^o, h^o, and f^{n+1}, it is apparent that $\widetilde{K}_v$ (and therefore $\widetilde{K}$) are contained in the space, $x^o \ge 0$, $x^{n+1} \ge 0$ of R^{n+2}.

<u>Theorem 1</u>.

Consider the controlled process, $\widetilde{S}$ as above, with initial point $(o, x_o, 0) = \widetilde{x}_o$, set of attainability, $\widetilde{K}$, at time, t_1, and compact convex controller restraint set, $\Omega \subset R^m$. Then, the vertical saturation, $\widetilde{K}_v$, of $\widetilde{K}$ is a closed convex set in R^{n+2}.

<u>Proof</u>:

To show that $\widetilde{K}_v$ is closed, consider a sequence of points, $\widetilde{y}_k$, in $\widetilde{K}_v$ converging to $\widetilde{y}$ in R^{n+2}. Since $\widetilde{K}_v$ is the vertical saturation of $\widetilde{K}$, we can find a sequency of controllers, $u^{(k)}(t)$, on t_o, t_1, with responses, $\widetilde{x}_k(t)$, such that $x_k(t_1) = y_k$, $x_k^o(t_1) \le y_k^o$, and $x_k^{n+1}(t_1) \le y_k^{n+1}$. We can further suppose that a subsequence, $u^{(k)}(t)$, still denoted by k, converges weakly to an admissible controller, $u(t) \subset \Omega$, and the corresponding responses, $x_k(t)$, converge to $x(t)$, as is done in problems of time-optimal control [26].

But also $y^o \ge \liminf_{k \to \infty} x_k^o(t_1) \ge x^o(t_1)$ and $y^{n+1} \ge \liminf_{k \to \infty} x_k^{n+1}(t_1) \ge x^{n+1}(t_1)$ which follow, as in the proof of Theorem 8 of Chapter III of Reference 26 and is a general property of convex functions. Hence, the response, $[x^o(t), x(t), x^{n+1}(t)]$, leads to an end point, $[x^o(t_1), y, x^{n+1}(t_1)]$, of $\widetilde{K}$. Thus, (y^o, y, y^{n+1}) lies in $\widetilde{K}_v$ and so $\widetilde{K}_v$ is closed in R^{n+2}.

Note the orthogonal projection of $\tilde{K}$ on the x-space, R^n, is just the compact convex domain, K, of the time-optimal problem [26].

To prove the convexity, we will show that if $\tilde{x}_1$ and $\tilde{x}_2$ are only two points belonging to $\tilde{K}_v$, then the point, $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$, is contained in $\tilde{K}_v$ for $0 \leq \lambda \leq 1$.

Let, then, $\tilde{x}_1$ and $\tilde{x}_2$ belong to $\tilde{K}$ (if they belong to $\tilde{K}_v$ and not $\tilde{K}$, take points of $\tilde{K}$ for which they are the vertical saturation) and suppose $u_1(t)$ and $u_2(t)$ steer to $\tilde{x}_1$ and $\tilde{x}_2$, respectively.

Define the controller, $u_\lambda(t) = \lambda u_1(t) + (1-\lambda)u_2(t)$, and calculate

$$x_\lambda(t_1) = \Phi(t)x_o + \int_{t_o}^{t_1} \Phi(t)\,\Phi(t)^{-1}\,B(t)\,u_\lambda(t)\,dt$$

$$= \lambda\, x_1(t_1) + (1-\lambda)\, x_2\,(t_1) \quad .$$

Further

$$x_\lambda^{\,o}(t_1) = \int_{t_o}^{t_1} f^o\,[x_\lambda(t)] + h^o\,[u_\lambda(t)]\,dt$$

$$\leq \lambda\, x^o_{\;1} + (1-\lambda)\, x_2^{\,o}$$

and

$$x_\lambda^{n+1}(t_1) = \int_{t_o}^{t_1} f^{n+1}\,[x_\lambda(t)]\,dt$$

$$\leq \lambda x_1^{\,n+1} + (1-\lambda)x_2^{\,n+1}$$

because of the convexity of f^o, h^o, f^{n+1}.

Using the property of the vertical saturation, we conclude that there do not exist two points, $(\tilde{x}_1, \tilde{x}_2)$, with $\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$ outside the set, $\tilde{K}_v$, as needed for nonconvexity.

It is shown later that the optimum points of the vertical saturation set for our problem are boundary points. We now characterize the boundary points of $\tilde{K}_v$ which belong to $\tilde{K}$ in terms of maximal controllers.

Define u(t) on $[t_o, t_1]$ to be a maximal controller in case there exists a nonvanishing, adjoint response, $\tilde{\eta}(t) = [\eta_o(t), \eta(t), \eta_{n+1}(t)]$ of:

$$\begin{aligned}
\eta_o &\equiv \text{constant} \leq 0 \\
\dot{\eta} &= -\eta\, A(t) - \eta_o \frac{\partial f^o}{\partial x}(x(t), t) - \eta_{n+1} \frac{\partial f^{n+1}}{\partial x}(x(t)). \\
\eta_{n+1} &\equiv \text{constant} \leq 0
\end{aligned}$$

such that

$$\eta_o h^o[u(t), t] + \eta(t)B(t)u(t) = \underset{u \in \Omega}{\mathrm{Max}} \{\eta_o h^o(u, t) + \eta(t)B(t)u\}$$

almost always on $[t_o, t_1]$. Here, x(t) is the response of $\tilde{S}$) corresponding to the controller, u(t) on $[t_o, t_1]$ and the initial data, (o, x_o, o). A controller steering to a boundary point of a set, $\tilde{N} \subset R^{n+2}$, is termed an extremal controller for $\tilde{N}$.

Theorem 2.

Consider the controlled system, $\tilde{S}$) as above, with initial point $\tilde{x}_o$, at t_o, compact convex controller restraint set, Ω, and closed convex vertical saturation set, $\tilde{K}_v$. If u(t) is a maximal controller, then u(t) is an extremal controller for both $\tilde{K}_v$ and $\tilde{K}$. Conversely, if $\tilde{x}_1$ is boundary point of $\tilde{K}_v$ belonging to $\tilde{K}$, the controller which steers from $\tilde{x}_o$ to $\tilde{x}_1$ is necessarily a maximal controller.

Proof :

Assume the admissible controller, $u(t)$, steers $\tilde{x}(t)$ from $\tilde{x}_o$ to a boundary point, $\tilde{x}_1$, of $\tilde{K}_v$. There exists a support plane, π, of $\tilde{K}_v$, and hence $\tilde{K}$ at $\tilde{x}_1$. Choose $\tilde{\eta}_1$ to be a nonzero vector normal to π and directed into the halfspace defined by π which does meet $\tilde{K}_v$ (and hence does not meet $\tilde{K}$). Let $\tilde{\eta}(t)$ on $[t_o, t_1]$ with $\tilde{\eta}(t) = \tilde{\eta}_1$ be the adjoint response corresponding to the admissible controller, $u(t)$. We wish to show that

$$\eta_o h^o [u(t), t] + \eta(t)B(t)u(t) = \underset{u \epsilon \Omega}{\mathrm{Max}} (\eta_o h^o(u, t) + \eta(t)B(t)u)$$

almost everywhere on t_o, t_1 . The proof will be by contradiction, by supposing that $u(t)$ fails to satisfy this maximum condition on some closed set of positive duration in $[t_o, t_1]$.

Define $\tilde{u}(t)$ on $[t_o, t_1]$ by

$$\eta_o h^o [\tilde{u}(t), t] + \eta(t)B(t)\tilde{u}(t) = \underset{u \epsilon \Omega}{\mathrm{Max}} (\eta_o h^o(u, t) + \eta(t)B(t)u).$$

It is apparent that $\tilde{u}(t)$ is bounded and can be chosen to be measurable as in the Appendix of Chapter II of Reference 26. Let I_1 be a compact subset of positive duration in t_o, t_1 , where $u(t)$ and $\tilde{u}(t)$ are continuous and where

$$\delta + \eta_o h^o [u(t), t] + \eta(t)B(t)u(t) < \eta_o h^o [\tilde{u}(t), t] + \eta(t)B(t)\tilde{u}(t)$$

for some $\delta > 0$. Pick a time, $\tau \epsilon I_1$, so that the set, $(\tau, \tau + \epsilon) \cap I_1$, has measure, $\epsilon(1 + 0(\epsilon))$, for all small $\epsilon > 0$. For given $\delta > o$, consider the modified controller

$$u_\epsilon(t) = \tilde{u}(t) \text{ on } (\tau, \tau + \epsilon) \cap I_1$$
$$= u(t) \text{ elsewhere on } t_o t_1 .$$

Calculate $\int_{t_o}^{t_1} \frac{d\tilde{\eta}\tilde{x}_\epsilon}{dt} dt = \int_{t_o}^{t_1} [\dot{\tilde{\eta}} \tilde{x}_\epsilon + \tilde{\eta}\dot{\tilde{x}}_\epsilon] dt = \tilde{\eta}(t_1) \tilde{x}_\epsilon(t_1) \qquad - \tilde{\eta}(t_o)\tilde{x}_o$

$$= \int_{t_o}^{t_1} \left[\left\{ -\eta(t)A(t) - \eta_o \frac{\partial f^o}{\partial x} [x(t), t] - \eta_{n+1} \frac{\partial f^{n+1}}{\partial x} [x(t)] \right\} x_\epsilon(t) \right.$$

$$+ \eta_o \left\{ f^0 \left[x_\epsilon(t), t\right] + h^o \left[u_\epsilon(t), t\right] \right\} + \eta(t) \left[A(t)x_\epsilon(t) + B(t)u_\epsilon(t)\right\}$$

$$\left. + \eta_{n+1} \left\{ f^{n+1}\left[x_\epsilon(t)\right]\right\}\right] dt \quad ,$$

and
$$\int_{t_o}^{t_1} \frac{d\tilde{\eta}\tilde{x}}{dt}\, dt = \int_{t_o}^{t_1} \left[\dot{\tilde{\eta}}\tilde{x} + \tilde{\eta}\dot{\tilde{x}}\right] dt = \tilde{\eta}(t_1)\ \hat{x}(t_1) - \tilde{\eta}(t_o)\ \tilde{x}_o$$

$$= \int_{t_o}^{t_1} \left[\left\{ -\eta(t)A(t) - \eta_o \frac{\partial f^o}{\partial x}\left[x(t), t\right] - \eta_{n+1} \frac{\partial f^{n+1}}{\partial x}\left[x(t)\right]\right\} x(t)\right.$$

$$+ \eta_o \ f^o\left[x(t), t\right] + h^o \left[u(t), t\right] \Big\} + \eta(t)\left\{A(t)x(t)+B(t)u(t)\right\}$$

$$\left. + \eta_{n+1} \left\{ f^{n+1}\left[x(t)\right]\right\}\right] dt .$$

Thus
$$\tilde{\eta}(t_1)\ \tilde{x}_\epsilon(t_1) - \tilde{\eta}(t_1)\ \tilde{x}(t_1) > \delta\epsilon\left[1 + 0(\epsilon)\right]$$

$$+ \int_{t_o}^{t_1} \left[\eta_o \ f^o\left[x_\epsilon(t), t\right] - \frac{\partial f^o}{\partial x}\left[x(t), t\right]\left[x_\epsilon(t) - x(t)\right] - f^o\left[x(t), t\right]\right.$$

$$\left. + \eta_{n+1}\left\{ f^{n+1}\left[x_\epsilon(t)\right] - \frac{\partial f^{n+1}}{\partial x}\left[x(t)\right]\left[x_\epsilon(t) - x(t)\right] - f^{n+1}\left[x(t)\right]\right\}\right] dt$$

Therefore, for ϵ sufficiency small

$$\tilde{\eta}(t_1)\ \tilde{x}_\epsilon(t_1) > \tilde{\eta}(t_1)\ \tilde{x}(t_1) = \tilde{\eta}(t_1)\ \tilde{x}_1$$

contradicting the construction of $\tilde{\eta}(t_1) = \eta_1$ as the outward normal of $\tilde{K}_v$ at $\tilde{x}_1$.

Conversely, calculate $\frac{d\tilde{\eta}\tilde{x}}{dt}$ and $d\,\frac{\tilde{\eta}\tilde{x}_u}{dt}$, as above, and use the assumed convexity of f^o and f^{n+1} to find

$$\tilde{\eta}(t_1)\tilde{x}(t_1) > \tilde{\eta}(t_1)\tilde{x}_u(t_1)$$

where x_u is a response corresponding to any admissible controller, $u(t) \subset \Omega$, on $\left[t_o, t_1\right]$, and $\tilde{x}(t)$ is a response corresponding to a maximal controller. Thus, for $\eta_o \leq 0$, $\eta_{n+1} \leq 0$, $\tilde{x}(t_1)$ belongs to the boundary of $\tilde{K}$, where the exterior normal vector $\tilde{\eta}(t_1)$ has $\eta_o \leq 0$, $\eta_{n+1} \leq 0$, and therefore to the boundary of $\tilde{K}_v$. Q. E. D.

Lemma

The set of attainability, $\tilde{K}(t_1)$, lies in some sphere, $S(r) = \{\tilde{x} \mid \tilde{x} < r\}$, of finite radius, r, for $0 \leq t_2 \leq \bar{t} < \infty$.

Proof:

It follows immediately from the assumed conditions on A(t), B(t), h^o, f^o, f^{n+1}, and the compactness of the controller restraint set, Ω.

Let $r^o = \sup_{\tilde{y} \epsilon \tilde{K}} y^o$ and $r^{n+1} = \sup_{\tilde{y} \epsilon K} y^{n+1}$

By the Lemma, both r^0 and r^{n+1} are finite. Defining the set,

$$\tilde{M} = \left\{ x^o, x, x^{n+1} \mid \tilde{x} \epsilon K_v, \; x^o \leq r^o, \; x^{n+1} \leq r^{n+1} \right\} \subset R^{n+2},$$

clearly, $\tilde{K} \subset \tilde{M}$, and $\tilde{M}$ is compact.

Theorem 3.

Consider the soft bound controlled process, $\tilde{S}$), as given above. Assume there exists one admissible controller, $u(t) \subset \Omega$, on $[0, t_1]$, steering $\tilde{x}^o$ to the closed target set, $\tilde{G} = \{x^o, x, x^{n+1} \mid x^o < \infty, x \epsilon G, x^{n+1} < \beta\}$, at time t_1. Then, there exists an optimum controller steering $\tilde{x}^o$ to $\tilde{G}$, minimizing the cost functional of control, $C(u) = g\left[x(t_1)\right] + x^o(t_1)$.

Proof

The continuous function, $g(x) + x^o$, assumes a minimum on the compact set, $\tilde{M} \cap \tilde{G}$, at, say, the point, $\tilde{x}^* = (x^{o*}, x^*, x^{n+1*})$. Since $\tilde{K} \subset \tilde{M}$, we must only establish that $\tilde{x}^* \epsilon \tilde{K}$. Fix x and x^{n+1} at x^*, $(x^{n+1})^*$ and consider the function, $g(x^*) + x^o$, on the set, $\tilde{M} \cap \tilde{G}$. The minimum point. $\tilde{x}^*$, must be at a boundary point of the convex set, $\tilde{K}_v$, and hence $\tilde{M}$, since $\tilde{G}$ involves no constraint on x^o. Because of the definition of the vertical saturation set, $\tilde{K}_v$, it is true that this boundary point belongs to $\tilde{K}$ (see Figure 13) (the vertical saturation set contains everything to the right and above the point, $\tilde{x}^*$).

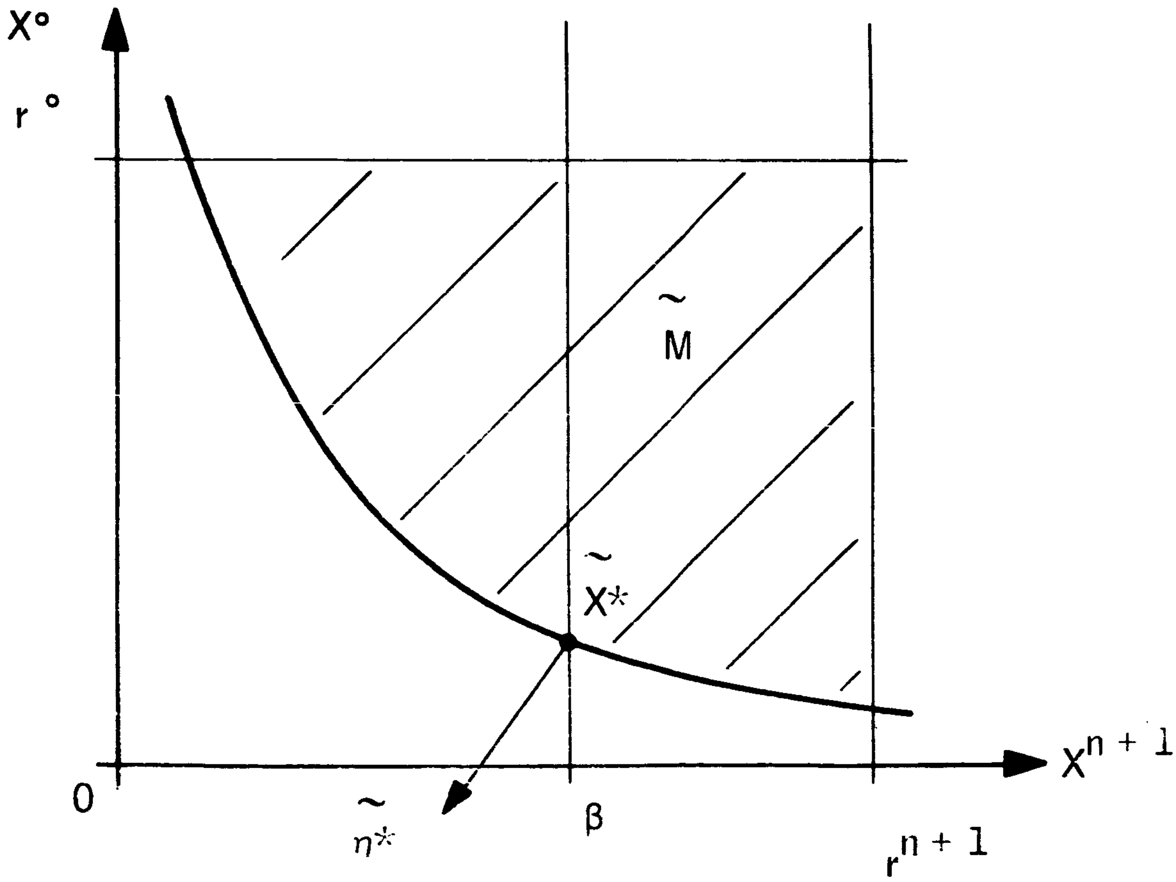

Figure 13. Illustration of the Set of Attainability And An Outward Normal Vector

Theorem 4.

Consider the above soft bound controlled process, $\widetilde{S}$), with cost functional of control, $C(u) = g[x(t_1)] + \int_{t_o}^{t_1} [f^o + h^o]\, dt$

and restraint coordinate

$$x^{n+1}(t_1) = \int_{t_o}^{t_1} f^{n+1}\, dt \leq \beta, \ \beta \geq 0.$$

Assume $g(x) \in C^1$ is convex in R^n and there exists an admissible controller, $u(t)$, on $[t_o, t_1]$, with response, $\tilde{x}(t)$, of $\widetilde{S}$), with $\tilde{x}(t_o) = \tilde{x}_o$ such that $x^{n+1}(t_1) \leq \beta$. Then, there exists a solution, $x^*(t)$, $\eta^*(t)$ of

$$\dot{x} = A(t)x + B(t)\, u^*(t, \eta)$$

$$\dot{\eta} = -\eta_o \frac{\partial f^o}{\partial x}(t, x) - \eta_{n+1} \frac{\partial f^{n+1}}{\partial x}(t, x) - \eta A(t)$$

with $\eta(t_1) = -\operatorname{grad} g[x(t_1)]$, $x(t_o) = x_o$

and either $\eta_o \leq 0$, $\eta_{n+1} < 0$, $x^{n+1}(t_1) = \beta$ or $\eta_{n+1} = 0$, $\eta_o < 0$, $x^{n+1}(t_1) \leq \beta$.

Here, $u^*(t, \eta)$ is defined by the maximum condition

$$\eta_o h^o(u^*, t) + \eta B(t) u^* = \operatorname*{Max}_{u \in \Omega} \left[\eta_o h^o(u, t) + \eta B(t) u\right].$$

An optimum controller is $u^*(t) = u^*\{t, \eta^*(t)\}$ with corresponding optimum response $x^*(t)$.

Proof.

Consider the hypersurfaces, S_c: $g(x) + x^o = c$, in R^{n+2}, of the halfspace $x^{n+1} \leq \beta$. There exists a unique-hypersurface, S_m, of this family such that S_m is tangent to $\widetilde{M}$, defined above, in the halfspace, $x^{n+1} \leq \beta$, and m is the optimal cost. Let $\tilde{\eta}^*(t_1) = \underset{\sim}{[}\eta_o, \eta^*(t_1), \eta_{n+1}]$ be a nonzero vector normal to S_m at some point, $p \in S_m \cap K$, of the halfspace, $x^{n+1} \leq \beta$, and let $\eta^*(t_1)$ be the outward normal to $\tilde{M}$ at p. As in the proof of Theorem 3, it is apparent that such a point, p, exists. Then, $\eta_o \leq 0$, $\eta_{n+1} \leq 0$, (see Figure 13). But, also, $p \in \partial \tilde{K}$ and $p \in \partial \tilde{K}_V$. Thus, $u^*(t)$, the controller steering to p, is a maximal controller by Theorem 2. A careful consideration of Figure 13 gives the two conditions on the end point. Q. E. D.

Remarks

In the case when we have steering to a target, $\tilde{G} = \{x^o, x, x^{n+1} \mid o < x^o < \infty, x \in G, x^{n+1} \leq \beta\}$, with G not the whole space as in Theorem 4, one proceeds in a similar manner to find necessary and sufficient conditions for optimum control. The details are identical to those of Chapter III of Reference 26 for the same type of problem.

7.4 CONCLUSION

In this chapter, the existance as well as necessary and sufficient conditions for the optimum controller with integral cost are established. As in Chapter 4, the method relies on measuring the positive constant, β, and the problem under discussion is of soft bound type. In the discussions, the controller restraint set, Ω, was assumed to be compact. However, it also appears that results along the above lines can be obtained in the case when Ω is not compact. The study of such cases is now underway.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDIES

8.1 SUMMARY OF RESULTS

A fundamental requirement in the proofs of the existence theorems of optimal controls in bounded phase-coordinate systems proposed by Chang [10] is the compactness of the set of all allowed "control-and-path" pairs. This subject was not clearly presented by Chang. A rigorous re-examination within the framework of current mathematics revealed that sequential compactness is guaranteed if the control restraint set is compact and the phase-coordinate restraint set is compact and arc-wise connected. For linear time-optimal control problems, a sufficiency condition* involving the maximum principle and jump conditions in the modified adjoint solution was established. By the utilization of this theorem, the optimal trajectories in the phase-coordinates can be obtained by the usual "backing out" procedure. Conditions under which the optimal solution is unique were established for the case when the phase-coordinate constraint sets are strictly convex but not necessarily compact and also for the case when they are polyhedra.

By introducing a positive constant which measures the excursions of the phase-coordinate trajectories outside their closed convex restraint sets, a method of approximate solution was derived for the time-optimal problem. The excursion constant can be made as small as desired so that the resulting solution approximates the solution that would be obtained by the method of using jump conditions in the modified adjoint solution. The necessary and sufficient conditions were established which are relatively easy to apply. The approximation to the optimal control problem with integral cost functional was also discussed and an existence theorem as well as necessary and sufficient conditions for the control were derived.

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* A necessary condition can be deduced from Neustadt's recent results [27], which is identical with the sufficient condition.

An analog computer program to implement Neustadt's algorithm [18] was developed. The program worked well in the sense of on-line computation for the time-optimal control problem with no constraints in the phase-coordinates, but failed for problems with constaints. An analysis of the program and the bounded phase-coordinate control problem revealed that the difficulty lies in the existence of singular arcs in the adjoint solution which correspond to the segments of the phase-coordinate trajectories lying on the boundries of their restraint sets. Such a situation requires a trial computation procedure with extremely rapid repeated rate which rules out the possiblity of on-line operation for space vehicles using currently available facilities.

8.2 RECOMMENDATIONS FOR FURTHER STUDIES OF THE BOUNDED PHASE-COORDINATE CONTROL PROBLEMS

As shown in Chapter 3, the time-optimal trajectories in the bounded phase-coordinates can be obtained for linear systems. A natural research problem, then, is to extend the results to linear optimal control systems with integral cost and to nonlinear time-optimal systems. New necessary and sufficient conditions for optimal control, from which the optimal trajectories of these controlled systems may be obtained, are thus required. Any results generated from these studies would have vast applications. Moreover, it may serve as a method to check the approximate solution shown in Chapter 7. Another area open for investigation is to develop computational algorithms for optimal controls for practical usage. Unless this is done, relaistic design of the controller is not efficient. This problem is now under investigation at Honeywell Inc., Systems and Research Division.

As to the approximate solution, a problem which involves the non-compact restraint set for the controller is also of importance. The study of this problem is also in process. For the implementation of the method, an investigation of the computational algorithms for a hybrid computer is recommended. The results may lead to a possible on-line operation.

Finally, a discrete approximation of the problem also has potential application. This field is far less developed. Nagata, et al., [17] studied the time-optimal problem by extending the results of Desoer and Wing [28, 29]. The method is quite involved. On the other hand, methods utilizating quadratic functions were developed for the case where the phase-cordinates are not bounded [30, 31, 32]. An extension of this method to bound phase-coordinate systems is well worth investigating.

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