Technical Memorandum No. 33-96

Flow of Plasma in a Slightly Rotational Magnetic Field

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PASADENA, CALIFORNIA

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ABSTRACT

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A solution for a flow of plasma away from the Sun in a slightly rotational magnetic field has been determined, assuming that the plasma is incompressible and inviscid and has infinite electrical conductivity. The magnetic field vector is shown to be a constant multiple of the velocity vector, and the problem is reduced to finding the solutions to the vorticity equation and inhomogeneous elliptic-type partial differential equation. An analytic solution is obtained for the case of constant rotation, and streamlines are plotted showing the influence of a rotational field.

I. INTRODUCTION

The general magnetic field of the Sun approximates an axisymmetric dipole with its axis in the north-south direction. The deviation from the true dipole may, however, be such that the magnetic field takes on a rotational character. The steady flow of plasma away from the Sun under the influence of a rotational magnetic field is the subject of the present study. The irrotational magnetic field case was studied previously (Ref. 1).

It is shown that here, as in the irrotational-flow case, the magnetic field vector is a constant multiple of the velocity vector; the problem reduces to finding the solutions to the vorticity equation and inhomogeneous elliptic-type partial differential equation. An analytic solution is obtained for the case of constant rotation, and streamlines are plotted showing the influence of a rotational field.

The assumptions made are as follows:

(1) The plasma is incompressible and inviscid and has infinite electrical conductivity.

(2) The magnetic field is represented as the sum of irrotational and rotational components, where the latter is very much smaller than the former.

(3) The influence of the rotational component of the magnetic field on each of the other field variables is expressed by small perturbations about the irrotational-flow solution.

(4) The planar-flow case is studied since it simplifies the mathematical treatment considerably while retaining the physical phenomenon of interest here.

II. SMALL PERTURBATION EQUATIONS

The equations describing the steady, continuum flow of incompressible and inviscid plasma in a magnetic field are given below. All variables are expressed in emu.

The continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

The momentum equation

$$\rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{2}$$

Maxwell's equations

$$\nabla \times \mathbf{B} = 4\pi \mathbf{J} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \times \mathbf{E} = 0 \tag{5}$$

Ohm's Law

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{6}$$

where

$\mathbf{u}$ = velocity of fluid

p = static pressure

ρ = density of fluid

$\mathbf{B}$ = magnetic field strength

$\mathbf{J}$ = current density

σ = electrical conductivity

$\mathbf{E}$ = electric field intensity

All the field variables are assumed representable by first-order perturbations about the irrotational-flow solution; hence, let

$$\mathbf{u} = \mathbf{u}_o + \mathbf{u}'$$

$$\mathbf{B} = \mathbf{B}_o + \mathbf{B}'$$

$$p = p_o + p'$$

$$\mathbf{J} = \mathbf{J}_o + \mathbf{J}' = \mathbf{J}'$$

$$\mathbf{E} = \mathbf{E}_o + \mathbf{E}' = \mathbf{E}'$$

where primed quantities denote the small perturbations and subscript o represents the irrotational-flow solution.

Upon substituting these quantities into the continuity, momentum, and Maxwell's equations and the Ohm's Law (neglecting the second-order perturbation terms), the following equations result:

The continuity equation

$$\nabla \cdot \mathbf{u}' = 0 \tag{7}$$

The momentum equation

$$(\nabla \times \mathbf{u}') \times \mathbf{u}_o = -\nabla \left(\mathbf{u}' \cdot \mathbf{u}_o + \frac{p'}{\rho} \right) + \frac{1}{\rho} \mathbf{J}' \times \mathbf{B}_o \tag{8}$$

Maxwell's equations

$$\nabla \times \mathbf{B}' = 4\pi \mathbf{J}' \tag{9}$$

$$\nabla \cdot \mathbf{B}' = 0 \tag{10}$$

$$\nabla \times \mathbf{E}' = 0 \tag{11}$$

Ohm's Law

$$\mathbf{J}' = \sigma (\mathbf{E}' + \mathbf{u}_o \times \mathbf{B}' + \mathbf{u}' \times \mathbf{B}_o) \tag{12}$$

For a fluid with infinite electrical conductivity, Ohm's Law simplifies to

$$\mathbf{E}' = -(\mathbf{u}_o \times \mathbf{B} + \mathbf{u}' \times \mathbf{B}_o)$$

Furthermore, for steady planar flow, $\mathbf{E}' = 0$; hence,

$$\mathbf{u}_o \times \mathbf{B}' + \mathbf{u}' \times \mathbf{B}_o = 0 \tag{13}$$

For the irrotational-flow case, it was found that the magnetic field was a constant multiple of the velocity field, or

$$\mathbf{B}_o = \lambda_o \mathbf{u}_o \tag{14}$$

where λ_o is a constant. Upon substitution in Eq. (13), the following results:

$$(\lambda_o \mathbf{u}' - \mathbf{B}') \times \mathbf{u}_o = 0 \tag{15}$$

Using the infinite electrical conductivity and planar flow conditions, Ohm's Law given in the form of Eq. (6) can also be written as

$$\mathbf{B}_o + \mathbf{B}' = \lambda (\mathbf{u}_o + \mathbf{u}') \tag{16}$$

where λ is an arbitrary scalar function of position. After substituting for $\mathbf{B}_o$ from Eq. (14) and simplifying,

$$\mathbf{B}' = (\lambda - \lambda_o) \mathbf{u}_o + \lambda \mathbf{u}' \tag{17}$$

Eliminating $\mathbf{B}'$ from Eqs. (17) and (15),

$$[(\lambda_o - \lambda)\,\mathbf{u}' - (\lambda - \lambda_o)\,\mathbf{u}_o] \times \mathbf{u}_o = 0$$

or

$$(\lambda_o - \lambda)\,\mathbf{u}' \times \mathbf{u}_o = 0$$

Aside from the special case of the perturbed velocity being everywhere parallel to the irrotational-flow velocity, this equation is satisfied if

$$\lambda = \lambda_o$$

From Eq. (16), there results

$$\mathbf{B}' = \lambda_o \mathbf{u}' \tag{18}$$

It is thus concluded that the magnetic field vector is everywhere a constant multiple of the velocity vector.

The vorticity equation is obtained by eliminating $\mathbf{J}'$ and $\mathbf{B}'$ from the momentum equation (Eq. 8) by using Eqs. (9), (14), and (18) and then taking the curl of both sides. After simplification, the following equation results:

$$\nabla \times [\mathbf{\Omega} \times \mathbf{u}_o]\left(1 - \frac{\lambda_o^2}{4\pi\rho}\right) = 0 \tag{19}$$

where

$$\mathbf{\Omega} = \nabla \times \mathbf{u} = \nabla \times \mathbf{u}' \tag{20}$$

is the vorticity vector. Since λ_o is arbitrary, Eq. (19) reduces to

$$\nabla \times [\mathbf{\Omega} \times \mathbf{u}_o] = 0 \tag{21}$$

Expanding and noting that div $\mathbf{u}_o = 0$ and $\mathbf{\Omega} = \Omega \mathbf{i}_z$, where $\mathbf{i}_z$ is a unit vector normal to the plane containing the velocity vector, Eq. (21) becomes

$$u_{r_o}\frac{\partial \Omega}{\partial r} + \frac{u_{\theta_o}}{r}\frac{\partial \Omega}{\partial \theta} = 0 \tag{22}$$

The solution to this equation, being a linear partial differential equation of the first order, can be expressed in terms of its characteristics.

$$\frac{dr}{u_{r_o}} = \frac{r d\theta}{u_{\theta_o}} \tag{23}$$

where u_{r_o} and u_{θ_o}, the irrotational-flow velocity components, are assumed known. Equation (23) is also recognized as the equation for the streamlines of the irrotational-flow solution. Letting

$$f(r, \theta) = \psi_n$$

be the parametric solution of Eq. (23) with the parameter ψ_n defined by $f(r_a, \theta_k) = \psi_k$, the solution of Eq. (22) can be expressed as

$$\Omega = \Omega(f - \psi_n) \tag{24}$$

The vorticity, therefore, remains constant along a streamline. If the vorticity is specified along a boundary, $r = r_a$, then its value is given everywhere by Eq. (24).

III. SOLUTION FOR ROTATIONAL VELOCITY FIELD

Considering a region bounded by $r_a \leqq r \leqq r_b$ and $0 \leqq \theta \leqq \pi/2$, the problem is reduced to finding the velocity at any interior point, knowing the boundary conditions and the vorticity distribution in the region. The equations to be solved simultaneously for the velocity components are the continuity equation and the defining equation for vorticity which are, in component form,

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \tag{25}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \Omega \tag{26}$$

By solving for u_θ between the two equations, the following inhomogeneous partial differential equation of the elliptic type results:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^3 \frac{\partial u_\theta}{\partial r}\right) + u_\theta + \frac{\partial^2 u_\theta}{\partial \theta^2} = f(r,\theta) \tag{27}$$

where

$$f(r,\theta) = r^2 \frac{\partial \Omega}{\partial r} + 2r\Omega \tag{28}$$

The boundary conditions are

$$\begin{aligned} u_\theta(r_a,\theta) &= u_{\theta_a}(\theta) & u_\theta(r_b,\theta) &= 0 \\ u_\theta(r,0) &= 0 & u_\theta(r,\pi/2) &= 0 \end{aligned} \tag{29}$$

where $u_{\theta_a}(\theta)$ is assumed known for the time being and r_b is taken large enough so that the velocity is essentially radial beyond this radius.

To show explicitly the dependence of the solution on the boundary conditions and the vorticity distribution, Green's method is used in solving Eq. (27). Green's function, $G(r,\theta\,|\,r_o,\theta_o)$, satisfies the following differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r^3 \frac{\partial G(r,\theta\,|\,r_o,\theta_o)}{\partial r}\right] + G(r,\theta\,|\,r_o,\theta_o) + \frac{\partial^2 G(r,\theta\,|\,r_o,\theta_o)}{\partial \theta^2} = \frac{\delta(r-r_o)\,\delta(\theta-\theta_o)}{r} \tag{30}$$

where the subscript o denotes the source coordinates and δ, the Dirac delta function. The boundary conditions are

$$\begin{aligned} G(r,\theta\,|\,r_a,\theta_o) &= 0 & G(r,\theta\,|\,r_b,\theta_o) &= 0 \\ G(r,\theta\,|\,r_o,0) &= 0 & G(r,\theta\,|\,r_o,\pi/2) &= 0 \end{aligned} \tag{31}$$

After multiplying Eq. (27) by $G(r,\theta\,|\,r_o,\theta_o)$ and Eq. (30) by $u_\theta(r,\theta)$ and subtracting, then the resulting equation is integrated over the source coordinates, keeping in mind Green's theorem and the symmetry property of Green's function. After insertion of the boundary conditions and simplification, the θ component of the velocity becomes

$$u_\theta(r,\theta) = \int_{r_a}^{r_b}\int_0^{\pi/2} f(r_o,\theta_o)\,G(r,\theta\,|\,r_o,\theta_o)\,r_o\,dr_o\,d\theta_o - \int_0^{\pi/2} u_{\theta_a}(\theta_o)\,r_a^3 \left.\frac{\partial G(r,\theta\,|\,r_o,\theta_o)}{\partial r_o}\right|_{r_a} d\theta_o \tag{32}$$

It is assumed that $u_{\theta_a}(\theta)$ is known; however, in most physical situations $u_{r_a}(\theta)$ is given instead. The two variables are connected by the continuity equation. Inserting Eq. (32) into the continuity equation and integrating, the following relation is obtained:

$$\begin{aligned} u_r(r,\theta) = \frac{g(\theta)}{r} &- \frac{1}{r}\int \frac{\partial}{\partial \theta}\int_{r_a}^{r_b}\int_0^{\pi/2} f(r_o,\theta_o)\,G(r,\theta\,|\,r_o,\theta_o)\,r_o\,dr_o\,d\theta_o\,dr \\ &+ \frac{1}{r}\int \frac{\partial}{\partial \theta}\int_0^{\pi/2} u_{\theta_a}(\theta_o)\,r_a^3 \left.\frac{\partial G(r,\theta\,|\,r_o,\theta_o)}{\partial r_o}\right|_{r_o=r_a} d\theta_o\,dr \end{aligned} \tag{33}$$

where $g(\theta)$ is an arbitrary function. Letting $r = r_a$ in Eq. (33) and interchanging the order of integration, the following linear integral equation results:

$$u_{r_a}(\theta) = \frac{g(\theta)}{r_a} + \int_0^{\pi/2} u_{\theta_a}(\theta_o)\,r_a^2 \times \int^{r_a} \left.\frac{\partial^2 G(r,\theta\,|\,r_o,\theta_o)}{\partial \theta\,\partial r_o}\right|_{r_o=r_a} dr\,d\theta_o + \frac{g(\theta)}{r_a} \tag{34}$$

This equation will be solved subsequently for a specific example.

To solve for Green's function, which is a solution of Eq. (30), it is assumed that the θ dependence is expressible as a Fourier series, or

$$G(r, \theta | r_o, \theta_o) = \sum_{n=1}^{\infty} g_n(r|r_o) \frac{4}{\pi} \sin 2n\theta_o \sin 2n\theta \tag{35}$$

where the coefficients g_n are functions of r only. Substituting in Eq. (30) and simplifying,

$$\sum_{n=1}^{\infty} \frac{4}{\pi} \sin 2n\theta_o \sin 2n\theta \left\{ \frac{1}{r} \frac{d}{dr} \left(r^3 \frac{dg_n}{dr} \right) + (1 - 4n^2) g_n \right\} = \frac{\delta(r - r_o)\, \delta(\theta - \theta_o)}{r}$$

After multiplying both sides by $\sin 2m\theta$ and integrating over the range 0 to $\pi/2$, a linear nonhomogeneous differential equation of the Sturm–Liouville type results.

$$\frac{1}{r} \frac{d}{dr} \left(r^3 \frac{dg_n}{dr} \right) + (1 - 4n^2) g_n = \frac{\delta(r - r_o)}{r} \tag{36}$$

The boundary conditions follow from Eq. (31) and are

$$g_n(r_a | r_o) = 0 \qquad g_n(r_b | r_o) = 0 \tag{37}$$

Following Morse and Feshbach (Ref. 2) the solution to Eq. (36) is given by

$$g_n(r|r_o) = \frac{1}{r_o^3 \Delta(y_1 y_2)} \begin{cases} y_1(r)\, y_2(r_o); & r \leq r_o \\ y_2(r)\, y_1(r_o); & r \geq r_o \end{cases} \tag{38}$$

where $y_1(r)$ and $y_2(r)$ are two independent solutions of the following homogeneous equation:

$$\frac{1}{r} \frac{d}{dr} \left(r^3 \frac{dg}{dr} \right) + (1 - 4n^2) g = 0$$

and $\Delta(y_1 y_2)$ is the Wronskian. These values are given by

$$y_1(x) = r_a^{-1-2n}\, r^{-1+2n} - r_a^{-1+2n}\, r^{-1-2n}$$

$$y_2(x) = r_b^{-1-2n}\, r^{-1+2n} - r_b^{-1+2n}\, r^{-1-2n}$$

$$\Delta(y_1 y_2) = \frac{4n}{r^3 r_a r_b} \left[\left(\frac{r_b}{r_a} \right)^{2n} - \left(\frac{r_a}{r_b} \right)^{2n} \right]$$

Substitute in Eq. (38) and then in Eq. (35). The final form of Green's function becomes

$$G(r, \theta | r_o, \theta_o) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin 2n\theta_o \sin 2n\theta \frac{r_a r_b}{\left[\left(\frac{r_b}{r_a} \right)^{2n} - \left(\frac{r_a}{r_b} \right)^{2n} \right]}$$
$$\times \begin{cases} (r_a^{-1-2n}\, r^{-1+2n} - r_a^{-1+2n}\, r^{-1-2n}) \\ (r_b^{-1-2n}\, r^{-1+2n} - r_b^{-1+2n}\, r^{-1-2n}) \end{cases}$$
$$\times \begin{cases} (r_b^{-1-2n}\, r_o^{-1+2n} - r_b^{-1+2n}\, r_o^{-1-2n}) & r \leq r_o \\ (r_a^{-1-2n}\, r_o^{-1+2n} - r_a^{-1+2n}\, r_o^{-1-2n}) & r \geq r_o \end{cases} \tag{39}$$

The derivative occurring in Eq. (32) becomes

$$\left. \frac{\partial G(r, \theta | r_o, \theta_o)}{\partial r_o} \right|_{r_a} = \sum_{n=1}^{\infty} \frac{4}{\pi} \sin 2n\theta_o \sin 2n\theta \frac{\left(\frac{r}{r_b} \right)^{2n} - \left(\frac{r_b}{r} \right)^{2n}}{r_a^2 r \left[\left(\frac{r_b}{r_a} \right)^{2n} - \left(\frac{r_a}{r_b} \right)^{2n} \right]} \tag{40}$$

and the kernal occurring in Eq. (34) becomes

$$\int^{r_a} \left. \frac{\partial^2 G(r_a, \theta | r_o, \theta_o)}{\partial \theta\, \partial r_o} \right|_{r_o = r_a} dr$$
$$= \sum_{n=1}^{\infty} \frac{4}{\pi} \sin 2n\theta_o \cos 2n\theta \frac{\left[\left(\frac{r_a}{r_b} \right)^{2n} + \left(\frac{r_b}{r_a} \right)^{2n} \right]}{r_a^2 \left[\left(\frac{r_b}{r_a} \right)^{2n} - \left(\frac{r_a}{r_b} \right)^{2n} \right]} \tag{41}$$

In conclusion, it is seen that the specification of vorticity distribution and tangential or radial component of velocity on the inner boundary determines the problem completely. The computational procedure is summarized as follows:

(1) If given for the inner boundary, the vorticity distribution is computed everywhere in the region using Eq. (24).

(2) The source function, $f(r, \theta)$, is next computed from Eq. (28).

(3) The velocity distribution is then computed from Eqs. (32) and (33). If the radial velocity is specified on the inner boundary, Eq. (34) is used to generate the tangential component of velocity.

(4) The magnetic field is finally computed from Eq. (16).

IV. APPLICATIONS

A. Irrotational Flow

On the inner boundary, $r = r_a$, it is assumed that the rotation vector Ω is zero, and the radial velocity component is given by

$$u_{r_a}(\theta) = \frac{\mu \sin\theta}{r_a^2}$$

where μ is the strength of a two-dimensional dipole. The outer boundary is placed far enough from the inner boundary so that the flow field is essentially radial beyond this radius. Here, it is placed at $r_b = 4r_a$.

The rotation vector is seen to be zero everywhere from Eq. (24); hence, the source function, $f(r,\theta)$, is also zero from Eq. (28).

The tangential component of the velocity on the inner boundary, $u_{\theta_a}(\theta)$, is computed from Eq. (34). Inserting the value of the kernal (Eq. 41) into Eq. (34), the following linear integral equation is obtained:

$$\frac{\mu\sin\theta}{r_a^2} = \frac{g(\theta)}{r_a} + \int_0^{\pi/2} u_{\theta_a}(\theta_o) \sum_{n=1}^{\infty} \frac{4}{\pi} \sin 2n\theta_o \cos 2n\theta \, d\theta_o \frac{\left(\frac{r_a}{r_b}\right)^{2n} + \left(\frac{r_b}{r_a}\right)^{2n}}{\left(\frac{r_b}{r_a}\right)^{2n} - \left(\frac{r_a}{r_b}\right)^{2n}} \tag{42}$$

Assume that $u_{\theta_a}(\theta_o)$ can be expanded in a Fourier series.

$$u_{\theta_a}(\theta_o) = \sum_{k=1}^{\infty} A_k \sin 2k\theta_o \tag{43}$$

Substituting for $u_{\theta_a}(\theta_o)$ in Eq. (42) and performing the integration, the following equation results:

$$\frac{\mu\sin\theta}{r_a^2} = \frac{g}{r_a} + \sum_{n=1}^{\infty} A_n \cos 2n\theta \frac{\left(\frac{r_a}{r_b}\right)^{4n} + 1}{1 - \left(\frac{r_a}{r_b}\right)^{4n}}$$

Multiplying both sides by $\cos 2n\theta$ and integrating from $\theta = 0$ to $\pi/2$, the coefficient A_n becomes

$$A_n = \frac{4\mu}{\pi r_a^2} \frac{1}{1-4n^2} \frac{1 - \left(\frac{r_a}{r_b}\right)^{4n}}{1 + \left(\frac{r_a}{r_b}\right)^{4n}}$$

and

$$g = \frac{2\mu}{\pi r_a}$$

Hence, the tangential component of velocity on the inner boundary becomes

$$u_{\theta_a}(\theta) = \sum_{n=1}^{\infty} \frac{4\mu}{\pi r_a^2} \frac{1}{1-4n^2} \frac{1 - \left(\frac{r_a}{r_b}\right)^{4n}}{1 + \left(\frac{r_a}{r_b}\right)^{4n}} \sin 2n\theta \tag{44}$$

The velocity distribution is finally obtained by inserting Eq. (44) into Eqs. (32) and (33) and performing the indicated operations. The velocity components become

$$u_r(r,\theta) = \frac{2\mu}{\pi r_a r} + \sum_{n=1}^{\infty} \frac{4\mu}{\pi(1-4n^2)} \frac{1}{r_a r} \frac{\left(\frac{r_a r}{r_b^2}\right)^{2n} + \left(\frac{r_a}{r}\right)^{2n}}{1 + \left(\frac{r_a}{r_b}\right)^{4n}} \cos 2n\theta \tag{45}$$

$$u_\theta(r,\theta) = -\sum_{n=1}^{\infty} \frac{4\mu}{\pi(1-4n^2)} \frac{1}{r_a^2} \frac{\left(\frac{r_a r}{r_b^2}\right)^{2n} - \left(\frac{r_a}{r}\right)^{2n}}{1 + \left(\frac{r_a}{r_b}\right)^{4n}} \sin 2n\theta \tag{46}$$

B. Rotational Flow

The inner boundary is assumed to have a tangential velocity distribution (as given by Eq. 44) equal to that for the irrotational-flow case. Also, on the inner boundary the rotation vector is assumed constant, which means that it is constant everywhere by Eq. (24). The outer boundary is again taken at $r_b = 4r_a$.

Substituting Eqs. (28), (39), (40), and (44) into Eqs. (32) and (33) and performing the indicated operations, the velocity components become

$$u_\theta(r,\theta) = \sum_{n=1}^{\infty} \frac{4\mu}{\pi r_a^2} \frac{1}{4n^2-1} \frac{\left(\frac{r_a r}{r_b^2}\right)^{2n} - \left(\frac{r_a}{r}\right)^{2n}}{1+\left(\frac{r_a}{r_b}\right)^{4n}} \sin 2n\theta$$

$$+ \sum_{n\ \mathrm{odd}} \frac{2\Omega A_n}{\pi\left[1-\left(\frac{r_a}{r_b}\right)^{4n}\right]} \sin 2n\theta \qquad (47)$$

$$u_r(r,\theta) = \frac{2\mu}{\pi r_a r} + \sum_{n=1}^{\infty} \frac{4\mu}{\pi r_a r} \frac{1}{(1-4n^2)} \frac{\left(\frac{r_a r}{r_b^2}\right)^{2n} + \left(\frac{r_a}{r}\right)^{2n}}{1+\left(\frac{r_a}{r_b}\right)^{4n}} \cos 2n\theta$$

$$- \sum_{n\ \mathrm{odd}} \frac{4\Omega B_n}{\pi n\left[1-\left(\frac{r_a}{r_b}\right)^{4n}\right]} \cos 2n\theta \qquad (48)$$

where

$$A_1 = \left[1-\left(\frac{r_a}{r_b}\right)^4\right] r \ln r + \left[\left(\frac{r_a}{r_b}\right)^4 \ln r_a - \ln r_b\right] r + \frac{r_a^4}{r^3} \ln \frac{r_b}{r_a}$$

$$A_n = \left[1-\left(\frac{r_a}{r_b}\right)^{4n}\right] \frac{nr}{1-n^2} + \left[1-\left(\frac{r_a}{r_b}\right)^{2n+2}\right]\left(\frac{n}{n^2-1}\right)\frac{r^{2n-1}}{r_b^{2n-2}}$$
$$+ \left[1-\left(\frac{r_a}{r_b}\right)^{2n-2}\right]\left(\frac{n}{n^2-1}\right)\frac{r_a^{2n+2}}{r^{2n+1}} \qquad n>1$$

$$B_1 = \left[1-\left(\frac{r_a}{r_b}\right)^4\right] r \left[\frac{\ln r}{2} - \frac{1}{4}\right]$$
$$+ \left[\left(\frac{r_a}{r_b}\right)^4 \ln r_a - \ln r_b\right]\frac{r}{2} - \frac{r_a^4}{2r^3} \ln \frac{r_b}{r_a}$$

$$B_n = \left[1-\left(\frac{r_a}{r_b}\right)^{4n}\right] \frac{nr}{2(1-n^2)} + \left[1-\left(\frac{r_a}{r_b}\right)^{2n+2}\right]\left(\frac{1}{n^2-1}\right)\frac{r^{2n-1}}{2r_b^{2n-2}}$$
$$- \left[1-\left(\frac{r_a}{r_b}\right)^{2n-2}\right]\left(\frac{1}{n^2-1}\right)\frac{r_a^{2n+2}}{2r^{2n+1}} \qquad n>1$$

The flow field showing the effect of rotation on the streamlines obtained from Eqs. (47) and (48) is shown in Fig. 1 (a–d). Here, μ and r_a were taken to be unity.

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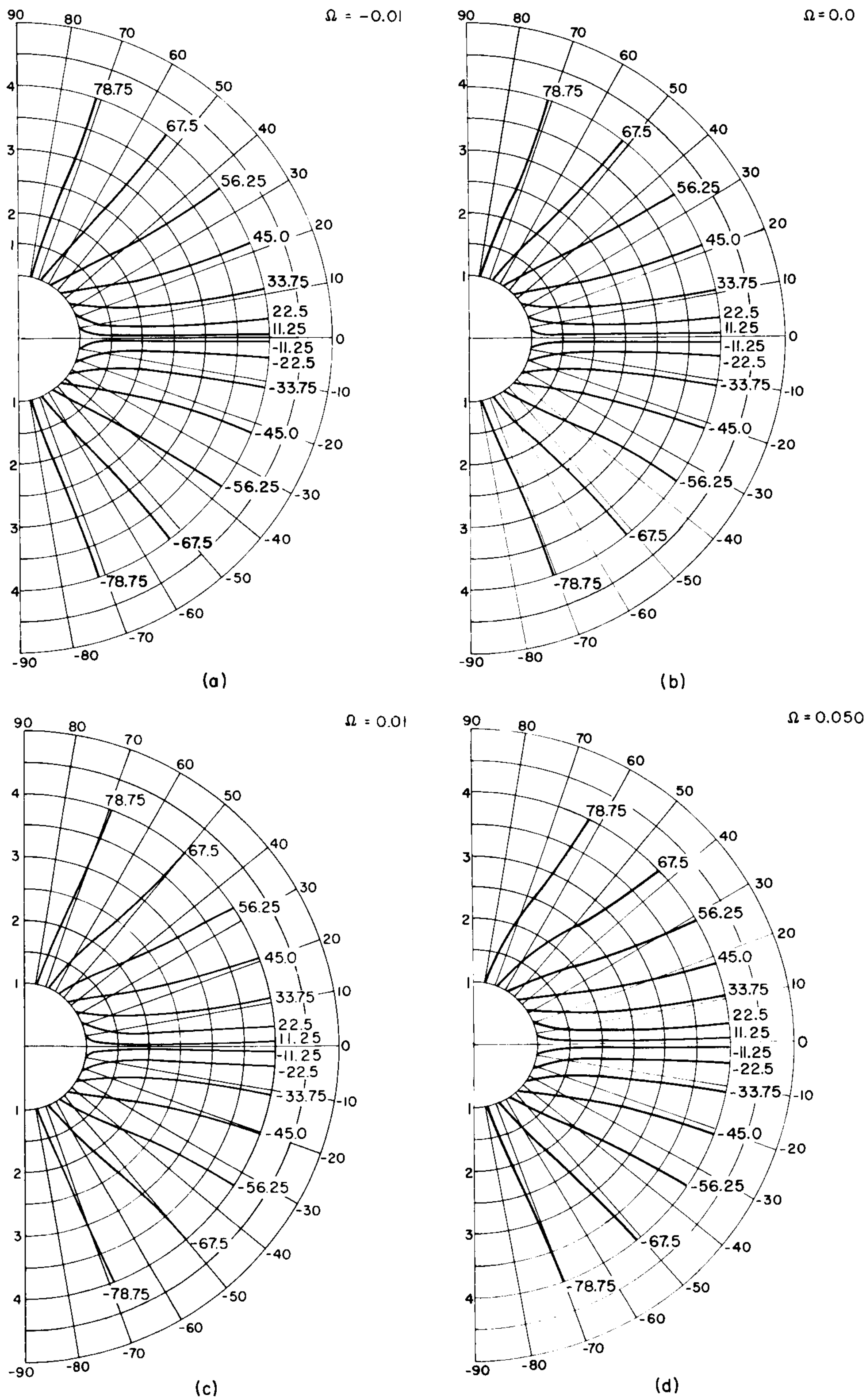

Fig. 1. Flow field showing the effect of rotation on streamlines (a) $\Omega = -0.01$, (b) $\Omega = 0.0$, (c) $\Omega = 0.01$, (d) $\Omega = 0.050$