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ON THE MICHAILOV CRITERION FOR EXPONENTIAL POLYNOMIALS

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PENNSYLVANIA STATE UNIVERSITY
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On the Michailov Criterion for Exponential Polynomials

by

Allan M. Krall[1]

We consider an equation of the form

$$F(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} \tilde{a}_{ij} z^j e^{\omega_i z} = 0 \qquad (1)$$

where $0 \leqq \omega_1 < \omega_2 \ldots < \omega_m$ are real numbers and $\tilde{a}_{ij}$, $i = 1, \ldots m$, $j = 1, \ldots n$ are complex numbers. We assume that there is at least one coefficient a_{ij} different from zero when $i \geqq 1$.

In many applications it is necessary to know whether or not $F(z)$ has any zeros in the right half plane. The best known technique for answering this question is due to Pontrjagin [1] and [2]. Pontrjagin's criterion, however, is very difficult to apply. In fact it has been found unsatisfactory except as a theoretical result (see [3, page 420]). Popov, [3, page 420]), states that to provide a more useful criterion, A. A. Sokolòv [4] and N. N. Miasnikov extended the Michailov criterion to cover exponential polynomials such as $F(z)$. Unfortunately no reference is given for

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Miasnikov's work and Sokolov's is inaccessible in this country. Since the Michailov criterion is easily stated, easily proved and easy to apply, it seems advantageous even at the risk of duplication to present it here.

Dividing by $e^{\omega_m z}$ and letting $\tilde{a}_{ij} = a_{m-i+1,j}$ we transform (1) into

$$G(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} z^j e^{-r_i z} = 0. \qquad (2)$$

Here $r_i = \omega_m - \omega_{m-i+1} > 0$ for $i = 2, 3, \ldots m$ and $r_1 = 0$. The zeros of $F(z)$ and $G(z)$ are the same, and it is $G(z)$ we will consider from now on.

In general $G(z)$ will contain an infinite number of zeros which occur in chains having the following properties.

1. The imaginary parts of the zeros are $O(k)$ for $k = \pm 1, \pm 2, \ldots \pm n, \ldots$
2. The real parts of the zeros are $O(\log k)$, (advanced type), constant, (neutral type) or $- O(\log k)$ (retarded type).

Any combination of advanced, neutral or retarded types may occur.

The term $a_{1n} z^n$ is said to be the principle term of $G(z)$. If the principle term is present, there are no zeros of advanced type. However, there still may be an infinite number of zeros in the right half plane because of the presence of zeros of neutral type.

We will need the following hypotheses in what follows.

H 1. $|a_{1n}| > \sum_{i=2}^{m} |a_{in}|$.

H 2. $a_{in} = 0, \ i = 2, 3, \ldots m.$

H 3. All the coefficients a_{ij} are real.

<u>Theorem</u> 1. <u>Let H 1 hold. Let</u>

$$D = |a_{1n}| - \sum_{i=2}^{m} |a_{in}|, \tag{3}$$

$$M = \sup_{j=0,1,\ldots n-1} \sum_{i=2}^{m} |a_{ij}| \,/\, D. \tag{4}$$

<u>Then in the right half plane $G(z)$ has at most a finite number of zeros all of which lie inside a semicircle of radius $M + 1$ centered at the origin.</u>

Proof. Let $|z| > M + 1$. Then

$$|G(z)| \geq |a_{1n}| \; |z|^n - \sum_{i=2}^{m} |a_{in}| \; |z|^n - \sum_{j=0}^{n-1} \left(\sum_{i=1}^{m} |a_{ij}|\right) |z|^j$$

$$\geq D[|z|^n - M \sum_{j=0}^{n-1} |z|^j]$$

$$\geq D[|z|^n - M (|z|^n - 1)/(|z| - 1)]$$

$$\geq D[|z|^n \{|z| - (M + 1)\} + M] / (|z| - 1)$$

$$> 0.$$

If $G(z)$ had an infinite number of zeros in the right half plane, they would approach infinity giving a contradiction.

We write

$$G(z) = a_{1n} z^n (1 + \phi(z) + \Psi(z)), \qquad (5)$$

where

$$\phi(z) = \sum_{i=2}^{m} (a_{in}/a_{1n}) e^{-r_i z} \qquad (6)$$

and

$$\Psi(z) = \sum_{i=2}^{m} \sum_{j=0}^{n-1} (a_{ij}/a_{1n}) z^{j-n} e^{-r_i z} \quad . \qquad (7)$$

Note that in the right half plane including the imaginary axis $|\phi(z)| < 1$ and $\Psi(z) = O(1)$ as $|z| \to \infty$ whenever H 1 holds.

Theorem 2. (Michailov Criterion). Let H 1 hold. As z varies along the imaginary axis from $-i\infty$ to $i\infty$ then $G(z)$ passes through the origin each time $G(z)$ has an imaginary zero. If $G(z)$ has no imaginary zeros, then

$$N = \frac{n}{2} - \frac{1}{2\pi} \Delta (-iy, iy) \arg G(z) + \qquad (8)$$

$$+ \frac{1}{2\pi} [\arg (1 + \phi(iy) + \Psi(iy)) - \arg (1 + \phi(-iy)$$

$$+ \Psi(-iy))],$$

<u>where N is the number of zeros of G(z) in the right half plane, y is any number greater than M + 1 of theorem 1, large enough so that $|\phi(z) + \Psi(z)| < 1$ when $|z| \geq y$ and $\Delta (iy, -iy) \arg G(z)$ is the net change in arg G(z) as z varies from -iy to iy</u>.

Proof. We choose a semicircular contour C varying from -iy to iy along a circle centered at the origin with radius y and then along the imaginary axis from iy to -iy. If G(z) has no zeros on the imaginary axis then it is well known that

$$N = \frac{1}{2\pi} \Delta_c \arg G(z) - \frac{1}{2\pi} \Delta (-iy, iy) \arg G(z) \qquad (9)$$

where Δ_c denotes the net change in arg G(z) along the semicircle. We see from (7) that

$$\Delta_c \arg G(z) = n \Delta_c \arg z + \Delta_c \arg (1 + \phi(z) + \Psi(z)). \qquad (10)$$

$$\Delta_c \arg G(z) = n\pi + \arg (1 + \phi(iy) + \Psi(iy)) -$$

$$- \arg (1 + \phi(-iy) + \Psi(-iy)), \qquad (11)$$

since $1 + \phi(z) + \Psi(z)$ cannot wind around the origin. Inserting (11) in (9) achieves the result.

Corollary 1. If H 1 holds, and $G(z)$ has no imaginary zeros, then $N = 0$ if and only if

$$\Delta(-iy, iy) \arg G(z) = n\pi + \arg(1 + \phi(iy) + \Psi(iy)) - \arg(1 + \phi(-iy) + \Psi(-iy)). \tag{12}$$

Corollary 2. If H 1 and H 3 hold, and $G(z)$ has no imaginary zeros, then

$$N = \frac{n}{2} - \frac{1}{\pi}\Delta(0, iy) \arg G(z) + \frac{1}{\pi} \arg(1 + \phi(iy) + \Psi(iy)). \tag{13}$$

Proof. When H 3 holds the argument for $\overline{y}$ are the negative of those for y.

Corollary 3. If H 1 and H 3 hold, and $G(z)$ has no imaginary zeros, then $N = 0$ if and only if

$$\Delta(0, iy) \arg G(z) = \frac{n\pi}{2} + \frac{1}{\pi} \arg(1 + \phi(iy) + \Psi(iy)). \tag{14}$$

Corollary 4. If H 1 and H 2 hold, and $G(z)$ has no imaginary zeros, then

$$N = \frac{n}{2} - \frac{1}{2\pi} \Delta(-i\infty, i\infty) \arg G(z). \tag{15}$$

Proof. When H 2 holds, $\phi(z) \equiv 0$.
Since $\Psi(z) = O(1)$, $1 + \Psi(z) \to 1$ as $|z| \to \infty$. Thus the last two terms of (8) approach 0.

Corollary 5. If H 1 and H 2 hold, and $G(z)$ has no imaginary zeros, then $N = 0$ if and only if

$$\Delta\,(-i\infty, i\infty) \arg G(z) = n\pi \tag{16}$$

Corollary 6. If H 1, H 2 and H 3 hold, and G(z) has no imaginary zeros, then

$$N = \frac{n}{2} - \frac{1}{\pi}\,\Delta\,(0, i\infty) \arg G(z). \tag{17}$$

Corollary 7. If H 1, H 2 and H 3 hold, and G(z) has no imaginary zeros, then N = 0 if and only if

$$\Delta\,(0, i\infty) \arg G(z) = \frac{n\pi}{2}. \tag{18}$$

If these statements are compared to those of Pontrjagin, it is easy to see that they imply each other. The main advantage to the Michailov criterion is the removal of the perpetual oscillation present in Pontrjagin's criterion.

It is a fairly simple matter to see if arg G(z) varies through the appropriate number of quadrants or not. This procedure should prove quite useful in application.

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