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The Variance of Spectral Estimates

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ABSTRACT

In this Report the main result proved is a generalization of a formula of Goldstein (Ref. 1), who showed that if the estimate $\hat{S}(\omega)$ for the spectral density is computed by the use of the function $y(x) = \mathrm{sgn}(x)$, and if the spectrum is flat, then the dominant term in the variance of $\hat{S}(\omega)$ is $\frac{1}{2}\pi^2 K/N$. Theorem 3 evaluates this term for non-flat spectra and for more general functions $y(x)$.

This analysis shows that the loss in accuracy caused by working with $y(x)$ instead of with x itself can be decreased considerably by using, for $y(x)$, a step function with more than two values.

I. INTRODUCTION

The spectral density $S(\omega), |\omega| \leq \pi$, of a discrete stationary Gaussian process $\{x_k\}$, $-\infty < k < \infty$, of mean zero, can be expressed in terms of the correlations $R_x(k) = E(x_n x_{n+k})$ by

$$S(\omega) = \sum_{k=-\infty}^{\infty} e^{ik\omega} R_x(k) = R_x(0) + 2\sum_{k=1}^{\infty} \cos k\omega\, R_x(k).$$

An estimate of $S(\omega)$ can be obtained from observations of $\{x_k\}$ by truncating the series and replacing the quantities $R_x(k)$ with appropriate estimates $\hat{R}_x(k)$. If we assume that $R_x(0) = E(x_k^2)$ is known, then the x_k's can be normalized so that $R_x(0) = 1$, and the estimate for $S(\omega)$ is

$$\hat{S}(\omega) = 1 + 2\sum_{k=1}^{K} \cos k\omega\, \hat{R}_x(k). \tag{1}$$

A simple choice of $\hat{R}_x(k)$ is

$$\hat{R}_x(k) = \frac{1}{N}\sum_{n=1}^{N} x_n x_{n+k}, \tag{2}$$

where N is large. Each term in the sum has the expected value $R_x(k)$ and the variance of this expression approaches zero as $N \to \infty$. Hence, for large N, the value of this expression is close to $R_x(k)$, with a high probability. However, for large N the evaluation of the sum can be quite time-consuming.

It has been observed (Ref. 1) that if

$$y_i = \begin{cases} +1, x_i \geq 0, \\ -1, x_i < 0, \end{cases}$$

then $R_y(k) = E(y_n y_{n+k})$ satisfies the relation

$$R_x(k) = \sin\left[\frac{\pi}{2} R_y(k)\right].$$

This suggests using

$$\left.\begin{aligned} \hat{R}_y(k) &= \frac{1}{N}\sum_{n=1}^{N} y_n y_{n+k}, \\ \hat{R}_x(k) &= \sin\left[\frac{\pi}{2}\hat{R}_y(k)\right]. \end{aligned}\right\} \quad (3)$$

For large N, and for K small compared with N, this formula for $\hat{R}_x(k)$ can be evaluated much more rapidly than (2) (Ref. 1). When $\hat{S}(\omega)$ is evaluated by (1) and (3), the problem of estimating its mean and variance arises. It is hoped that the mean is close to $S(\omega)$ and the variance is small.

A more general method is considered here. We take $y_i = y(x_i)$, where $y(x)$ is any odd, bounded, non-decreasing function, normalized so that $E(y_i^2) = 1$. It is shown by Lemma 3 that there is a function $F(t)$ such that

$$R_x(k) = F[R_y(k)].$$

In general, $F(t)$ is not an entire function, but is analytic in a region of the complex plane including the open interval $-1 < t < 1$. For the present purposes, $F(t)$ may be extended in any way to a continuous function on $-\infty < t < \infty$. We take

$$\hat{R}_y(k) = \frac{1}{N}\sum_{n=1}^{N} y_n y_{n+k},$$

$$\hat{R}_x(k) = F[\hat{R}_y(k)].$$

With this definition, it is shown (Theorem 3) that, except for a term of the order $1/N$, $E[S(\omega)]$ is

$$S_K(\omega) = 1 + 2\sum_{k=1}^{K} \cos k\omega\, R_x(k),$$

and $E\{\hat{S}(\omega)^2\}$ is approximately $S_K(\omega)^2$, with a leading error term of the order K/N, which is given explicitly, and another error term $o(K/N)$, the exact order of which depends on the degree of regularity assumed for $S(\omega)$. This result is obtained for a large class of summation methods (or windows) substituted for Eq. (1), including Césaro sums.

The hypothesis on $S(\omega)$ in Theorem 3 is satisfied if $S(\omega)$ is a periodic function of bounded variation which satisfies a Lipshitz condition of order α, $\alpha > 0$ (Ref. 2).

II. ESTIMATES FOR THE MOMENTS OF $\hat{R}_y(k)$

Lemma 1. Let $f(z)$, $z = (z_1, \cdots, z_n)$ be analytic in a convex region D containing the origin, with $|f(z)| \leq M$. Let $\{\pi_k(z)\}$ be a set of products of the z_j's such that in the power series expansion of $f(z)$ at $(0, \cdots, 0)$, every term is divisible by one of the π_k's. If ζ is a point whose δ-neighborhood

$$|z_j - \zeta_j| < \delta, \qquad j = 1, \cdots, n,$$

is in D, then

$$|f(\zeta)| \leq 2^n M \sum_k \left(\frac{3}{\delta}\right)^{d_k} |\pi_k(\zeta)|,$$

where d_k is the degree of π_k.

Proof. Suppose p of the ζ_j's have absolute value less than $\delta/3$. By renaming the coordinates, these may be taken to be $\zeta_1, \zeta_2, \cdots, \zeta_p$. Then if

$$\left.\begin{aligned} |z_j| &< \frac{2\delta}{3}, \qquad & j &= 1, \cdots, p, \\ |z_j - \zeta_j| &< \delta, \qquad & j &= p+1, \cdots, n, \end{aligned}\right\} \quad (4)$$

z is in the region D. It follows that the power series expansion

$$f(z) = \sum_{k_1, \cdots, k_p = 0}^{\infty} g_{k_1 \cdots k_p}(z_{p+1}, \cdots, z_n)\, z_1^{k_1} \cdots z_p^{k_p} \quad (5)$$

converges in the region (4), and

$$|g_{k_1\cdots k_p}(z_1,\cdots,z_p)|\leqq M\left(\frac{3}{2\delta}\right)^{k_1+\cdots+k_p}$$

Because of the convexity of D, there is a sequence of overlapping regions of this type converging to zero, in which $f(z)$ is analytic and Eq. (5) is valid. Hence (5) is also valid in a neighborhood of the origin. There we have

$$g_{k_1\cdots k_p}(z_{p+1},\cdots,z_n)=\sum_{k_{p+1},\cdots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_{p+1}^{k_{p+1}}\cdots z_n^{k_n}$$

$$f(z)=\sum_{k_1,\cdots,k_n=0}^{\infty} a_{k_1\cdots k_n} z_1^{k_1}\cdots z_n^{k_n}. \tag{6}$$

If $\pi_k'(z)$ is obtained from $\pi_k(z)$ by omitting the factors z_j, with $j>p$, then each term in Eq. (6) must be divisible by one of the quantities π_k'. It follows that the same is true in Eq. (5). The sum of the absolute values of all terms in (5) which are divisible by one of the products $\pi_k'(z)$ is dominated by

$$M\left(\frac{3}{2\delta}\right)^{d'_k}|\pi_k'(z)|\prod_{j=1}^{p}\left(1-\frac{3}{2\delta}|z_j|\right)^{-1},$$

where d_k' is the degree of $\pi_k'(z)$. At $z=\zeta$, this is, at most,

$$M\left(\frac{3}{2\delta}\right)^{d'_k}|\pi_k'(\zeta)|\left(1-\frac{1}{2}\right)^{-p}=2^p M\left(\frac{3}{2\delta}\right)^{d'_k}|\pi_k'(\zeta)|$$
$$\leqq 2^n M\left(\frac{3}{\delta}\right)^{d'_k}|\pi_k'(\zeta)|\leqq 2^n M\left(\frac{3}{\delta}\right)^{d_k}|\pi_k(\zeta)|,$$

since, in the last step, only factors of the form $3/\delta\,|z_j|$, $j\geqq p+1$, were added.

Summing over the π_k's accounts for each term in Eq. (5) at least once. Hence, the result follows.

Lemma 2. Let $\{x_i\}$ be a stationary Gaussian process with mean zero and variance 1, and a spectral density $S(\omega)$, $|\omega|\leqq\pi$, which is integrable.

Then for any positive m, there is a constant $\mu_m>0$ such that any $m\times m$ covariance matrix $[R_x(p_j-p_k)]_{j,k=1,\cdots,m}$, $p_1<p_2<\cdots<p_m$, has its eigenvalues $\geqq\mu_m$.

Proof. We will use induction on m, and let μ_m be the infimum of eigenvalues for matrices of rank m. It will be shown that $\mu_m>0$, and $\mu_m\leqq\mu_{m-1}$ for $m\geqq 2$.

For $m=1$, the matrix is the identity. Hence, $\mu_1=1$.

Suppose we have determined that $\mu_1,\cdots,\mu_{m-1}$ are positive. We have

$$\mu_m=\inf_{\substack{a_1^2+\cdots+a_m^2=1\\ p_1<p_2<\cdots<p_m}}\sum_{j,k=1}^{m} a_j a_k R_x(p_j-p_k).$$

By putting $a_m=0$, μ_{m-1} can be obtained from this expression. Hence $\mu_m\leqq\mu_{m-1}$. If $\mu_m=\mu_{m-1}$, it also is positive. Therefore we may assume $\mu_m<\mu_{m-1}$. Expressing $R_x(p_j-p_k)$ in terms of $S(\omega)$,

$$\mu_m=\inf\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\left|\sum_{k=1}^{m}a_k e^{ip_k\omega}\right|^2.$$

By the Riemann-Lebesgue lemma,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\,e^{in\omega}=0.$$

Hence there is a number n_0 such that if $n>n_0$, this integral has an absolute value less than $(\mu_{m-1}-\mu_m)/4m$. If one of the differences $p_{l+1}-p_l$, $1\leqq l\leqq m-1$, is greater than n_0,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\left|\sum_{k=1}^{m}a_k e^{ip_k\omega}\right|^2$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\left\{\left|\sum_{k=1}^{l}a_k e^{ip_k\omega}\right|^2+\left|\sum_{k=l+1}^{m}a_k e^{ip_k\omega}\right|^2\right\}$$
$$+2\sum_{j=1}^{l}\sum_{k=l+1}^{m}a_j a_k\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\,e^{i\omega(p_k-p_j)}$$
$$\geqq\mu_l\sum_{k=1}^{l}a_k^2+\mu_{m-l}\sum_{k=l+1}^{m}a_k^2-2\sum_{j,k=1}^{m}|a_j||a_k|\frac{\mu_{m-1}-\mu_m}{4m}$$
$$\geqq\mu_{m-1}-2m\cdot\frac{\mu_{m-1}-\mu_m}{4m}=\frac{\mu_{m+1}+\mu_m}{2}>\mu_m.$$

It follows that such sets of p_j's need not be considered in finding μ_m, and

$$\mu_m=\inf_{\substack{a_1^2+\cdots+a_m^2=1\\ 0\leqq p_1<\cdots<p_m\leqq mn_0}}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\left|\sum_{k=1}^{m}a_k e^{ip_k\omega}\right|^2$$
$$=\min_{0\leqq p_1<\cdots<p_m\leqq mn_0}\mu(m;p_1,\cdots,p_m),$$

where

$$\mu(m;p_1,\cdots,p_m)$$
$$=\min_{a_1^2+\cdots+a_m^2=1}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\,S(\omega)\left|\sum_{k=1}^{m}a_k e^{ip_k\omega}\right|^2.$$

This is the minimum eigenvalue of a certain matrix, and it is the value of the integral when $(a_1, \cdots, a_m)$ is the corresponding eigenvector. This value is positive, since $S(\omega) > 0$ on a set of positive measure, and the other factor in the integrand has only a finite number of zeros. Hence $\mu_m > 0$.

Remark: If $S(\omega)$ is bounded below by a positive constant $\underline{S}$, we may take $\mu_m = \underline{S}$ for all m.

Lemma 3. *Let $y(x)$ be an odd, monotonic increasing function of x, with $y(x) = O(x^n)$ as $x \to \infty$ for some power n. Let x_1 and x_2 be random variables with mean 0 and variance 1, and a bivariate Gaussian distribution. Let $y_i = y(x_i)$, $i = 1, 2$, and assume $E(y_i^2) = 1$. There is a function $f(z)$ of the complex variable z and an inverse function $F(z)$, depending only on the function $y(x)$, such that $E(y_1 y_2) = f\{E(x_1 x_2)\}$, $E(x_1 x_2) = F\{E(y_1 y_2)\}$. The functions f and F are odd, and are analytic in a region of the complex plane containing the open interval $-1 < z < 1$. Also, $f(\pm 1) = \pm 1$, $F(\pm 1) = \pm 1$, and $f(z)$ and $F(z)$ are continuous, increasing functions of real z on $-1 \leq z \leq 1$.*

Proof. Define

$$f(z) = \frac{1}{2\pi(1-z^2)^{1/2}} \int dx_1\, dx_2\, y(x_1)\, y(x_2) \exp\left[- \frac{x_1^2 + x_2^2 - 2zx_1x_2}{2(1-z^2)} \right] \qquad (7)$$

for $-1 < \operatorname{Re} z < 1$, taking the branch of the square root which is positive for z real. The integral converges uniformly in any compact subset of this strip; hence it defines an analytic function there. Define $f(1) = 1$, $f(-1) = -1$.

Differentiating,

$$f'(z) = \frac{1}{2\pi(1-z^2)^{1/2}} \int dx_1\, dx_2\, y(x_1)\, y(x_2) \times \left[\frac{z}{1-z^2} + \frac{(x_1 - zx_2)(x_2 - zx_1)}{(1-z^2)^2} \right] \exp\left[- \frac{x_1^2 + x_2^2 - 2zx_1x_2}{2(1-z^2)} \right]$$

and using integration by parts,

$$f'(z) = \frac{1}{2\pi(1-z^2)^{1/2}} \int dy(x_1)\, dy(x_2) \exp\left[- \frac{x_1^2 + x_2^2 - 2zx_1x_2}{2(1-z^2)} \right],$$

which is positive for $-1 < z < 1$. It follows that $f(z)$ has an inverse $F(z)$, which is analytic in a neighborhood of the image under f of $\{-1 < z < 1\}$.

We must show

$$\lim_{z \to \pm 1} f(z) = \pm 1,$$

taking the limit through real z with $|z| < 1$. In Eq. (7), put $x_2 = zx_1 + t(1-z^2)^{1/2}$:

$$f(z) = \frac{1}{2\pi} \int dx_1\, y(x_1)\, e^{-\frac{1}{2}x_1^2} \int dt\, e^{-\frac{1}{2}t^2} y\left[zx_1 + t(1-z^2)^{1/2}\right].$$

Since $y[zx_1 + t(1-z^2)^{1/2}] = O(1 + |x_1|^n + |t|^n)$, then, by the dominated convergence theorem,

$$\lim_{z \to 1-} f(z) = \frac{1}{2\pi} \int dx_1\, y(x_1)\, e^{-\frac{1}{2}x_1^2} \int dt\, e^{-\frac{1}{2}t^2} y(x_1) = 1.$$

Since $f(z)$ is clearly an odd function,

$$\lim_{z \to -1+} f(z) = -1.$$

Hence the analytic inverse function $F(z)$ is defined in a region which intersects the real axis on $-1 < z < 1$. If the definition $F(\pm 1) = \pm 1$ is used, all the conclusions of the lemma follow.

Hypothesis A. *The sequence $\{x_i\}$ is a Gaussian process with $E(x_i) = 0$, $E(x_i^2) = 1$ for all i, such that for any set of distinct integers $i_1, i_2, \cdots, i_m$ the covariance matrix $[R_x(i_j, i_k)]_{j,k=1,\cdots,m}$ is positive definite with its minimum eigenvalue at least μ_m, a positive constant.*

The function $y(x)$ is an odd, bounded, nondecreasing function on $-\infty < x < \infty$ with $E[y(x_i)^2] = 1$. The random process $\{y_i\}$ is defined by $y_i = y(x_i)$.

This hypothesis will be used in several lemmas. However, in some of the lemmas, the full strength of the hypothesis is not necessary. For example, in Lemma 4, $y(x)$ may be any bounded measurable function.

Lemma 4. Assume Hypothesis A, with $|y(x)| \leq Y$. Let $n_1, \cdots, n_p$ be integers, of which only $n_{i_1}, \cdots, n_{i_q}$ are distinct. Then there is an analytic function

$$F_{n_1 \cdots n_p}(\{z_{kj}\}_{1 \leq k < j \leq q})$$

of $\frac{1}{2}q(q-1)$ complex variables such that

$$E(y_{n_1} \cdots y_{n_p}) = F_{n_1 \cdots n_p}(\{R_x(n_{i_k}, n_{i_j})\}_{1 \leq k < j \leq q}). \tag{8}$$

The function $F_{n_1 \cdots n_p}$ depends only on $y(x)$ and on the coincidences in the sequence $n_1, \cdots, n_p$, and is analytic in the convex region D_p formed by the union of the regions

$$|z_{kj} - \rho_{kj}| < \mu[(\rho_{mn})]/4p, \qquad 1 \leq k < j \leq q, \tag{9}$$

where (ρ_{mn}) is any positive definite symmetric $q \times q$ matrix with 1's along the diagonal, and $\mu[(\rho_{mn})]$ is its minimum eigenvalue. In D_p,

$$|F_{n_1 \cdots n_p}(\{z_{kj}\})| < (2^{1/2} Y)^p. \tag{10}$$

In particular, this inequality is valid if

$$|z_{kj} - R_x(n_{i_k}, n_{i_j})| < \mu_p/4p, \qquad 1 \leq k < j \leq q.$$

Proof. Define the $q \times q$ matrix M by

$$M_{ii} = 1, \qquad i = 1, \cdots, q,$$
$$M_{ij} = M_{ji} = z_{ij}, \qquad 1 \leq i < j \leq q.$$

Let

$$F_{n_1 \cdots n_p}(\{z_{kj}\})$$
$$= \frac{1}{(2\pi)^{q/2}(\det M)^{1/2}} \int dt_1 \cdots dt_q \, y(t_1)^{l_1} \cdots y(t_q)^{l_q}$$
$$\exp\left[-\frac{1}{2}\sum_{i,j=1}^{q} t_i t_j (M^{-1})_{ij}\right], \tag{11}$$

where l_j is the number of appearances of n_{i_j} among $n_1, \cdots, n_p$. Let $\{\rho_{kj}\}$ be a value of $\{z_{kj}\}$ at which M is positive definite. It will be shown that in the region (9), $\det(M) \neq 0$, and the integral in Eq. (11) converges uniformly. Since the union of all these regions is a convex set, there is a unique continuation of $[\det(M)]^{1/2}$ throughout D_p, subject to the condition $[\det(M)]^{1/2} > 0$ if M is positive definite. The bound of (10) will be established in the region (9).

Let M_0 be the value of M when $\{z_{kj}\} = \{\rho_{kj}\}$. Then if we set

$$M = M_0 + M_1,$$

the elements of M_1 have absolute value less than $\mu/4p$ in (9), where $\mu = \mu[(\rho_{kj})]$. M_0 has a positive definite square root $M_0^{1/2}$. In terms of the norm

$$\|u\| = \left(\sum_{j=1}^{q} |u_j|^2\right)^{1/2}$$

of a complex q-vector u, we have

$$\|M_0^{-1/2} u\| \leq \mu^{-1/2}\|u\|,$$

and in (9)

$$\|M_1 u\| \leq \left(\sum_{j=1}^{q}\left[\frac{\mu}{4p}\sum_{k=1}^{q}|u_k|\right]^2\right)^{1/2} \leq \frac{\mu}{4}\|u\|.$$

Hence

$$\|M_0^{-1/2} M_1 M_0^{-1/2} u\| \leq \frac{1}{4}\|u\|.$$

This shows that the expansion

$$M^{-1} = M_0^{-1/2}\sum_{n=0}^{\infty}(-1)^n (M_0^{-1/2} M_1 M_0^{-1/2})^n M_0^{-1/2}$$

converges, and

$$\|M_0^{1/2}(M^{-1} - M_0^{-1}) M_0^{1/2} u\|$$
$$= \left\|\sum_{n=1}^{\infty}(-1)^n (M_0^{-1/2} M_1 M_0^{-1/2})^n u\right\|$$
$$\leq \sum_{n=1}^{\infty} 4^{-n}\|u\| = \frac{1}{3}\|u\|.$$

Two consequences are

$$\mathrm{Re}\left\{\sum_{i,j=1}^{q} u_i (M_0^{1/2} M^{-1} M_0^{1/2})_{ij} u_j\right\} \geq \frac{2}{3}\sum_{j=1}^{q} u_j^2 \tag{12}$$

for real u, and

$$\|M_0^{1/2} M_1^{-1} M_0^{1/2} u\| \leq \frac{4}{3}\|u\|.$$

By the last inequality,

$$|\det(M_0^{1/2} M_1^{-1} M_0^{1/2})| \leq \left(\frac{4}{3}\right)^q. \tag{13}$$

In Eq. (11), put $t = M_0^{1/2} u$. We then have

$$|F_{n_1} \cdots {}_{n_p}| \leq \frac{|\det M^{-1}|^{1/2}}{(2\pi)^{q/2}} Y^p \int dt_1 \cdots dt_q \exp\left[-\frac{1}{2} \operatorname{Re} \sum_{i,j=1}^{q} t_i (M^{-1})_{ij} t_j\right]$$
$$= \frac{|\det (M_0^{1/2} M_1^{-1} M_0^{1/2})|^{1/2}}{(2\pi)^{q/2}} Y^p \int du_1 \cdots du_q \exp\left[-\frac{1}{2} \operatorname{Re} \sum_{i,j=1}^{q} u_i (M_0^{1/2} M_1^{-1} M_0^{1/2})_{ij} u_j\right].$$

Using (12) and (13), the integral converges uniformly in (9), and

$$|F_{n_1} \cdots {}_{n_p}| \leq \frac{(4/3)^{q/2}}{(2\pi)^{q/2}} Y^p \int du_1 \cdots du_q \exp\left(-\frac{1}{3} \sum_{i=1}^{q} u_i^2\right)$$
$$= 2^{q/2} Y^p \leq (2^{1/2} Y)^p.$$

Lemma 5. Assume Hypothesis A. Let p be a fixed positive integer, and let $n_j, m_j, j = 1, \cdots, p$ be integers, with $m_j > n_j$. It follows that

$$E\left\{\prod_{j=1}^{p} [y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})]\right\} = O(\Sigma |\pi^{(p)}|)$$

where the sum is over all products $\pi^{(p)}$ of correlations $R_x(l_1, l_2)$, with l_1, l_2 taken from $\{n_1, \cdots, m_p\}$ and $l_1 \neq l_2$, which have the following properties (and are minimal in this respect):

(i). Each of the 2p letters appears an odd number of times.

(ii). For each pair (n_j, m_j), there is an even number of factors–at least two–with exactly one index in the pair (n_j, m_j).

Proof. First assume that $n_1, \ldots, m_p$ are all distinct. Expand the product

$$P = \prod [y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})]$$

into a sum of products of y_k's and expectations. Taking the expected value term by term, we find by Lemma 4 that $E(P)$ is the value of a function $G_{n_1} \cdots {}_{m_p}(\{z_{kj}\})$ which is analytic in the region D_{2p}, with

$$|G_{n_1} \cdots {}_{m_p}| < 2^p (2^{1/2} Y)^{2p}.$$

Lemma 1 will be applied to this function. A product which is minimal with respect to (i) and (ii) has a degree less than 3 $\binom{2p}{2}$, since no correlation may appear more than three times. Hence it suffices to show that in the expansion of $G_{n_1} \cdots {}_{m_p}$ about the origin, every term possesses these two properties. For simplicity, we may consider the corresponding expansion of $E(P)$, which is valid if all the correlations are small.

Replacing any one of the variables y_{n_j} (or y_{m_j}) with its negative changes the sign of $E(P)$. This may be accomplished by replacing x_{n_j} with $-x_{n_j}$, which changes the sign of $R_x(n_j, l)$ for $l \neq n_j$. Hence $E(P)$ is odd in these correlations, which shows that each term has the first property.

Now, for a given $j \leq p$, consider the correlations $R_x(n_j, l)$, $R_x(m_j, l)$ for $l \neq n_j$ or m_j. Replacing x_{n_j} and x_{m_j} with $-x_{n_j}$ and $-x_{m_j}$ changes the signs of these correlations, and leaves $E(P)$ unchanged. Therefore, $E(P)$ is an even function of these correlations. Also, if all of these correlations were zero, we would have

$$E(P) = E\{y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})\}$$
$$\times E\left\{\prod_{k \neq j} [y_{n_k} y_{m_k} - E(y_{n_k} y_{m_k})]\right\} = 0.$$

Hence each term is of positive degree in $\{R_x(n_j, l), R_x(m_j, l), l \neq n_j \text{ or } m_j\}$, so satisfying the second property.

If the values of $n_1, \cdots, m_p$ are not all distinct, the above procedure shows that

$$E(P) = O(\Sigma |\pi'^{(p)}|),$$

where, instead of satisfying (i) and (ii), each product satisfies the following properties:

(i′). Any number which is the value of an odd (or even) number of subscripts appears an odd (or even) number of times.

(ii′). Each pair (n_j, m_j) which is distinct from the other subscripts satisfies (ii).

In one of the factors $R_x(q_1, q_2)$ of one of the products $\pi'^{(p)}$, there is no unique way of assigning one of the letters $n_1, \cdots, m_p$ to q_1, unless q_1 is the value of only one letter. Suppose that this assignment is made in any permissible way. Then $\pi'^{(p)}$ can be modified, without changing its value, so that it satisfies (i) and (ii).

First, (i) will be satisfied. Let $l_1, \cdots, l_m$ be a set of equal letters. For $1 \leqq j \leqq m-1$, insert into $\pi'^{(p)}$ the factor $R_x(l_j, l_m)$ $(=1)$, if necessary, to make l_j appear an odd number of times. Then by (i′), l_m appears an odd number of times. After this procedure is applied to each set of equal letters, (i) is satisfied.

Property (i) implies that for each j, $\pi'^{(p)}$ contains an even number of the factors $R_x(n_j, l)$, $R_x(m_j, l)$, with $l \neq n_j$ or m_j. If this number is zero, then by (ii′) one of the letters—e.g., n_j—is equal to another letter l_1. Insert into $\pi'^{(p)}$ the factor $R_x(n_j, l_1)^2$. If this is done for each j, (ii) is satisfied.

Lemma 6. *Assume Hypothesis A. Let $\{x_i\}$ be stationary, with a spectral density $S(\omega)$, $|\omega| \leqq \pi$, which is in L^2. Define*

$$R_y(k) = E(y_n y_{n+k}),$$

$$\hat{R}_y(k) = \frac{1}{N} \sum_{n=1}^{N} y_n y_{n+k}.$$

Suppose p is a fixed positive integer. Then for large N and for arbitrary positive $k_1, \cdots, k_p$,

$$E\left\{\prod_{j=1}^{p} [\hat{R}_y(k_j) - R_y(k_j)]\right\} = O(N^{-p/2}).$$

Proof. By definition,

$$\hat{R}_y(k_j) - R_y(k_j) = \frac{1}{N} \sum_{n=1}^{N} [y_n y_{n+k_j} - E(y_n y_{n+k_j})].$$

Hence,

$$E\left\{\prod_{j=1}^{p} [\hat{R}_y(k_j) - R_y(k_j)]\right\}$$
$$= \frac{1}{N^p} \sum_{n_1, \cdots, n_p = 1}^{N} E\left\{\prod_{j=1}^{p} [y_{n_j} y_{n_j+k_j} - E(y_{n_j} y_{n_j+k_j})]\right\}.$$

Applying Lemma 5, with $m_j = n_j + k_j$, it follows that this quantity is

$$O\left(\sum_{\pi^{(p)}} \frac{1}{N^p} \sum_{n_1, \cdots, n_p = 1}^{N} |\pi^{(p)}|\right).$$

The number of products $\pi^{(p)}$ depends only on p. Thus it suffices to show

$$\frac{1}{N^p} \sum_{n_1, \cdots, n_p = 1}^{N} |\pi^{(p)}| = O(N^{-p/2}).$$

Since $S(\omega)$ is in L^2,

$$\sigma = \sum_{-\infty}^{\infty} |R_x(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega)^2 \, d\omega < \infty.$$

For $1 \leqq j \leqq p$, if l_1, l_3 are selected from n_j, m_j, and l_2, l_4 from the other subscripts, by Schwarz' inequality

$$\sum_{n_j=1}^{N} |R_x(l_1 - l_2) R_x(l_3 - l_4)|$$
$$\leqq \left[\sum_{n_j=1}^{N} R_x(l_1 - l_2)^2 \sum_{n_j=1}^{N} R_x(l_3 - l_4)^2\right]^{1/2} \leqq \sigma, \tag{14}$$

and similarly,

$$\sum_{n_j=1}^{N} |R_x(l_1 - l_2)| \leqq \sigma^{1/2} N^{1/2}. \tag{15}$$

If necessary, let some of the factors of $\pi^{(p)}$ be removed to make it a product π', which is a minimal product with the property that for each j, the total number of appearances of n_j and m_j is at least two, and none appear in the combination $R_x(n_j - m_j)$. Note that the degree of π' is at least p.

Consider

$$\sum_{n_1, \cdots, n_p = 1}^{N} |\pi'|.$$

For any j, if n_j or m_j occur in exactly two factors of π', we may estimate the sum from above by summing over n_j and applying (14). If there is only one such factor, (15) may be applied. This gives a sum over less indices of a product of lower degree. Reiteration of this procedure eventually leads to a product in which, for any j such that n_j or m_j appears, there are at least three factors containing n_j or m_j. By the minimal property of π', there are no factors remaining.

Let ν_2 be the number of times Eq. (14) was applied, and ν_1 the number of times Eq. (15) was applied. Then

$$\sum_{n_1, \cdots, n_p = 1}^{N} |\pi'| \leqq \sigma^{\nu_2} (\sigma N)^{\nu_1/2} N^{p - \nu_1 - \nu_2}$$
$$= \sigma^{\nu_2 + \nu_1/2} N^{p - (\nu_2 + \nu_1/2)}$$

The degree of π' is $\nu_1 + 2\nu_2 \geqq p$; also, $\nu_1 + \nu_2 \leqq p$. Therefore we conclude that

$$\frac{1}{N^p} \sum_{n_1, \cdots, n_p = 1}^{N} |\pi^{(p)}| \leqq \frac{1}{N^p} \sum_{n_1, \cdots, n_p = 1}^{N} |\pi'|$$

$$\leqq \sigma^{\nu_2 + \nu_1/2} N^{-(\nu_2 + \nu_1/2)} \leqq \sigma^p N^{-p/2}.$$

Lemma 7. In Lemma 6, if we add the condition

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

it follows that

$$E\left\{\prod_{j=1}^{p} [\hat{R}_y(k_j) - R_y(k_j)]\right\} = O(N^{-[(p+1)/2]}),$$

where $[(p+1)/2]$ denotes the integral part of $(p+1)/2$.

Proof. In the proof of Lemma 6, whenever a factor $N^{1/2}$ is introduced into the estimate by the use of Eq. (15),

$$\sum_{n_j=1}^{N} |R_x(l_1 - l_2)| = O(1)$$

could be used instead. Thus, in the estimate $O(N^{-p/2})$, if $p/2$ is not an integer, we may replace it by the next larger integer $[(p+1)/2]$.

Lemma 8. Under the hypotheses of Lemma 6, for fixed m and δ, and for arbitrary $k > 0$,

$$\Pr\{|\hat{R}_y(k) - R_y(k)| > \delta\} = O(N^{-m}).$$

Proof. If we take $p = 2m$ in Lemma 6. and $k_1, \cdots, k_p = k$, we obtain

$$E\{|\hat{R}_y(k) - R_y(k)|^{2m}\} = O(N^{-m}).$$

This expected value is at least as large as

$$\delta^{2m} \Pr\{|\hat{R}_y(k) - R_y(k)| > \delta\}.$$

Thus we obtain the desired result.

Theorem 1. *Let $G(z_1, \cdots, z_m)$ be analytic in a domain of complex m-space which contains the set*

$$-1 < z_j < 1, \qquad j = 1, \cdots, m,$$

and assume that $G(z_1, \cdots, z_m)$ is defined and bounded when all the z_j's are real. Let $\{x_i\}$ be a stationary Gaussian process with mean 0 and variance 1, and a spectral density $S(\omega)$ in L^2. Let $y(x)$ be an odd, bounded, nondecreasing function with $E[y(x_i)^2] = 1$. Put

$$y_i = y(x_i),$$

$$R_y(k) = E(y_n y_{n+k}),$$

$$\hat{R}_y(k) = \frac{1}{N} \sum_{n=1}^{N} y_n y_{n+k}.$$

Then for fixed p, large N, and arbitrary $k_1, \cdots, k_m > 0$,

$$E\{G[\hat{R}_y(k_1), \cdots, \hat{R}_y(k_m)]\} = G[R_y(k_1), \cdots, R_y(k_m)]$$
$$+ \sum_{2 \leqq q_1 + \cdots + q_m \leqq p} a_{q_1 \cdots q_m} E\left\{\prod_{j=1}^{m} [\hat{R}_y(k_j) - R_y(k_j)]^{q_j}\right\} + O(N^{-(p+1)/2}), \tag{16}$$

where

$$a_{q_1 \cdots q_m} = \frac{1}{q_1! \cdots q_m!} \left(\frac{\partial}{\partial z_1}\right)^{q_1} \cdots \left(\frac{\partial}{\partial z_m}\right)^{q_m} G(z_1, \cdots, z_m)\bigg|_{z_j = R_y(k_j),\ j = 1, \cdots, m}.$$

Proof. Let μ_2 be the quantity of Lemma 2. Then for $k > 0$, and for real t_1, t_2,

$$t_1^2 + t_2^2 + 2t_1 t_2 R_x(k) \geqq \mu_2 (t_1^2 + t_2^2).$$

This implies $|R_x(k)| \leqq 1 - \mu_2$. By Lemma 3, $|R_y(k)| \leqq f(1 - \mu_2) < 1$.

For sufficiently small ρ, the set of $(z_1, \cdots, z_m)$ such that

$$|z_j - \zeta_j| < \rho, \qquad j = 1, \cdots, m$$

for some $\zeta_1, \cdots, \zeta_m$ with

$$-f(1-\mu_2) \leq \zeta_j \leq f(1-\mu_2), \qquad j = 1, \cdots, m$$

lies in the region of analyticity of $G(z_1, \cdots, z_m)$. Thus it follows that for any $k_1, \cdots k_m$, the set

$$|z_j - R_y(k_j)| < \rho, \qquad j = 1, \cdots, m$$

lies in this region.

Suppose

$$|z_j - R_y(k_j)| \leq \rho/2, \qquad j = 1, \cdots, m. \tag{17}$$

Then

$$G(z_1, \cdots, z_m) \doteq G[R_y(k_1), \cdots, R_y(k_m)] + \sum_{1 \leq q_1 + \cdots + q_m \leq p} a_{q_1 \cdots q_m} \prod_{j=1}^{m} [z_j - R_y(k_j)]^{q_j} + O\left(\sum_{j=1}^{m} |z_j - R_y(k_j)|^{p+1} \right).$$

By Lemma 8, the probability that (17) is not satisfied by $z_j = \hat{R}_y(k_j)$ for all j is $O(N^{-(p+1)/2})$. Hence

$$E\left\{ G[\hat{R}_y(k_1), \cdots, \hat{R}_y(k_m)] - G[R_y(k_1), \cdots, R_y(k_m)] - \sum_{1 \leq q_1 + \cdots + q_m \leq p} a_{q_1 \cdots q_m} \prod_{j=1}^{m} [\hat{R}_y(k_j) - R_y(k_j)]^{q_j} \right\}$$

$$= O\left[N^{-(p+1)/2} + \sum_{j=1}^{m} E\{ |\hat{R}_y(k_j) - R_y(k_j)|^{p+1} \} \right]. \tag{18}$$

By Schwarz's inequality and Lemma 6,

$$E\{ |\hat{R}_y(k_j) - R_y(k_j)|^{p+1} \} \leq (E\{ [\hat{R}_y(k_j) - R_y(k_j)]^{2p+2} \})^{1/2} = O(N^{-(p+1)/2}).$$

Thus, since the terms with $\Sigma q_j = 1$ have expectation 0, taking the expected value of the left members in Eq. (18) term by term yields the desired result.

Theorem 1a. *If, in addition to the hypotheses of Theorem 1, we assume*

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

the error term in Eq. (16) is $O(N^{-[p/2]-1})$.

Proof. The proof is similar to that of Theorem 1; however Lemma 7 is used instead of Lemma 6.

III. THE MEANS OF FUNCTIONS OF THE $\hat{R}_y(k)$

Lemma 9. *Assume Hypothesis A.*

(*a*).
$$E(y_{n_1}y_{n_2}y_{n_3}y_{n_4}) = R_y(n_1,n_2)R_y(n_3,n_4) + R_y(n_1,n_3)R_y(n_2,n_4) + R_y(n_1,n_4)R_y(n_2,n_3) + O(\Sigma|\pi_3|),$$

where the sum is over all products π_3 of three distinct correlations $R_x(n_j - n_{j'})$ with $j \neq j'$ such that each index $n_1, \cdots, n_4$ occurs in the product.

(*b*). *Let* $v_j = y_{n_j}y_{m_j} - E(y_{n_j}y_{m_j})$, $j = 1, \cdots, 4$. *Then*

$$E(v_1v_2v_3v_4) = E(v_1v_2)E(v_3v_4) + E(v_1v_3)E(v_2v_4) + E(v_1v_4)E(v_2v_3) + O(\Sigma|\pi_3^*|),$$

where the sum is over all products π_3^* of three distinct correlations $R_x(q_j, q_{j'})$ with $q_j = n_j$ or m_j, $q_{j'} = n_{j'}$ or $m_{j'}'$, such that the three pairs (j, j') include 1, 2, 3 *and* 4.

Proof. (a). Assume first that $n_1, \ldots, n_4$ are distinct. Consider the expansion of $E(y_1y_2y_3y_4)$ in a power series when the correlations are all small. Since $y(x)$ is odd, each term involves every subscript. Therefore the only terms not divisible by one of the products π_3 are those which contain only two distinct correlations. There are three possible choices for these correlations: $R_x(n_1,n_2)$ and $R_x(n_3,n_4)$, and the two other pairs obtained by permutation of the indices. The same applies to the quantity

$$Q = E(y_{n_1}y_{n_2}y_{n_3}y_{n_4}) - E(y_{n_1}y_{n_2})E(y_{n_3}y_{n_4}) - E(y_{n_1}y_{n_3})E(y_{n_2}y_{n_4}) - E(y_{n_1}y_{n_4})E(y_{n_2}y_{n_3}).$$

But if all correlations are zero except $R_x(n_1,n_2)$ and $R_x(n_3,n_4)$, or one of the similar pairs, Q is zero. Hence (a) is true by Lemma 1.

If $n_1 = n_2 \neq n_3$, $n_2 \neq n_4 \neq n_3$,

$$Q = E(y_{n_1}^2 y_{n_3}y_{n_4}) - E(y_{n_3}y_{n_4}) - 2E(y_{n_1}y_{n_3})E(y_{n_1}y_{n_4}).$$

For small values of $R_x(n_1,n_3)$, $R_x(n_1,n_4)$, $R_x(n_3,n_4)$, Q may be expanded in a power series in these correlations. Consider first the terms which involve only one of these correlations. The terms involving only $R_x(n_1,n_3)$ give the value of Q when $R_x(n_1,n_4) = R_x(n_3,n_4) = 0$, which is zero. Consequently there are no such terms. Similarly, there are no terms which do not contain at least two distinct correlations. Thus, by Lemma 1,

$$Q = O(|R_x(n_1,n_3)R_x(n_1,n_4)| + |R_x(n_1,n_3)R_x(n_3,n_4)| + |R_x(n_1,n_4)R_x(n_3,n_4)|).$$

Inserting the factor $R_x(n_1,n_2)$ $(=1)$ in each term gives $O(\Sigma|\pi_3|)$.

The cases with more than one pair of equal subscripts may be treated similarly.

(b). The proof of Lemma 9b is analogous to that of 9a. Therefore, only the case of distinct subscripts will be considered.

In the expansion of

$$Q^* = E(v_1v_2v_3v_4) - E(v_1v_2)E(v_3v_4) - E(v_1v_3)E(v_2v_4) - E(v_1v_4)E(v_2v_3),$$

every term which is not divisible by one of the products π_3^* contains only correlations which may be put into two classes: After a permutation of subscripts, one class of correlations depends on n_1, m_1, n_2, m_2; the other class depends on n_3, m_3, n_4, m_4. However, setting all correlations zero which are not in one of these classes makes $Q^* = 0$, since then

$$E(v_1v_2v_3v_4) = E(v_1v_2)E(v_3v_4),$$

$$E(v_1v_3) = E(v_1v_4) = 0.$$

Consequently there are no such terms. The result follows by Lemma 1.

Lemma 10. *Assume Hypothesis A, with $\{x_i\}$ stationary and*

$$\sum_{\infty}^{\infty} |R_x(k)| < \infty.$$

Let

$$R_y(k) = E(y_n y_{n+k}), \qquad \hat{R}_y(k) = \frac{1}{N}\sum_{n=1}^{N} y_n y_{n+k}.$$

Then, for arbitrary $N, K > 0$,

$$\sum_{k,l=1}^{K} E\{[\hat{R}_y(k) - R_y(k)]^3 [\hat{R}_y(l) - R_y(l)]\} = O\left(\frac{K}{N^2} + \frac{K^2}{N^3}\right); \tag{19}$$

for $m = 3$ or 4 and $0 \leqq t \leqq m$,

$$\sum_{k,l=1}^{K} [|R_y(k)| + |R_y(l)|]\, E\{[\hat{R}_y(k) - R_y(k)]^t [\hat{R}_y(l) - R_y(l)]^{m-t}\} = O\left(\frac{K}{N^2}\right); \tag{20}$$

if $|a_k|, |b_k| \leqq 1, k = 1, \cdots, K$,

$$\sum_{k,l=1}^{K} a_k b_l E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}$$

$$= \sum_{k,l=1}^{K} a_k b_l \frac{1}{N^2} \sum_{n_1,n_2=1}^{N} [R_y(n_1 - n_2) R_y(n_1 - n_2 + k - l) + R_y(n_1 - n_2 + k) R_y(n_1 - n_2 - l)] + O(N^{-1}), \tag{21}$$

$$\sum_{k,l=1}^{K} (|R_y(k)| + |R_y(l)|)\, E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} = O(N^{-1}), \tag{22}$$

$$\sum_{k,l=1}^{K} |R_y(k) R_y(l)|\, E\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}), \tag{23}$$

and

$$\sum_{k=1}^{K} |R_y(k)|\, E\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}). \tag{24}$$

Proof. By Lemma 9a, with $m_j = n_j + k$, $j = 1, 2, 3$, $m_4 = n_4 + l$,

$$E\{[\hat{R}_y(k) - R_y(k)]^3 [\hat{R}_y(l) - R_y(l)]\} = \frac{1}{N^4} \sum_{n_1,\cdots,n_4=1}^{N} E\left\{\prod_{j=1}^{4} [y_{n_j} y_{m_j} - E(y_{n_j} y_{m_j})]\right\}$$

$$= 3E\{[\hat{R}_y(k) - R_y(k)]^2\} E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}$$

$$+ \frac{1}{N^4} \sum_{n_1,\cdots,n_4=1}^{N} O(\Sigma |\pi_3^*|),$$

where each product π_3^* is a product of three distinct correlations involving different n_j's, such that $n_1, \cdots, n_4$ all occur. By summing in the proper order, using estimates such as

$$\sum_{n_1=1}^{N} |R_x(n_1 - n_2)| \leq \sum_{-\infty}^{\infty} |R_x(n_1)| = O(1) \tag{25}$$

we find that for any such product,

$$\frac{1}{N^4} \sum_{n_1, \cdots, n_4=1}^{N} |\pi_3^*| = O(N^{-3}).$$

By the application of Lemma 7,

$$E\{[\hat{R}_y(k) - R_y(k)]^3 [\hat{R}_y(l) - R_y(l)]\} = O(N^{-3} + N^{-1} | E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}|). \tag{26}$$

We have

$$E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\} = \frac{1}{N^2} \sum_{n_1, n_2=1}^{N} [E(y_{n_1} y_{n_1+k} y_{n_2} y_{n_2+l}) - E(y_{n_1} y_{n_1+k}) E(y_{n_2} y_{n_2+l})].$$

Therefore, using Lemma 9a,

$$\sum_{k,l=1}^{K} |E\{[\hat{R}_y(k) - R_y(k)][R_y(l) - \hat{R}_y(l)]\}|$$

$$\leq \frac{1}{N^2} \sum_{k,l=1}^{K} \sum_{n_1, n_2=1}^{N} \{|R_y(n_1 - n_2) R_y(n_1 - n_2 + k - l)| + |R_y(n_1 - n_2 + k) R_y(n_1 - n_2 - l)| + O(\Sigma |\pi_3|)\}, \tag{27}$$

where each product π_3 is the product of three distinct correlations of pairs of the variables x_{n_1}, x_{n_1+k}, x_{n_2}, x_{n_2+l}, such that all subscripts occur. By summing in the proper manner, using estimates such as (25),

$$\frac{1}{N^2} \sum_{k,l=1}^{K} \sum_{n_1, n_2=1}^{N} |\pi_3| = O(N^{-1}).$$

The contribution of the first two terms in the sum on the right in (27) is $O(KN^{-1})$; since $R_y(j) = O(|R_x(j)|)$. Thus

$$\sum_{k,l=1}^{K} |E\{[\hat{R}_y(k) - R_y(k)][\hat{R}_y(l) - R_y(l)]\}| = O\left(\frac{K}{N}\right),$$

and summing (26) yields (19). Similarly, an equation analogous to (27) shows that (21) is true.

For the left side of (22), instead of Eq. (27), we now have

$$\sum_{k,l=1}^{K} (|R_y(k)| + |R_y(l)|)\,|E\,\{[\hat{R}_y(k) - R_y(k)]\,[\hat{R}_y(l) - R_y(l)]\}|$$

$$= O\left[\frac{1}{N^2}\sum_{k,l=1}^{K}\sum_{n_1,n_2=1}^{N}(|R_y(k)| + |R_y(l)|)\{|R_y(n_1-n_2)\,R_y(n_1-n_2+k-l)| + |R_y(n_1-n_2+k)\,R_y(n_1-n_2-l)| + \sum|\pi_3|\}\right].$$

Since $R_y(j) = O(|R_x(j)|)$, each R_y on the right may be replaced by R_x. The term $\sum|\pi_3|$ contributes $O(N^{-1})$, as before. Expanding the rest of the summand into a sum of products, we obtain a sum of terms of the type π_3. Hence (22) is true.

The remaining relations, (20), (23), and (24), depend only on Lemma 7:

$$E\,\{[\hat{R}_y(k) - R_y(k)]^t\,[\hat{R}_y(l) - R_y(l)]^{m-t}\} = O(N^{-2}), \qquad m = 3, 4,$$

$$E\,\{[\hat{R}_y(k) - R_y(k)]^2\} = O(N^{-1}),$$

and

$$\sum_{-\infty}^{\infty} |R_y(k)| < \infty.$$

Theorem 2. *Let $G(z)$ and $H(z)$ be odd functions, analytic in a region including the interval $-1 < z < 1$ and defined and bounded for all real z. Let a_k, b_k, $k = 1, 2, \cdots$ be numbers of absolute value at most 1.*

Assume that $\{x_i\}$ is a stationary Gaussian process with mean 0 and variance 1, with

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty.$$

Let $y(x)$ be an odd, bounded, nondecreasing function with $E\,\{y(x_i)^2\} = 1$.

Define

$$y_i = y(x_i),$$

$$R_y(k) = E(y_n y_{n+k}),$$

$$\hat{R}_y(k) = \frac{1}{N}\sum_{n=1}^{N} y_n y_{n+k}.$$

Then for arbitrary positive integers K, N,

$$E\left\{\sum_{k=1}^{K} a_k G[\hat{R}_y(k)]\right\} = \sum_{k=1}^{K} a_k G[R_y(k)] + O(N^{-1} + KN^{-2}), \tag{28}$$

$$E\left\{\sum_{k,l=1}^{K} a_k b_l G[\hat{R}_y(k)] H[\hat{R}_y(l)]\right\} = \sum_{k,l=1}^{K} a_k b_l G[R_y(k)] H[R_y(l)] + G'(0) H'(0) \sum_{k,l=1}^{K} a_k b_l \frac{1}{N^2} \sum_{n_1,n_2=1}^{N}$$

$$\times [R_y(n_1 - n_2) R_y(n_1 - n_2 + k - l) + R_y(n_1 - n_2 + k) R_y(n_1 - n_2 - l)]$$

$$+ O(N^{-1} + K^2N^{-3}). \tag{29}$$

Proof. Apply Theorem 1a to the function $G(z_1) H(z_2)$. For $p = 4$ we have

$$E\{G[\hat{R}_y(k)] H[(\hat{R}_y(l)]\} = G[R_y(k)] H[R_y(l)] + \sum_{m=2}^{4} \sum_{t=0}^{m} \frac{G^{(t)}[R_y(k)] H^{(m-t)}[R_y(l)]}{t!(m-t)!}$$

$$\times E\{[\hat{R}_y(k) - R_y(k)]^t [\hat{R}_y(l) - R_y(l)]^{m-t}\} + O(N^{-3}). \tag{30}$$

Similarly,

$$E\{G[\hat{R}_y(k)]\} = G[R_y(k)] + \frac{1}{2} G''[R_y(k)] E\{[\hat{R}_y(k) - R_y(k)]^2\} + O(N^{-2}). \tag{31}$$

The expressions on the left in (28) and (29) may be expressed in terms of (30) and (31). The O-terms in these equations contribute $O(K^2N^{-3})$ to Eq. (29) and $O(KN^{-2})$ to Eq. (28). The contributions of the other terms will now be investigated.

For any derivative of G (or H) which occurs, we have

$$G^{(t)}[R_y(k)] = \begin{cases} G^{(t)}(0) + O(|R_y(k)|), & t \text{ odd}, \\ O(|R_y(k)|), & t \text{ even}, \end{cases}$$

since $G(z)$ is odd. Eliminate the derivatives in Eq. (30) in this way, multiply by $a_k b_l$, and sum over k and l. By applying Eqs. (19) through (23) of Lemma 10, Eq. (29) results.

Similarly, by the use of Eq. (24), Eq. (28) follows from Eq. (31).

Lemma 11. *Suppose $g(\omega)$, $|\omega| \leq \pi$, has the Fourier series*

$$g(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega}.$$

Let $f(z)$ be analytic in $|z| < \rho$, with $f(0) = f'(0) = f''(0) = 0$. Let b_k, $-\infty < k < \infty$, be such that $b_k = f(a_k)$ if $|a_k| < \rho$. If $g(\omega)$ has bounded variation, and

$$\sum_{-\infty}^{\infty} |a_k| < \infty,$$

then the function

$$h(\omega) = \sum_{k=-\infty}^{\infty} b_k e^{ik\omega}$$

has a continuous second derivative.

Proof. Choose r between 0 and ρ. There are at most a finite number of values of k for which $|a_k| > r$; removing the corresponding terms from the series for $g(\omega)$ and $h(\omega)$ cannot affect the conclusion. From this we may assume that $|a_k| \leq r$ for all k. Then $b_k = O(|a_k|^3)$.

For $k \neq 0$,

$$|a_k| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\omega} g(\omega)\, d\omega \right| = \left| \frac{1}{2\pi k} \int_{-\pi}^{\pi} e^{ik\omega}\, dg(\omega) \right|$$

$$\leq \frac{1}{2\pi k} \int_{-\pi}^{\pi} |dg(\omega)|.$$

Hence

$$a_k = O(1/k),$$

and

$$k^2 b_k = O(k^2 |a_k|^3) = O(|a_k|).$$

This shows that

$$-\sum k^2 b_k e^{ik\omega}$$

is a continuous function, since the series converges uniformly. The function $h(\omega)$ is obtained by integrating twice.

IV. THE MAIN THEOREM

Theorem 3. Assume that $\{x_i\}$ is a stationary Gaussian process with $E(x_i) = 0$, $E(x_i^2) = 1$,

$$\sum_{-\infty}^{\infty} |R_x(k)| < \infty,$$

and a spectral density $S(\omega)$, $|\omega| \leq \pi$, which is a function of bounded variation. Let $y(x)$ be an odd, bounded, nondecreasing function on $-\infty < x < \infty$ such that $E[y(x_i)^2] = 1$. Define $y_i = y(x_i)$ and

$$\hat{R}_y(k) = \frac{1}{N} \sum_{n=1}^{N} y_n y_{n+k}.$$

Let $S_y(\omega)$ be the spectral density of $\{y_i\}$, and $F(z)$ the function of Lemma 3, with its definition extended to all real z so that $F(z)$ is bounded for z real.

Let $\{c_k, k = 1, \cdots, K\}$ be a nonincreasing sequence of numbers with $c_1 \leq 1$, $c_K \geq 0$.

Define

$$\hat{S}(\omega) = 1 + 2 \sum_{k=1}^{K} c_k \cos k\omega\, F[\hat{R}_y(k)],$$

$$S_K(\omega) = 1 + 2 \sum_{k=1}^{K} c_k \cos k\omega\, R_x(k).$$

Then for any positive integers K, N with $K \leq N$,

$$E\{\hat{S}(\omega)\} = S_K(\omega) + O(N^{-1}) \tag{32}$$

and for $0 < \omega < \pi$

$$E\{\hat{S}(\omega)^2\} = S_K(\omega)^2 + \frac{2\gamma K}{N} F'(0)^2 S_y(\omega)^2 + \mathcal{E}, \tag{33}$$

where

$$\gamma = \frac{1}{K} \sum_{k=1}^{K} c_k^2$$

and $\mathcal{E}$ satisfies the following bounds:

(i). For ω such that

$$S(\omega') = S(\omega) + O(|\omega - \omega'|^{\alpha}), \tag{34}$$

where $0 < \alpha \leq 1$,

$$\mathcal{E} = \begin{cases} O(K^{1-\alpha} N^{-1}), & 0 < \alpha < 1, \quad (35) \\ O([1 + \log K]\, N^{-1}), & \alpha = 1. \quad (36) \end{cases}$$

(ii). If ω is such that the derivative $S'(\omega)$ exists, and

$$S(\omega') = S(\omega) + (\omega' - \omega) S'(\omega) + O(|\omega' - \omega|^{1+\beta}), \tag{37}$$

where $\beta > 0$,

$$\mathcal{E} = O(N^{-1}) \tag{38}$$

Proof. By applying Lemma 3 and Lemma 11, $S_y(\omega)$ satisfies all the hypotheses for $S(\omega)$.

In Theorem 2, take $a_k = b_k = c_k \cos k\omega$, $G(z) = H(z) = F(z)$. Using the series for $\hat{S}(\omega)$, we find that (32) is true for $K \leq N$, and

$$E\{\hat{S}(\omega)^2\} = S_K(\omega)^2 + 4F'(0)^2 \sum_{k,\,l=1}^{K} c_k c_l \cos k\omega \cos l\omega \frac{1}{N^2} \sum_{n_1,\,n_2=1}^{N} \{R_y(n_1 - n_2) R_y(n_1 - n_2 + k - l)$$

$$+ R_y(n_1 - n_2 + k) R_y(n_1 - n_2 - 2)\} + O(N^{-1}).$$

This sum may be expressed in terms of $S_y(\omega)$ by the relation

$$R_y(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega' \, S_y(\omega') \, e^{\pm ij\omega'} .$$

We find that (33) is true, with

$$\mathcal{E} = -\frac{2\gamma K}{N} F'(0)^2 S_y(\omega)^2 + \frac{F'(0)^2}{\pi^2} \iint d\omega' d\omega'' \, S_y(\omega') \, S_y(\omega'') \, \mathcal{H}(\omega, \omega', \omega'') + O(N^{-1}),$$

where

$$\mathcal{H}(\omega, \omega', \omega'') = \sum_{k,\,l=1}^{K} c_k c_l \cos k\omega \cos l\omega \frac{1}{N^2} \sum_{n_1,\,n_2=1}^{N} \{e^{i\omega'(n_1 - n_2) - i\omega''(n_1 - n_2 + k - l)} + e^{i\omega'(n_1 - n_2 + k) - i\omega''(n_1 - n_2 - l)}\}.$$

Integrating term by term,

$$\iint d\omega' d\omega'' \, \mathcal{H}(\omega, \omega', \omega'') = \frac{4\pi^2}{N} \sum_{k=1}^{K} c_k^2 \cos^2 k\omega$$

$$= \frac{2\pi^2}{N} \sum_{k=1}^{K} c_k^2 + \frac{2\pi^2}{N} \sum_{k=1}^{K} c_k^2 \cos 2k\omega.$$

By the second mean value theorem (Ref. 3), since c_k^2 is monotonic, for some index $K' \leq K$

$$\sum_{k=1}^{K} c_k^2 \cos 2k\omega = c_1^2 \sum_{k=1}^{K'} \cos 2k\omega + c_K^2 \sum_{k=K'+1}^{K} \cos 2k\omega + O(1) = O(1),$$

for $0 < \omega < \pi$. Hence

$$\iint d\omega' d\omega'' \, \mathcal{H}(\omega, \omega', \omega'') = \frac{2\pi^2 K}{N} \gamma + O(N^{-1}),$$

and

$$\mathcal{E} = \frac{F'(0)^2}{\pi^2} \iint d\omega' d\omega'' \, [S_y(\omega') \, S_y(\omega'') - S_y(\omega)^2] \, \mathcal{H}(\omega, \omega', \omega'') + O(N^{-1}).$$

Let $d_k = c_k - c_{k+1}$, $k \leq K - 1$, $d_K = c_K$. Then

$$\mathcal{H}(\omega, \omega', \omega'') = \sum_{j,\,m=1}^{K} d_j d_m \, \mathcal{H}_{jm}(\omega, \omega', \omega''),$$

where

$$\mathcal{H}_{jm}(\omega,\omega',\omega'') = \sum_{k=1}^{j}\sum_{l=1}^{m} \cos k\omega \cos l\omega \frac{1}{N^2}\sum_{n_1,n_2=1}^{N} \{e^{i\omega'(n_1-n_2)-i\omega''(n_1-n_2+k-l)} + e^{i\omega'(n_1-n_2+k)-i\omega''(n_1-n_2-l)}\}. \tag{39}$$

By virtue of the identities

$$\sum_{n_1,n_2=1}^{N} e^{i(\omega'-\omega'')(n_1-n_2)} = \frac{\sin^2\frac{N}{2}(\omega'-\omega'')}{\sin^2\frac{1}{2}(\omega'-\omega'')},$$

$$2\sum_{k=1}^{K}\cos k\omega\, e^{i\omega_1 k} = e^{i[(K+1)/2](\omega_1-\omega)}\frac{\sin\frac{K}{2}(\omega_1-\omega)}{\sin\frac{1}{2}(\omega_1-\omega)} + e^{i[(K+1)/2](\omega_1+\omega)}\frac{\sin\frac{K}{2}(\omega_1+\omega)}{\sin\frac{1}{2}(\omega_1+\omega)},$$

we have

$$\mathcal{H}_{jm}(\omega,\omega',\omega'') = \frac{1}{N^2}\frac{\sin^2\frac{N}{2}(\omega'-\omega'')}{\sin^2\frac{1}{2}(\omega'-\omega'')} O\left[\sum_{\omega_1=\pm\omega',\pm\omega''}\sum_{p=j,m}\frac{\sin^2\frac{p}{2}(\omega_1-\omega)}{\sin^2\frac{1}{2}(\omega_1-\omega)}\right]. \tag{40}$$

Also,

$$\mathcal{E} = \frac{F'(0)^2}{\pi^2}\sum_{j,m=1}^{K} d_j d_m \mathcal{E}_{jm} + O(N^{-1}),$$

where

$$\mathcal{E}_{jm} = \int\int d\omega' d\omega''\,[S_y(\omega')S_y(\omega'') - S_y(\omega)^2]\,\mathcal{H}_{jm}(\omega,\omega',\omega'').$$

Since

$$\sum_{j,m=1}^{K}|d_j d_m| = c_1^2 \leq 1,$$

it is sufficient to show that $\mathcal{E}_{jm}$ satisfies (35), (36), or (38).

By (40),

$$\mathcal{E}_{jm} = O\left[\frac{1}{N^2}\int\int d\omega' d\omega''\,|S_y(\omega')S_y(\omega'') - S_y(\omega)^2|\,\frac{\sin^2\frac{N}{2}(\omega'-\omega'')}{\sin^2\frac{1}{2}(\omega'-\omega'')}\sum_{\omega_1,p}\frac{\sin^2\frac{p}{2}(\omega_1-\omega)}{\sin^2\frac{1}{2}(\omega_1-\omega)}\right].$$

Using the fact that $S_y(\omega)$ is an even function,

$$\mathcal{E}_{jm} = O\left[\frac{1}{N^2}\int\int d\omega' d\omega''\,|S_y(\omega')S_y(\omega'') - S_y(\omega)^2|\,\frac{\sin^2\frac{N}{2}(\omega'-\omega'')}{\sin^2\frac{1}{2}(\omega'-\omega'')}\sum_{p=j,m}\frac{\sin^2\frac{p}{2}(\omega'-\omega)}{\sin^2\frac{1}{2}(\omega'-\omega)}\right]. \tag{41}$$

If we assume (34), it follows that

$$S_y(\omega')\,S_y(\omega'') - S_y(\omega)^2 = O\left\{\left|\sin\frac{1}{2}(\omega'-\omega'')\right|^{\alpha} + \left|\sin\frac{1}{2}(\omega'-\omega)\right|^{\alpha}\right\}.$$

Eq. (41) may be estimated by using the relations

$$\int_0^{\pi} \frac{\sin^2\frac{n}{2}\omega'}{\sin^2\frac{1}{2}\omega'}\,d\omega' = \pi n,$$

$$\int_0^{\pi} \frac{\sin^2\frac{n}{2}\omega'}{\left|\sin\frac{1}{2}\omega'\right|^{2-\alpha}}\,d\omega' = O\left[\int_0^{1/n} n^{2-\alpha}d\omega' + \int_{1/n}^{\pi}\frac{d\omega'}{\omega'^{2-\alpha}}\right] = \begin{cases} O(n^{1-\alpha}), & 0<\alpha<1,\\ O(1+\log n), & \alpha = 1.\end{cases}$$

We find for $0 < \alpha < 1$,

$$\mathcal{E}_{jm} = O\left[\sum_{p=j,m}(N^{-1}p^{1-\alpha} + pN^{-1-\alpha})\right] = O(K^{1-\alpha}N^{-1}),$$

and for $\alpha = 1$,

$$\mathcal{E}_{jm} = O([1+\log K]\,N^{-1}),$$

verifying (35) and (36).

If we now assume (37), this implies that

$$S_y(\omega') = S_y(\omega) + \frac{\cos\omega - \cos\omega'}{\sin\omega}S_y'(\omega) + O(|\cos\omega-\cos\omega'|^{1+\beta}),$$

$$S_y(\omega')\,S_y(\omega'') - S_y(\omega)^2 = S_y(\omega)\,S_y'(\omega)\frac{2\cos\omega - \cos\omega' - \cos\omega''}{\sin\omega}$$

$$+ O(|\cos\omega-\cos\omega'|^{1+\beta} + |\cos\omega-\cos\omega''|^{1+\beta}$$

$$+ |\cos\omega-\cos\omega'|\,|\cos\omega-\cos\omega''|).$$

By a procedure analogous to that above, using the estimate

$$\int_0^{\pi}\frac{\sin^2\frac{n}{2}\omega'\,d\omega'}{\left|\sin\frac{1}{2}\omega'\right|^{1-\beta}} = O(1),$$

we find that

$$\mathcal{E}_{jm} = \frac{S_y(\omega)\,S_y'(\omega)}{\sin\omega}\iint d\omega'd\omega''\,(2\cos\omega - \cos\omega' - \cos\omega'')\,\mathcal{K}_{jm}(\omega,\omega',\omega'') + O(N^{-1}).$$

This integral may be evaluated by integrating the series for $\mathcal{K}_{jm}$ term by term. Most of the terms give no contribution. We have

$$\frac{1}{4\pi^2}\iint d\omega' d\omega'' \, 2\cos\omega - \cos\omega' - \cos\omega'') \, \mathcal{K}_{jm}(\omega, \omega', \omega'')$$

$$= \frac{2}{N}\cos\omega \sum_{k=1}^{\min(j,m)} \cos^2 k\omega$$

$$- \frac{2N-1}{2N^2}\left[\sum_{k=2}^{\min(j,m+1)} \cos k\omega \cos(k-1)\omega + \sum_{k=1}^{\min(j,m-1)} \cos k\omega \cos(k+1)\omega\right]$$

$$= \frac{2}{N}\sum_{k=1}^{\min(j,m)} [\cos\omega\cos^2 k\omega - \cos k\omega\cos(k+1)\omega] + O(N^{-1} + KN^{-2})$$

$$= \frac{1}{N}\sin\omega \sum_{k=1}^{\min(j,m)} \sin 2k\omega + O(N^{-1}) = O(N^{-1}).$$

Hence $\mathcal{E}_{jm} = O(N^{-1})$.

Remark: For simple choices of $S(\omega)$, a more explicit formula for the variance of $\hat{S}(\omega)$ may be given. For example, if

$$S(\omega) = 1 + 2R_x(1)\cos\omega,$$

in the terms of order N^{-1} only the quantities

$$\rho = E(y_1 y_2),$$

$$\sigma = E(y_1 y_2 y_3 y_4),$$

$$\nu = E(y_1^2 y_2^2),$$

$$\tau = E(y_1 y_2^2 y_3),$$

$$\tau' = E(y_1^2 y_2 y_3)$$

occur. For ordinary partial sums $(c_1, \cdots, c_K = 1)$, we find that

$$\frac{1}{\pi}\int_0^\pi \mathrm{Var}[\hat{S}(\omega)]\,d\omega = \frac{2K}{N}F'(0)^2 \cdot \frac{1}{\pi}\int_0^\pi S_y(\omega)^2\,d\omega$$

$$- \frac{1}{N}F'(0)^2[2 + 8\rho + 2\nu + 4\tau + 8\sigma - 18\rho^2]$$

$$- \frac{1}{N}[F'(0)^2 - F'(\rho)^2][2\nu + 4\tau + 4\sigma - 10\rho^2] + O(KN^{-2}).$$

For small values of $R_r(1)$, the terms following the first on the right decrease the average variance.

V. THE VALUE OF $F'(0)$

If $\hat{S}(\omega)$ is evaluated by the first method described in the introduction (using $y(x) = x$), it is easily verified that the conclusion of Theorem 3 applies: e.g., if $S(\omega)$ satisfies (ii) of Theorem 3,

$$E\{\hat{S}(\omega)^2\} = S_K(\omega)^2 + \frac{2\gamma K}{N} S(\omega)^2 + O(N^{-1}),$$

for $0 < \omega < \pi$. The term of the order K/N given in Theorem 3 differs from this in the replacement of $S(\omega)^2$ with $S_y(\omega)^2$ and the introduction of the factor $F'(0)^2$.

It can be shown that $S_y(\omega)$ always lies between the same bounds as $S(\omega)$, and has the same average value. Thus the effect of the factor $S_y(\omega)$ is to increase the variance of $\hat{S}(\omega)$ in some places, and decrease it in others.

The factor $F'(0)^2$ gives a uniform increase in variance for all ω. By differentiating Eq. (7) in the proof of Lemma 3, we have

$$F'(0) = 1/f'(0) = E\{xy(x)\}^{-2}.$$

In particular, for $y(x) = x$, $F'(0) = 1$, and for $y(x) = \mathrm{sgn}(x)$, $F'(0) = \pi/2$. It can be shown that for any other choice of $y(x)$ which is an odd nondecreasing function such that $E\{y(x)^2\} = 1$, $F'(0)$ lies between these limits.

It is advantageous to choose a function for $y(x)$ such that $\hat{R}_y(k)$ can be rapidly computed and $F'(0)$ is close to 1. The computation is simple if $y(x)$ is a step function taking only a few values. This leads us to consider a function of the type

$$y(x) = a_j, \qquad c_{j-1} < x < c_j, \qquad j = 1, \cdots, n,$$
$$y(-x) = -y(x),$$

where $0 = c_0 < c_1 < \cdots < c_n = +\infty$, $0 \leqq a_1 \cdots < a_n$. For a given value of n, the numbers a_j, c_j may be chosen so as to minimize $F'(0)$, given $E[y(x)^2] = 1$. This must be done numerically for $n \geqq 2$. The best choices for $n = 2, 3$ are:

$n = 2$: $\quad a_1 = 0.482,$
$a_2 = 1.608,$
$c_1 = 0.981, \qquad F'(0) = 1.133.$

$n = 3$: $\quad a_1 = 0.327,$
$a_2 = 1.030,$
$a_3 = 1.951,$
$c_1 = 0.659,$
$c_2 = 1.447, \qquad F'(0) = 1.062.$

If these numbers are compared with the value $F'(0) = 1.571$ for $n = 1$, it is seen that taking $n = 2$ reduces $F'(0)$ most of the way to 1.

In the construction of an autocorrelator using such a step function, it is convenient if all nonzero values of $y(x)$ have ratios which are powers of 2. If we modify the function given above for $n = 2$ by making $a_2 = 4a_1$, and then choose the best value of c_1, we get the function

$$y(x) = \begin{cases} 1.608, & x > 0.943, \\ 0.402, & 0 < x < 0.943, \\ -y(-x), & x < 0, \end{cases}$$

for which $F'(0) = 1.137$. This is almost the same value as that given above, showing that the value of $F'(0)$ is not very sensitive to small changes in the constants. The function $F(z)$ is plotted for this case in Fig. 1. Note that this function is essentially linear until $|z|$ is close to 1.

Another choice of $y(x)$ which is easier to work with is given by choosing $n = 2$, $a_1 = 0$. The minimum of $F'(0)$ is 1.232, occurring for the function

$$y(x) = \begin{cases} 1.36, & x > 0.612, \\ 0, & |x| < 0.612, \\ -1.36, & x < -0.612. \end{cases}$$

For $n = 2$, $a_2 = 2a_1$, the corresponding values are $a_1 = 0.7095$, $c_1 = 0.9765$, $F'(0) = 1.188$.

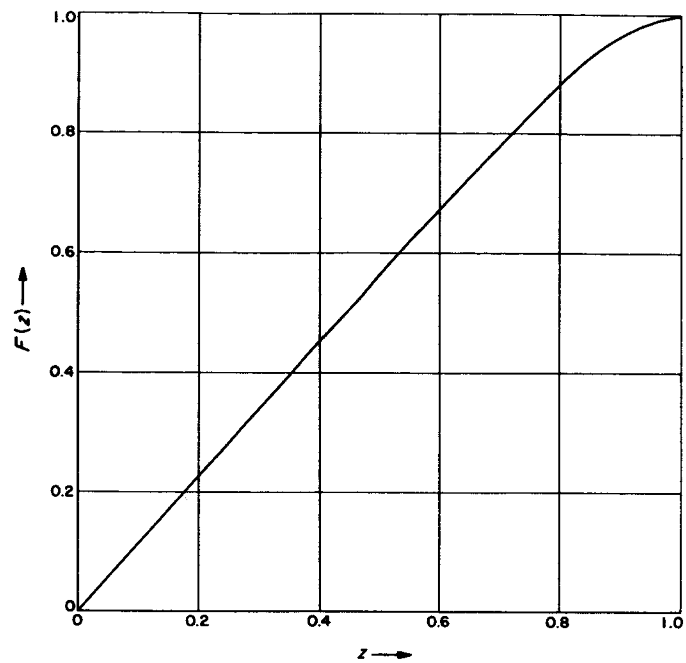

Fig. 1. The function *F* (z) when y (x) takes four values with a ratio of 1:4

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