NONSYMMETRIC INFLATION OF A MAGNETIC DIPOLE*

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Abstract

The basic theoretical properties of nonsymmetric inflation of a dipole magnetic field by ionized gas are pointed out. It is shown that the distortion $\Delta \mathbf{B}$ of the field in the neighborhood of the dipole depends very much upon whether there is a nonconducting layer present and upon the configuration of the inflating gas, in contrast to symmetric inflation, where $\Delta \mathbf{B}$ depends only upon the total energy of the inflating gas. Dissipation of the nonsymmetric portion of the inflating gas is rapid because nonsymmetric inflation drives $\nabla \times \mathbf{B}$, or current, down along the magnetic lines of force and across the ionosphere. Dissipation is also rapid because of the nonsymmetric distortion of the field, whose pattern rotates with the azimuthal drift of the inflating gas. The high rate of dissipation implies that nonsymmetric inflation can probably be important only during the active phase of a magnetic storm.

I. Introduction

The linear theory of the distortion of a dipole magnetic field inflated with an ionized gas is now well known (Dessler and Parker, 1959; Parker, 1962; Apel, Singer, and Wentworth, 1962; Sckopke, 1966) for gas distributions with rotational symmetry. The theory shows that the distortion $\Delta \mathbf{B}$ in the neighborhood of the dipole is independent of everything except the total thermal energy $\mathcal{E}$ of the inflating gas. The distortion $\Delta \mathbf{B}$ is uniform in the vicinity of the dipole, points in the direction of the dipole, and has a magnitude given by

$$\frac{\Delta B}{B_0} = \frac{2\mathcal{E}}{3\mathcal{E}_M} \tag{1}$$

where B_0 is the unperturbed dipole field intensity at the equatorial plane at a radial distance $r = a$, and $\mathcal{E}_M \equiv \frac{1}{3} B_0^2 a^2$ is the magnetic energy of the dipole field outside the sphere $r = a$. In particular, the result given by (1) is independent of whether there are nonconducting regions, such as the terrestrial atmosphere, in the dipole field. The field is stationary everywhere in space and there are no dissipative effects (except for inelastic collisions of the inflating particles).

This paper presents the additional effects which arise in the distortion of a dipole field by a <u>non</u>symmetric gas distribution, pointing out that a nonconducting shell, such as the terrestrial atmosphere, may then have profound effects on the dissipation rate and on the worldwide average distortion ΔB produced by a given gas energy $\mathcal{E}$. In order to explain the new effects which arise from the combination of a nonsymmetric distribution and a nonconducting shell $r = a$, consider first a nonsymmetric gas distribution inflating a dipole field pervaded everywhere with a tenuous (zero stress) highly conducting fluid. Once this case is worked out, the consequences of a nonconducting shell are immediately evident. To keep the exposition as simple as possible, consider the hypothetical situation that the gas pressure which distorts the dipole is the result of N particles distributed uniformly along an azimuthal segment $\Delta\phi$ at a radial distance R in the equatorial plane, as sketched in Fig. 1. Let all the particles have the same velocity w perpendicular to the magnetic field, so that the segment drifts around the dipole with azimuthal velocity u_ϕ but does not disperse in time. The azimuthal drift velocity u_ϕ is then $3c\mu/qR$ where q is the charge of each particle and μ is the diamagnetic moment $\frac{1}{2}Mw^2/B(R)$ of each particle. The total particle energy in the segment is $\mathcal{E} = \frac{1}{2}NMw^2$ and the total diamagnetic moment is $N\mu = \mathcal{E}/B(R)$.

The magnetic field is distorted by the pressure of the particles in the segment. In particular, the magnetic field surrounding the segment of particles tends to squeeze the diamagnetic particles out of the dipole field. The particles are, of course, tied to the lines of force through the segment, with the result that the lines of force through the segment are distorted outward. The outward distortion of the lines of force permits some outward expansion of the field nearer the dipole, just as when the particles are distributed with axial symmetry (Dessler and Parker, 1959). But the calculation of the resulting $\Delta\mathbf{B}$ from the particle stresses is a little more complicated than in the case of axial symmetry because $\nabla \times \mathbf{B}$ is not zero outside the region of particles.

A brief review of the basic theoretical formulation of the problem will help to see our way through. The point is that $\nabla \times \mathbf{B}$ is determined by the particle pressure, and the formal theory gives the distortion $\Delta \mathbf{B}$ as a volume integral over $\nabla \times \mathbf{B}$. So $\nabla \times \mathbf{B}$ is the vector on which the theory is built. Formally the distortion $\Delta \mathbf{B}(\mathbf{r})$ at any point $\mathbf{r}$ is related to the $\nabla \times \mathbf{B}$ (produced by the pressure) through the integral relation (Parker, 1962)

$$\Delta \mathbf{B}(\mathbf{r}) = \frac{1}{4\pi} \int d^3\mathbf{r}' \frac{[\nabla' \times \mathbf{B}(\mathbf{r}')] \times \boldsymbol{\xi}}{\xi^3} \tag{2}$$

$$= \frac{1}{c} \int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}') \times \boldsymbol{\xi}}{\xi^3} \tag{3}$$

where $\boldsymbol{\xi} = \mathbf{r} - \mathbf{r}'$ *. The quantity $\nabla \times \mathbf{B}$ is calculated from the equation $\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$ for hydrostatic equilibrium between the particle pressure p and the magnetic stress $(\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$ exerted on the medium.** In the axially symmetric case the $\nabla \times \mathbf{B}$ produced by p is everywhere perpendicular to $\mathbf{B}$, so that taking the vector product with $\mathbf{B}$ yields $\nabla \times \mathbf{B} = 4\pi \nabla p \times \mathbf{B}/B^2$, permitting

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** If the pressure is not isotropic, then the hydrostatic equation must be written in tensor form, $\partial p_{ij}/\partial x_j = \partial M_{ij}/\partial x_j$ where p_{ij} is the pressure tensor and $M_{ij} = -\delta_{ij} B^2/8\pi + B_i B_j/4\pi$ is the usual Maxwell stress tensor for a magnetic field.

* The $\nabla \times \mathbf{B}$ and the current density $\mathbf{j}$ are interchangeable through Maxwell's equation $c \nabla \times \mathbf{B} = 4\pi \mathbf{j}$ for quasi-steady fields.

calculation of $\nabla \times \underset{\sim}{\mathbf{B}}$ for any given pressure distribution p. The vector $\nabla \times \underset{\sim}{\mathbf{B}}$ is in the azimuthal direction and the lines of $\nabla \times \underset{\sim}{\mathbf{B}}$ form closed circles around the dipole.

In the nonsymmetric case $\nabla \times \underset{\sim}{\mathbf{B}}$ can be traced out only by some further considerations. The condition $\nabla p = (\nabla \times \underset{\sim}{\mathbf{B}}) \times \underset{\sim}{\mathbf{B}}/4\pi$ gives the perpendicular component of $\nabla \times \underset{\sim}{\mathbf{B}}$ alright, but there is in general a parallel component of $\nabla \times \underset{\sim}{\mathbf{B}}$ too, which may in turn lead to additional perpendicular $\nabla \times \underset{\sim}{\mathbf{B}}$ elsewhere. We are considering a dipole field filled with a tenuous Ohmic fluid exerting no stresses on the field. Currents, and hence $\nabla \times \underset{\sim}{\mathbf{B}}$, are free to flow anywhere and in any direction throughout the field. In the segment of particles the net field distortion $\nabla \times \underset{\sim}{\mathbf{B}}$ (or $\underset{\sim}{\mathbf{j}}$) is non-vanishing and is a vector pointing forward (if the particles are positively charged) along the segment. Since the divergence of $\nabla \times \underset{\sim}{\mathbf{B}}$ (or $\underset{\sim}{\mathbf{j}}$) vanishes everywhere, the vector $\nabla \times \underset{\sim}{\mathbf{B}}$ (or $\underset{\sim}{\mathbf{j}}$) cannot terminate at the ends of the segment, so $\nabla \times \underset{\sim}{\mathbf{B}}$ must extend into space beyond the segment. Beyond the ends of the segment we have $(\nabla \times \underset{\sim}{\mathbf{B}}) \times \underset{\sim}{\mathbf{B}} = 0$ because no forces are exerted. Hence $\nabla \times \underset{\sim}{\mathbf{B}}$ flows away from the segment into the origin along the dipole lines of force as indicated by the arrows in Fig. 1. It is this unavoidable streaming of $\nabla \times \underset{\sim}{\mathbf{B}}$ inward along the magnetic lines of force when the inflation is nonsymmetric that makes the great difference from the symmetric case.

So long as we confine our attention to $\Delta \underset{\sim}{\mathbf{B}}$ at the origin, it is readily shown, from the familiar right hand rule and the symmetry of the dipole lines of force, that the $\nabla \times \underset{\sim}{B}$ along the dipole lines of force produces no effect. Thus, in spite of the more complicated pattern of $\nabla \times \underset{\sim}{B}$, only the $\nabla \times \underset{\sim}{B}$ in the segment of particles contributes to $\Delta \underset{\sim}{B}$ at the origin. The distortion $\Delta \underset{\sim}{B}$ at the origin is still given correctly by (1)*

Now insert a nonconducting shell at $r = a$. The change is that $\nabla \times \underset{\sim}{B}$ (or $\underset{\sim}{j}$) cannot penetrate inward along the magnetic lines of force across the nonconducting shell. So the vector circuit for $\nabla \times \underset{\sim}{B}$ must close between the lines of force before reaching $r = a$, leading to a new pattern for $\nabla \times \underset{\sim}{B}$, one in which $(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} \neq 0$. The fact that the pattern of $\nabla \times \underset{\sim}{B}$ is changed, with $\nabla \times \underset{\sim}{B}$ crossing over between lines of force in $r > a$, alters the distortion $\Delta \underset{\sim}{B}$ at the origin, so that (1) is not generally valid. Further, the nonvanishing of $(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$ where $\nabla \times \underset{\sim}{B}$ crosses over means that the magnetic stresses drive the background fluid into some motion $\underset{\sim}{v}$, providing an additional source of dissipation.

In the simple case that the conductivity perpendicular to the magnetic lines of force is largest just outside $r = a$ (as in the actual case of the ionosphere above the nonconducting atmosphere) the transfer of $\nabla \times \underset{\sim}{B}$ (or $\underset{\sim}{j}$)

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*There are some differences, of course. The distortion $\Delta \underset{\sim}{B}$ is no longer uniform in the vicinity of the origin. But since $\nabla^2 \Delta \underset{\sim}{B} = 0$ in the neighborhood of the origin, it follows that the field $\Delta \underset{\sim}{B}$, computed at the origin from (1), represents the mean $\Delta \underset{\sim}{B}$ over any small sphere centered on the origin. Hence $\Delta \underset{\sim}{B}(0)$ is still an approximate representation of a worldwide average $\Delta \underset{\sim}{B}$ produced by the stresses of the particle segment at $r = R$. Finally, there may be new sources of dissipation as a consequence of the drift of the particle segment and the resulting change of $\Delta \underset{\sim}{B}$ with time at points removed from the origin. More will be said on this later.

between lines of force takes place at A and C in Fig. 1, just outside $r = A$. For simplicity suppose that the transfer follows the shortest path between the lines of force, indicated by the short connecting segments in Fig. 1.

Consider the net distortion $\Delta \underset{\sim}{B}$ at the origin. The contribution of $\nabla \times \underset{\sim}{B}$ along the dipole lines of force is again zero, for reasons of symmetry. The segment of particles again contributes the amount given by equation (1), but it is instructive to note that this can be broken down into two contributions (Dessler and Parker, 1959): The drift u_ϕ gives a current $I = Nqu_\phi / R\Delta\phi$ associated with a field $I\Delta\phi / cR$ in the direction of the dipole at the origin; the diamagnetic moment $N\mu$ gives a field $-N\mu / R^3$ at the origin. In addition to the particle segment the crossover segments at A and C give a contribution to $\Delta \underset{\sim}{B}$ at the origin. Each crossover segment carries a current $I/2$. If θ_0 is the polar angle of the radius vector drawn to the crossover segment at A, then the length of the segment is $\Delta\phi \; a \sin\theta_0$. The A segment intersects the dipole line of force crossing the equator at $r = R$, so that a and θ_0 are related by $a = R \sin^2\theta_0$. The current $I/2$ in the segment is associated with the field $I\Delta\phi \sin\theta_0 / 2ca$ at the origin, with the component $-I\Delta\phi \sin^2\theta_0 / 2ca = -I\Delta\phi / 2cR$ in the direction of the dipole. The crossover at C gives an equal contribution. But the particle segment gave $I\Delta\phi / cR$, so the algebraic sum of all three segments is zero. There remains only the diamagnetic contribution $\Delta B = -N\mu / R^3$ of the particle segment, which can be written

$$\Delta B / B_0 = - \mathcal{E} / 3 \mathcal{E}_M \tag{4}$$

the minus sign implying that $\Delta \mathbf{B}$ is directed opposite to the dipole. This relation, in which ΔB is negative, replaces (1), in which ΔB was positive!

As a matter of fact, the result (4) is correct for the general case that the transfer of $\nabla \times \mathbf{B}$ is distributed continuously along the lines of force. The general calculation is obvious once the special case is worked out, so there is no need to give it here.

The calculated $\Delta \mathbf{B}$ in (4) represents an <u>increase</u> of the horizontal component at the surface of Earth which is half as large in magnitude as the <u>decrease</u> produced by a symmetric ring of particles.

Now the precise form of (4) is dependent on the idealization that the transfer of $\nabla \times \mathbf{B}$ (or $\mathbf{j}$) between lines of force is restricted to a simple linear path, whereas in the actual case the transfer will be broader. We have used the simple linear path only to establish the important qualitative differences which are introduced by the combination of nonsymmetric inflation and a nonconducting sphere. The next section takes up the complication that the crossover goes both east and west between the magnetic lines of force.

II. Inflation by Several Segments of Particles

Consider $\nabla \times \mathbf{B}$ when there are n equal spaced azimuthal segments of particles distributed around the equatorial plane at a radial distance R as sketched in Fig. 2. Denote the length of each segment by $\Delta \phi_1$ and the separation between their ends by $\Delta \phi_2$. It is shown in Appendix I that the crossover of $\nabla \times \mathbf{B}$ between lines of force at $r = a$ may be thought of as being restricted to the segments shown in Fig. 2. For convenience we deal with the currents rather than $\nabla \times \mathbf{B}$, leaving the correct physical interpretation of the electrical impedance to Appendix I. Denote the current in each particle segment by I, and denote the impedance of the plasma between the lines of force separated by $\Delta\phi_1$ and $\Delta\phi_2$ by $Z_1(\Delta\phi_1)$ and $Z_2(\Delta\phi_2)$ respectively. Under steady conditions there is no net flow of magnetic lines of force toward or away from the axis of the dipole, so the line integral of the electric field must vanish around either of the circles formed by the crossover paths at $r = a$. Hence if I_1 is the current crossing over $\Delta\phi_1$ and I_2 over $\Delta\phi_2$, it follows that $I_1 Z_1 = I_2 Z_2$, taking all current magnitudes to be positive. Conservation of current requires that $I = 2(I_1 + I_2)$. It is readily shown, from symmetry considerations that the total field distortion $\Delta \underset{\sim}{\mathbf{B}}$ at the origin is in the direction of the dipole. The diamagnetic moment of the particles contributes

$$\frac{\Delta B_1}{B_0} = \frac{-nN\mu}{B_0 R^3} = -\frac{\mathcal{E}}{3\mathcal{E}_M} \tag{5}$$

at the origin. The streaming of $\nabla \times \underset{\sim}{\mathbf{B}}$ (or $\underset{\sim}{\mathbf{I}}$) along the lines of force contributes nothing. The $\nabla \times \underset{\sim}{\mathbf{B}}$ (or $\mathbf{I}$) in the particle segments, and in the crossover segments at $r = a$, contributes a total of

$$\frac{\Delta B_2}{B_0} = + \frac{n I Z_1 (\Delta\phi_1 + \Delta\phi_2)}{B_0 c R (Z_1 + Z_2)}$$

$$= \frac{1 + \Delta\phi_2/\Delta\phi_1}{1 + Z_2/Z_1} \frac{\mathcal{E}}{\mathcal{E}_M} . \tag{6}$$

The total $\Delta \mathbf{B}$ at the origin is in the direction of the dipole with a magnitude $\Delta B = \Delta B_1 + \Delta B_2$. If we were to make the assumption that the electrical impedance between two tubes of force passing through the ionosphere were proportional to the separation, it is evident that $\Delta B_2 = B_0 \, \mathcal{E}/\mathcal{E}_M$ so that the total distortion is given by (1) again. But in fact Z is more nearly proportional to the logarithm of the separation, because of the geometry of the two dimensional flow between widely separated points. This is shown in Appendix I, together with a discussion of the physical significance of Z in the present hydromagnetic context.

The two dimensional electrical impedance Z between two <u>small</u> regions of dimension s separated in a uniform resistive medium by a large distance D is proportional to $\ln D/s$. With $D = a \Delta\phi \sin\theta_0$ and $(a/s) \sin\theta_0$, we have $Z \propto \ln m\Delta\phi$ where $m\Delta\phi >> 1$ as a consequence of the smallness of s. Then

$$\frac{\Delta B_2}{B_0} = + \frac{\mathcal{E}}{\mathcal{E}_M} \frac{1 + \Delta\phi_2/\Delta\phi_1}{1 + \ln(m\Delta\phi_2)/\ln(m\Delta\phi_1)} \,. \qquad (7)$$

To exhibit the theoretical range of variation of $\Delta B_2/B_0$ suppose that $\nu \equiv \Delta\phi_2/\Delta\phi_1$ becomes large without limit as a consequence of making $\Delta\phi_1$ small.* In this limit

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* With s sufficiently small that $m\Delta\phi_1$ remains large compared to one, of course.

$$\frac{\Delta B_2}{B_0} \sim \frac{\nu}{\ln \nu} \ln(m\,\Delta\phi_1)\,\frac{\mathcal{E}}{\mathcal{E}_M} \tag{8}$$

which increases without bound, (as a consequence of increasing I. Thus in the limit of short particle segments <u>the distortion field at the origin can be made arbitrarily large</u>. The direction of ΔB is such as to represent an average decrease of the horizontal component over the surface of any small sphere enclosing the dipole.

The opposite limit, that ν becomes very small, with both $m\,\Delta\phi_1$ and $m\,\Delta\phi_2$ remaining large, leads to

$$\frac{\mathcal{E}}{2\mathcal{E}_M} < \frac{\Delta B_2}{B_0} < \frac{\mathcal{E}}{\mathcal{E}_M}$$

depending upon the magnitude of $m\,\Delta\phi_1$. The total field is then

$$\frac{\mathcal{E}}{6\mathcal{E}_M} < \frac{\Delta B}{B_0} < \frac{2\mathcal{E}}{3\mathcal{E}_M} .$$

In this case the net distortion ΔB may be rather less than the value given by (1).

These hypothetical extreme examples show that there is a wide range of theoretical possibilities.* Altogether, it is evident that nonsymmetric inflation

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* The example of the previous section, in which the total ΔB actually is negative, is recovered by making $Z_2/Z_1 \gg 1$ here.

in combination with a nonconducting atmosphere leads to a greater range of distortions than the symmetric inflation described by (1).

III. Discussion

The exposition has pointed out the basic theoretical effects of nonsymmetric inflation of a dipole magnetic field. In particular, a nonconducting atmosphere surmounted by an ionosphere permits a wide range for the worldwide average of the magnetic distortion $\Delta B / B_0$. In one extreme the inflation could produce a worldwide increase, described by (4), though we do not urge that this extreme is ever realized in nature. Generally speaking, an average worldwide decrease of the horizontal component is expected, ranging from a fraction to rather more than the value given by (1).

The reader should be aware that a number of serious idealizations have been introduced in order to make possible a concise presentation of the principal effects. The basic principles which we have pointed out are essentially unaffected by the idealizations. But, of course, the complications of the real situation must be included when one eventually constructs models for quantitative comparison with actual geomagnetic storms. To mention a few of the idealizations, we have neglected the Hall effect and ambipolar diffusion. We have made no attempt to follow the dispersion of a bunch of particles arising from the velocity spread of the particles.

The discussion has been limited to $\Delta \mathbf{B}$ at the origin because the value at the origin is equal to the average over any small sphere centered on the origin, and hence is a first approximation to the worldwide average $\Delta \mathbf{B}$.

But we have ignored the large local effects that occur around Earth in the nonsymmetric inflation. Finally, we have swept the fringing fields under the rug in Appendix I. As pointed out in Appendix I, the idealized particle segments would soon be deformed into arcs by the dissipative effects in the ionosphere, to which the nonsymmetric portion of the inflation is strongly coupled.

Returning to the qualitative points established in the text, consider the dissipation of the thermal energy of the nonsymmetric inflation. The portion of the inflating gases that is symmetric about the dipole axis is subject to no dissipation apart from atomic collisions and perhaps some internal plasma instabilities. The theoretical and observed dissipation time is typically 10^5 sec (Dessler and Parker, 1959). In contrast, the nonsymmetric portion of the inflating gas is obliged to drive currents and winds in the ionosphere, where the Cowling conductivity is as small as $10^6 - 10^8$ esu (Hanson, 1961). Consequently the dissipation time must be very short, probably of the order of $10^2 - 10^4$ sec.* A simple rectangular example of the dissipation is outlined in Appendix II, illustrating the outward motion of the inflating particles as they lose energy to the ionosphere.

When we look into the local magnetic effects of nonsymmetric inflation there are obviously additional dissipative effects. As noted earlier $\Delta \mathbf{B}(\mathbf{r})$ is not a uniform field in the vicinity of the origin. The local inhomogeneities, which

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* Alternatively the ionospheric currents may be viewed as the result of the diamagnetic buoyancy of the particles. The dissipation in the ionosphere permits the buoyancy of the particles to convect the particles, and the associated lines of force, outward through the geomagnetic field. This brings us to the viewpoint expressed first by Gold (1959; Chang et al., 1965).

can be very large, drift around with the inflating particles, producing local contortions in the ionospheric gases coupled to the total magnetic field. These are in addition to the ionospheric currents discussed so far.

In addition to dissipation, there is the deformation of the particle segments, caused by the fringing field (see Appendix I), and there is dispersion, as a consequence of the different particle velocities. A particle with 5 kev per unit electron charge has a drift velocity of about 1.5 km/sec at a distance of 4 Earth's radii, moving 5×10^3 km in one hour. The drift velocity is proportional to the particle energy, so that azimuthal inhomogeneities in the gas are probably smeared at some rate of the order of 1 km/sec, giving a life of the order of minutes to hours.

Altogether, then, nonsymmetric inflation is rather quickly dissipated and dispersed. Hence significant nonuniform inflation of the geomagnetic field is expected only during the active phase of a magnetic storm. It may be that the deep portion of some main phases which relaxes quickly after the active part of the storm is the result of nonsymmetric inflation. The basic theoretical properties of nonsymmetric inflation have been pointed out in this paper and it remains now to show from observational analysis of local variations at the surface of Earth and in space the extent to which nonsymmetric inflation contributes to the worldwide average decrease of the horizontal component during a magnetic storm.

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Appendix I. Motions Driven by Particle Segments

The text of this paper has ignored the spreading of $\nabla \times \mathbf{B}$ in crossing between magnetic lines of force through the ionosphere. The crossover was represented by a line segment with an electrical impedance Z. But the problem is really hydromagnetic in character so that impedance and emf are secondary quantities, in spite of their occasional convenience, and it remains for us to explain and to justify the treatment used in the text.

Consider, then, the physical significance of the impedance through the ionosphere between two tubes of flux separated by a distance $\Delta\phi$. The buoyant force exerted by the particle segment at $r = R$ tends to carry the lines of force outward, and the feet of the lines of force poleward. The viscosity of the ionosphere, and the nonconducting atmosphere beneath, resist the poleward drift of the feet of the lines of force. The force exerted on the ionosphere by the magnetic field is $(\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$, or, in terms of the current induced by $\nabla \times \mathbf{B}$, the force is $\mathbf{j} \times \mathbf{B}/c$. Now for stationary conditions there can be no net poleward drift of magnetic lines of force. So if the feet of some lines of force are being moved poleward by the buoyant forces of particles trapped in the field, there must be other lines of force being squeezed out of the polar regions and pushed toward the equator. The viscosity of the ionosphere resists the motion of the feet toward the equator, so $(\nabla \times \mathbf{B}) \times \mathbf{B}$ will be non-vanishing there too. The precise distribution of $\nabla \times \mathbf{B}$ over the regions moving toward the poles and toward the equator depends, obviously, upon the worldwide variation of viscosity and resistivity over the ionosphere. (See Gold, 1959; Axford and Hines, 1961; Chang et al., 1965). From the connection

$4\pi \mathbf{j} = c\, \nabla \times \mathbf{B}$ between $\nabla \times \mathbf{B}$ and the current density, it is evident that the electrical impedance is small in those regions of the ionosphere where $\nabla \times \mathbf{B}$ is large, and large where $\nabla \times \mathbf{B}$ is small. Thus $1/Z$ is a measure of the local force exerted by the field on the ionosphere.

It is instructive to treat in a rigorous manner one simple hypothetical example of the crossover of $\nabla \times \mathbf{B}$ through the ionosphere between magnetic lines of force. The example shows the deformation of the driving particle segments, which must be added to the dissipation and dispersion already mentioned. In addition the example demonstrates the validity of approximating the initial cross-over paths as simple segments. Represent the ionosphere by a thin plane sheet $z = 0$, of thickness ϵ, with surface conductivity σ and surface density ρ. Take the ionospheric fluid to be incompressible with a possible velocity distribution $\mathbf{v}(x, y)$ as a consequence of pressure p and magnetic forces $\mathbf{F}$. Then $\nabla \cdot \mathbf{v} = 0$. Viscous interaction with the nonconducting atmosphere beneath produces a drag which, as a first approximation, can be represented by $-K\mathbf{v}$ where K is a constant.

The two dimensional equation of motion is, then

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \frac{K}{\rho}\mathbf{v} + \frac{\mathbf{F}}{\rho}$$

Represent the geomagnetic field by the uniform vertical field $\mathbf{B} = \mathbf{e}_z B_0$.

The interaction of the ionosphere with the vertical field produces small magnetic perturbations $\mathbf{b}$, which are associated with the ionospheric current density

$\underset{\sim}{j} = c\,\nabla \times \underset{\sim}{b}/4\pi$. Integrating $\underset{\sim}{j}$ over the ionospheric thickness yields the surface current density $\underset{\sim}{i}$ where, incidentally, $\underset{\sim}{i}\,(x, y) = \underset{\sim}{e}_z \Delta \underset{\sim}{b}$ where $\Delta \underset{\sim}{b}$ is the change in $\underset{\sim}{b}$ across the ionospheric layer of small thickness ϵ . We have neglected the term $\epsilon \nabla \times \underset{\sim}{e}_z b_z$ assuming that ϵ is small compared to the horizontal scale of b_z . In terms of the surface current $\underset{\sim}{i}$, the magnetic force $\underset{\sim}{F}$ is $\underset{\sim}{i} \times \underset{\sim}{e}_z B_0/c$ to first order in the magnetic perturbation $\underset{\sim}{b}$.

The current density $\underset{\sim}{i}$ is given by Ohm's law as

$$\underset{\sim}{i} = \sigma\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{e}_z B_0/c\right).$$

For quasi-stationary conditions $\nabla \times \underset{\sim}{E} = 0$ and $\nabla \cdot \underset{\sim}{i} = 0$. It is readily shown that $\nabla \times \underset{\sim}{i} = 0$, so that $\underset{\sim}{i}$ can be written as $-\nabla \phi$. It then follows that $\nabla \times \underset{\sim}{F} = \nabla \cdot \underset{\sim}{F} = 0$, and the curl of the equation of motion gives $d\,\nabla \times \underset{\sim}{v}/dt = -(K/\rho)\nabla \times \underset{\sim}{v} + [(\nabla \times \underset{\sim}{v}) \cdot \nabla]\underset{\sim}{v}$ so that we may put $\nabla \times \underset{\sim}{v} = 0$. Since $\nabla \cdot \underset{\sim}{v} = 0$, write

$\underset{\sim}{v} = \nabla \times \underset{\sim}{e}_z A(x,y)$ where $\nabla^2 A = 0$. To determine the pressure p, take the divergence of the equation of motion, obtaining

$$\nabla^2 (p + \tfrac{1}{2}\rho v^2) = 0$$

after using the vector identity $(\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v} = \nabla v^2/2$ when $\nabla \times \underset{\sim}{v} = 0$. If the velocity vanishes and the ambient pressure is p_0 outside the region of flow, the solution is

$$p = p_0 - \tfrac{1}{2}\rho v^2 .$$

The equation of motion may now be written

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \frac{K}{\rho}\underset{\sim}{v} = \underset{\sim}{F} .$$

For quasi-steady conditions $\partial \underset{\sim}{v}/\partial t \cong 0$, so that

$$\underset{\sim}{v} = \frac{\rho B_0}{Kc}\, \underset{\sim}{i} \times \underset{\sim}{e}_z = \frac{\rho B_0}{Kc}\, \underset{\sim}{e}_z \times \nabla \phi = -\frac{\rho B_0}{Kc} \nabla \times \underset{\sim}{e}_z \phi .$$

But $\underset{\sim}{v} = \nabla \times \underset{\sim}{e}_z A$, so

$$A = -\frac{\rho B_0}{Kc}\, \phi .$$

Since $\rho B_0/Kc$ is a constant, it follows that the streamlines and the equipotential surfaces coincide, just as in a nondissipative medium where $\mathbf{v} = c\,\mathbf{E}\times\mathbf{B}/B^2$. It follows that $\mathbf{v}\times\mathbf{B}$, and hence $\mathbf{j}$ and $\nabla\times\mathbf{b}$ are parallel to $\mathbf{E}$.

Consider, then, the flow as a consequence of $\nabla\times\mathbf{B}$ or $\mathbf{j}$ (driven by particle segments elsewhere along the magnetic lines of force) entering the ionospheric sheet $z = 0$ in a small neighborhood of each of the points $x = \pm 2nh$, $y = 0$ where $n = 0, 1, 2, \cdots$, and leaving at $x = (\pm 2n+1)h$, $y = 0$. The equipotentials and streamlines are sketched in Fig. 3. It is a simple matter to prove that the disturbance driven by the sources of $\nabla\times\mathbf{B}$ at $x = \pm nh$ drops off exponentially at large distance with a characteristic length h. If the sources and sinks are equivalent to the charges of $\pm q$, respectively, then

$$E_x(x,y) = 2q(xS_1 - hS_2),\quad E_y(x,y) = 2qyS_1$$

where

$$S_1 = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(x-nh)^2+y^2},\quad S_2 = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n n}{(x-nh)^2+y^2}.$$

The series are readily summed using the contour integrals

$$J_1 = \int \frac{dw\,\operatorname{csch}(\pi w/h)}{y^2-(w-ix)^2},\quad J_2 = \int \frac{dw\;w\,\operatorname{csch}(\pi w/h)}{y^2-(w-ix)^2}$$

in the complex w - plane. The integrals have poles at $w = \pm i n h$ and $ix \pm y$. The result is

$$E_x(x,y) = \frac{2\pi q}{h} \frac{\sin(\pi x/h)\cosh(\pi y/h)}{\sin^2(\pi x/h) + \sinh^2(\pi y/h)}$$

$$E_y(x,y) = \frac{2\pi q}{h} \frac{\cos(\pi x/h)\sinh(\pi y/h)}{\sin^2(\pi x/h) + \sinh^2(\pi y/h)} .$$

It is evident that the field declines as $\exp(-\pi|y|/h)$ at large $|y|$. This fact is important because it means that the effects of $\underset{\sim}{j}$ or $\nabla \times b$ at distances large compared to h may be calculated as though $\underset{\sim}{j}$ etc where confined to the real axis. This is the formal justification for treating the crossover paths in the text as simple line segments. It is also the justification for ignoring the crossover between the northern and southern hemispheres.

The magnetic lines of force of B_o are essentially equipotentials. Hence, the electric field configuration sketched in Fig. 3 maps out of the ionosphere along the magnetic lines of force, having the same pattern elsewhere along the lines of force as in the ionosphere (see general discussion in Axford and Hines, 1961). Hence the flow $c\underset{\sim}{E} \times \underset{\sim}{B}/B^2$ of the driving particles at large r in the equatorial plane is essentially the same as in the ionosphere, with the result that the particle segments are steadily deformed and soon are not simple segments. The particle segments in the equatorial plane are deformed in the manner illustrated by the streamlines in Fig. 3. The particle segments, responsible for driving the system, not only dissipate and disperse, but deform as well. The next appendix gives a simple rectangular system in which the deformation of the particle distribution is absent, permitting the dissipation to be studied in a particularly simple manner.

Appendix II. Dissipation from Nonuniform Rotation

Consider a simple example of the dissipation and outward convection of clumps of particles inflating the geomagnetic field. The actual dipole geometry of the geomagnetic field produces so many complications in the calculation that it obscuressome of the physical effects, so we choose here a simplified, albeit artificial, geometry. The deformation of the particle distribution is eliminated at the expense of quantitative similarity to the actual geomagnetic situation. To accomplish the simplification, straighten out the magnetic lines of force keeping $z = 0$ as the equatorial plane. Transform the ionosphere into the planes $z = \pm h$ and the field into $\underset{\sim}{B} = e_z B_0$. The magnetic connection of the equatorial plane and ionosphere is thereby maintained. Place the particles in the $z = 0$ equatorial plane, distributed with surface density $n(x)$ in strips of width αl extending in the x-direction and separated by empty strips of width $(1-\alpha) l$, as illustrated in Fig. 4. Here α is a fixed fraction less than one. The strips of uniform width would correspond to radial sectors in the equatorial plane of the geomagnetic dipole. The particles are all assumed to have the same diamagnetic moment μ, so that if $B(x)$ is taken to be $B_0 \exp(x/L)$, the particles all have the same drift in the positive y-direction (perpendicular to the strips) of $v_y = \mu c/qL$. This corresponds to the azimuthal drift in a dipole field. It follows that if the particle strips were located in $nl < y < (n+\alpha)l$ initially, where $n = \cdots -2, -1, 0, +1, +2, \cdots$, the strips subsequently occupy $nl + v_y t < y < (n+\alpha)l + v_y t$.

The ionosphere, folded around into $z = \pm h$, is represented by a sheet of high conductivity.

Assume that the entire structure is repeated over z from $-\infty$ to $+\infty$ (from $z = h$ to $3h$, $-h$ to $-3h$, etc.) thereby eliminating the complication of fringing fields at $z = \pm h$. Make $h << L$ so that the unperturbed field lines are straight lines parallel to the z-axis, as assumed at the outset. Assume that the particle density $n(x)$ is sufficiently small that the particle pressure does not greatly deform the magnetic field, so that the linearized form of the theory can be used.

The vectors $\nabla \times \underset{\sim}{B}$ (or $\underset{\sim}{j}$) produced by the particle stresses are indicated by the arrows in Fig. 4, flowing forward in the y-direction across the particle strips, where the surface current density is denoted by $I = n(x) v_y q$. The flow divides at the leading edge of each strip, streaming to the ionospheric planes $z = \pm h$ along the magnetic lines of force. At $z = \pm h$ the vectors again split, flowing back the distance αl and forward the distance $(1 - \alpha) l$, and then returning along the magnetic lines of force to the rear edge of a particle strip. Denote the surface current density and electric field by I_1 and E_1 in the strip αl at $z = \pm h$, and by I_2 and E_2 in the strip $(1 - \alpha) l$. The current densities I_1 and I_2 are confined to $\pm h$ because of the assumed high surface conductivity across the magnetic lines of force, though the conductivity is everywhere in $-h < z < +h$ very high along the lines of force. The fields E_1 and E_2 are in the y-direction since $l << L$. They are uniform over z from $-h$ to $+h$ as a consequence of the high electrical conductivity along the lines of force. The electric fields are also uniform over y since $l << L$. Note that E_1 and I_1 are in the negative y-direction, and E_2 and I_2 are in the positive y-direction.

Conservation of current (or $\nabla \times \mathbf{B}$) requires that

$$I = 2(I_1 + I_2), \tag{II 1}$$

The electric potential difference over one period l in the y-direction is

$$\begin{aligned} \Phi &= -\alpha l E_1 + (1-\alpha) l E_2 \\ &= \frac{l}{\sigma}\left[(1-\alpha) I_2 - \alpha I_1\right], \end{aligned} \tag{II 2}$$

where σ is the surface conductivity of the ionospheric sheets. There is no net x-transport of magnetic lines of force, so $\Phi = 0$. Then

$I_1/I_2 = E_1/E_2 = 1/\alpha - 1$ and $I_2 = \alpha I/2$,

$I_1 = (1-\alpha)I/2$.

Since the electric field is uniform over z , the electric drift of the particles at $z = 0$ follows from E_1 computed at $z = \pm h$. The drift is in the negative x-direction with the speed

$$\begin{aligned} v_x &= c\,\frac{E_1}{B} \\ &= \frac{(1-\alpha) q c v_y n(x)}{2\sigma(x) B(x)} \end{aligned} \tag{II 3}$$

.

For steady conditions the particle flux $n(x)\, v_x$ must be independent of x. Hence make $n^2/\sigma B$ independent of x. It is sufficient to choose $\sigma(x)$ constant and

$$n(x) = n(0)\, \exp\left(x/2L\right) \tag{II 4}$$

to achieve stationary conditions. It follows that*

$$v_x(x) = \frac{(1-\alpha)\, q\, c\, v_y\, n(0)}{2\sigma B_0}\, \exp\left(-\frac{x}{2L}\right). \tag{II 5}$$

This velocity is in the negative x-direction, indicating that the ionospheric dissipation convects the particles out of the field.

The x coordinate of a particle at $x = 0$ when $t = 0$ is

$$x = \frac{L}{2} \ln\left[1 - \frac{(1-\alpha)\mu\, c^2 n(0)}{L^2 \sigma B_0}\, t\right] \tag{II 6}$$

and the particle energy is

$$\tfrac{1}{2} M w^2 = \mu\, B(x)$$

$$= \mu B_0 \left[1 - \frac{(1-\alpha)\mu\, c^2 n(0)}{L^2 \sigma B_0}\, t\right]^2. \tag{II 7}$$

*For as far in the negative x-direction as the guiding center approximation can be applied.

Hence the characteristic dissipation orescape time of the individual particles is

$$\tau = \frac{L^2 \sigma B_0}{(1-\alpha)\mu c^2 n(0)} \tag{II 8}$$

Note that the dissipation rate $1/\tau$ is proportional to the kinetic energy density of the particles in space and to the width of the gap between particle strips. This general property applies to the dissipation and escape of bunches of particles trapped in the geomagnetic dipole. The rate of energy loss for a given geometry is proportional to the square of the particle energy, and the energy loss decreases as the width of the gap between bunches of particles decreases, giving zero energy loss for an unbroken (axially summetric) distribution.

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Figure Captions

Fig. 1. Sketch of the two dipole lines of force through the ends of the azimuthal segment of particles in the equatorial plane. The drift velocity of the particles is v_ϕ. The flow of $\nabla \times \mathbf{B}$, or the current I, is indicated by the arrows. The crossover segments at $r = a$ and $\theta_0, \pi - \theta_0$ are labeled A and C. The geomagnetic dipole is directed downward in the sketch.

Fig. 2. Sketch of the dipole lines of force through the ends of a sequence of azimuthal segments of particles in the equatorial plane. The arrows indicate the flow of $\nabla \times \mathbf{B}$ along and across the magnetic lines of force. The geomagnetic dipole is directed upward in the sketch.

Fig. 3. The solid lines represent the flow of $\nabla \times \mathbf{B}$ (or current) through the ionosphere between the feet of equally spaced tubes of flux connecting into the ends of particle segments. The broken lines represent the associated drift $c\,\mathbf{E} \times \mathbf{B}/B^2$ of the magnetic lines of force and particles.

Fig. 4. Sketch of the rectangular geometry of strips of particles in the magnetic field $B(x)$ in the z-direction. The strips drift to the right as a consequence of dB/dx and into the page as a consequence of the dissipation in the ionospheric planes $z = \pm h$. The arrows indicate the direction of streaming of $\nabla \times \mathbf{B}$.

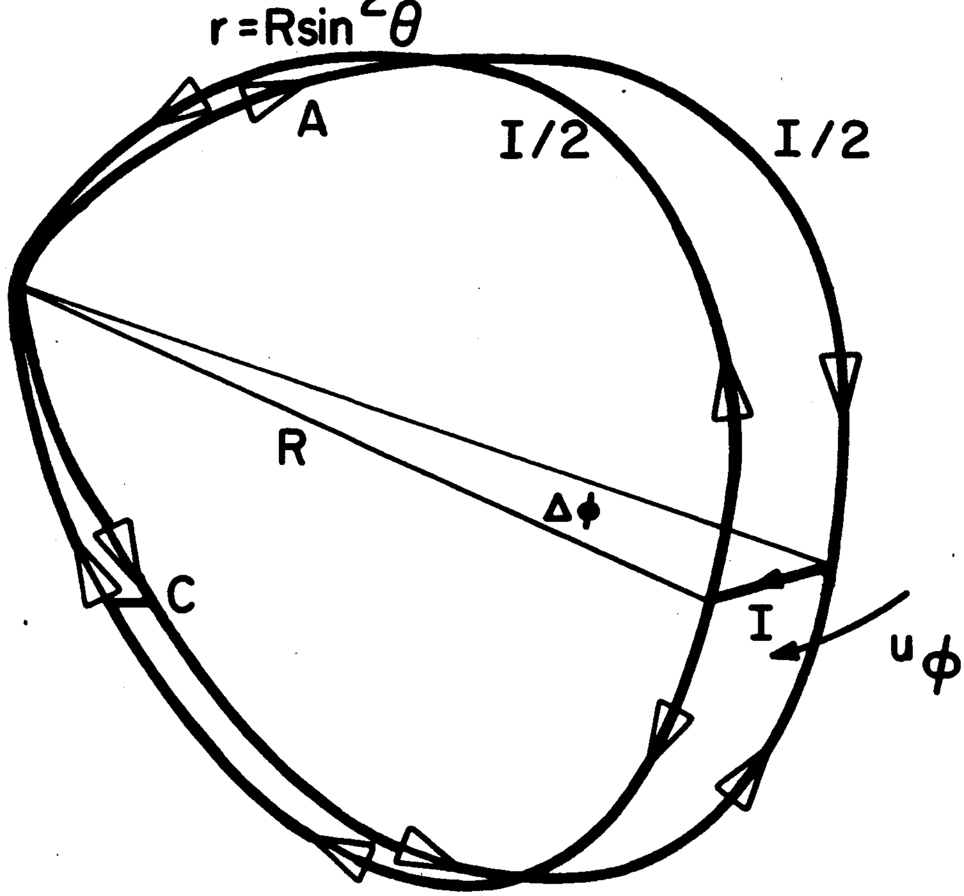

Fig. 1

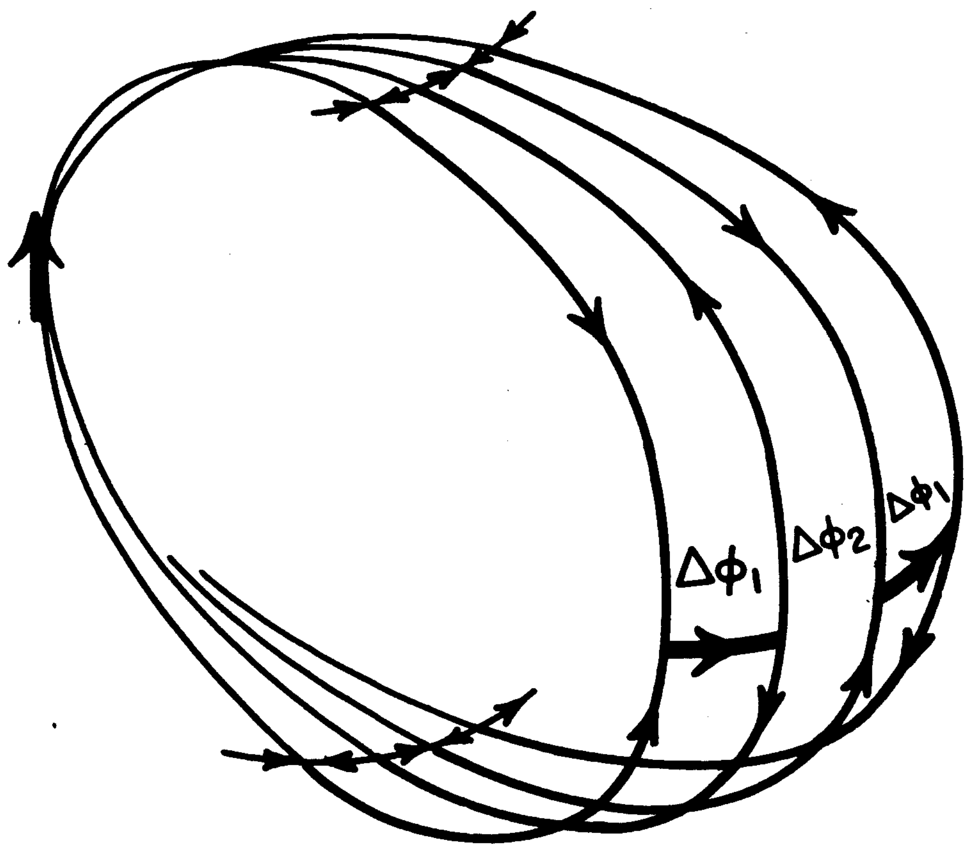

Fig. 2

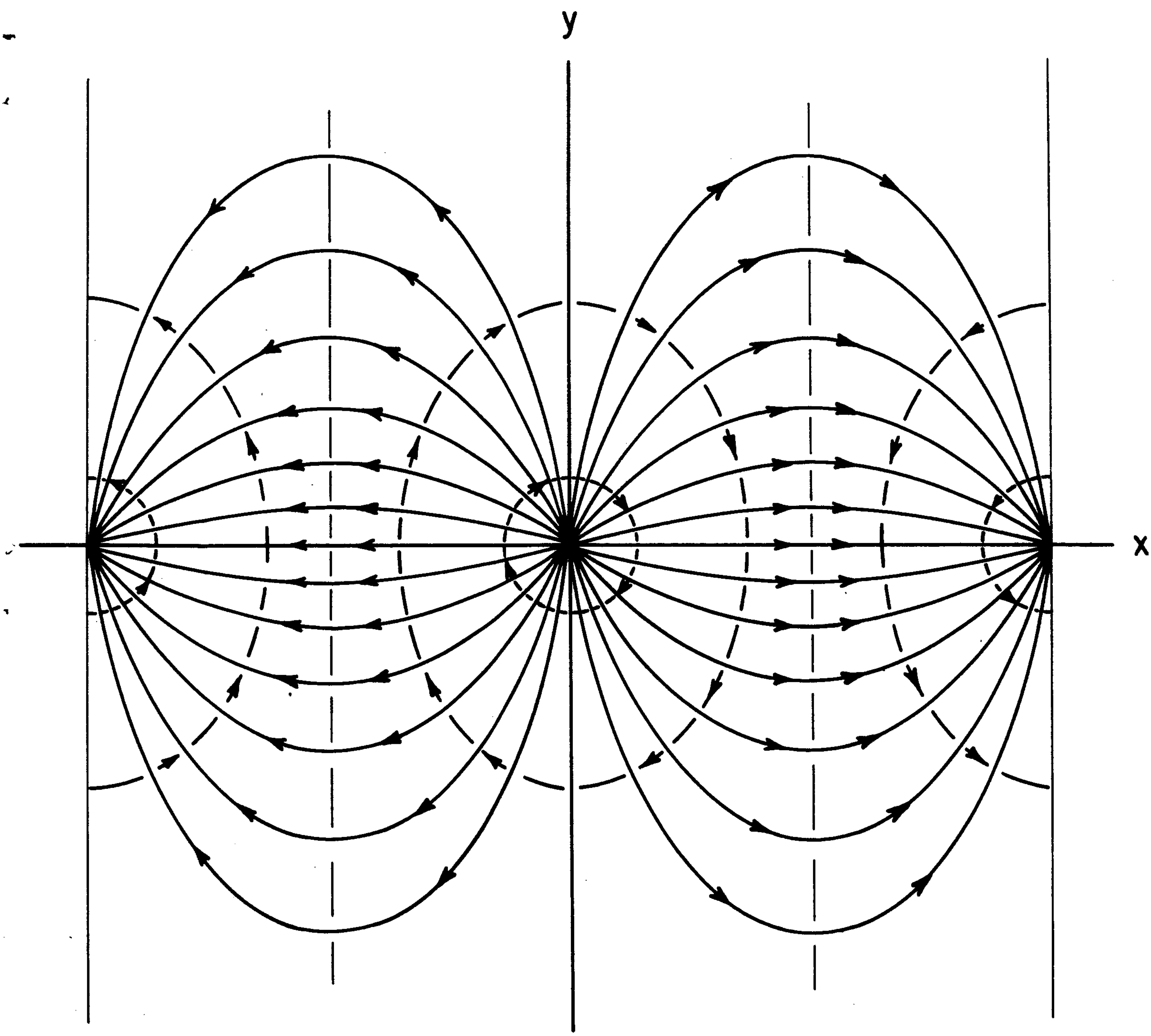

Fig. 3

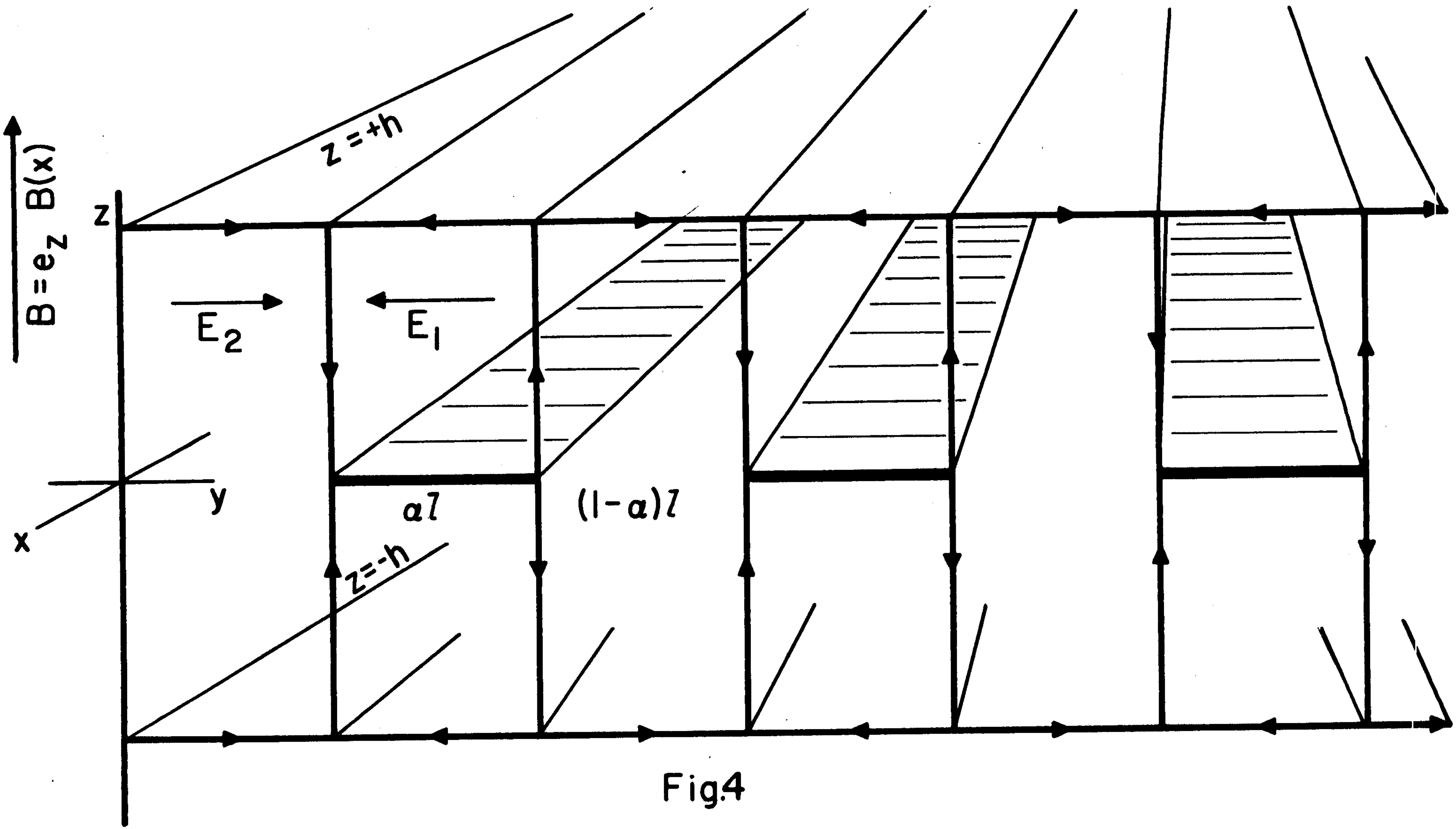

Fig.4