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THE OPTIMIZATION OF A RELAY CONTROLLED SYSTEM SUBJECTED TO RANDOM DISTURBANCES

by Gregory August Hyver

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. INTRODUCTION

Fundamental to the difficulty of the control optimization problem is the physical realization of this control once the characteristics of the optimum control are known. For example, it is well known that optimal solutions resulting from the use of the Maximum Principle (Ref. 1) usually involve far-end terminal conditions on the adjoint variable. This solution then requires that the control be a function of both the present state of the dynamical system and of this terminal condition of the adjoint system. Physical realization requires that the control be a function of the state of the dynamical system or of some estimate of this state; therefore, if the control is to be realized there must exist some transformation which takes the adjoint terminal condition to a state initial condition. Unfortunately for most optimization problems this transformation either cannot be found or if it can be found it is usually too unwieldly to use. As a result of these difficulties various quasi-optimal schemes have been devised.

Flügge-Lotz and Craig (Ref. 2) have devised a scheme for the minimum effort, regulator control using a bounded control. The time required to zero the error is a free parameter in the problem; however, for certain fixed times the scheme gives a true optimal solution. Various assumptions are made about the anticipated errors so that this time is chosen optimally. This solution is basically open loop so that the control objective is sensitive to errors made in state measurements and to disturbances acting on the system once the measurements are made; however, by periodically sampling the state and treating each new sampled state as an initial state, the loop is closed. If some of the characteristics of the disturbances were known this type of mechanization may not and probably would not be better than that mechanization which utilizes those known properties of the disturbances.

In an effort to take advantage of the fact that certain characteristics of the disturbances are known, A. M. Hopkins and P. K. C. Wang (Ref. 3) have realized a quasi-optimal solution utilizing these known characteristics. Their work treats the design of a relay control system required to zero the input-output error when the input to the control system is a random process. Essentially, the input process is predicted and the switching times found so that the error between the output and the predicted input, and all of the error derivatives are zeroed in a minimum of time. This requires among other things, that the input process has to be differentiable and that the error state is near the origin, for in this case advantage is taken of the fact that when the error state is near the origin the minimal trajectories have at most N-1 switchings, where N is the order of the plant. This then requires that the control system operate in two modes; namely, one mode being used when the error state is near the origin and the above criterion is used and the other mode being used when the error state is distant from the origin and then only requiring that the control system be used to zero the error and the error rate.

In both the solutions of Flügge-Lotz/Craig and Hopkins/Wang, the scheme for the realization of the quasi-optimal control has been the same; that is, from the properties of the true optimal solutions enough information is extracted so that a reasonably good quasi-optimal realization results. The degree of "goodness" of this quasi-optimal scheme is limited by the allowable degree of complexity of the control system hardware.

In this study a method of solution is advanced in which certain statistical properties of the optimal control are determined. Since this method is different from those in use at present, it is felt that additional information about the properties of the optimal solution will result when this method is used and that this additional information can be used to an advantage in the realization of a quasi-optimal scheme.

In this study it is assumed that the plant is subjected to random disturbances so that at any time the state of the plant is a random variable. It will be shown that, in certain cases, the optimal solution requires that the control signal be a function of the state of the plant; therefore, in this study it is assumed that the state of the plant can always be accurately determined.

The approach used in this study is to require that the control signal be the output of a relay, and then to find the optimal control signal satisfying this requirement. This approach may be used to re-evaluate those systems which result in a relay control due to certain approximations made during the course of the solution. In Ref. 4, for example, the plant's impulse response is expanded into a truncated Taylor's series and as a result of this truncation, the optimal control becomes a relay control. If a relay control would have been assumed at the outset, this assumption would have allowed more information about the actual physical system to be used during the course of the solution. This study attempts to take advantage of this additional information.

II. PROBLEM DEFINITION

2.1 STATEMENT OF THE PROBLEM

In the formulation of the problem, a general description of the system is given and the problem is described in a general manner; however, the solution is studied for various specialized versions of both the system and the stated problem in order that a maximum amount of information may be extracted from this study.

When used in this report, the expression "system" is used to denote the plant which is to be controlled and also the inputs effecting the performance of the plant. Among these inputs is the control itself. The expression "dynamical system" is understood to be the system exclusive of the inputs.

In this investigation, the design problem is to zero a given state in a prescribed time T while requiring that a certain performance criterion be minimized during the process. The dynamical system is assumed to be linear with additive noise at the input. It is apparent that it is not possible to zero the state at time T unless the effects of the additive noise were exactly known during the interval of control; therefore, the design criterion will be such that for a given initial state, the average value, over a large number of trials, of the state at time T be zero. In an effort to keep the final state within a small hypersphere around the origin, a certain multiple of the variance of the final state may be added to the original performance index which in this study is given by

$$E\left\{\int_{t_o}^{T}\sum_{i=1}^{m}|u^i(t)|dt\right\}.$$

Mathematically, this design criterion requires that

$$E[\underline{x}(T)] = 0 \tag{2.1a}$$

while the performance criterion

$$J = E\left\{\int_{t_o}^{T}\sum_{i=1}^{m}|u^i(t)|dt + \underline{x}'(T)Q\underline{x}(T)\right\} \tag{2.1b}$$

is minimized, where

$\underline{x}(T)$ = state vector at time T $(n \times 1)$.

$u^i(t)$ = i-th element of the control vector.

Q = constant matrix $(n \times n)$.

E = expectation operator.

2.2 THE SYSTEM

The differential equations determining the response of the system are given by the following expression

$$\dot{\underline{x}}(t) = F\,\underline{x}(t) + D\,\underline{u}(t) + B\underline{w}(t) \tag{2.2}$$

where

$\underline{x}(t)$ = state vector $(n \times 1)$

$\underline{u}(t)$ = control vector $(m \times 1\ ,\ m < n)$

$\underline{w}(t)$ = random noise vector $(r \times 1\ ,\ r < n)$

F = constant matrix $(n \times n)$

D = constant matrix $(n \times m)$

B = constant matrix $(n \times r)$

Throughout this paper, the state $\underline{x}(t)$ is considered to be an error state so that the expressions "state" and "error state" are used interchangeably.

It is assumed that all of the states of the dynamical system are available for measurement and that the measurements of these states are not noisy.

2.3 THE CONTROL SIGNAL

Since the control signal is restrained to be the output of a relay, the amplitude of this signal is restricted to a finite number of values. At any given time this amplitude is a random variable as are the switching times of this signal. In this sense, the control signal can be considered to be an ensemble sample from a random process.

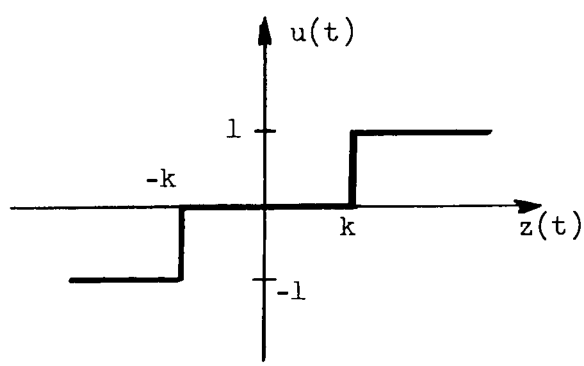
For convenience, only symmetrical relays will be considered. The mathematical description of this relay is given by

$$u(t) = +1 \quad \text{for} \quad z(t) \geq +k \tag{2.3a}$$

$$= -1 \quad \text{for} \quad z(t) \leq -k \tag{2.3b}$$

$$= 0 \quad \text{for} \quad |z(t)| < k \tag{2.3c}$$

where $k \geq 0$. This relay is shown in Fig. 2.1 where the signal $z(t)$ is the input to the relay.



MODEL OF THE RELAY

FIGURE 2.1

The fact that the "on" magnitude has unity value is again only in the interest of convenience. This restriction for a system with a scalar control or for a system with multiple controls is no restriction at all providing the "on" values of each of the controls has the same magnitude. This can be seen from an examination of Eq. 2.2. This equation can be normalized by defining new state variables and a new noise vector such that the "on" magnitude of the control signal has the value of unity.

If the input signal to the relay, $z(t)$, is a sample from a random process then there exist relationships defined on the probability distribution of $z(t)$ such that the stochastic properties of the control signal may be determined. For example using the fact that $u(t)$ is constrained to be either 0, $+1$, or -1 then

$$
\begin{aligned}
E[u(t)] &= \sum_{j=1}^{3} u_j(t)p[u_j(t)] \\
&= p_1(t) - p_2(t)
\end{aligned}
\tag{2.4a}
$$

$$
\begin{aligned}
E[u^2(t)] &= \sum_{j=1}^{3} u_j^2(t)p[u_j(t)] \\
&= p_1(t) + p_2(t)
\end{aligned}
\tag{2.4b}
$$

$$
\begin{aligned}
E[|u(t)|] &= \sum_{j=1}^{3} |u_j(t)|p[u_j(t)] \\
&= p_1(t) + p_2(t)
\end{aligned}
\tag{2.4c}
$$

where E is the expectation operator and

$u_j(t)$ = the j-th value of the control signal at time "t", where $u_1(t) = +1,\ u_2(t) = -1,\ u_3(t) = 0.$

$p_1(t)$ = the probability that at time "t" the control signal has the value $+1$.

$p_2(t)$ = the probability that at time "t" the control signal has the value -1.

Since the input signal to the relay is $z(t)$, these expectations can be written in terms of the probability distribution of $z(t)$ by using the mathematical description of the relay given in Eqs. 2.3.

$$E[u(t)] = p_1(t) - p_2(t)$$

$$= p[z(t) \geq +k] - p[z(t) \leq -k] \qquad (2.5a)$$

$$E[u^2(t)] = p_1(t) + p_2(t)$$

$$= p[z(t) \geq +k] + p[z(t) \leq -k] \qquad (2.5b)$$

$$E[|u(t)|] = p_1(t) + p_2(t)$$

$$= [z(t) \geq +k] + p[z(t) \leq -k] \qquad (2.5c)$$

Other relationships can be obtained in a similar manner.

In the ensuing discussion it will be found necessary to determine such stochastic properties of the control signal as

1. $E[u(t)]$
2. $E[u(t_1)\, u(t_2)]$
3. $E[u(t_1)\, w(t_2)]$

where $E[u(t_1)\, u(t_2)]$ is the auto-correlation function of the control signal and $E[u(t_1)\, w(t_2)]$ is the cross-correlation between the control signal at time t_1, and the noise at time t_2.

2.4 THE PERFORMANCE INDEX

As stated in Section 2.1, the performance index is such that some combination of total control effort and final state will be considered. Because both the control effort and the final state are random variables, this performance index must be based on an average cost; therefore, performance is defined as

$$J = E\left\{\int_{t_o}^{T} \sum_{i=1}^{m} |u^i(t)|dt\right\} + E[\underline{x}'(T)\, Q\underline{x}(T)] \qquad (2.6)$$

where the prime denotes transpose. The matrix Q is a constant $n \times n$ matrix, and $u^i(t)$ is the i-th element of the control vector.

2.5 ASSUMPTIONS

The development presented in this study will assume that both the additive noise and the control signal are scalar quantities, so that Eq. 2.2 may be written

$$\dot{\underline{x}}(t) = F\underline{x}(t) + Du(t) + Bw(t) \tag{2.7}$$

where D and B are vectors and $w(t)$ is a random process.

The system (Eq. 2.7) has the well known solution, (Ref. 5)

$$\underline{x}(t) = e^{F(t-t_o)}\underline{x}(t_o) + \int_{t_o}^{t} e^{F(t-\tau)}[Du(\tau)+Bw(\tau)]d\tau \tag{2.8}$$

where $e^{F(t-t_o)}$ is the transition matrix of the linear dynamical system. An element $\phi_{ij}(t-t_o)$ of this matrix is the time response of the i-th state due to an initial condition on the j-th state.

The integral on the right hand side of Eq. 2.8 is a function of the stochastic processes $u(t)$ and $w(t)$; therefore, it is necessary to ask if the integral exists and in what sense. For the purposes of this paper, it is sufficient to require that the integral exists in the sense that it can be represented by an approximating sum.

Since $u(t)$ is constrained to be either $+1$, -1 , or 0 , it is apparent that the part of the integral which involves the $u(t)$ process exists as the limit of an approximating sum, for if the integral is represented by an approximating sum and the subdivisions occur at the switching times, then the value of the approximating sum is exactly that value of the integral.

In order to show that the part of the integral involving the process $w(t)$ also exists as the limit of approximating sums, it is sufficient to show that the process satisfies the hypotheses of a theorem from

Parzen (Ref. 6) which requires that the stochastic process have the properties:

1) continuous parameter
2) mean value function $m(t) = E[w(t)]$ is continuous in t.
3) covariance kernel $K(s,t) = \text{Cov}\,[w(s), w(t)]$ is continuous in both s and t.

Therefore, in this study it will be assumed that $w(t)$ has the above properties. Then from Parzen's theorem

$$E\left[\int_a^b w(t)dt\right] = \int_a^b m(t)dt \tag{2.9a}$$

$$\text{Var}\left[\int_a^b w(t)dt\right] = \int_a^b\int_a^b K(s,t)dsdt \tag{2.9b}$$

$$K(s,t) = E[w(s)w(t)] - m(s)m(t) \tag{2.9c}$$

In this study it will be assumed that $m(t)$ is identically zero, so that Eq. 2.9 reduces to

$$E\left[\int_a^b w(t)dt\right] = 0 \tag{2.10a}$$

$$\text{Var}\left[\int_a^b w(t)dt\right] = \int_a^b\int_a^b K(s,t)dsdt \tag{2.10b}$$

$$K(s,t) = E[w(s)w(t)] \tag{2.10c}$$

The investigation in this study will be concerned with the solution to the optimization problem when various assumptions are made about the system and the performance index.

III. MINIMUM FUEL EXPENDITURE

3.1 DEVELOPMENT

In this chapter it is assumed that the matrix Q in Eq. 2.6 is the null matrix. The performance criterion reduces to the minimization of the function:

$$J = E\int_{t_o}^{T} |u(t)|dt \tag{3.1}$$

such that $E[x(T)] = 0$. From Eq. 2.8 it follows that

$$E\left\{e^{F(T-t_o)}\underline{x}(t_o) + \int_{t_o}^{T} e^{F(T-t)}[Du(t)+Bw(t)]dt\right\} = 0 \tag{3.2}$$

Since the state vector $\underline{x}(t_o)$ is assumed to be known, then

$$E[e^{F(T-t_o)}\underline{x}(t_o)] = e^{F(T-t_o)}\underline{x}(t_o) \tag{3.3}$$

Upon using the summation property of the expectation operator and also using Eq. 2.10a and Eq. 3.3, Eq. 3.2 can be written

$$\int_{t_o}^{T} e^{F(T-t)}DE[u(t)]dt = -e^{F(T-t_o)}\underline{x}(t_o) \tag{3.4}$$

or

$$\int_{t_o}^{T} e^{-Ft}DE[u(t)]dt = -e^{-Ft_o}\underline{x}(t_o) \tag{3.5}$$

From Eq. 2.4a, Eq. 3.5 can be written

$$\int_{t_o}^{T} e^{-Ft}D[p_1(t)-p_2(t)]dt = -e^{-Ft_o}\underline{x}(t_o) \tag{3.6}$$

and by using Eq. 2.4c, Eq. 3.1 becomes

$$J = \int_{t_o}^{T} [p_1(t) + p_2(t)]\, dt \tag{3.7}$$

In this formulation, $p_1(t)$ and $p_2(t)$ can be considered the control variables. Since $p_1(t)$ and $p_2(t)$ are probability functions, these functions must satisfy the following constraints

$$p_1(t) \geq 0 \tag{3.8a}$$

$$p_2(t) \geq 0 \tag{3.8b}$$

$$p_1(t) + p_2(t) \leq 1 \tag{3.8c}$$

The region of allowable values of these variables, which is now also the region of admissible controls, is shown in Fig. 3.1.

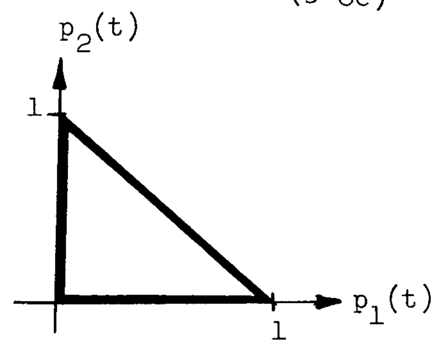


FIG. 3.1. REGION OF ADMISSIBLE CONTROLS.

The problem can now be restated in terms of Eq. 3.6 and Eq. 3.7 by defining a matrix C, such that

$$C\underline{p}(t) = D[p_1(t) - p_2(t)] \tag{3.9}$$

and by defining the new system as

$$\dot{\tilde{\underline{x}}}(t) = F\tilde{\underline{x}}(t) + C\underline{p}(t) \tag{3.10}$$

where $\tilde{\underline{x}}(t)$ is the state of this equivalent system. By using the constraints in Eq. 3.8, Eq. 3.7 can be expressed in the following form

$$J = \int_{t_o}^{T} \sum_{j=1}^{2} |p^j(t)|\, dt \tag{3.11}$$

With Eq. 3.10 as system equation, Eq. 3.11 as the performance index, and 3.8 as the constraints on the controls, the problem is now in a form where the Maximum Principle can be used to an advantage.

3.2 EXAMPLE

As an application of this method the second order plant

$$\ddot{x}(t) = u(t) + w(t) \tag{3.12}$$

is considered. One defines as usual

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = u(t) + w(t)$$

then the following system evolves

$$\dot{\underline{x}}(t) = F\underline{x}(t) + Du(t) + \underline{B}w(t) \tag{3.13}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e^{Ft} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

From Eq. 3.9 it follows that

$$C = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

therefore Eq. 3.10 can be written

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \tag{3.14}$$

and from 3.11 it follows that

$$J = \int_{t_o}^{T} [|p_1(t)| + |p_2(t)|]dt \tag{3.15}$$

or

$$J = \int_{t_o}^{T} [p_1(t) + p_2(t)]dt \tag{3.16}$$

For this system the Maximum Principle (Ref. 1), yields the Hamiltonian

$$H = - [p_1 + p_2] + \lambda_1\tilde{x}_2 + \lambda_2 [p_1 - p_2]$$

$$= \lambda_1\tilde{x}_2 - p_1(1-\lambda_2) - p_2(1 + \lambda_2) \tag{3.17}$$

where it is understood that λ_1 , λ_2 , p_1 , and p_2 may all be time dependent.

The adjoint variables are determined by the following equation

$$\mathring{\lambda}_1 = - \frac{\partial H}{\partial \tilde{x}_1} = 0 \tag{3.18a}$$

$$\mathring{\lambda}_2 = - \frac{\partial H}{\partial \tilde{x}_2} = -\lambda_1 \tag{3.18b}$$

which yield

$$\lambda_2 = -\lambda_1(t_o)t + \lambda_2(t_o) \tag{3.19}$$

The Hamiltonian H is maximized by the following choice of $p_1(t)$ and $p_2(t)$:

$$\begin{aligned} p_1(t) &= 0 \quad \text{if} \quad \lambda_2 < 1 \\ &= 1 \quad \text{if} \quad \lambda_2 > 1 \\ p_2(t) &= 1 \quad \text{if} \quad \lambda_2 < -1 \\ &= 0 \quad \text{if} \quad \lambda_2 > -1 \end{aligned} \tag{3.20}$$

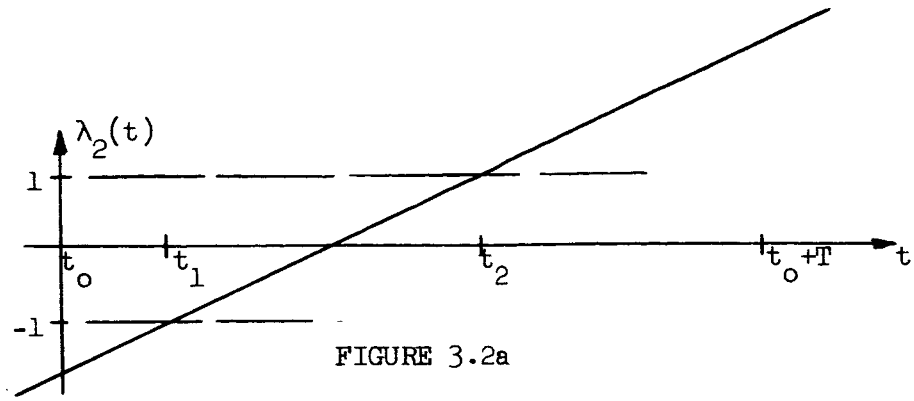


FIGURE 3.2a

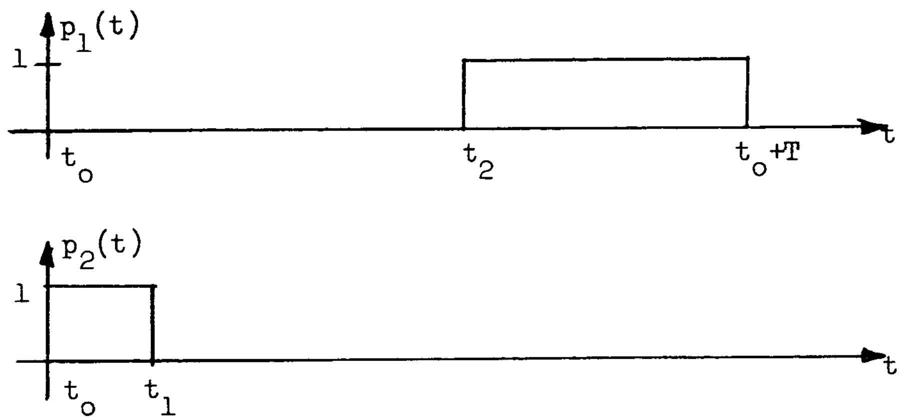


FIGURE 3.2b

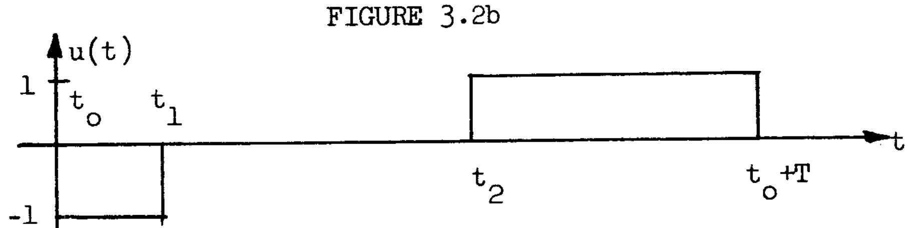


FIGURE 3.2c

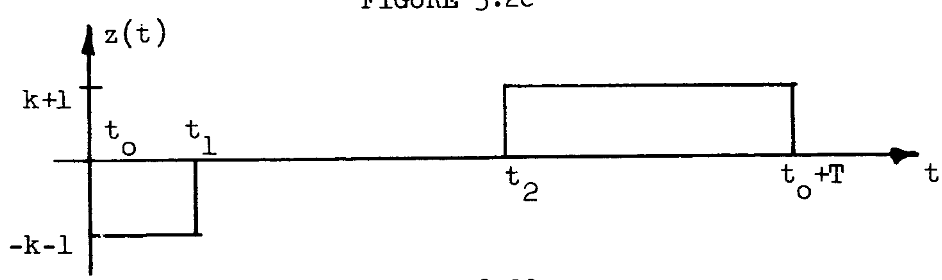


FIGURE 3.2d

FIGURE 3.2 TIME HISTORIES OF THE VARIOUS VARIABLES DERIVED IN THIS SECTION

These values of $p_1(t)$ and $p_2(t)$ are within the allowable region as set forth in the constraint Eq. 3.8 .

Figure 3.2b shows the "controls" $p_1(t)$ and $p_2(t)$ for the adjoint, $\lambda_2(t)$, shown in Fig. 3.2a.

Therefore; with probability one, the control signal to the actual dynamical system will be as shown in Fig. 3.2c. It is necessary to use the qualifying phrase "with probability one" since the only requirement on the control signal is that its probability distribution be as shown in Fig. 3.2b.

It is to be noted that this is the same control signal which would result if the disturbance $w(t)$ was ignored at the outset. This result is essentially a consequence of the requirement that the control only zeros the expected value of the final error state. There is no guarantee that this control signal will zero this final state; rather, there is only a guarantee that this control signal will zero the average value of the final state.

Incidentally, Fig. 3.2d shows a signal which can be used as an input signal to the relay if the control signal is to be the one shown in Fig. 3.2c. The value of k in Fig. 3.2d corresponds to the value of k in Fig. 2.1. There are an infinite number of input signals that would also give this same control signal. In this case the determination of an input to the relay, $z(t)$, given the required stochastic properties of the output is a trivial problem.

Figure 3.3 shows the physical realization of this control scheme assuming that the input to the relay, $z(t)$, is known. There is again the well known problem of determining the initial conditions of the adjoint variable. The system is open loop in that $z(t)$ is a programmed input.

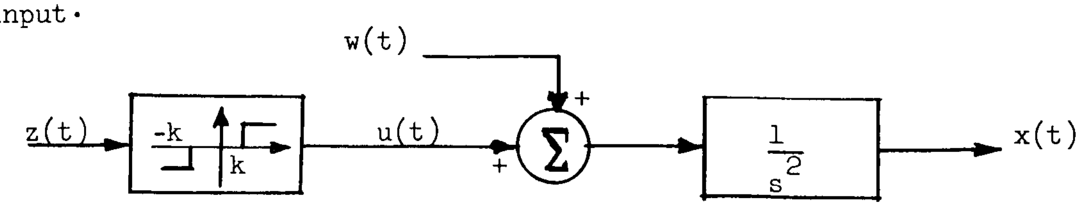


FIG. 3.3 REALIZATION OF THE SCHEME OF SECTION 3.2

3.3 EVALUATION OF CONTROL SCHEME OR SECTION 3.2

In order to check the "goodness" of this control it is necessary to determine the variance of this final state.

In this case, $E[u(t)] = u(t)$. When the constraint as given by Eq. 3.4 is satisfied, Eq. 2.8 can be written, (for $t = T$)

$$\underline{x}(T) = \int_0^T e^{F(T-\tau)} Bw(\tau)d\tau \tag{3.21}$$

Therefore,

$$E[\underline{x}(T)\underline{x}'(T)] = \int_0^T\int_0^T e^{F(T-\tau)} BB'e^{F'(T-\xi)} \overline{w(\tau)w(\xi)}d\tau d\xi \tag{3.22}$$

where the bar over the variable denotes the expectation operator and the prime denotes the transpose of the matrix. From Eq. 3.13, the above expression may be written out in its entirety.

$$\begin{bmatrix} \overline{x_1^2(T)} & \overline{x_1(T)x_2(T)} \\ \overline{x_2(T)x_1(T)} & \overline{x_2^2(T)} \end{bmatrix} = \int_0^T\int_0^T \begin{bmatrix} (T-\tau)(T-\xi) & (T-\xi) \\ (T-\tau) & 1 \end{bmatrix} \overline{w(\tau)w(\xi)}d\tau d\xi \tag{3.23}$$

Therefore the variance of the final state is given by,

$$\overline{x_1^2(T)} = \int_0^T\int_0^T (T-\tau) \quad (T-\xi) \quad \overline{w(\tau)w(\xi)}d\tau d\xi \tag{3.24a}$$

$$\overline{x_2^2(T)} = \int_0^T\int_0^T \overline{w(\tau)w(\xi)}d\tau d\xi \tag{3.24b}$$

For example, assuming $E[w(\tau)w(\xi)] = G\delta(\tau-\xi)$, so that $w(t)$ is white noise of power intensity G , then

$$\overline{x_1^2(T)} = G\int_0^T\int_0^T (T-\tau)\ (T-\xi)\delta(\tau-\xi)d\tau d\xi$$

$$= \frac{GT^3}{3} \tag{3.25a}$$

$$\overline{x_2^2(T)} = G\int_0^T\int_0^T \delta(\tau-\xi)d\tau d\xi$$

$$= GT \tag{3.25b}$$

The variance increases with the noise power and with the time T of the problem. Since the variance increases with T, an obvious method of reducing this variance is by reducing this time T. If T is a free parameter, it should be chosen as the minimum time required to zero the state. With this choice for T then a scheme similar to that presented by Hopkins and Wang (Ref. 3) can be used since that scheme zeros an error state in minimum time.

There are however, problems in which T is fixed; for example, a rendezvous problem in which the rendezvous time is given. In this case T is not a free parameter, so that if it is required that the variance be smaller, there must be a corresponding increase of fuel expenditure. It is to this class of problems that this investigation is next directed.

In the next chapter it is assumed that the cost of the fuel is negligible compared to the cost resulting from errors at the final time and that this final time is fixed and is greater than the minimum time mentioned in the preceding discussion.

IV. MINIMUM FINAL STATE VARIANCE

4.1 THE PERFORMANCE CRITERION

In this chapter, the problem in which the cost of control effort adds only a negligible amount to the performance index is considered. In this case, it can be assumed that the matrix Q (Eq. 2.6) is purposely chosen so that the integral in Eq. 2.6 has little effect upon the total cost. For this particular assumption Eq. 2.6 reduces to

$$J = E[\underline{x}'(T)Q\underline{x}(T)] \tag{4.1}$$

Again with the assumption that the additive noise has zero mean and that the state $\underline{x}(t_o)$ can be accurately determined, the constraint equation (Eq. 3.5) holds and is rewritten here for convenience.

$$\int_{t_o}^{T} e^{-Ft} DE[u(t)]dt = -e^{-Ft_o}\underline{x}(t_o) \tag{3.5}$$

By the use of Eq. 2.8, Eq. 4.1 may be expanded such that this cost is expressed in terms of the control effort and the additive noise, each of which is again assumed to be a scalar quantity.

$$J = E\Bigg\{[\underline{x}'(t_o)e^{F'(T-t_o)} + \int_{t_o}^{T} [u(\xi)D' + w(\xi)B']e^{F'(T-\xi)}d\xi]\cdot Q\cdot$$

$$[e^{F(T-t_o)}\underline{x}(t_o) + \int_{t_o}^{T} e^{F(T-\tau)}[Du(\tau) + Bw(\tau)]d\tau]\Bigg\} \tag{4.2}$$

At this point, the matrix Q is assumed to be a constant times the identity matrix; so that this matrix need not be considered further.

After expanding the right hand side of Eq. 4.2, it follows that

$$J = E\Bigg\{\underline{x}'(t_o)e^{F'(T-t_o)}e^{F(T-t_o)}\underline{x}(t_o)$$

(continued)

$$+ \underline{x}'(t_o)e^{F'(T-t_o)} \int_{t_o}^{T} e^{F(T-\tau)}Du(\tau)d\tau$$

$$+ \underline{x}'(t_o)e^{F'(T-t_o)} \int_{t_o}^{T} e^{F(T-\tau)}Bw(\tau)d\tau$$

$$+ \left[\int_{t_o}^{T} u(\xi)D'e^{F'(T-\xi)}d\xi \right] e^{F(T-t_o)}\underline{x}(t_o)$$

$$+ \left[\int_{t_o}^{T} w(\xi)B'e^{F'(T-\xi)}d\xi \right] e^{F(T-t_o)}\underline{x}(t_o)$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\xi)}e^{F(T-\tau)}D\ u(\xi)\ u(\tau)\ d\xi d\tau$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\xi)}e^{F(T-\tau)}B\ u(\xi)\ w(\tau)\ d\xi d\tau$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} B'e^{F'(T-\xi)}e^{F(T-\tau)}D\ w(\xi)\ u(\tau)\ d\xi d\tau$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} B'e^{F'(T-\xi)}e^{F(T-\tau)}\ B\ w(\xi)\ w(\tau)\ d\xi d\tau \Bigg\} \qquad (4.3)$$

Upon using the summation properties of the expectation operator and the assumption that $E[w(t)] = 0$, Eq. 4.3 is reduced to

$$J = \underline{x}'(t_o)\ e^{F'(T-t_o)}\ e^{F(T-t_o)}\ \underline{x}(t_o)$$

$$+ x'(t_o)\ e^{F'(T-t_o)} \int_{t_o}^{T} e^{F(T-\tau)}\ D\ \overline{u(\tau)}\ d\tau \qquad \text{(Continued)}$$

$$+\left[\int_{t_o}^{T} \overline{u(\xi)}\, D' e^{F'(T-\xi)} d\xi\right] e^{F(T-t_o)} \underline{x}(t_o)$$

$$+\int_{t_o}^{T}\int_{t_o}^{T} B' e^{F'(T-\xi)} e^{F(T-\tau)} B\, \overline{w(\xi)\, w(\tau)}\, d\xi d\tau$$

$$+\int_{t_o}^{T}\int_{t_o}^{T} D' e^{F'(T-\xi)} e^{F(T-\tau)} B\, \overline{u(\xi)\, w(\tau)}\, d\xi d\tau$$

$$+\int_{t_o}^{T}\int_{t_o}^{T} B' e^{F'(T-\tau)} e^{F(T-\tau)} D\, \overline{w(\xi)\, u(\tau)}\, d\xi d\tau$$

$$+\int_{t_o}^{T}\int_{t_o}^{T} D' e^{F'(T-\xi)} e^{F(T-\tau)} D\, \overline{u(\xi)\, u(\tau)}\, d\xi d\tau \qquad (4.4)$$

where the bar denotes the expectation operator.

This equation can be further simplified by noting that certain terms are independent of the control and therefore, need not be considered in the minimization of J .

For example, the double integral which involves only the autocorrelation of $w(\tau)$ is independent of the control and hence need not be considered in the minimization of J . Because of Eq. 3.5, the integral in the second term on the right hand side of Eq. 4.4 is simply $-e^{F(T-t_o)}\underline{x}(t_o)$ which is therefore only a function of the initial conditions; in fact when one uses Eq. 3.5 it can be shown that

$$\underline{x}'(t_o) e^{F'(T-t_o)} \int_{t_o}^{T} e^{F(T-\tau)}\, \overline{Du(\tau)}\, d\tau$$

(continued)

$$+\left[\int_{t_o}^{T} \overline{u(\xi)} \quad D'e^{F'(T-\xi)}d\xi\right]e^{F(T-t_o)} \underline{x}(t_o)$$

$$= - 2\underline{x}'(t_o) e^{F'(T-t_o)} e^{F(T-t_o)} \underline{x}(t_o) \tag{4.5}$$

The control dependent terms may be collected for convenience by defining

$$J^* = J + \underline{x}'(t_o) e^{F'(T-t_o)} e^{F(T-t_o)} \underline{x}(t_o)$$

$$- \int_{t_o}^{T}\int_{t_o}^{T} B'e^{F'(T-\xi)}e^{F(T-\tau)} B \overline{w(\xi) w(\tau)} \, d\xi d\tau \tag{4.6a}$$

With this definition for J^*, the control which minimizes J^* also minimizes J; therefore, J^* will be written so that its dependence on the control can be seen,

$$J^* = \int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\xi)}e^{F(T-\tau)}B \overline{u(\xi) w(\tau)} \, d\xi d\tau$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} B'e^{F'(T-\xi)} e^{F(T-\tau)}D \overline{u(\tau) w(\xi)} \, d\xi d\tau$$

$$+ \int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\xi)}e^{F(T-\tau)}D \overline{u(\tau) u(\xi)} \, d\xi d\tau \tag{4.6b}$$

Since each of the integrals of Eq. 4.6b is a scalar quantity and since the two integrals involving the noise process, $w(\cdot)$, are the transpose of each other, Eq. 4.6b can be rewritten

$$J^* = \int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\tau)}e^{F(T-\xi)}D \overline{u(\tau) u(\xi)} \, d\xi d\tau \qquad \text{(continued)}$$

$$+ 2\int_{t_o}^{T}\int_{t_o}^{T} D'e^{F'(T-\tau)}e^{F(T-\xi)} B\,\overline{u(\tau)\,w(\xi)}\,d\xi d\tau \tag{4.7}$$

It will now be assumed that the noise is white. Insofar as the statistics of the noise are concerned, this assumption presents no real restriction if the noise is gaussian since a linear filter can be added to the dynamical system such that the output of this filter is colored and gaussian when the input is white and gaussian. However for those systems where the noise is not gaussian, the assumption that the noise is white may impose a restriction on the class of problems for which the following development is applicable. The assumption that the noise is white must therefore be considered when the system equations are derived.

With this assumption, it follows that the control signal is independent of the future values of the noise since the future values are not predictable; therefore, since the noise has zero mean, the average, $\overline{u(\tau)\,w(\xi)}$ is zero for $\xi > \tau$. With this assumption concerning the spectral density of the noise and the fact that the first term on the right hand side of Eq. 4.7 is symmetrical in τ and ξ, this equation can be rewritten

$$J^* = 2\int_{t_o}^{T}\int_{t_o}^{\tau} D'e^{F'(T-\tau)}e^{F(T-\xi)}D\,\overline{u(\xi)\,u(\tau)}\,d\xi d\tau$$

$$+ 2\int_{t_o}^{T}\int_{t_o}^{\tau} D'e^{F'(T-\tau)}e^{F(T-\xi)}B\,\overline{w(\xi)\,u(\tau)}\,d\xi d\tau \tag{4.8}$$

or

$$J^* = 2E\left[\int_{t_o}^{T} u(\tau)d\tau \int_{t_o}^{\tau} [D'e^{F'(T-\tau)}e^{F(T-\xi)}D\,u(\xi)\right.$$

$$\left. + D'\,e^{F'(T-\tau)}e^{F(T-\xi)}B\,w(\xi)]\,d\xi\right] \tag{4.9}$$

Equation 4.9 can be more simply expressed by defining a random variable, $y(\tau)$, such that

$$y(\tau) = \int_{t_o}^{\tau} [D' e^{F'(T-\tau)} e^{F(T-\xi)} D\, u(\xi) + D' e^{F'(T-\tau)} e^{F'(T-\xi)} Bw(\xi)] d\xi \quad (4.10)$$

With this definition for $y(\tau)$, Eq. 4.9 can be rewritten

$$J^* = 2 \int_{t_o}^{T} \overline{u(\tau)\, y(\tau)}\, d\tau \quad (4.11)$$

Minimization of J^* as given by Eq. 4.11, subject to the constraints as given by Eq. 3.5 is the primary objective of this research. This minimization must consider the fact that $u(\tau)$ is constrained to be the output of a relay.

The performance index, J^* , as given by Eq. 4.11 can be minimized if the function, $\overline{u(\tau)\, y(\tau)}$, is minimized at each instant of time. Minimization of $\overline{u(\tau)\, y(\tau)}$ is discussed in the following section.

4.2 MINIMIZATION OF $\overline{u(\tau)\, y(\tau)}$

Minimization of the average, $\overline{u(\tau)\, y(\tau)}$, can be first investigated by expressing this average in the form

$$\overline{u(\tau)\, y(\tau)} = \rho[u(\tau) , y(\tau)]\sigma[u(\tau)]\sigma[y(\tau)] + \overline{u(\tau)}\; \overline{y(\tau)} \quad (4.12)$$

where

$\rho[u(\tau),y(\tau)]$ is the normalized correlation coefficient

$\sigma[y(\tau)]$ is the standard deviation of $y(\tau)$

$\sigma[u(\tau)]$ is the standard deviation of $u(\tau)$.

Since the average value of the noise is zero, the function, $\overline{y(\tau)}$, in Eq. 4.12 is given by the expression

$$\overline{y(\tau)} = \int_{t_o}^{\tau} D' e^{F'(T-\tau)} e^{F(T-\xi)} D\, \overline{u(\xi)}\, d\xi \quad (4.13)$$

Therefore when Eq. 4.12 and Eq. 4.13 are substituted into Eq. 4.11, the performance index becomes

$$J^*=2\int_{t_o}^{T} \rho[u(\tau),y(\tau)]\ \sigma[u(\tau)]\ \sigma[y(\tau)]\ d\tau$$

$$+\ 2\int_{t_o}^{T}\int_{t_o}^{\tau} D' e^{F'(T-\tau)} e^{F(T-\xi)} D\ \overline{u(\tau)}\ \overline{u(\xi)}\ d\xi d\tau \tag{4.14a}$$

or

$$J^* = 2\int_{t_o}^{T} \rho[u(\tau),\ u(\tau)]\ \sigma[u(\tau)]\ \sigma[y(\tau)]\ d\tau$$

$$+\left[\int_{t_o}^{T} \overline{u(\tau)}\ D' e^{F'(T-\tau)} d\tau\right]\left[\int_{t_o}^{T} e^{F(T-\xi)} D\ \overline{u(\xi)} d\xi\right] \tag{4.14b}$$

When Eq. 3.4 and Eq. 4.14b are combined, the resulting equation for J^* is given by

$$J^* = 2\int_{t_o}^{T} \rho[u(\tau)\ ,\ y(\tau)]\ \sigma[u(\tau)]\ \sigma[y(\tau)]\ d\tau$$

$$+\ \underline{x}'(t_o)\ e^{F'(T-t_o)} e^{F(T-t_o)} \underline{x}'(t_o) \tag{4.15}$$

The last term of Eq. 4.15 is independent of the control; therefore in order to minimize Eq. 4.15 it is only necessary to minimize the value of the integral. This integral will assume its most negative value if the correlation coefficient, $\rho[u(\tau),y(\tau)]$, assumes its most negative value at each instant of time independent of the standard deviations $\sigma[u(\tau)]$ and $\sigma[y(\tau)]$. However, it is well known, (Ref. 7, p. 263, for example) that the only cases for which perfect correlation such as this exists are for those cases where the two random variables are linearly related. Since $u(\tau)$ has at most three values and $y(\tau)$ can have possible an infinite number, it is obvious that the two random variables can not be linearly related and thus $\overline{u(\tau)y(\tau)}$ can not attain this lower bound. The problem is now to find the attainable minimum and having found it, to determine the conditions on the control so that this minimum may be realized.

Justification for the manipulations used in the following presentation can be found in any good book on probability theory (Ref. 7, for example).

The average, $\overline{u(\tau)\ y(\tau)}$, is given by

$$\overline{u(\tau)\ y(\tau)} = \int_0^1 y(\tau)\ E[u(\tau)|y(\tau)]\ dF[y(\tau)] \tag{4.16}$$

where $E[u(\tau)|y(\tau)]$ is the expectation of $u(\tau)$ conditioned on $y(\tau)$ and $F[y(\tau)]$ is the cumulative distribution fuction of $y(\tau)$. When it is assumed that the probability density function of $y(\tau)$ contains no delta functions, then the differential, $dF[y(\tau)]$, is given by

$$dF[y(\tau)] = p[y(\tau)]\ dy(\tau) \tag{4.17}$$

Also, the average, $\overline{u(\tau)}$, is given by

$$\overline{u(\tau)} = \int_0^1 E[u(\tau)|y(\tau)]\ dF[y(\tau)] \tag{4.18}$$

Eq. 3.5 imposes an integral constraint on the values of $\overline{u(\cdot)}$; therefore, the expression for $\overline{u(\tau)}$ as given by Eq. 4.18 imposes a constraint on the minimum value of $\overline{u(\tau)\ y(\tau)}$, since the function $E[u(\tau)|y(\tau)]dF[y(\tau)]$ which determines the minimum of $\overline{u(\tau)y(\tau)}$ also determines the value of $\overline{u(\tau)}$. This can be easily seen in the event it is required that $\overline{u(\tau)}$ be +1 . For in this case it is necessary that the function $E[u(\tau)|y(\tau)]$ be +1 .

Therefore the problem of minimizing the function $\overline{u(\tau)\ y(\tau)}$ will be rephrased as follows:

"For a fixed but as yet unknown value of $\overline{u(\tau)}$ find the function, $E[u(\tau)|y(\tau)]$, such that the average, $\overline{u(\tau)\ y(\tau)}$, is minimized."

For convenience it is assumed that the density function of $y(\tau)$ contains no delta functions. In forming $y(\tau)$, the two signals,

$u(\xi)$ and $w(\xi)$, are smoothed by the integration process; therefore, besides being convenient, this assumption is also reasonable. When this assumption is used, Eq. 4.16 and Eq. 4.18 can be written

$$\overline{u(\tau)\ y(\tau)} = \int_{-\infty}^{\infty} y(\tau)\ E[u(\tau)|y(\tau)]\ p[y(\tau)]dy(\tau) \tag{4.19}$$

and

$$\overline{u(\tau)} = \int_{-\infty}^{\infty} E[u(\tau)|y(\tau)]\ p[y(\tau)]\ dy(\tau) \tag{4.20}$$

Minimization of Eq. 4.19 subject to the constraint given by Eq. 4.20 can be considered as an optimization problem in which the Maximum Principle can be effectively used. In this formulation, the dynamical system is defined by the differential equation

$$\frac{d\ \overline{u(\tau)}}{d\ y(\tau)} = E[u(\tau)|y(\tau)]\ p[y(\tau)] \tag{4.21}$$

and Eq. 4.19 is the appropriate performance criterion. The "control", which for this system is $E[u(\tau)|y(\tau)]$ is bounded by ± 1 . The Hamiltonian for this system becomes

$$\begin{aligned} H[y(\tau)] &= -y(\tau)\ E[u(\tau)|y(\tau)]\ p[y(\tau)] \\ &\quad + \lambda_1[y(\tau)]\ E[u(\tau)|y(\tau)]\ p[y(\tau)] \end{aligned} \tag{4.22}$$

where

$$\dot{\lambda}_1[y(\tau)] = -\frac{\partial H[y(\tau)]}{\partial E[u(\tau)]} = 0$$

which implies that at each fixed time, $\lambda_1[y(\tau)]$ is a constant.

Since $p[y(\tau)]$ is a positive quantity and since $E[u(\tau)|y(\tau)]$ is bounded by ± 1 , the Hamiltonian is maximized by making

$$E[u(\tau)|y(\tau)] = \mathrm{SGN}\left\{\lambda_1[y(\tau)] - y(\tau)\right\} \tag{4.23}$$

where

$$\mathrm{SGN}\left\{x\right\} = \begin{cases} +1 & \text{for } x > 0 \\ -1 & x < 0 \end{cases}$$

Therefore, $E[u(\tau)|y(\tau)]$ is always either $+1$ or -1. The value of $y(\tau)$ where this function changes sign is that value where $y(\tau)$ is equal to $\lambda_1[y(\tau)]$. By letting this value of $y(\tau)$ be denoted by $\hat{y}(\tau)$, Eq. 4.20 can be written

$$\overline{u(\tau)} = \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau)]dy(\tau) - \int_{\hat{y}(\tau)}^{\infty} p[y(\tau)]\,dy(\tau) \tag{4.24}$$

Equation 4.24 is the rule which is used to find the value of $\hat{y}(\tau)$ in terms of the fixed but as yet unknown value of $\overline{u(\tau)}$. This procedure requires that the probability density function of $y(\tau)$ is known.

In a physical relay it must be admitted that there is a slight delay in the action of the relay. Equation 4.23 specifies that the control signal, $u(\tau)$, be correlated with the random variable $y(\tau)$. Because of the slight delay in the action of the relay, the control signal, $u(\tau)$, will actually be correlated with the random variable, $y(\tau-\epsilon)$ where ϵ can be considered to be an infinitesimal quantity. Since $y(\tau)$ is a smooth function, the assumption is made that the density function of $y(\tau)$ and the density function of $y(\tau-\epsilon)$ are very nearly identical so that Eq. 4.24 may be written

$$\overline{u(\tau)} = \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau-\epsilon)]d\{y(\tau-\epsilon)\} - \int_{-\infty}^{\infty} p[y(\tau-\epsilon)]\,d\{y(\tau-\epsilon)\} \tag{4.24a}$$

Since $y(\tau-\epsilon)$ is composed of the past values of the noise and the control signal, it is therefore possible to determine the distribution of this function. The distribution of the function $y(\tau)$ can be assumed to be identical to this distribution of $y(\tau-\epsilon)$. Therefore, for a given value of $\overline{u(\tau)}$, Eq. 4.24 may be used to compute $\hat{y}(\tau)$.

Once the optimizing value of $\overline{u(\tau)}$ is known and $\hat{y}(\tau)$ is computed, there is a straightforward rule for specifying the control law such that the "best" correlation is obtained. This rule requires that

$$E[u(\tau)|y(\tau)] = 1 \quad \text{if} \quad y(\tau) < \hat{y}(\tau) \tag{4.25a}$$

$$= -1 \quad \text{if} \quad y(\tau) > \hat{y}(\tau) \tag{4.25b}$$

But $E[u(\tau)|y(\tau)] = 1$ implies that

$$p[u(\tau) = 1 \mid y(\tau)] = 1$$

$$p[u(\tau) = 0 \mid y(\tau)] = 0 \tag{4.26}$$

$$p[u(\tau) = -1 \mid y(\tau)] = 0$$

and $E[u(\tau)|y(\tau)] = -1$ implies that

$$p[u(\tau) = 1 \mid y(\tau)] = 0$$

$$p[u(\tau) = 0 \mid y(\tau)] = 0 \tag{4.27}$$

$$p[u(\tau) = -1 \mid y(\tau)] = 1$$

and thus given $y(\tau)$ and $\hat{y}(\tau)$, $u(\tau)$ is known to be almost always either $+1$ or -1 ; the dead zone in the relay is thus unnecessary since $p[u(\tau) = 0 \mid y(\tau)] = 0$ always.

Now that $\overline{u(\tau)\, y(\tau)}$ has been minimized with respect to the correlation between $u(\tau)$ and $y(\tau)$, it is only necessary to optimize the unknown value $\overline{u(\tau)}$.

When the optimizing value for $E[u(\tau)|y(\tau)]$ is substituted into Eq. 4.19, this equation becomes

$$\overline{u(\tau)\, y(\tau)}_{min} = \int_{-\infty}^{\hat{y}(\tau)} y(\tau)\, p[y(\tau)] d\, y(\tau) - \int_{\hat{y}(\tau)}^{\infty} y(\tau)\, p[y(\tau)] d\, y(\tau) \tag{4.28}$$

Since it was previously assumed that the probability density of $y(\tau)$ is known and is not dependent on the control at time τ , the value of the control, $\overline{u(\tau)}$, appears only in the limits of the integrals of Eq. 4.28 since $\overline{u(\tau)}$ is a function of $\hat{y}(t)$. Therefore, minimization of J^* can be further affected by the proper determination of either the function $\hat{y}(\cdot)$ or the function $\overline{u(.)}$.

This optimal value of the function, $\overline{u(\cdot)}$, must also satisfy the constraints given by Eq. 3.5; therefore, at this point the Maximum Principle can be used to find this optimal function.

A new state variable is defined by the equation

$$\dot{\underline{\tilde{x}}}(t) = F\underline{\tilde{x}}(t) + D\, \overline{u(t)} \tag{4.29}$$

where the F and the D are identical to the F and the D of the original system equations. It is obvious that this new state variable, $\underline{\tilde{x}}(t)$, is actually the average value of the state of the original system.

With Eq. 4.29 as the system equation, the constraint on the original system given by Eq. 3.5 reduces to the requirement that the state $\underline{\tilde{x}}(T)$ is zero.

The performance criterion for this equivalent system is of the form

$$J^* = 2 \int_{t_o}^{T} f[u(\tau)]\, d\tau \tag{4.30a}$$

where

$$f[u(\tau)] = \int_{-\infty}^{\hat{y}(\tau)} y(\tau)\, p[y(\tau)] d\, y(\tau) - \int_{\hat{y}(\tau)}^{\infty} y(\tau)\, p[y(\tau)] d\, y(\tau) \tag{4.30b}$$

The function $f[\overline{u(\tau)}]$ is a function of $\overline{u(\tau)}$ since the limits of the integrals are functions of $\overline{u(\tau)}$.

In this reformulation of the problem, the Maximum Principle is used to find the optimal value of $\overline{u(\tau)}$ which minimizes Eq. 4.30a subject to the constraint that the final state, $\tilde{x}(T)$, is zero.

4.3 SUMMARY

In this chapter a procedure is presented which derives a rule for determining the magnitude of the control signal such that the variance of the final state is minimized. This procedure requires that a function of the past values of the control signal and the noise be formed and that this function be compared to a predetermined function, $\hat{y}(\tau)$, and on the basis of this comparison, the control signal is generated. This value of $\hat{y}(\tau)$ is the value of $y(\tau)$ where the function, $E[u(\tau)|y(\tau)]$, changes sign from +1 to -1 .

The mathematical procedure for determining this value of $\hat{y}(\tau)$ is summarized as follows:

1) By defining an equivalent system (Eq. 4.29), the optimal value

of $\overline{u(\tau)}$ is determined by the application of the Maximum Principle.

2) The value of $\hat{y}(\tau)$ is computed from this value of $\overline{u(\tau)}$ by the use of Eq. 4.24.

Application of these procedures is presented in Chapter V.

V. APPLICATIONS TO PHYSICAL SYSTEMS

5.1 INTRODUCTION

In this chapter it is shown that when the input noise is gaussian, the minimization procedure described in Chapter IV can give a very good quasi-optimal solution to the relay control problem. The emphasis is on the $1/s^2$ plant; however, toward the end of this chapter the $1/(s^2+1)$ plant is briefly considered.

5.2 A QUASI-OPTIMAL SOLUTION FOR THE $1/s^2$ PLANT

The procedure described in Chapter IV will now be used in order to determine a quasi-optimal control law for the system described by the differential equation

$$\frac{d^2x(t)}{dt^2} = u(t) + w(t) \tag{5.1}$$

where the noise, $w(t)$, is assumed to be a sample from a stationary, zero mean, gaussian random process with auto-correlation function given by $G\delta(t)$ where $\delta(t)$ is the Dirac delta function.

In this case the state equations are given by Eq. 3.13. From Eq. 3.13 it is seen that the vectors B and D are identical; therefore the random variable $y(\tau)$ as defined by Eq. 4.10 is given by

$$y(\tau) = \int_0^{\tau} D' e^{F'(T-\tau)} e^{F(T-\xi)} D[u(\xi) + w(\xi)]\, d\xi \tag{5.2}$$

where for convenience, it is assumed that the lower limit, t_o, is zero.

By defining a new random variable $v(\xi)$ such that

$$v(\xi) = u(\xi) + w(\xi) \tag{5.3}$$

then Eq. 5.2 can be written

$$y(\tau) = \int_0^{\tau} D' e^{F'(T-\tau)} e^{F(T-\xi)} D\, v(\xi) d\xi \tag{5.4}$$

With this substitution, the average, $\overline{u(\tau)\ y(\tau)}$, is given by

$$\overline{u(\tau)\ y(\tau)} = \int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D\ \overline{u(\tau)\ v(\xi)}\ d\xi \tag{5.5}$$

The average, $\overline{u(\tau)\ v(\xi)}$, appearing in the integrand of Eq. 5.5 can be expanded as follows

$$\begin{aligned}\overline{u(\tau)\ v(\xi)} &= \overline{u(\tau)\ [u(\xi) + w(\xi)]} \\ &= \overline{u(\tau)\ u(\xi)} + \overline{u(\tau)\ w(\xi)} \\ &= \overline{u(\tau)}\ \overline{u(\xi)} + \overline{u(\tau)\ w(\xi)} + \rho[u(\tau)\ ,\ u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]\end{aligned} \tag{5.6}$$

where $\rho[u(\tau)\ ,\ u(\xi)]$ = correlation coefficient

$\sigma[u(\cdot)]$ = standard deviation of $u(\cdot)$

When Eq. 5.5 and Eq. 5.6 are substituted into the performance criterion, J^*, as given by Eq. 4.11, this equation becomes

$$\begin{aligned}J^* &= 2\int_0^T\int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D\ \overline{u(\tau)}\ \overline{u(\xi)}\ d\xi d\tau \\ &+ 2\int_0^T\int_0^\tau D' e^{F'(T-\tau)} e^{F(\tau-\xi)} D\ \overline{u(\tau)\ w(\xi)}\ d\xi d\tau \\ &+ 2\int_0^T\int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D\rho[u(\tau)\ ,\ u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]\ d\xi d\tau\end{aligned} \tag{5.7a}$$

This equation can be further simplified by rewriting the first term on the right hand side so that

$$\begin{aligned}J^* &= \underline{x}'(t)\ e^{F'T} e^{FT} \underline{x}(o) \\ &+ 2\int_0^T\int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D\ \overline{u(\tau)\ w(\xi)}\ d\xi d\tau \\ &+ 2\int_0^T\int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D\rho[u(\tau)\ ,\ u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)] d\tau d\xi\end{aligned} \tag{5.7b}$$

The term in Eq. 5.7b which involves the correlation coefficient, $\rho[u(\tau) , u(\xi)]$, is bounded from above since each of the factors of the integrand are bounded from above; that is, the factor $D'e^{F'(T-\tau)}e^{F(T-\xi)}D$ is bounded since the dynamical system is assumed to be stable and also each of the factors involving the control signal is bounded from above by $+1$.

Since the contribution of this term to the performance criterion, J^* , is bounded and the contribution from the term involving $\overline{u(\tau)w(\xi)}$ may be unbounded, there are cases where the effects of the factor, $\rho[u(\tau) , u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]$, are insignificant compared to the effects of the factor $\overline{u(\tau)w(\xi)}$. These cases are investigated in this section and then in a later section the more general case is investigated where the effects of the factor $\rho[u(\tau) , u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]$ are included in the study.

The justification for neglecting this correlation term can also be seen if it is noted that the random variable, $v(\xi)$, is the sum of the random variable, $u(\xi)$, whose amplitude is either $+1$ or -1 and the random variable, $w(\xi)$, whose variance is infinite.

Since $w(\xi)$ is a zero mean, gaussian random variable with infinite variance, then it can be assumed that the distribution of $v(\xi)$ is gaussian with infinite variance since the addition of the finite term, $u(\xi)$, can hardly change the shape of the distribution. The mean value of $v(\xi)$ must be given by $\overline{u(\xi)}$ since $v(\xi) = w(\xi) + u(\xi)$ and the mean value of $w(\xi)$ is zero.

Insofar as the statistics of $v(\xi)$, are concerned, one may therefore consider the statistics of $\tilde{v}(\xi)$ where

$$\tilde{v}(\xi) = \overline{u(\xi)} + w(\xi) \tag{5.8a}$$

since $\tilde{v}(\xi)$ is gaussian with mean given by $\overline{u(\xi)}$ and infinite variance. With this assumption, an approximation to the average, $\overline{u(\tau)\, v(\xi)}$, can be found as follows:

$$\overline{u(\tau)\ v(\xi)} \cong \overline{u(\tau)\ \tilde{v}(\xi)}$$

$$\cong \overline{u(\tau)\ [\overline{u(\xi)} + w(\xi)]}$$

$$\tilde{=} \overline{u(\tau)}\ \overline{u(\xi)} + \overline{u(\tau)\ w(\xi)} \qquad (5.8b)$$

When one compares Eq. 5.8b to Eq. 5.6, it can be seen that the above assumptions neglects the effects of the term involving the autocorrelation of the control signal.

This study will now continue using the assumption that, insofar as the performance is concerned, the effects of the term involving the autocorrelation of the control signal are negligible compared to the effects of the term involving $\overline{u(\tau)\ w(\xi)}$ and thus this former term can be omitted from the performance criterion. This relative significance will be investigated toward the end of this Section and also in Section 5.3.

The original performance criterion, J, can be expressed as a function of the above mentioned variables by substituting Eq. 5.7b into Eq. 4.4 giving

$$J = \int_0^T\int_0^T B'e^{F'(T-\tau)}e^{F(T-\xi)}B\ \overline{w(\xi)\ w(\tau)}\ d\xi d\tau$$

$$+ 2\int_0^T\int_0^\tau D'e^{F'(T-\tau)}e^{F(T-\xi)}D\ \overline{u(\tau)\ w(\xi)}\ d\xi d\tau$$

$$+ 2\int_0^T\int_0^\tau D'e^{F'(T-\tau)}e^{F(T-\xi)}D\rho[u(\tau)\ ,\ u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]d\xi d\tau \qquad (5.9a)$$

Therefore when the term involving the autocorrelation of the control signal is neglected and when the autocorrelation, $G\delta(\tau-\xi)$ is substituted for $\overline{w(\xi)\ w(\tau)}$, the approximation to the performance criterion, J, is given by $\tilde{J}$ where

$$\tilde{J} = G \int_0^T B' e^{F'(T-\tau)} e^{F(T-\tau)} B \, d\tau$$

$$+ 2 \int_0^T \int_0^\tau D' e^{F'(T-\tau)} e^{F(T-\xi)} D \, \overline{u(\tau)\, w(\xi)} \, d\xi d\tau \qquad (5.9b)$$

From the state equations, Eq. 3.13, it follows that

$$D' e^{F'(T-\tau)} e^{F(T-\xi)} D = (T-\tau)\,(T-\xi) + 1 \qquad (5.10a)$$

and

$$B' e^{F'(T-\tau)} e^{F(T-\tau)} B = (T-\tau)^2 + 1 \qquad (5.10b)$$

Equation 5.10 is substituted into Eq. 5.9b yielding the equation

$$\tilde{J} = G \left[\frac{T^3}{3} + T \right] + 2 \int_0^T \int_0^\tau [(T-\tau)\,(T-\xi) + 1]\, \overline{u(\tau)\, w(\xi)} \, d\xi d\tau \qquad (5.11a)$$

Minimization of $\tilde{J}$ with respect to the control is equivalent to the minimization of J_1 where

$$J_1 = 2 \int_0^T \int_0^\tau [(T-\tau)\,(T-\xi) + 1]\, \overline{u(\tau)\, w(\xi)} \, d\xi d\tau \qquad (5.11b)$$

In order to use the procedure of Chapter IV for minimizing J_1, it is necessary that the random variable, $y(\tau)$, be defined by

$$y(\tau) = \int_0^\tau h(\tau,\xi)\, w(\xi)\, d\xi \qquad (5.12a)$$

where

$$h(\tau,\xi) = (T-\tau)\,(T-\xi) + 1 \qquad (5.12b)$$

so that J_1 is given by

$$J_1 = 2 \int_0^T \overline{u(\tau)\, y(\tau)} \, d\tau \qquad (5.12c)$$

Since $y(\tau)$ as defined by Eq. 5.12a is essentially the output of a linear filter when the input is gaussian noise, then $y(\tau)$ is also gaussian. The mean and the variance of $y(\tau)$ are determined from the equations

$$E[y(\tau)] = \int_0^\tau h(\tau,\xi)\, \overline{w(\xi)}\, d\xi \qquad (5.13a)$$

$$= 0$$

and

$$\sigma^2[y(\tau)] = E[y^2(\tau)] = \int_0^\tau \int_0^\tau h(\tau,\varphi)h(\tau,\xi)\, \overline{w(\varphi)\; w(\xi)}\, d\varphi d\xi$$

$$= G \int_0^\tau h^2(\tau,\xi)\, d\xi$$

$$= \frac{G}{3} \Big\{ 3\tau + 3T^2(T-\tau) + T^3\, (T-\tau)^2 - 3(T-\tau)^3 - (T-\tau)^5 \Big\} \qquad (5.13b)$$

Therefore $y(\tau)$ is gaussian with zero mean and variance given by Eq. 5.13b. The procedure developed in Chapter IV can now be used to minimize J_1.

From Eq. 4.24 it follows that

$$E[u(\tau)] = \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau)]\, dy(\tau) - \int_{\hat{y}(\tau)}^{\infty} p[y(\tau)]\, dy(\tau)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma[y(\tau)]} \int_{-\infty}^{\hat{y}(\tau)} \exp\left\{-\frac{y^2(\tau)}{2\sigma^2[y(\tau)]}\right\} dy(\tau)$$

$$- \frac{1}{\sqrt{2\pi}\,\sigma[y(\tau)]} \int_{\hat{y}(\tau)}^{\infty} \exp\left\{-\frac{y^2(\tau)}{2\sigma^2[y(\tau)]}\right\} dy(\tau)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{k(\tau)} \exp\left\{-\frac{x^2}{2}\right\} dx \qquad (5.14a)$$

where $k(\tau) = \hat{y}(\tau)/\sigma[y(\tau)]$

$\hat{y}(\tau)$ = switching point for $E[u(\tau)|y(\tau)]$

$\sigma[y(\tau)]$ = standard deviation of $y(\tau)$

From Eq. 4.28 it also follows that

$$\overline{u(\tau)y(\tau)}_{min} = \int_{-\infty}^{\hat{y}(\tau)} y(\tau)p[y(\tau)]dy(\tau) - \int_{\hat{y}(\tau)}^{\infty} y(\tau)p[y(\tau)]dy(\tau)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma[y(\tau)]} \int_{-\infty}^{\hat{y}} y(\tau) \exp\left\{-\frac{y^2(\tau)}{2\sigma^2[y(\tau)]}\right\} dy(\tau)$$

$$- \frac{1}{\sqrt{2\pi}\,\sigma[y(\tau)]} \int_{\hat{y}(\tau)}^{\infty} y(\tau) \exp\left\{-\frac{y^2(\tau)}{2\sigma^2[y(\tau)]}\right\} dy\,(\tau)$$

$$= \frac{\sigma[y(\tau)]}{\sqrt{2\pi}} \left[\int_{-\infty}^{k(\tau)} xe^{-\frac{x^2}{2}} dx - \int_{k(\tau)}^{\infty} xe^{-\frac{x^2}{2}} dx\right]$$

$$\overline{u(\tau)y(\tau)}_{min} = -\frac{2\sigma[y(\tau)]}{\sqrt{2\pi}} e^{-\frac{k^2(\tau)}{2}} \tag{5.14b}$$

When this minimum value of $\overline{u(\tau)y(\tau)}$ is substituted into Eq. 5.12c, the performance criterion, J_1, is given by

$$J_1 = -2\int_0^T \sqrt{\frac{2}{\pi}}\, \sigma[y(\tau)] e^{-\frac{k^2(\tau)}{2}} d\tau \tag{5.15}$$

The minimum value of J_1 can now be determined by finding the optimal value of $\overline{u(\tau)}$ (or $k(\tau)$) which minimizes J_1 and also satisfies the constraints given by Eq. 3.5. The control, $\overline{u(\tau)}$, is that function of $k(\tau)$ as given by Eq. 5.14a.

The standard deviation, $\sigma[y(\tau)]$, is given by the square root of the variance given by Eq. 5.13b, or

$$\sigma[y(\tau)] = \sqrt{\frac{G}{3}} \left\{ 3\tau + 3T^2(T-\tau) + T^3(T-\tau)^2 - 3(T-\tau)^3 - (T-\tau)^5 \right\}^{1/2} \tag{5.16}$$

When an equivalent system is defined as in Eq. 4.29, the Maximum Principle yields the Hamiltonian, $H(\tau)$, given by

$$H(\tau) = \sqrt{\frac{2}{\pi}}\ \sigma[y(\tau)] e^{-\frac{k^2(\tau)}{2}} + \lambda_2(\tau) E[u(\tau)] + \lambda_1(\tau)\tilde{x}_2(\tau) \tag{5.17}$$

where $\lambda_2(\tau)$ is again linear in τ .

In order to maximize $H(\tau)$ with respect to the control, $E[u(\tau)]$, the expression for $H(\tau)$ as given by Eq. 5.17 can be differentiated with respect to $E[u(\tau)]$ and the result set equal to zero. From an inspection of Eq. 5.17 it can be seen that this differentiated equation will not contain $E[u(\tau)]$ explicitly but will contain $k(\tau)$ explicitly. Since $k(\tau)$ and $E[u(\tau)]$ are related in a one-to-one manner by Eq. 5.14a, this differentiated equation can be solved for the optimizing value of $k(\tau)$ instead of the optimizing value of $E[u(\tau)]$. It is to be noted that even though the domain of the function $E[u(\tau)]$ is limited, the domain of the function $k(\tau)$ includes all of the real numbers so that one need not be concerned with the possibility that this maximizing value of $k(\tau)$ so obtained falls outside of the allowable domain of $k(\tau)$.

When the Hamiltonian, Eq. 5.17, is differentiated with respect to $E[u(\tau)]$, the resulting equation becomes

$$\frac{dH(\tau)}{dE[u(\tau)]} = -\sqrt{\frac{2}{\pi}}\ \sigma[y(\tau)]\ k(\tau)\ \frac{dk(\tau)}{dE[u(\tau)]}\ e^{-\frac{k^2(\tau)}{2}} + \lambda_2(\tau) \quad (5.18)$$

After both sides of Eq. 5.14a are differentiated with respect to $E[u(\tau)]$, one obtains the equation

$$\sqrt{\frac{2}{\pi}}\ \frac{dk(\tau)}{dE[u(\tau)]}\ e^{-\frac{k^2(\tau)}{2}} = 1 \quad (5.19)$$

Substitution of Eq. 5.19 into Eq. 5.18 yields the equation

$$\frac{dH(\tau)}{dE[u(\tau)]} = -\ \sigma[y(\tau)]\ k(\tau) + \lambda_2(\tau) \quad (5.20)$$

Also,

$$\frac{d^2H(\tau)}{dE^2[u(\tau)]} = -\ \sigma[y(\tau)]\ \frac{dk(\tau)}{dE[u(\tau)]} \quad (5.21)$$

From Eq. 5.19, it is obvious that $\frac{dk(\tau)}{dE[u(\tau)]}$ is always positive; therefore, since $\sigma[y(\tau)]$ is always positive, Eq. 5.21 is always negative, which is the requirement for a maximum. When Eq. 5.20 is equated to zero, the value of $k(\tau)$ which maximizes $H(\tau)$ is found.

$$k(\tau) = \frac{\lambda_2(\tau)}{\sigma[y(\tau)]} \quad (5.22)$$

This value of $k(\tau)$ when substituted into Eq. 5.15, results in the minimum of this function, J_1 . Since $k(\tau) = \hat{y}(\tau)/\sigma[y(\tau)]$, it follows that $\hat{y}(\tau) = \lambda_2(\tau)$; therefore, when this system is realized, $y(\tau)$ can be compared directly to $\lambda_2(\tau)$.

From Eq. 5.16, $\sigma[y(\tau)]$ can be found

$$\sigma[y(\tau)] = G^{1/2}g(\tau) \quad (5.23)$$

where

$$g(\tau) = \left\{\tau + T^2(T-\tau) - (T-\tau)^3 + \frac{1}{3}\left[T^3(T-\tau)^2 - (T-\tau)^5\right]\right\}^{1/2}$$

With this expression for $\sigma[y(\tau)]$, Eq. 5.15 becomes

$$J_1 = -\sqrt{\frac{8G}{\pi}} \int_0^T g(\tau)\, e^{-\frac{k^2(\tau)}{2}}\, d\tau \tag{5.24}$$

Since $k(\tau)$ is only a function of $E[u(\tau)]$, which is a function of the initial state; for a given initial state, Eq. 5.24 becomes more negative as G becomes larger.

The neglected term in Eq. 5.6 is bounded by unity; therefore, the maximum amount added to the cost, J, by this term is bounded and independent of G. Therefore, for G large enough, the effect of the maximum possible cost increase due to this neglected term is not significant compared to the cost savings due to J_1 as given by Eq. 5.24. It is to be noted that Eq. 5.24 always results in a cost savings since the integrand of this equation is always positive and the minus sign is before the integral.

For this class of problems, this scheme is a good approximation to the optimal control. In a later section, the noise is not assumed to be large and for this case another scheme results.

There is, however, the same difficulty when trying to mechanize this control as there is in the deterministic case; namely, the solution depends on the initial conditions of the adjoint variables and not directly on the initial conditions of the state variables. However, assuming that the proper adjoint is found for a given initial state, then $u(\tau)$ is generated as a function of the output of the time varying filter given by Eq. 5.12a. If this output is greater than $\hat{y}(\tau)$, which from Eq. 5.22 and Eq. 5.14a is seen to be $\lambda_2(\tau)$, $u(\tau)$ is made -1 and if this output is less than $\lambda_2(\tau)$, $u(\tau)$ is made $+1$.

The bound of unity on the term $\rho[u(\tau), u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]$ is in most cases, conservative. The error resulting from neglecting this term can be accurately computed. The computational scheme is presented in Section 5.3.

Figure 5.1 shows a method which may be used to realize this control. This realization assumes that the noise is available for measurement.

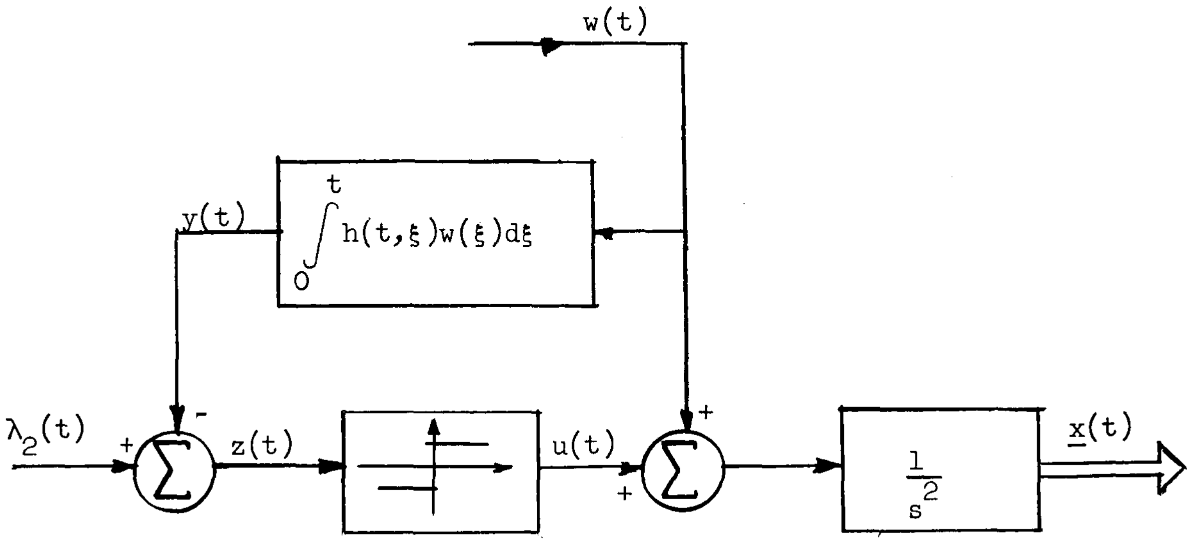


FIGURE 5.1
REALIZATION OF THE CONTROL USING A FIRST
APPROXIMATION TO THE OPTIMAL CONTROL

5.3 A METHOD FOR COMPUTING THE FINAL STATE VARIANCES WHEN USING THE CONTROL SCHEME OF SECTION 5.2

This section offers a computational scheme for computing the error in the cost resulting from neglecting the term $\rho[u(\tau),u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]$ in Eq. 5.9a.

Since

$$\rho[u(\tau),u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)] = \overline{u(\tau)u(\xi)} - \overline{u(\tau)}\,\overline{u(\xi)} \qquad (5.25)$$

the cost for the system of Section 5.2 resulting from this term is

$$J_2 = \int_0^T \int_0^T [(T-\tau)\ (T-\xi) + 1]\ [\overline{u(\tau)u(\xi)} - \overline{u(\tau)}\,\overline{u(\xi)}]\ d\tau d\xi \qquad (5.26)$$

or

$$J_2 = \int_0^T \int_0^T [(T-\tau)(T-\xi) + 1]\ \overline{u(\tau)u(\xi)}\ d\tau d\xi - \underline{x}'(0)e^{F'T}e^{FT}\underline{x}(0) \qquad (5.27)$$

For this second order plant

$$- \underline{x}'(o)e^{F'T}e^{FT}\underline{x}(o) = - [T\, x_2(o) + x_1(o)]^2 - x_2^2(o) \tag{5.28}$$

Therefore, in order to evaluate J_2, only the double integral is unknown and will now be computed.

The autocorrelation function of $u(\cdot)$ may be expanded as follows.

$$\begin{aligned} E[u(\tau)u(\xi)] = {} & p[u(\tau) = 1\,,\ u(\xi) = 1] + p[u(\tau) = -1,\ u(\xi) = -1] \\ & - p[u(\tau) = 1,\ u(\xi) = -1] - p[u(\tau) = -1,\ u(\xi) = 1] \end{aligned}$$

or

$$\begin{aligned} E[u(\tau)u(\xi)] = {} & p[y(\tau) < \hat{y}(\tau), y(\xi) < \hat{y}(\xi)] \\ & + p[y(\tau) > \hat{y}(\tau),\ y(\xi) > \hat{y}(\xi)] \\ & - p[y(\tau) < \hat{y}(\tau),\ y(\xi) > \hat{y}(\xi)] \\ & - p[y(\tau) > \hat{y}(\tau),\ y(\xi) < \hat{y}(\xi)] \end{aligned} \tag{5.29}$$

When one uses the relationship that

$$p[y(\tau),\ y(\xi)] = p[y(\tau)|y(\xi)]p[y(\xi)]$$

then, Eq. 5.29 can be written

$$\begin{aligned} E[u(\tau)u(\xi)] = {} & \int_{-\infty}^{\hat{y}(\xi)} \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau)|y(\xi)]p[y(\xi)]dy(\tau)dy(\xi) \\ & + \int_{\hat{y}(\xi)}^{\infty} \int_{\hat{y}(\tau)}^{\infty} p[y(\tau)|y(\xi)]p[y(\xi)]dy(\tau)dy(\xi) \\ & - \int_{-\infty}^{\hat{y}(\xi)} \int_{\hat{y}(\tau)}^{\infty} p[y(\tau)|y(\xi)]p[y(\xi)]dy(\tau)dy(\xi) \\ & - \int_{\hat{y}(\xi)}^{\infty} \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau)|y(\xi)]p[y(\xi)]dy(\tau)dy(\xi) \end{aligned}$$

or

$$E[u(\tau)u(\xi)] = \int_{-\infty}^{\hat{y}(\xi)} I(\tau,\xi)p[y(\xi)]dy(\xi)$$

$$- \int_{\hat{y}(\xi)}^{\infty} I(\tau,\xi)p[y(\xi)]dy(\xi) \qquad (5.30)$$

where

$$I(\tau,\xi) = \int_{-\infty}^{\hat{y}(\tau)} p[y(\tau)|y(\xi)]dy(\tau)$$

$$- \int_{\hat{y}(\tau)}^{\infty} p[y(\tau)|y(\xi)]dy(\tau) \qquad (5.31)$$

The function $I(\tau,\xi)$ given by Eq. 5.31 will now be studied. Since $y(\tau)$ and $y(\xi)$ are normal random variables with zero means and variances given by $\sigma^2[y(\tau)]$ and $\sigma^2[y(\xi)]$ respectively, then the distribution of $y(\tau)$ given $y(\xi)$ is normal with mean given by "m" and variance given by σ_1^2 where

$$m = \frac{\overline{y(\tau)y(\xi)}}{\sigma^2[y(\xi)]} y(\xi) \qquad (5.32)$$

$$\sigma_1^2 = \sigma^2[y(\tau)] \left\{ 1 - \left\{ \frac{\overline{y(\tau)\ y(\xi)}}{\sigma[y(\tau)]\sigma[y(\xi)]} \right\}^2 \right\} \qquad (5.33)$$

These facts concerning the parameters of the gaussian conditional probability distribution can be found on page 54 of Ref. 6.

Equation 5.31 can now be evaluated.

$$I(\tau,\xi) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\hat{y}(\tau)} \exp\left\{- \frac{(y(\tau)-m)^2}{2\sigma_1^2}\right\} dy(\tau)$$

$$- \frac{1}{\sqrt{2\pi}\sigma_1} \int_{\hat{y}(\tau)}^{\infty} \exp\left\{- \frac{(y(\tau)-m)^2}{2\sigma_1^2}\right\} dy(\tau) \qquad (5.34)$$

With a change of variable, Eq. 5.34 simplifies to

$$I(\tau;\xi) = \frac{2}{\sqrt{2\pi}} \int_0^f \exp\left\{-\frac{x^2}{2}\right\} dx \tag{5.34a}$$

where

$$f = \frac{\hat{y}(\tau)-m}{\sigma_1}$$

When Eq. 5.34a is substituted into Eq. 5.30, the expectation, $E[u(\tau)u(\xi)]$, becomes

$$E[u(\tau)u(\xi)] = 2\int_{-\infty}^{\hat{y}(\xi)} p[y(\xi)]dy(\xi) \int_0^f \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$- 2\int_{\hat{y}(\xi)}^{\infty} p[y(\xi)]dy(\xi) \int_0^f \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \tag{5.35}$$

where

$$p[y(\xi)] = \frac{1}{\sqrt{2\pi}\,\sigma[y(\xi)]} \exp\left\{-\frac{y^2(\xi)}{2\sigma^2[y(\xi)]}\right\} \tag{5.36}$$

With the introduction of the new variable, $z(\xi)$, where

$$z(\xi) = \frac{y(\xi)}{\sigma[y(\xi)]}$$

Eq. 5.35 reduces to

$$E[u(\tau)u(\xi)] = \frac{1}{\pi} \int_{-\infty}^{k(\xi)} \exp\left\{-\frac{z^2(\xi)}{2}\right\} dz(\xi) \int_0^f \exp\left\{-\frac{x^2}{2}\right\} dx$$

$$- \frac{1}{\pi} \int_{k(\xi)}^{\infty} \exp\left\{-\frac{z^2(\xi)}{2}\right\} dz(\xi) \int_0^f \exp\left\{-\frac{x^2}{2}\right\} dx \tag{5.37}$$

where with the substitution of $z(\xi)$, f becomes

$$f = \frac{\hat{y}(\tau) - \dfrac{\overline{y(\tau)y(\xi)}}{\sigma[y(\xi)]}\, z(\xi)}{\sigma[y(\tau)]\left\{1 - \dfrac{\overline{y(\tau)y(\xi)}^2}{\sigma^2[y(\tau)]\sigma^2[y(\xi)]}\right\}^{1/2}} \tag{5.38}$$

It remains to evaluate $\overline{y(\tau)y(\xi)}$ which for $\tau = \xi$ will give $\sigma^2[y(\tau)]$ or equivalently $\sigma^2[y(\xi)]$. From Eq. 5.12a and from the fact that $w(.)$ is white with autocorrelation, $G\delta(t)$, it follows that

$$\overline{y(\tau)y(\xi)} = \int_0^\tau\int_0^\xi h(\tau,\varphi_1)h(\xi,\varphi_2)\overline{w(\varphi_1)w(\varphi_2)}\, d\varphi_1 d\varphi_2 \tag{5.39a}$$

or

$$= G\int_0^\tau\int_0^\xi h(\tau,\varphi_1)h(\xi,\varphi_2)\delta(\varphi_1-\varphi_2)\, d\varphi_2 d\varphi_1 \tag{5.39b}$$

When the properties of the delta function, $\delta(\varphi_1-\varphi_2)$, are considered, Eq. 5.39b can be written

$$\overline{y(\tau)y(\xi)} = G\int_0^{\min(\tau,\xi)} h(\tau,\varphi)h(\xi,\varphi)d\varphi \tag{5.40}$$

When the expression, $(T-\tau)(T-\xi) + 1$, is substituted for $h(\tau,\xi)$, and Eq. 5.40 is integrated, the average, $\overline{y(\tau)y(\xi)}$, is computed to be

$$\overline{y(\tau)y(\xi)} = \frac{G}{6}\Big\{6\tau + 3T^2(2T-\tau-\xi) + 2T^3(T-\tau)(T-\xi) - 2(T-\xi)(T-\tau)^4 - 3(2T-\tau-\xi)(T-\tau)^2\Big\} \text{ for } \tau<\xi \tag{5.41a}$$

$$= \frac{G}{6}\Big\{6\xi + 3T^2(2T-\tau-\xi) + 2T^3(T-\tau)(T-\xi) - 2(T-\tau)(T-\xi)^4 - 3(2T-\xi-\tau)(T-\xi)^2\Big\} \text{ for } \tau>\xi \tag{5.41b}$$

$$= \frac{G}{3}\Big\{3\tau + 3T^2(T-\tau) + T^3(T-\tau)^2 - (T-\tau)^5 - 3(T-\tau)^3\Big\} \text{ for } \tau=\xi \tag{5.41c}$$

The double integral required for the evaluation of J_2 given by Eq. 5.27, is now found by substituting Eq. 5.37 into this integral and performing the indicated integration.

In summary, the procedure for evaluating the cost due to the factor $\rho[u(\tau),u(\xi)]\sigma[u(\tau)]\sigma[u(\xi)]$ is as follows:

1. $E[u(\tau)u(\xi)]$ is evaluated from Eq. 5.37 where

   a) f is given by Eq. 5.38 where

      1) $\vartheta(\tau) = \lambda_2(\tau)$

      2) $\overline{y(\tau)y(\xi)}$ is given by either Eq. 5.41a or 5.41b

      3) $\sigma[y(\tau)]$ is given by the square root of Eq. 5.41c.

2. The above expression for $E[u(\tau)u(\xi)]$ is substituted into the integral of Eq. 5.27 and the integral is evaluated.

3. The term given by Eq. 5.28 is computed.

4. The sum of the results of step 2 and step 3 is the resulting cost.

Because $\exp(-x^2/2)$ is not integrable in closed form, numerical methods must be used to evaluate the double integral of Eq. 5.27.

5.4 CRITICAL EVALUATION OF THE CONTROL SCHEME PROPOSED IN SECTION 5.2

In order to test the advantage of the scheme of Section 5.2 over the open loop scheme which does not consider the effects of the noise, it is only necessary to compare the savings in cost, J_1 , given by Eq. 5.15 to the cost incurred by the term involving the autocorrelation of the control signal which is given by Eq. 5.27.

It will now be shown that there are cases when this scheme results in a substantial reduction of the final state variance and that this scheme approaches the true optimal as the ratio of noise power to control power increases. Given an initial state, it is first necessary to investigate the effects of a change of G on the net cost when using the scheme presented in Section 5.2.

By considering first Eq. 5.27, which is the cost due to the fact that the control signal is correlated in time, one can see that only the double integral of this equation can possible change as a result of a change in G since the remaining term is independent of G ; therefore, only the double integral is now considered.

When the expression for the standard deviation $\sigma[y(\tau)]$, given by Eq. 5.23 is used, Eq. 5.22 can be written

$$k(\tau) = \frac{\lambda_2(\tau)}{g(\tau)\sqrt{G}} \qquad (5.42)$$

where $g(\tau)$ is a function not dependent on G . Since $\overline{u(\tau)}_{opt}$ depends only on $k(\tau)$, for a fixed initial state $\lambda_2(\tau)$ must be varied as G is varied so that Eq. 3.5 is still satisfied. One can define a new adjoint variable as $\tilde{\lambda}_2(\tau)$ where $\tilde{\lambda}_2(\tau) = \lambda_2(\tau)/\ G$. With this definition for $\tilde{\lambda}_2(\tau)$, Eq. 5.42 may be written

$$k(\tau) = \frac{\tilde{\lambda}_2(\tau)}{g(\tau)} \qquad (5.42a)$$

Since the initial conditions of $\tilde{\lambda}_2(\tau)$ determine $E[u(\tau)]$ which in turn determines the state $\underline{x}(o)$, it follows that $\tilde{\lambda}_2(\tau)$ or equivalently the ratio $\lambda_2(\tau)/\ G$ must be fixed for any given initial state $\underline{x}(o)$. Therefore in the following development, the initial conditions of the adjoint variable, $\lambda_2(\tau)$, are adjusted so that the ratio $\lambda_2(\tau)/\sqrt{G}$ is unchanged as G is varied.

It is now necessary to investigate the consequences of such a change in G on the other important variables.

When Eqs. 5.41a, 5.41b, and 5.41c are used, the equation for f , Eq. 5.38, can be rewritten

$$f = \frac{\frac{\mathfrak{y}(\tau)}{\sqrt{G}} - f_1(\tau,\xi)z(\tau)}{f_2(\tau,\xi)} \qquad (5.43)$$

where $f_1(\tau,\xi)$ and $f_2(\tau,\xi)$ are functions not involving G . Since $\mathfrak{y}(\tau) = \lambda_2(\tau)$, and since the ratio $\lambda_2(\tau)/\sqrt{G}$ remains constant, then the ratio $\mathfrak{y}(\tau)/\sqrt{G}$ also remains constant, so that as G is varied f remains constant. Therefore as G is varied and $\lambda_2(\tau)$ varied so as to satisfy the constraints given by the initial state, the cost from the double integral term involving the autocorrelation of the control signal is unchanged. The total cost caused by the autocorrelation of the control signal is thus seen to be independent of G for a given initial error state.

However, from Eq. 5.24 it can be seen that the change in cost as a result of correlating the signal with the noise becomes more negative as G increases. Therefore, there exists a critical value of G such that for lower values of G it would be best not to use this scheme since the open loop scheme results in a smaller variance; however, for larger values of G, the scheme of Section 5.2 results in a net decrease in the variance of the final error state.

Figure 5.2 shows the various costs as a function of G for the following condition:

$$x_1(o) = 13.60$$

$$x_2(o) = -3.16$$

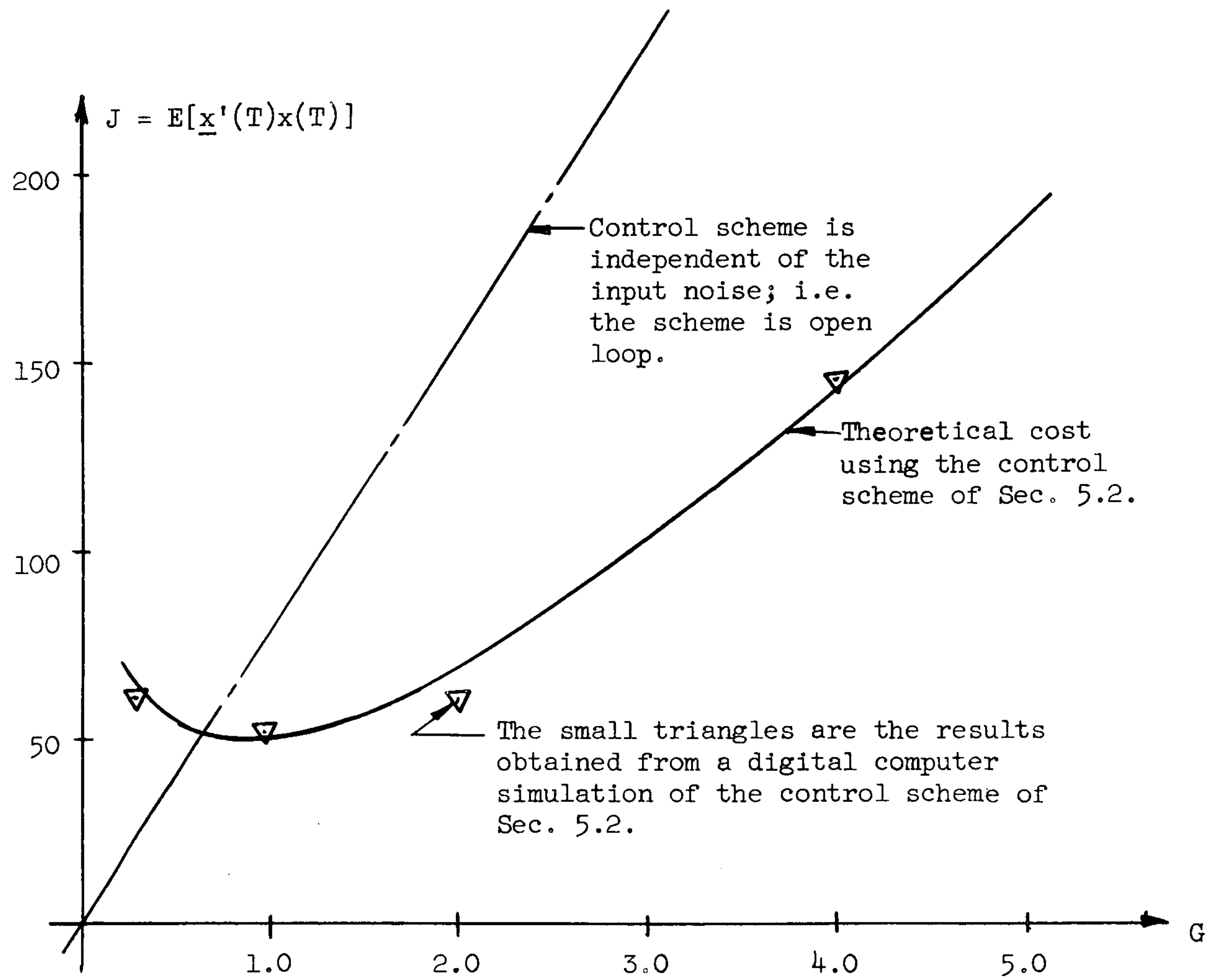
$$T = 6.00$$

where the "critical" point occurs at $G = 0.65$

In Figure 5.2, the solid line is drawn by using values computed from the method of Section 5.3. The experimental points, denoted by the small triangles, were obtained by using a digital computer simulation of the system. At each of the points, there were 150 runs used to determine the variance of the final state, which is the performance criterion for this system. The computer program was a digital simulation of the system of Fig. 5.1.

At this point it is conjectured, but not proven, that a better solution to this problem might be found by the use of the information gained during the computations of the various quantities described in Section 5.3. The control scheme of Section 5.2 neglected a term of the cost, J, since the probability distribution of $v(\xi)$ was not exactly known at that time. In this section, it is shown that this term of the cost contributed a fixed amount independent of the noise power, G, so that for small values of G the scheme of Section 5.2 results in a poor control scheme.

A better estimate of the probability distribution of $v(\xi)$ can be made if the information gained during the computations discussed in Section 5.3 is properly used. This better estimate will allow one to consider the effects of the neglected term.



SYSTEM: $\ddot{x}(t) = u(t) + w(t)$

DATA: $x_1(o) = 13.60$, $x_2(o) = -3.16$, $T = 6.00$, $\overline{w(\tau)w(\xi)} = G\delta(\tau-\xi)$

FIGURE 5.2 COMPARISON OF TWO CONTROL SCHEMES WHEN THE OBJECTIVE IS TO ZERO THE AVERAGE VALUE OF THE FINAL STATE AT A FIXED TIME

With this new estimate of the distribution of $v(\xi)$, a new control law can be found which minimizes the performance criterion, J^* . For this new control law, the resulting cost is computed. If there is an improvement, another iteration cycle might be tried. If the cost, computed by the use of Eq. 5.9a, is found to be within design specifications, then the problem is solved and that control scheme can be implemented. In a sense then, this procedure is equivalent to an iterative scheme with the solution of Section 5.2 used as an initial condition.

The following alternative method describes how the information obtained from the solution given in Section 5.2 can be used in an attempt to find a better control scheme; that is, to reduce the cost even more than the reduction, if any, accomplished in that section. It is shown by a digital computer simulation of the resulting system that this scheme also gives very good results when the noise power, G , is small.

For this development, the random variable $y(\tau)$ as given by Eq. 5.2 will include the effects of the term, $u(\xi)$. In order to differentiate this random variable from the one used in Section 5.2, the random variable $\tilde{y}(\tau)$ will be used where

$$\tilde{y}(\tau) = \int_0^{\tau} h(\tau,\xi)\,[u(\xi) + w(\xi)]d\xi \tag{5.44}$$

where $h(\tau,\xi)$ is given by Eq. 5.12b.

With this definition for $\tilde{y}(\tau)$, then that part of Eq. 5.9a which can be affected by minimization can be written

$$\tilde{J}_1 = \int_0^{T} \overline{u(\tau)\tilde{y}(\tau)}\, d\tau \tag{5.45}$$

where

$$\overline{u(\tau)\tilde{y}(\tau)} = \int_0^{\tau} h(\tau,\xi)\, \overline{[u(\xi) + w(\xi)]\, u(\tau)}\, d\xi \tag{5.46}$$

With this definition for $\tilde{y}(\tau)$, the total cost given by Eq. 5.9a is given by

$$J = G\left\{\frac{T^3}{3} + T\right\} + 2\int_0^T \overline{u(\tau)\,\tilde{y}(\tau)}\,d\tau - \underline{x}'(o)e^{F'(T)}e^{F(T)}\underline{x}(o) \qquad (5.47)$$

From the development in Chapter IV it is necessary that the probability distribution of $\tilde{y}(\tau)$ be known. It is again assumed that the random variable $u(\xi) + w(\xi)$ is nearly gaussian so that $\tilde{y}(\tau)$ is gaussian. It is important to note that this is a much weaker assumption then that one which is used in Section 5.2 where the average value of $u(\tau)$ is used in place of $u(\tau)$. In this case the assumption is made that $\tilde{y}(\tau)$ is gaussian and not that $u(\xi) + w(\xi)$ is gaussian. The signal $u(\xi) + w(\xi)$ is essentially passed through a filter with weighting function, $h(\tau,\xi)$. Since the action of this filter is similar to a summing process, the output tends to become more gaussian than the input (Ref. 8). The random variable, $\tilde{y}(\tau)$, is assumed to be gaussian with mean $\tilde{m}(\tau)$ and variance $\tilde{\sigma}^2(\tau)$ where,

$$\tilde{m}(\tau) = \int_0^\tau h(\tau,\xi)\,\overline{u(\xi)}\,d\xi \qquad (5.48)$$

and

$$\tilde{\sigma}^2(\tau) = \overline{\tilde{y}^2(\tau)} - \tilde{m}^2(\tau) \qquad (5.49)$$

The mean squared value, $\overline{\tilde{y}^2(\tau)}$, is given by

$$\overline{\tilde{y}^2(\tau)} = \int_0^\tau\int_0^\tau h(\tau,\xi)h(\tau,\varphi)\,\overline{[u(\xi)+w(\xi)]\,[u(\varphi)+w(\varphi)]}\,d\varphi d\xi \qquad (5.50)$$

which can be simplified to the expression

$$\overline{\tilde{y}^2(\tau)} = 2\int_0^\tau h(\tau,\xi)\int_0^\xi h(\tau,\varphi)\,\left[\overline{u(\varphi)u(\xi)} + \overline{w(\varphi)u(\xi)}\right] d\varphi d\xi$$

$$+ G\int_0^\tau h^2(\tau,\xi)d\xi \qquad (5.50a)$$

The procedure which will be used in an attempt to improve the system assumes that the mean and the variance of $\tilde{y}(\tau)$ are known and then uses the same method as was used in Section 5.2 to find the optimal control law. Since $\tilde{y}(\tau)$ is a function of this derived control law, the actual mean and variance of $\tilde{y}(\tau)$ can be computed once this control law is known. If these actual parameters agree with the assumed parameters then the problem is solved and the control scheme can be implemented. If these parameters disagree, then another iteration cycle may be used; however, in this cycle the computed parameters of the last iteration cycle are used. There is no proof of convergence given here; however, toward the end of this section a method is proposed so that this iteration scheme need not be used.

The procedure that one might use to estimate the parameters of the distribution of $\tilde{y}(\tau)$ will now be discussed.

A rather crude estimate of these parameters can be made if one uses the function $\overline{u(\cdot)}$ given in the previous cycle. With this function, the mean value function, $\tilde{m}(\tau)$, can be obtained from Eq. 5.48. An estimate of $\overline{y^2(\tau)}$ can be made by only considering the last term of Eq. 5.50a. These estimates will be very close to the correct values when the noise power, G , is large. When G is small, the above estimate of $\tilde{y}^2(\tau)$ may be very poor. In an attempt to make a better estimate, a method is proposed which uses a forward iteration scheme. Since it is felt that the methods used for obtaining these estimates are not essential to this study, this procedure is presented in Appendix A.

At this time, it is assumed that a very good estimate of the mean value function, $\tilde{m}(\tau)$, and the variance $\tilde{\sigma}(\tau)$, has been made and these estimates will now be used in the ensuing development.

Since it is assumed that the distribution for $\tilde{y}(\tau)$ is known, the procedure developed in Chapter IV and later used in Section 5.2 can be used to minimize the performance criterion given by Eq. 5.45.

The expression for $E[u(\tau)]$ given by Eq. 3.44a can be written

$$E[u(\tau)] = \frac{1}{\sqrt{2\pi\tilde{\sigma}(\tau)}} \int_{-\infty}^{\infty} \exp\left[-\frac{[\tilde{y}(\tau) - \hat{m}(\tau)]^2}{2\tilde{\sigma}^2(\tau)}\right] E[u(\tau)|\tilde{y}(\tau)]\, d\tilde{y}(\tau) \quad (5.51)$$

The new variable, $z(\tau) = [\tilde{y}(\tau) - \tilde{m}(\tau)]/\tilde{\sigma}(\tau)$, reduces Eq. 5.51 to

$$E[u(\tau)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2(\tau)}{2}\right] E[u(\tau)|\tilde{\sigma}(\tau)z(\tau) + \tilde{m}(\tau)]dz(\tau) \qquad (5.52)$$

From Eq. 4.19, it follows that

$$\overline{u(\tau)\tilde{y}(\tau)} = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}(\tau)} \int_{-\infty}^{\infty} \tilde{y}(\tau) \exp\left[-\frac{[\tilde{y}(\tau) - \tilde{m}(\tau)]^2}{2\tilde{\sigma}^2(\tau)}\right] E[u(\tau)|\tilde{y}(\tau)] \; d\tilde{y}(\tau) \qquad (5.53)$$

The same substitution that led to Eq. 5.52 can be used to reduce Eq. 5.53. With this substitution, Eq. 5.53 becomes

$$\overline{u(\tau)\tilde{y}(\tau)} = \tilde{m}(\tau) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-z^2(\tau)/2\right] E[u(\tau)|\tilde{\sigma}(\tau)z(\tau) + \tilde{m}(\tau)] \, dz(\tau)$$

$$+ \frac{\tilde{\sigma}(\tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(\tau) \exp\left[-\frac{z^2(\tau)}{2}\right] E[u(\tau)|\tilde{\sigma}(\tau)z(\tau) + \tilde{m}(\tau)] \; dz(\tau)$$

$$= \tilde{m}(\tau)E[u(\tau)] + \frac{\tilde{\sigma}(\tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(\tau) \exp\left[-\frac{z^2(\tau)}{2}\right] E[u(\tau)|\tilde{\sigma}(\tau)z(\tau)+\tilde{m}(\tau)] \; dz(\tau) \qquad (5.54)$$

Since $\tilde{m}(\tau)$ and $\tilde{\sigma}(\tau)$ are assumed to be known functions, Eq. 5.54 may be first minimized with respect to the correlation by letting

$$\begin{aligned} E[u(\tau)|\tilde{\sigma}(\tau)z(\tau) + \tilde{m}(\tau)] &= -1 \quad \text{if} \quad z(\tau) > k^*(\tau) \\ &= +1 \quad \text{if} \quad z(\tau) < k^*(\tau) \end{aligned} \qquad (5.55a)$$

or equivalently

$$\begin{aligned} E[u(\tau)|\tilde{y}(\tau)] &= -1 \quad \text{if} \quad \frac{\tilde{y}(\tau) - \tilde{m}(\tau)}{\tilde{\sigma}(\tau)} > k^*(\tau) \\ &= +1 \quad \text{if} \quad \frac{\tilde{y}(\tau) - \tilde{m}(\tau)}{\tilde{\sigma}(\tau)} < k^*(\tau) \end{aligned} \qquad (5.55b)$$

where $k^*(\tau)$ must be determined from Eq. 5.52. With the function $E[u(\tau)|\tilde{\sigma}(\tau)z(\tau) + \tilde{m}(\tau)]$ given by Eq. 5.55a, Eq. 5.52 becomes

$$E[u(\tau)] = \sqrt{\frac{2}{\pi}} \int_0^{k*(\tau)} \exp\left[-\frac{z^2(\tau)}{2}\right] dz(\tau) \qquad (5.56)$$

Therefore $k*(\tau)$ is that function of $E[u(\tau)]$ given by Eq. 5.56.

When Eq. 5.55a is substituted into Eq. 5.54, it follows that

$$\overline{u(\tau)\tilde{y}(\tau)} = \tilde{m}(\tau)\, E[u(\tau)] - \frac{2\tilde{\sigma}(\tau)}{\sqrt{2\pi}} \exp\left[-\frac{k*^2(\tau)}{2}\right] \qquad (5.57)$$

Since $\tilde{m}(\tau)$ and $\tilde{\sigma}(\tau)$ are assumed to be known functions and since $k*(\tau)$ is a function of $E[u(\tau)]$, it follows that Eq. 5.57 is a function of only $E[u(\tau)]$. Therefore, if this function, $u(\tau)\tilde{y}(\tau)$, is substituted into Eq. 5.45, the Maximum Principle can be used to minimize the performance criterion, $\tilde{J}_1$, by the proper choice of the function, $E[u(\tau)]$. The Hamiltonian for this system becomes

$$H(\tau) = -\tilde{m}(\tau)\, E[u(\tau)] + \frac{2\tilde{\sigma}(\tau)}{\sqrt{2\pi}} \exp\left[-\frac{k*^2(\tau)}{2}\right]$$

$$+ \lambda_2(\tau)\, E[u(\tau)] + \lambda_1(\tau)\tilde{x}_2(\tau) \qquad (5.58)$$

This Hamiltonian can now be differentiated and equated to zero to find the optimal value of $k*(\tau)$ which in turn is equivalent to finding the optimal value of $E[u(\tau)]$. Since the only restriction of $k*(\tau)$ is that it satisfies Eq. 5.56 and since the right hand side of this equation is bounded by +1 and -1, there is no restriction of the allowable domain of $k*(\tau)$.

$$\frac{dH(\tau)}{dE[u(\tau)]} = \lambda_2(\tau) - \tilde{m}(\tau) - \frac{2}{\sqrt{2\pi}}\tilde{\sigma}(\tau) \exp\left[-\frac{k*^2(\tau)}{2}\right] k*(\tau) \frac{dk*(\tau)}{dE[u(\tau)]} \qquad (5.59)$$

However, after both sides of Eq. 5.56 are differentiated with respect to $E[u(\tau)]$, one obtains the equation

$$\sqrt{\frac{2}{\pi}} \exp\left[-\frac{k*^2(\tau)}{2}\right] \frac{dk*(\tau)}{dE[u(\tau)]} = 1 \qquad (5.59a)$$

When Eq. 5.59a is substituted into Eq. 5.59, the expression for the optimal value of $k^*(\tau)$ is found to be

$$k^*(\tau) = \frac{\lambda_2(\tau) - \tilde{m}(\tau)}{\tilde{\sigma}(\tau)} \tag{5.60}$$

When Eq. 5.55b and Eq. 5.60 are combined, the optimal control law specifies that

$$E[u(\tau)|\tilde{y}(\tau)] = -1 \quad \text{if} \quad \tilde{y}(\tau) > \lambda_2(\tau) \tag{5.61a}$$

$$= +1 \quad \text{if} \quad \tilde{y}(\tau) < \lambda_2(\tau) \tag{5.61b}$$

It is thus seen that in order to generate a control signal having the required statistical properties, it is only necessary that the random variable $\tilde{y}(\tau)$ be compared to $\lambda_2(\tau)$ and the control $u(\tau)$, be determined on the basis of this comparison.

The correct initial conditions on the adjoint variable, $\lambda_2(\tau)$, are found by evaluating the integral of Eq. 5.56 with the value of $k^*(\tau)$ given by Eq. 5.60. With this substitution, Eq. 5.56 can be written

$$E[u(\tau)] = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{\lambda_2(\tau) - \tilde{m}(\tau)}{\tilde{\sigma}(\tau)}} \exp\left[-\frac{z^2(\tau)}{2}\right] dz(\tau)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\frac{\lambda_2(o) - \lambda_1(o)\tau - \tilde{m}(\tau)}{\tilde{\sigma}(\tau)}} \exp\left[-\frac{z^2(\tau)}{2}\right] dz(\tau) \tag{5.61c}$$

The two constants $\lambda_2(o)$ and $\lambda_1(o)$ are found by evaluating the constraint equations, Eq. 3.5, which for this $1/s^2$ plant become

$$\int_0^T E[u(\tau)]\, d\tau = -x_2(o) \tag{5.62a}$$

$$\int_0^T \tau E[u(\tau)]\, d\tau = x_1(o) \tag{5.62b}$$

When the assumed values, $\tilde{m}(\tau)$ and $\tilde{\sigma}(\tau)$, are used, the initial conditions of the adjoint variable can be found. Since this adjoint variable is used as a comparator for $\tilde{y}(\tau)$, the actual values of the mean and the variance of $\tilde{y}(\tau)$ can now be computed using the method proposed in Appendix A. If the agreement between the actual parameters and the assumed parameters is close enough, the problem is solved; if not, another iteration cycle may be used. Convergence of this procedure requires that the values of the assumed parameters be very close to the actual values.

A method of solution which does not require this estimate of $\tilde{m}(\tau)$ and $\tilde{\sigma}(\tau)$ will now be discussed. It is shown in this chapter that the adjoint variable completely specifies the control law; it is only necessary that the function, $\tilde{y}(\tau)$, be generated and be compared to the adjoint variable so that the value of the control signal may be determined. This fact together with the principle of superposition allows the space of initial conditions of the state variable to be mapped upon the space of initial conditions of the adjoint variable. The principle of superposition may be used since the dynamical system is linear.

The procedure used for this type of solution is as outlined below.

1. The state initial conditions are assumed to be zero, so that the system of Eq. 5.1 has the solution

$$\underline{x}(T) = \int_0^T e^{F(T-\tau)} D[u(\tau) + w(\tau)]\, d\tau \tag{5.62c}$$

2. For a given noise process, $w(\cdot)$, and an assumed adjoint variable, this system is simulated on a computer so that the control is optimal in that the requirements of this Section are satisfied.
3. For this same adjoint variable, enough runs are made so that the average value of the final state may be determined.
4. Since the actual problem required that this average value of the final state be zero, the principle of superposition can be used to determine the correct initial conditions of the states so that this final value is zero. From Eq. 3.5 it

therefore follows that the correct value of the initial state can be determined from the relation

$$\underline{x}(o) = - e^{-FT} \int_0^T e^{F(T-\tau)} D \, \overline{u(\tau)} \, d\tau \tag{5.62d}$$

where from Eq. 5.62c it follows that

$$\underline{x}(o) = - e^{-FT} \underline{x}^*(T) \tag{5.62e}$$

where $\underline{x}^*(T)$ is the average value of the final state as determined by the computer runs of Step 3.

This procedure was used in order to determine the goodness of this control scheme. Approximately 150 computer runs were made on each system for each assumed adjoint variable and for each value of the noise power, G. The output of the computer was the corresponding initial states and the variance of the final states.

The initial states found from the above computer study were then used to determine the adjoint variable for the control scheme of Section 5.2; that is, the control scheme which neglected the term involving the correlation of the control signal. In Table I, a summary of the results is presented. For the purpose of comparison the table is divided in two sections. The first section, which includes System "A" to "E" inclusive, is composed of those systems whose corresponding state initial conditions are very nearly the same. The initial conditions of those systems in the second section, Systems "F" to "K" inclusive, are likewise nearly the same but are substantially different from those of the first section. It can be noted that the systems of Case II behave poorly as G becomes smaller; this is in agreement with the presentation of Section 5.3. In contrast, the performance of the systems of Case III improves significantly for the lower values of G.

Figure 5.3 and Figure 5.4 are included so that the time histories of the various functions mentioned in this section can be observed.

TABLE I

SUMMARY OF THE RESULTS OF USING VARIOUS CONTROL SCHEMES WHEN THE INPUT NOISE IS WHITE*

System	G	T	Initial State		Adjoint Variable		Performance = $\overline{x'(T)x(T)}$			Percent Change†	
			$x_1(o)$	$x_2(o)$	Case II	Case III	Case I	Case II	Case III	Case II	Case III
A	4.00	6.0	12.623	-2.405	-26.97 +19.18t	-10.00 + 5.00t	312.0	137.2	91.43	-56.02	-70.70
B	2.00	6.0	13.514	-2.695	-20.31 +16.13t	-10.00 + 5.00t	156.0	66.7	32.49	-57.21	-79.17
C	1.00	6.0	13.513	-2.838	- 9.47 + 9.51t	-10.00 + 5.00t	78.0	49.3	12.79	-32.82	-83.60
D	0.50	6.0	14.361	-3.089	- 6.80 + 9.06t	-10.00 + 5.00t	39.0	39.4	4.85	+ 1.00	-85.00
E	0.24	6.0	14.629	-3.239	- 4.96 + 5.95t	-10.00 + 5.00t	18.7	50.9	1.94	+171.79	-89.64
F	4.00	6.0	4.741	-0.657	- 8.39 + 3.69t	-10.00 + 2.00t	312.0	118.1	74.00	-62.16	-76.28
G	2.00	6.0	3.402	-0.695	- 3.02 + 1.46t	-10.00 + 2.00t	156.0	72.0	18.56	-53.84	-88.10

(Continued on next page.)

TABLE I (Continued)

System	G	T	Initial State		Adjoint Variable		Performance = $\overline{x'(T)x(T)}$			Percentage Change+	
			$x_1(o)$	$x_2(o)$	Case II	Case III	Case I	Case II	Case III	Case II	Case III
H	1.00	6.0	5.740	-0.854	$-4.77 + 2.26t$	$-10.00 + 2.00t$	78.0	69.0	3.74	-11.52	-95.20
J	0.50	6.0	6.578	-1.033	$-3.62 + 1.86t$	$-10.00 + 2.00t$	39.0	81.3	0.81	+108.48	-97.92
K	0.24	6.0	6.662	-1.145	$-1.55 + 1.06t$	$-10.00 + 2.00t$	18.7	102.3	0.29	+446.69	-98.45

*The definitions of the various cases are as follows:

CaseI : In this scheme there is no attempt made to reduce to variance of the final state.

Case II : This is the scheme of Section 5.2; that is, the effects of the autocorrelation of the control signal are neglected.

Case III : This is the scheme of this section. The effects of the autocorrelation of the control signal are considered.

+Values in these columns are the percent changes caused by using the scheme of either Case II or Case III in place of the scheme of Case I. The minus sign denotes an improvement while the plus sign indicates a poorer performing system.

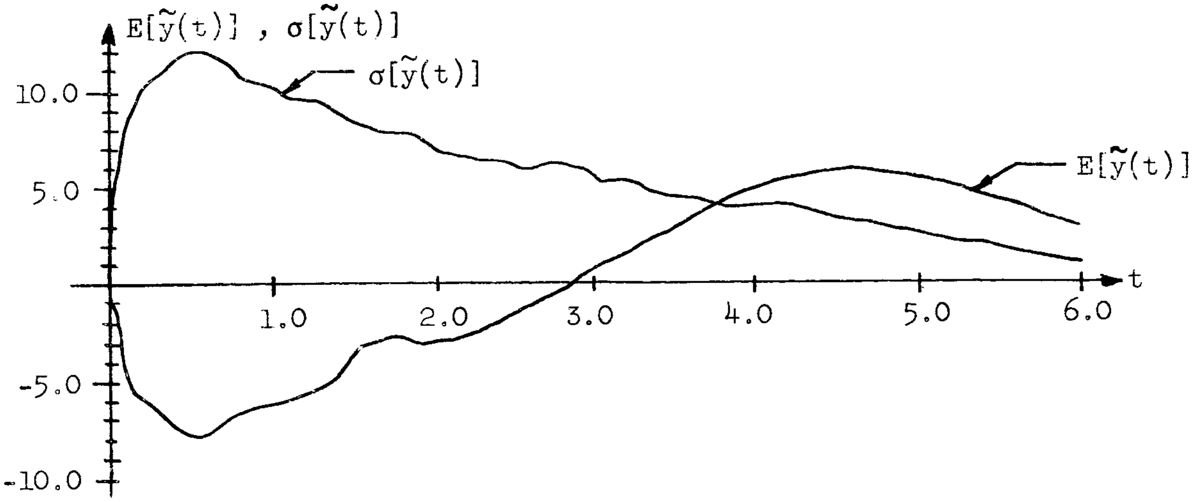


FIGURE 5.3a MEAN AND STANDARD DEVIATION OF $\tilde{y}(t)$

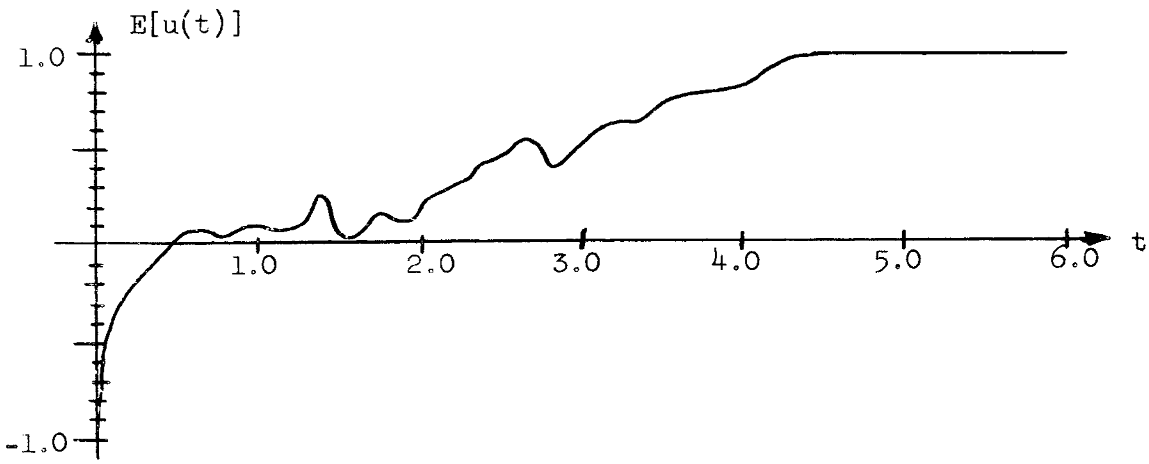


FIGURE 5.3b $E[u(t)]$

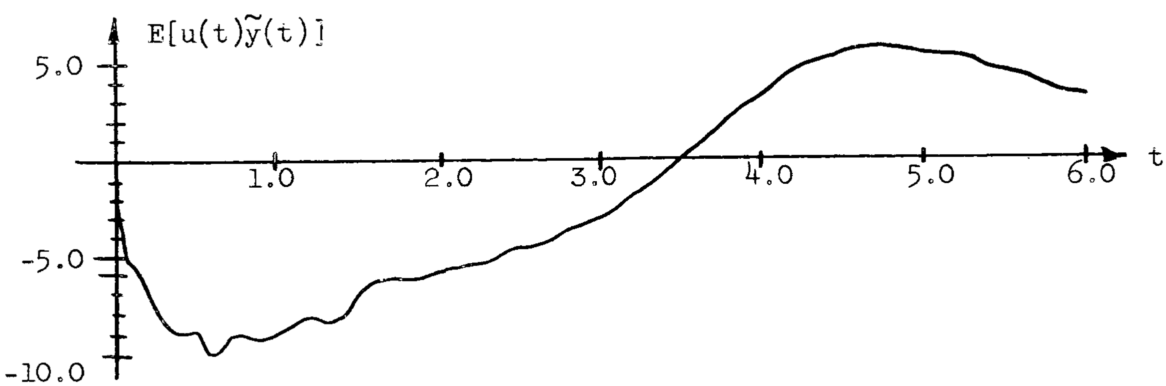


FIGURE 5.3c $E[u(t)\tilde{y}(t)]$

FIGURE 5.3 GRAPH OF THE VARIOUS TIME HISTORIES OF CASE III, SYSTEM "D" , TABLE I

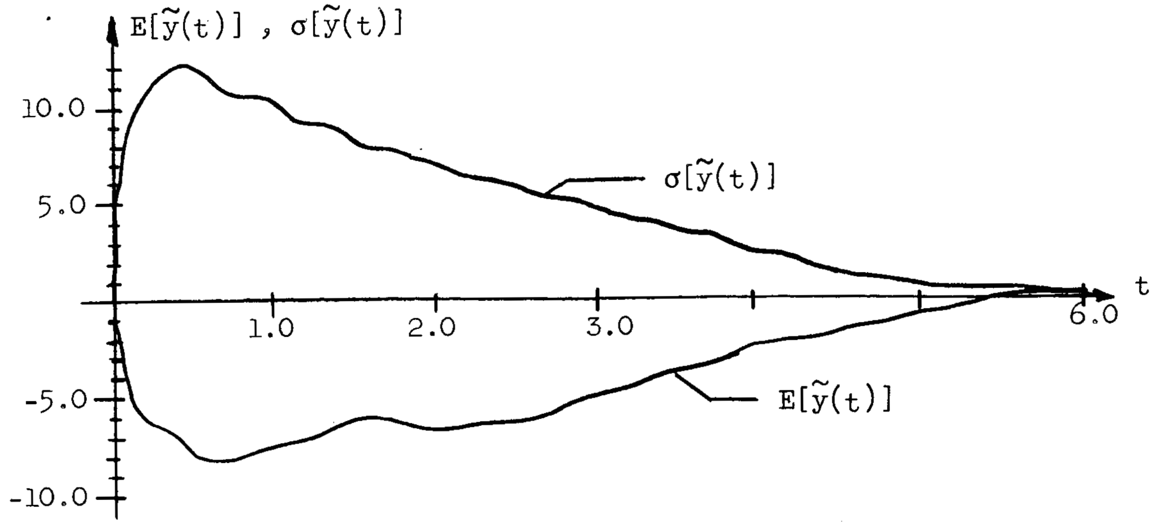


FIGURE 5.4a MEAN AND STANDARD DEVIATION OF $\tilde{y}(t)$

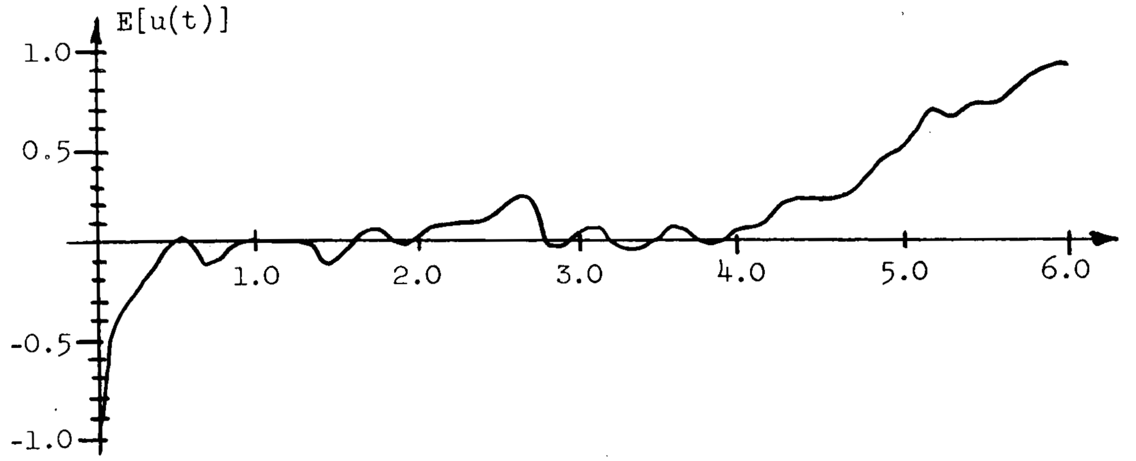


FIGURE 5.4b E[u(t)]

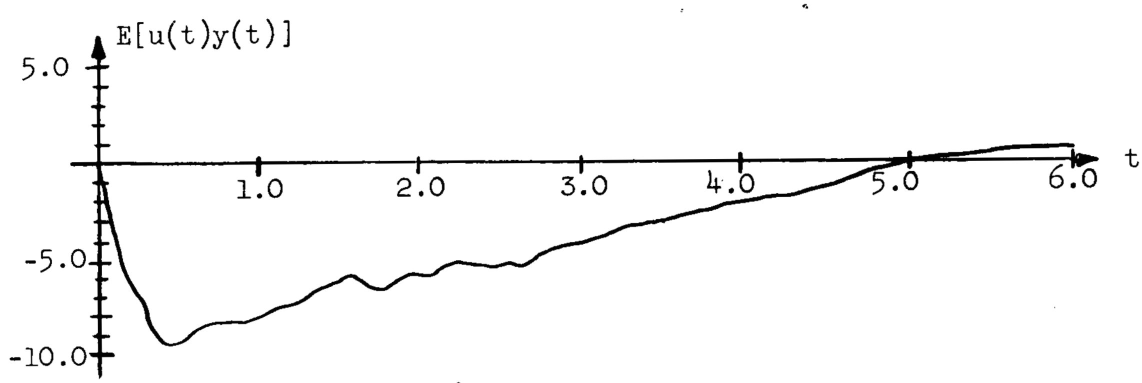


FIGURE 5.4c $E[u(t)\tilde{y}(t)]$

FIGURE 5.4 GRAPH OF THE VARIOUS TIME HISTORIES OF CASE III, SYSTEM "J" , TABLE I

In both of these figures the noise power is identical; however, the initial states are different.

5.5 PHYSICAL REALIZATION FOR THE $1/s^2$ PLANT.

Theoretically it appears that a reasonably good system has been derived by the procedure of Section 5.4; now the system must be realized. It is necessary that the signal $\tilde{y}(\tau)$ be made available, where $\tilde{y}(\tau)$ is given by Eq. 5.44 which is rewritten here for convenience,

$$\tilde{y}(\tau) = \int_0^\tau [(T-\tau)(T-\xi) + 1]' [u(\xi) + w(\xi)]\, d\xi$$

$$= (T-\tau) \int_0^\tau (T-\xi)\, [u(\xi) + w(\xi)] d\xi + \int_0^\tau [u(\xi) + w(\xi)]\, d\xi \qquad (5.63)$$

In order to realize this system, it will be advantageous to rewrite Eq. 5.63 in the form

$$\tilde{y}(\tau) = (T-\tau)y_1(\tau) + y_2(\tau) \qquad (5.64)$$

where

$$y_1(\tau) = \int_0^\tau (T-\xi)\, [u(\xi) + w(\xi)]\, d\xi$$

$$y_2(\tau) = \int_0^\tau [u(\xi) + w(\xi)]\, d\xi$$

For this second-order plant, the system equation, Eq. 2.8, can be evaluated giving the two equations for the state of the plant

$$x_2(\tau) = x_2(o) + \int_0^\tau [u(\xi) + w(\xi)]\, d\xi \qquad (5.65a)$$

$$x_1(\tau) = [x_1(o) + \tau x_2(o)] + \int_0^\tau (\tau-\xi)\, [u(\xi) + w(\xi)]\, d\xi \qquad (5.65b)$$

Therefore, with the use of Eq. 5.65a and Eq. 5.65b, the variables, $y_1(\tau)$

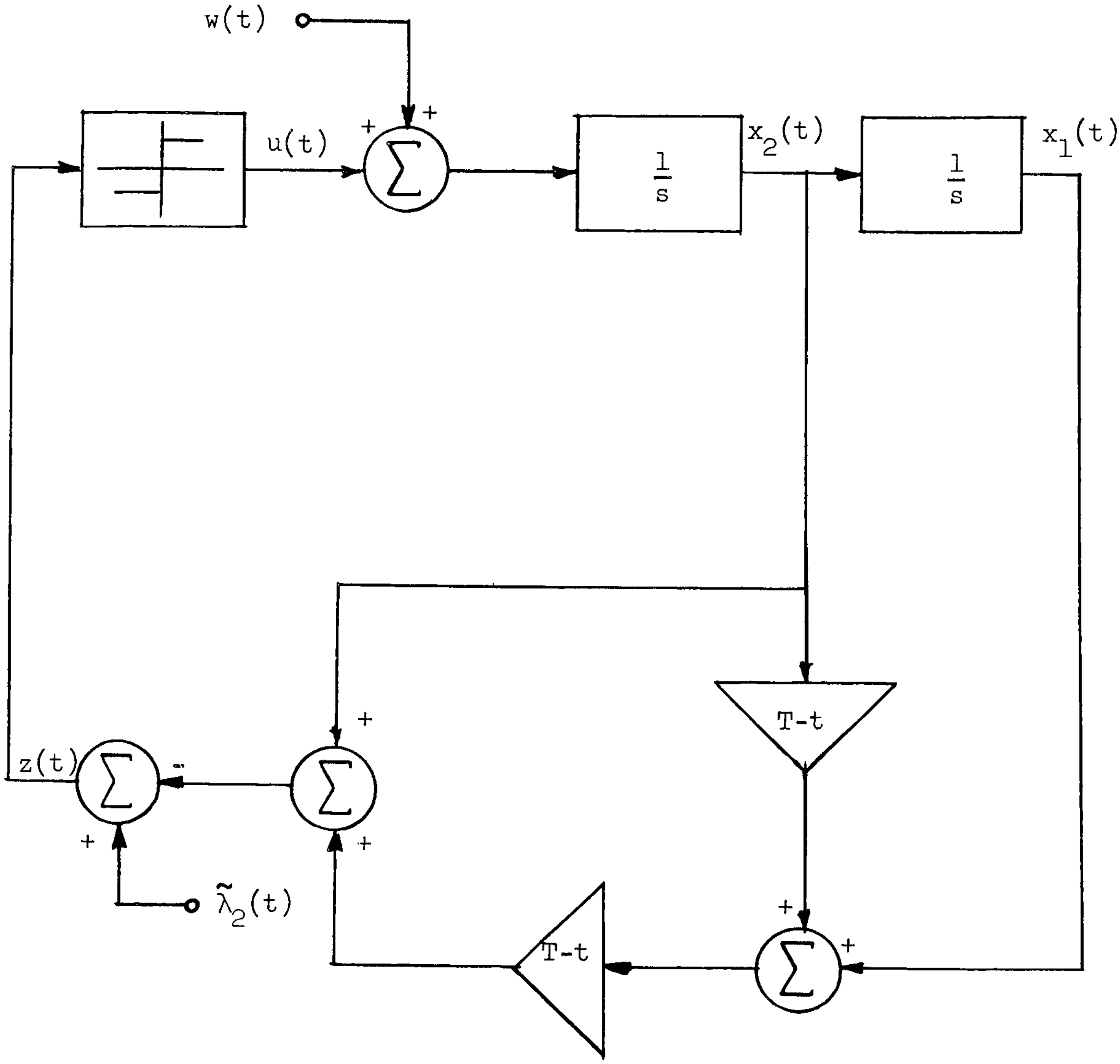


FIGURE 5.5 REALIZATION OF THE CONTROL SCHEME OF SECTION 5.5

and $y_2(\tau)$, can be expressed in terms of the state of the plant since

$$x_2(\tau) = x_2(o) + y_2(\tau) \tag{5.66a}$$

$$x_1(\tau) = x_1(o) + \tau x_2(o) + y_1(\tau) - (T-\tau)y_2(\tau) \tag{5.66b}$$

Then Eq. 5.66a and Eq. 5.66b are solved for $y_1(\tau)$ and $y_2(\tau)$, yielding

$$y_1(\tau) = x_1(\tau) - [x_1(o) + \tau x_2(o)] + (T-\tau)y_2(\tau) \tag{5.67a}$$

$$y_2(\tau) = x_2(\tau) - x_2(o) \tag{5.67b}$$

When the above equations are substituted into Eq. 5.64, the function $\tilde{y}(\tau)$ is given by

$$\begin{aligned}\tilde{y}(\tau) &= (T-\tau)x_1(\tau) + [(T-\tau)^2 + 1]\, x_2(\tau) \\ &\quad - [(T-\tau)x_1(o) + (T^2 - T\tau + 1)x_2(o)]\end{aligned} \tag{5.68}$$

Since the last term on the right hand side of Eq. 5.68 is linear in τ as is the adjoint variable, there would be no difference between comparing $\tilde{y}(\tau)$ with $\lambda_2(\tau)$ then there would be comparing $(T-\tau)x_1(\tau) + [(T-\tau)^2 + 1]x_2(\tau)$ to $\tilde{\lambda}_2(\tau)$ where

$$\tilde{\lambda}_2(\tau) = \lambda_2(\tau) + (T-\tau)x_1(o) + (T^2 - T\tau + 1)x_2(o) \tag{5.69}$$

It is to be noted that $\tilde{\lambda}_2(\tau)$ is linear in τ since $\lambda_2(\tau)$ is linear in τ. With this definition for $\tilde{\lambda}_2(\tau)$, the feedback realization of Fig. 5.5 results. Again it is assumed that the initial conditions of the adjoint variable, $\tilde{\lambda}_2(t)$, have been determined, possibly by the procedure discussed in Section 5.4.

5.6 $1/(s^2+1)$ PLANT

The differential equation for this system is given by the equation

$$\frac{d^2x(t)}{dt^2} + x(t) = u(t) + w(t) \tag{5.70}$$

or expressed in state notation

$$\dot{\underline{x}}(t) = F\underline{x}(t) + D\,u(t) + B\,w(t) \tag{5.71a}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \; ; \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.71b}$$

and

$$e^{Ft} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{5.72}$$

Therefore, it follows that

$$D' e^{F'(T-\xi)} e^{F(T-\tau)} D = \sin(T-\tau)\sin(T-\xi) + \cos(T-\tau)\cos(T-\xi)$$

$$= \cos\,(\tau-\xi) \tag{5.73}$$

With the assumption that the noise process, $w(\cdot)$, is again white, the modified performance criterion for the system is given by

$$J_1 = \int_0^T \overline{u(\tau)\,\tilde{y}(\tau)}\,d\tau \tag{5.74a}$$

where

$$\tilde{y}(\tau) = \int_0^\tau \cos\,(\tau-\xi)\,[u(\xi) + w(\xi)]\,d\xi \tag{5.74b}$$

The procedure used for the $1/s^2$ plant can now be used to find the optimal control signal for this plant. The only modification to that procedure is that $\cos(\tau-\xi)$ is used in place of $(T-\tau)(T-\xi) + 1$ for the definition of $h(\tau,\xi)$. Other systems can be similarly treated.

5.7 NON-WHITE GAUSSIAN NOISE AT THE INPUT TO THE PLANT

The procedure used in this chapter in order to find the optimal control signal when the input noise is gaussian and white can also be used to find this signal when the input is gaussian but not white.

Mathematically it is only necessary to replace the non-white noise by a linear filter with white noise at the input.

It is well known, that colored, gaussian noise can be generated by passing white gaussian noise through the proper linear filter. Therefore, instead of the system of Fig. 5.6a being used, where $\hat{w}(t)$ is colored, the system of Fig. 5.6b can be used where the filter H_1 is that filter which causes the process, $\tilde{w}(\cdot)$, to have the same statistical properties as the given noise process, $\hat{w}(\cdot)$, when the input, $w(t)$, to this filter is a gaussian white noise process with autocorrelation function $G\delta(t-\xi)$.

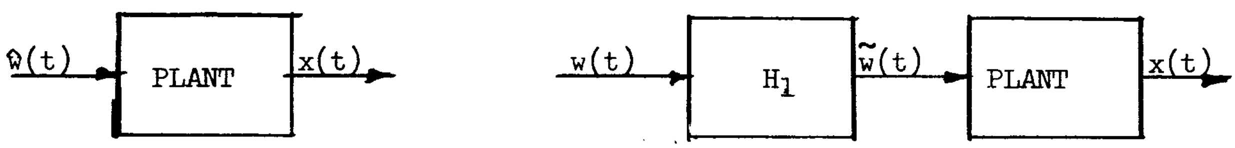


FIGURE 5.6a FIGURE 5.6b

FIGURE 5.6 STOCHASTICALLY EQUIVALENT SYSTEMS

When the representation of Fig. 5.6b is used, the procedure for finding the optimal control signal when the input noise is colored can be reduced to the procedure developed in this chapter for finding the optimal control signal when the input noise is white. This procedure will now be briefly described for the $1/s^2$ plant with exponentially correlated gaussian noise at the input.

The system to be studied is shown in Fig. 5.7a, where the process, $\hat{w}(\cdot)$, is exponentially correlated, zero mean gaussian noise. Figure 5.7b and Fig. 5.7c are systems which are stochastically equivalent to the system of Fig. 5.7c. The state, $x_3(t)$, is the value of the input noise at time t , therefore, the value of this state at the final time, T, is not to be a part of the performance criterion.

The system equations for the system of Fig. 5.7c are given by

$$\underline{\dot{x}}(t) = F\,\underline{x}(t) + D\,u(t) + B\,w(t) \tag{5.75}$$

where

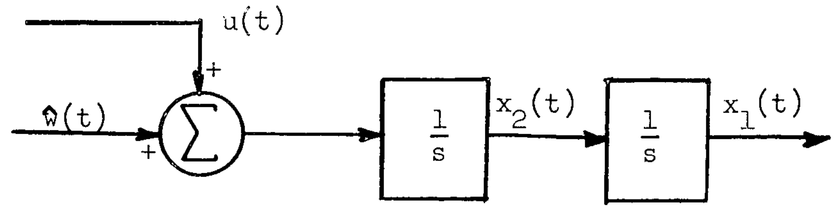


FIGURE 5.7a

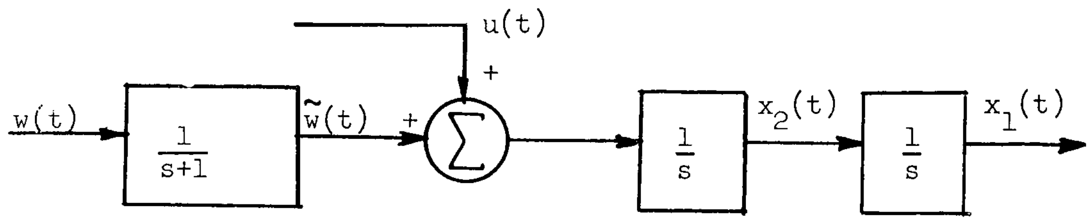


FIGURE 5.7b

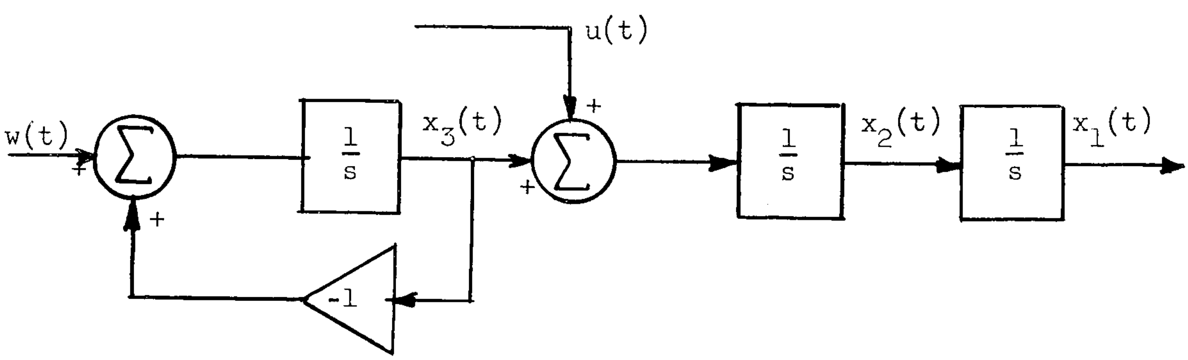


FIGURE 5.7c

FIGURE 5.7 STOCHASTICALLY EQUIVALENT REPRESENTATIONS

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \quad D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$e^{F(t-\tau)} = \begin{bmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) & \phi_{13}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) & \phi_{23}(t-\tau) \\ \phi_{31}(t-\tau) & \phi_{32}(t-\tau) & \phi_{33}(t-\tau) \end{bmatrix}$$

where $\phi_{ij}(t-\tau)$ is the output of the i'th integrator at time, t, due to a unit impulse applied at the input of the j'th integrator at time, τ, with zero initial conditions everywhere else. The values of these quantities can be calculated from an inspection of Fig. 5.7c.

$$\begin{aligned}
\phi_{11}(t-\tau) &= 1 \\
\phi_{12}(t-\tau) &= t-\tau \\
\phi_{13}(t-\tau) &= \exp\,[-(t-\tau)] + (t-\tau) - 1 \\
\phi_{21}(t-\tau) &= 0 \\
\phi_{22}(t-\tau) &= 1 \\
\phi_{23}(t-\tau) &= 1 - \exp\,[-(t-\tau)] \\
\phi_{31}(t-\tau) &= 0 \\
\phi_{32}(t-\tau) &= 0 \\
\phi_{33}(t-\tau) &= \exp\,[-(t-\tau)]
\end{aligned} \tag{5.76}$$

The performance index, J, is given by Eq. 3.28 where in this case

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.77}$$

The matrix, Q, is not the identity matrix since only the variance of of the states $x_1(T)$ and $x_2(T)$ are of any importance.

The constraint equation given by Eq. 3.5 is still applicable; however, since $\overline{x_3(t)}$ is zero by assumption, the vector equation, Eq. 3.5, gives only two non-trivial constraint equations. These two equations are identical to the constraint equations used in the previous sections of this chapter.

With $\tilde{y}(\tau)$ defined by Eq. 4.10, the procedure of Section 5.4 can now be used to find the optimal value of the comparator, $\hat{y}(\tau)$; which in this case will again be the adjoint variable, $\lambda_2(\tau)$. The procedure for determining the initial conditions of this adjoint variable is similar to that discussed in Section 5.4.

Thus the system has been optimized; however, the ficticious noise process, $w(\cdot)$, was used in the development. Realization of the system requires that the actual noise process, $\hat{w}(.)$, be used. This realization scheme will now be discussed.

The signal $\tilde{y}(\tau)$ given by Eq. 4.10 must be made available. With the help of Eq. 5.75, Eq. 4.10 can be expanded

$$\tilde{y}(\tau) = \int_0^\tau \Big\{ [\phi_{12}(T-\tau)\,\phi_{12}(T-\xi) + \phi_{22}(T-\tau)\,\phi_{22}(T-\xi)]\,u(\xi)$$

$$+ [\phi_{12}(T-\tau)\,\phi_{13}(T-\xi) + \phi_{22}(T-\tau)\,\phi_{23}(T-\xi)]\,w(\xi)\Big\}\,d\xi$$

$$= \phi_{12}^2(T-\tau)\int_0^\tau \phi_{12}(\tau-\xi)\,u(\xi)\,d\xi$$

$$+ \phi_{22}^2(T-\tau)\int_0^\tau \phi_{22}(\tau-\xi)\,u(\xi)\,d\xi$$

$$+ \phi_{12}(T-\tau)\,\phi_{13}(T-\tau)\int_0^\tau \phi_{13}(\tau-\xi)\,w(\xi)\,d\xi$$

$$+ \phi_{22}(T-\tau)\,\phi_{23}(T-\tau)\int_0^\tau \phi_{23}(\tau-\xi)\,w(\xi)\,d\xi \qquad (5.78)$$

The system equation, Eq. 2.7, has the solution given by the vector equation, Eq. 2.8. From this solution, the values of the states, $x_1(\tau)$

and $x_2(\tau)$ can be found.

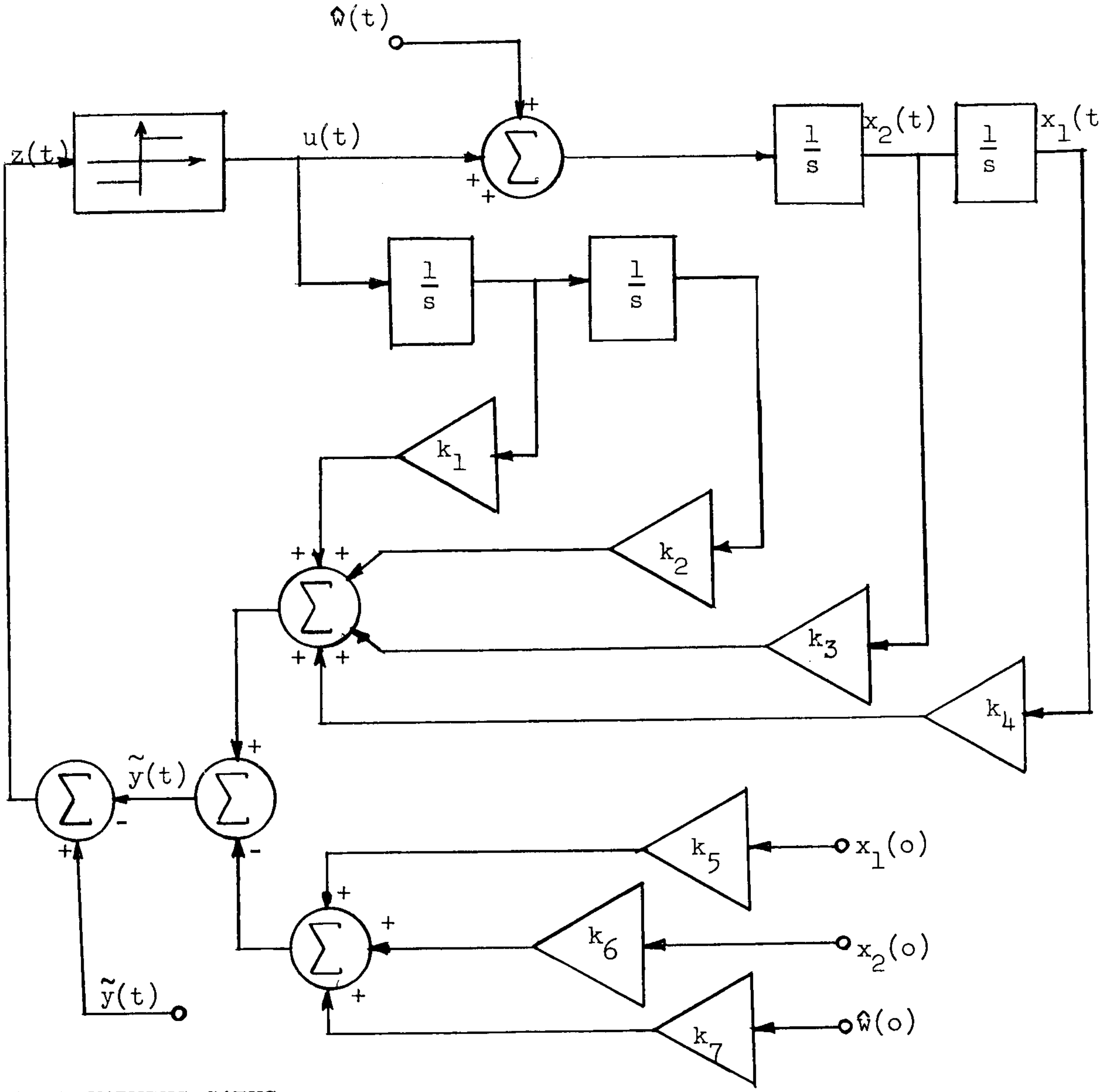
$$x_1(\tau) = \phi_{11}(\tau)\, x_1(o) + \phi_{12}(\tau)\, x_2(o) + \phi_{13}(\tau)\, x_3(o) + \int_0^{\tau} \phi_{12}\,(\tau-\xi)\, u(\xi)\, d\xi + \int_0^{\tau} \phi_{13}\,(\tau-\xi)\, w(\xi)\, d\xi \qquad (5.79a)$$

$$x_2(\tau) = \phi_{22}(\tau)\, x_2(o) + \phi_{23}(\tau)\, x_3(o) + \int_0^{\tau} \phi_{22}\,(\tau-\xi)\, u(\xi)\, d\xi + \int_0^{\tau} \phi_{23}(\tau-\xi)\, w(\xi)\, d\xi \qquad (5.79b)$$

Equation 5.78 and Eqs. 5.79 are combined to yield the equation for $\tilde{y}(\tau)$ in terms of the state of the plant and the initial conditions $x_1(o)$, $x_2(o)$, and $x_3(o)$. The state, $x_3(o)$, is the value of the noise at the initial time.

$$\begin{aligned}
\tilde{y}(\tau) = {} & \phi_{12}(T-\tau)\, \phi_{13}(T-\tau)\, x_1(\tau) + \phi_{22}(T-\tau)\, \phi_{23}(T-\tau)\, x_2(\tau) \\
& + \phi_{12}(T-\tau)[\phi_{12}(T-\tau) - \phi_{13}(T-\tau)] \int_0^{\tau} \phi_{12}(\tau-\xi)\, u(\xi)\, d\xi \\
& + \phi_{22}(T-\tau)\, [\phi_{22}(T-\tau) - \phi_{23}(T-\tau)] \int_0^{\tau} \phi_{22}(\tau-\xi)\, u(\xi)\, d\xi \\
& - \phi_{12}(T-\tau)\, \phi_{13}(T-\tau)\, [\phi_{11}(\tau)\, x_1(o) + \phi_{12}(\tau)\, x_2(o) + \phi_{13}(\tau)\, \hat{w}(o)] \\
& - \phi_{22}(T-\tau)\phi_{23}(T-\tau)\, [\phi_{22}(\tau)\, x_2(o) + \phi_{23}(\tau)\, \hat{w}(o)]
\end{aligned} \qquad (5.80)$$

The realization of the system can thus be based on Eq. 5.80. It is to be noted that the value of the noise at the initial time is required for this realization and therefore some means must be found for obtaining this value. In the white noise case, this quantity is not necessary since the future values of the noise are independent of this initial value; however, since the noise is now correlated, this value becomes a necessary part of the control scheme. This value essentially provides



TIME VARYING GAINS

$k_1 = 1 - \phi_{23}(T-t)$

$k_2 = \phi_{12}(T-t)\,[\phi_{12}(T-t) - \phi_{13}(T-t)]$

$k_3 = \phi_{23}(T-t)$

$k_4 = \phi_{12}(T-t)\,\phi_{13}(T-t)$

$k_5 = \phi_{12}(T-t)\,\phi_{13}(T-t)$

$k_6 = \phi_{12}(T)\,\phi_{13}(T-t) + \phi_{23}(T-t)$

$k_7 = \phi_{12}(T-t)\,\phi_{13}(T) + \phi_{23}(T)$

FIGURE 5.8 REALIZATION OF THE CONTROL SCHEME OF SECTION 5.6

the necessary bias that the function $\tilde{y}(\tau)$ must have in order that the effects of the future predictable part of the noise may be taken into account in the control scheme. It is important to note that this initial value of the noise is not needed in the computations for the optimal control law, it is only needed for the realization.

With this initial value of $\hat{\omega}(t)$ available, the system can be realized as shown in Fig. 5.8. The values of the impulse response functions, $\phi_{ij}(t)$, are given in Eq. 5.76.

VI. GENERAL PERFORMANCE CRITERION

6.1 Development

The investigations of Chapters IV and V treat those cases for which the performance criterion is either minimum fuel or minimum final state variance. In this chapter it will be shown that the techniques presented in the aforementioned chapters can be used to an advantage in the general case in which the performance criterion is a combination of both minimum fuel and minimum final state variance. The results of the preceding chapters are used as a guide in the development of the theory as presented in this chapter. The following theory should be treated as a conjecture.

The performance index is given by Eq. 2.6. Again in this chapter, a scalar control is assumed so that Eq. 2.6 can be written

$$J = E \int_0^T |u(t)| dt + E[\underline{x}'(T) Q \, \underline{x}(T)] \tag{6.1}$$

The constraint equation, Eq. 3.4, holds and is repeated here for convenience.

$$\int_0^T e^{F(T-t)} D \, \overline{u(t)} \, dt = - e^{FT} \underline{x}(o) \tag{6.2}$$

For simplicity it is assumed that the matrix, Q, is a diagonal matrix of identical diagonal elements so that Eq. 6.1 may be written

$$J = E\left[\int_0^T |u(t)| dt + \beta \underline{x}'(T) \, \underline{x}(T) \right] \tag{6.3}$$

where $\beta \geq 0$.

Using the procedure developed in Chapter V, Eq. 6.3 may be expressed in the following equivalent form

$$J^* = E\left[\int_0^T |u(t)|dt + \beta\int_0^T u(t)\, y(t)\, dt\right]$$

$$= E\left[\int_0^T [|u(t)| + \beta\, u(t)\, y(t)]\, dt\right] \tag{6.4}$$

where $y(t)$ is given by Eq. 4.10 repeated here for convenience

$$y(t) = \int_0^t [D'e^{F'(T-t)}e^{F(T-\xi)}D\, u(\xi)$$

$$+ D'e^{F'(T-t)}e^{F(T-\xi)}B\, w(\xi)]\, d\xi \tag{6.5}$$

Minimization of J with respect to the control is equivalent to the minimization of J^* with respect to the control; therefore Eq. 6.4 will be used as the performance index.

A procedure similar to that used in Chapter IV and V can now be used to minimize J^* . This procedure is demonstrated here by the following example.

6.2 Example

In order to demonstrate the procedure, the system given by Eq. 5.1 is used as an example. This equation is repeated here.

$$\ddot{x}(t) = u(t) + w(t) \tag{6.6}$$

When an equivalent system is defined as in Chapters IV and V , the Hamiltonian for the system of Eq. 6.6 and the performance criterion of Eq. 6.4 is given by

$$H(t) = -E[|u(t)|] - \beta E[u(t)y(t)] + \lambda_1(t)\, \tilde{x}_2(t) + \lambda_2(t)\, E[u(t)]$$

$$= E\left\{[\lambda_2(t) - \beta y(t)]u(t) - |u(t)|\right\} + \lambda_1(t)\, \tilde{x}_2(t) \tag{6.7}$$

It is again seen that $\lambda_2(t)$ is linear in time. In order to maximize $H(t)$, over all possible values of the control, it is only necessary that the first term of Eq. 6.7 be maximized; therefore by defining $H^*(t)$ such that

$$H^*(t) = E\left\{[\lambda_2(t) - \beta\, y(t)]\, u(t) - |u(t)|\right\} \tag{6.8}$$

then the maximization of $H(t)$ is equivalent to the maximization of $H^*(t)$ where $\lambda_2(t)$ is linear in time.

Equation 6.8 can also be written

$$H^*(t) = \int_{\infty}^{\infty} E\left\{\left\{[\lambda_2(t) - \beta\, y(t)]\, u(t) - |u(t)|\right\} |y(t)\right\} p[y(t)]dy(t) \tag{6.9}$$

From the definition of $y(t)$ as given by Eq. 6.5, it can be seen that $y(o)$ is zero; therefore $y(o)$ is deterministic so that the probability density function of $y(o)$ can be considered to be a unit delta function at $y(o)$ equal to zero. Equation 6.9 can thus be evaluated for t equal to zero.

$$H^*(o) = \lambda_2(o)\, u(o) - |u(o)| \tag{6.10}$$

In order to maximize $H^*(o)$, it is therefore necessary that

$$\begin{aligned} u(o) &= 1 && \text{when } \lambda_2(o) > 1 \\ u(o) &= -1 && \text{when } \lambda_2(o) < -1 \\ u(o) &= 0 && \text{when } |\lambda_2(o)| < 1 \end{aligned} \tag{6.11}$$

Therefore, for a given adjoint variable, the control signal at the initial time is deterministic. This fact allows the remaining statistical properties of the control signal to be computed.

Since $\tilde{y}(t)$ is a smooth function, it can be assumed that the statistical properties of $\tilde{y}(t)$ do not change too rapidly. This is the assumption used in Chapter V and shown to be true by a computer simulation of the system. Since the statistics of $y(o)$ are known, the statistics of $y(t)$ for any time can thus be computed in forward time by a method similar to that used in Chapter V. By assuming that the probability distribution of $y(t)$ is similar to the distribution of $y(t-\epsilon)$ and that the probability distribution of $y(t-\epsilon)$ is known, then in order to maximize $H^*(t)$, it is only necessary that the conditional expectation factor in the integrand of Eq. 6.9 be maximum for each $y(t)$. This can be accomplished by using the following scheme.

$$
\begin{aligned}
u(t) &= 1 \qquad \text{when} \qquad \lambda_2(t) - \beta\, y(t) > 1 \\
&= -1 \qquad \text{when} \qquad \lambda_2(t) - \beta\, y(t) < -1 \\
&= 0 \qquad \text{when} \qquad |\lambda_2(t) - \beta\, y(t)| < 1
\end{aligned}
\tag{6.12}
$$

or

$$
\begin{aligned}
u(t) &= 1 \qquad \text{when} \qquad y(t) < \frac{1}{\beta}\,[\lambda_2(t) - 1] \\
&= -1 \qquad \text{when} \qquad y(t) > \frac{1}{\beta}\,[\lambda_2(t) + 1] \\
&= 0 \qquad \text{when} \qquad \frac{1}{\beta}\,[\lambda_2(t) - 1] < y(t) < \frac{1}{\beta}\,[\lambda_2(t) + 1]
\end{aligned}
\tag{6.13}
$$

The expression for the average value of the control signal can be written

$$
\overline{u(t)} = \int_{-\infty}^{\infty} E[u(t)|y(t)]\; p[y(t)]\, dy(t)
$$

$$
= \int_{-\infty}^{\frac{1}{\beta}[\lambda_2(t) - 1]} p[y(t)]dy(t) - \int_{\frac{1}{\beta}[\lambda_2(t) + 1]}^{\infty} p[y(t)]dy(t) \tag{6.14}
$$

For a given adjoint variable, it is thus theoretically possible to compute the value of $\overline{u(t)}$, since the probability distribution of $y(t)$ can be computed by using the scheme similar to that scheme of Appendix A.

This computed value of $\overline{u(t)}$ can be used together with Eq. 6.2 in order to compute the initial states corresponding to that particular adjoint variable. Alternatively, the method of Chapter V which simulates the system on a computer can also be used to map the space of initial conditions of the adjoint variable on the space of initial conditions of the state variable.

At this point it will be conjectured that Eq. 6.12 represents the solution to not only the relay control problem but also to the more general class of problem which admits all bounded control functions. At no point in the derivation of the optimal control law was it assumed that the control signal was constrained to be the output of a relay. The only assumption needed, is that the control signal be bounded. Equation 6.12 requires that the optimal control be a relay control; however, if the performance index would be based on the square of the value of the control signal instead of the absolute value, the optimal solution would be given by

$$u(t) = \text{sat}\,[\lambda_2(t) - \beta\, y(t)]$$

where

$$\text{sat}\,[x] = \begin{cases} 1 & \text{for } x > 1 \\ -1 & \text{for } x < -1 \\ x & \text{for } |x| < 1 \end{cases}$$

It is also pointed out that the assumption that the noise is gaussian is not used in this chapter; therefore, the solution of this chapter holds for any noise process which can be represented as the output of a linear filter when the input is white. The assumption that the noise is gaussian is only needed in order to simplify computations.

To show that the results of this general procedure reduce to the results of the special cases covered in Chapter IV and V, it is only necessary to adjust the factor, β , of Eq. 6.3. When this factor approaches infinity, it can be seen that the cost from the fuel term given by the integral of Eq. 6.3 is negligible compared to the cost of final state errors so that Eq. 6.3 reduces to the performance index of Chapter V. The control signal is determined from Eq. 6.13, which for β approaching infinity becomes

$$u(t) = 1 \quad \text{when} \quad y(t) < \tilde{\lambda}_2(t)$$
$$= -1 \quad \text{when} \quad y(t) > \tilde{\lambda}_2(t) \qquad (6.15)$$

where $\tilde{\lambda}_2(t) = \lambda_2(t)/\beta$, and the term, $1/\beta$, is negligible. Since $\tilde{\lambda}_2(t)$ is linear in time and since the definition for $y(t)$ is identical in both Chapter V and this Chapter, this control scheme is identical to that scheme of Chapter V.

When the factor, β , is zero, Eq. 6.3 reduces to the performance index of Chapter IV. For β equal to zero, the control scheme can be found from Eq. 6.12 where

$$u(t) = 1 \quad \text{when} \quad \lambda_2(t) > 1$$
$$= -1 \quad \text{when} \quad \lambda_2(t) < -1 \qquad (6.16)$$
$$= 0 \quad \text{when} \quad |\lambda_2(t)| < 1$$

This is identical to the control scheme of Chapter IV.

6.3 Summary

In this chapter, a physical realization of a control scheme is conjectured such that this scheme is optimal in the sense that the performance criterion, based on both fuel expenditure and final state errors, is minimized. It is conjectured that the solution is not only applicable to those cases in which the control signal is constrained to be the output of a relay but also to those cases in which the only constraint on the control signal is that it be bounded. Also the system is assumed to be disturbed by noise which need not be gaussian; however when the noise is gaussian, the required computations are simplified. The realization again requires that the function, $y(t)$, be formed; however, for this case, $y(t)$ is compared to the sum of the adjoint variable and a term which is proportional to the relative significance of the term of the performance criterion involving the final state variance. The control signal is then generated on the basis of this comparison.

SUMMARY AND CONCLUSIONS

Pontryagin's Maximum Principle offers a feasible method for the solution of control optimization problems when the disturbances affecting the performance of the system are known functions of time. At this time there is not a satisfactory "Maximum Principle" available for those cases where these disturbances are random functions of time.

When the control signal is restricted to be the output of a relay, it is shown that in some cases the stochastic control problem can be reformulated so that Pontryagin's Maximum Principle can be effectively used.

This method of solution is applied to the problem of finding a physical realization of a relay control system such that this control drives the plant to a predetermined state at a given time in the future. During the interval of control, the plant is assumed to be disturbed by gaussian noise of known spectral density. The performance criterion in this case is the minimization of the variance of the final state.

The solution requires that a function, $\tilde{y}(\tau)$, be formed and that this function be compared to the predetermined adjoint variable of the Maximum Principle and on the basis of this comparison, the control signal is determined. In this scheme there is again the well known problem of determining the initial conditions of the adjoint variable. However, once these initial conditions have been determined, the realization of the system is a straight-forward procedure.

It is shown that the system can be realized as a very simple feedback control system. The system has time varying gains; however, these gains are easy to implement.

There is no guarantee that the resulting system is optimal, but the digital computer simulations indicate that the systems perform well.

It might be mentioned that even though the calculations that are required to determine the initial conditions of the adjoint are lengthy, the realization is a very simple system. Therefore the control system itself may be very small and can thus be used where size and weight are a criterion.

It is then conjectured that the techniques used in this research can be used to determine the statistical properties of an optimal control signal subject only to the constraint that the control signal be bounded; that is, the signal is not a priori assumed to be the output of a relay. Also, the system is assumed to be disturbed by noise which need not necessarily be gaussian. The performance criterion for this case is a combination of minimum fuel and minimum final state variance. The solution again requires that a function, $\tilde{y}(t)$, be formed; however, in this case $\tilde{y}(t)$ is compared to the sum of the adjoint variable and a term which is proportional to the relative significance of the term of the performance criterion involving the final state variance.

APPENDIX A

In this appendix, a computational scheme is proposed which will enable one to make an estimate of the parameters of the distribution of $\tilde{y}(\tau)$ as given in Section 5.5. This scheme uses a forward iteration procedure and therefore will ordinarily require the use of a digital computer. Since it was felt that an excessively large amount of computer time would be required if this scheme was used, this scheme was not used in the research connected with this thesis. However, for those interested in this approach, the procedure is briefly described in this appendix.

In order to use this procedure, it is necessary to assume a set of initial conditions of the adjoint variable, $\lambda_2(t)$. For the described procedure to converge, these assumed values should be close to the optimal values which will later be obtained during the minimization process. The values obtained when one uses the method of Section 5.2 may be used as a guide when these initial conditions are assumed.

From the definition of $\tilde{y}(t)$, given by Eq. 5.44, it is necessary that $\tilde{y}(o)$ is zero. Therefore, the control signal, $u(o)$, will be $+1$ if $\lambda_2(o)$ is less than zero and will be -1 if $\lambda_2(o)$ is greater than zero. For a given adjoint variable, the variable, $\tilde{y}(t)$, is a function of only the input noise up to time t. This functional dependence then gives the information needed in order to compute the mean and the variance of the distribution of $y(t)$.

With this mean and variance, the method of Section 5.5 is used in order to find the optimal adjoint variable. If the initial conditions of this adjoint variable are close to the assumed initial conditions, then the problem is solved. If the difference between these two adjoints is great enough then another iteration cycle may be used in order to attempt to obtain better agreement; however, the adjoint variable which was just derived is used in this cycle. There is no guarantee that this process will converge; however, it is felt that the process will converge if the assumed adjoint is close to the optimal adjoint. Just as in the deterministic case, there is here also the well-known problem of determining the initial conditions of the adjoint variable corresponding to a set of initial conditions of the state variable.

The function, $\tilde{y}(.)$, being the output of an integrator is relatively smooth; therefore, the control signal which occurs at time t , will be assumed to be a function of $\tilde{y}(t-\epsilon)$, where ϵ is small compared to the size of the time intervals used in the numerical iterations. This makes sense also for the physical system, since there must be a small delay in the action of the relay. The reader is advised to refer to Fig. A.1 in order to better follow the ensuing discussion.

As was previously mentioned, for a given adjoint variable the control signal, $u(o)$, is deterministic. This fact allows the iteration scheme to proceed forward. It follows that $\tilde{m}(o) = 0$, $\tilde{\sigma}(o) = 0$, $\overline{u(o)}$ is specified, $\overline{u^2(o)} = 1$, and $\overline{u(o)w(o)} = 0$ since $\overline{w(o)} = 0$.

In the following discussion it is assumed that the time scale is subdivided into equal time intervals and that the value of the time at the beginning of each of these intervals is simply denoted by the number of the interval, with zero being the number of the first interval. For example, the symbolism, $\tilde{m}(3)$, denotes the value of the mean of the function $\tilde{y}(.)$ at the beginning of the third time interval. The value of all variables are assumed to be constant during the interval. All integrations over time will therefore be accomplished by the use of an approximating sum.

Since $\overline{u(o)}$ is known, Eq. 5.48 can be used to find $\tilde{m}(1)$. The standard deviation, $\tilde{\sigma}(1)$, can be computed from Eq. 5.49 where $\overline{\tilde{y}^2(1)}$ is obtained from Eq. 5.50a. The values of $\overline{u(o)u(o)}$ and $\overline{u(o)w(o)}$ are the only terms required to evaluate Eq. 5.50a and these terms are known. Therefore, the mean and the variance of $\tilde{y}(1)$ is known, so that $\overline{u(1)}$ can be computed from Eq. 5.51 rewritten here for convenience in the simplified form.

$$E[u(\tau)] = \int_{-\infty}^{\infty} E[u(\tau)|\tilde{y}(\tau)]\, p[\tilde{y}(\tau)]\, d\tilde{y}(\tau)$$

where $E[u(\tau)|\tilde{y}(\tau)] = +1$ if $y(\tau) < \lambda_2(\tau)$

$= -1$ if $y(\tau) > \lambda_2(\tau)$

and where $\tilde{y}(\tau)$ is assumed to be gaussian. It therefore follows that $\tilde{m}(2)$ can be determined by the evaluation of Eq. 5.48.

It is more difficult to obtain the value of $\overline{\tilde{y}^2(2)}$ given by Eq. 5.50a, which is required in order to determine the standard deviation, $\tilde{\sigma}(2)$. The procedure which will now be explained can be used for determining the value of $\overline{\tilde{y}^2(i)}$ for any i. It can be seen from an examination of Eq. 5.50a that this calculation requires the terms $\overline{u(k)u(j)}$ where k and j are integers less than i.

The values of $\overline{u(k)u(j)}$ can be found by the use of the procedure developed in Section 5.3 for finding the effects of these correlation terms. That procedure required that the distribution of the functions $\tilde{y}(k)$ and $\tilde{y}(j)$ is known and that the average of the product $\tilde{y}(k)\tilde{y}(j)$ also is known. Since the mean and variance of the function $\tilde{y}(\cdot)$ have been determined for k and j less than i, the distribution is known. It will now be shown that there is also enough information to calculate the average, $\tilde{y}(k)\tilde{y}(j)$. This average can be written

$$\overline{\tilde{y}(k)\tilde{y}(j)} = \int_0^{k\Delta}\int_0^{j\Delta} h(k\Delta,\varphi_1)h(j\Delta,\varphi_2)\ \overline{[u(\varphi_1)+w(\varphi_1)]\,[u(\varphi_2)+w(\varphi_2)]}\ d\varphi_1 d\varphi_2$$

$$= \int_0^{k\Delta}\int_0^{j\Delta} h(k\Delta,\varphi_1)h(j\Delta,\varphi_2)\ \left[\overline{u(\varphi_1)u(\varphi_2)} + \overline{u(\varphi_1)w(\varphi_2)} + \overline{u(\varphi_2)w(\varphi_1)}\right] d\varphi_1 d\varphi_2$$

$$+ G\int_0^{k\Delta}\int_0^{j\Delta} h(k\Delta,\varphi_1)h(j\Delta,\varphi_2)\delta(\varphi_1-\varphi_2)\ d\varphi_1 d\varphi_2 \qquad \text{(A-1)}$$

where Δ is the length of the time interval. The value of the averages in the first double integral of Eq. A-1 have been previously determined; therefore, Eq. A-1 can be evaluated. Incidentally, this first double integral must be considered as a double sum.

It follows that $\overline{u(k)u(j)}$ can now be evaluated. Thus the variance of $\tilde{y}(i)$ can be determined if the term, $\overline{u(k)w(j)}$, can be computed. The following procedure shows that this term can also be computed. This average can be written in the following form.

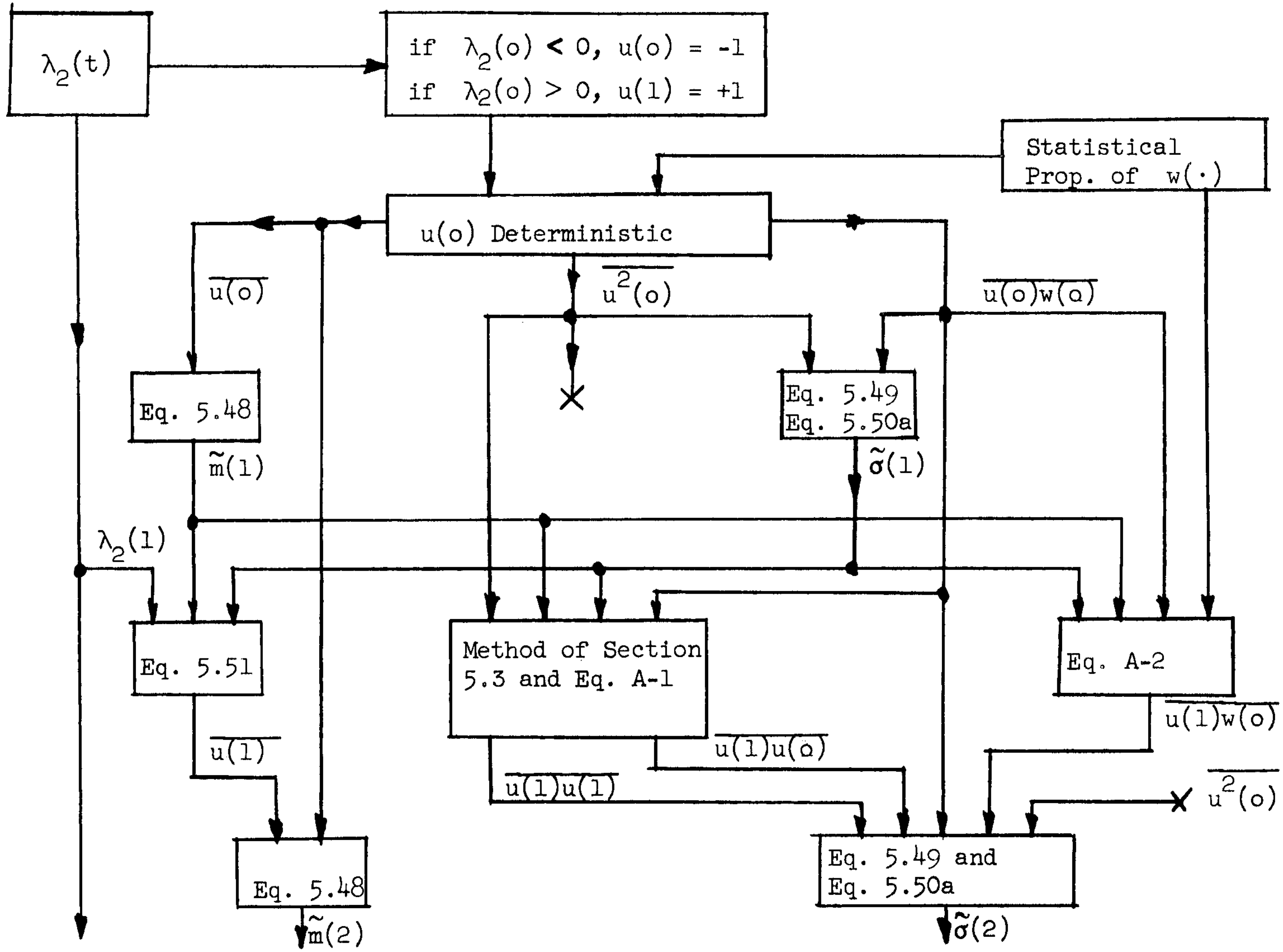


FIGURE A-1 COMPUTATIONAL SCHEME FOR COMPUTING THE MEAN VALUE FUNCTION AND THE VARIANCE OF THE RANDOM VARIABLE, $\tilde{y}(t)$

$$\overline{u(k)w(j)} = \int_{-\infty}^{\infty} w(j)\, E[u(k)|w(j)]\, p[w(j)]\, dw(j) \tag{A-2}$$

where it is known that $w(j)$ is gaussian with zero mean. Since the process $w(\cdot)$ is white, it is necessary that $E[u(k)|w(j)]$ be zero for j greater than k. For j less than k, this conditional expectation can be written

$$\begin{aligned} E[u(k)|w(j)] &= p[u(k) = 1\ |w(j)] - p[u(k) = -1\ |w(j)] \\ &= p[\tilde{y}(k) < \lambda_2(k)|w(j)] - p[\tilde{y}(k) > \lambda_2(k)|w(j)] \end{aligned} \tag{A-3}$$

Therefore in order to compute the value of Eq. A-3 which in turn allows Eq. A-2 to be evaluated, it is only necessary that the conditional distribution of $\tilde{y}(k)$ given $w(j)$ be known. Since $w(j)$ is gaussian and $\tilde{y}(k)$ is assumed to be gaussian, the conditional distribution is gaussian. In order to calculate the mean and the variance of this conditional distribution, it is necessary that the parameters of the distribution of $\tilde{y}(k)$ and $w(j)$ are known and that the average, $\overline{\tilde{y}(k)w(j)}$, is also known.

The parameters of the distribution of $\tilde{y}(k)$ have been previously determined and those of the distribution of $w(j)$ are given; therefore it is only necessary to investigate the average which can be written

$$\overline{\tilde{y}(k)w(j)} = \int_{0}^{k\Delta} [(T-k\Delta)\ (T-\varphi) + 1]\ \overline{[u(\varphi) + w(\varphi)]\ w(j)}\ d\varphi \tag{A-4}$$

Since $\overline{w(\varphi)w(j)}$ is known to be $G\delta(\varphi - j\Delta)$ and since $\overline{u(\varphi)w(j)}$ has been previously determined, Eq. A-4 can be evaluated and hence the average, $\overline{u(k)w(j)}$, can be determined.

It has now been shown that the necessary information is available for the computation of the variance of $\tilde{y}(i)$. The iteration procedure can thus be continued forward until the values of $\tilde{m}(i)$ and $\tilde{\sigma}(i)$ are known for all i.

Figure A-1 presents a summary of the computational scheme presented in this appendix.

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