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ON THE FEYNMAN INTEGRAL IN DYNAMICS

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ABSTRACT

The Lagrangian formulation of quantum dynamics in terms of the Feynman integral describes systems for which the Hamiltonian is classical in form and quantization is carried out in terms of commutators rather than anticommutators. The difficulty with this method is the actual evaluation of the Feynman integral itself. We give an explicit evaluation for classical wave motion in one dimension. This requires an extension of the Feynman method which was introduced by Tobocman and studied in detail by Davies.

We also discuss the work of Corson on the question of a unified formulation of dynamics.

ON THE FEYNMAN INTEGRAL IN DYNAMICS

A. M. Arthurs

1. Introduction

The Feynman integral approach to quantum mechanics [4] provides an alternative to the formulation based on the Schrödinger equation and the usual commutation rules. It does not however describe the Dirac field in which operators satisfy certain anti-commutation laws [5].

The difficulty with this method is the actual evaluation of the Feynman integral itself. Cases which have been explicitly evaluated are those corresponding to the free particle and to the harmonic oscillator [2]. A further example is provided by the Feynman integral formulation of classical wave motion in one dimension. To discuss this we first of all require an extension of the Feynman method which was introduced by Tobocman [5] and studied in detail by Davies [3].

2. Extension of Feynman method

This extension is based on the Hamiltonian of the system rather than on the Lagrangian. In it the time development of the wave function is given by

$$\psi(q'', T) = \int dq' \; K(q'', T\,; q', 0)\; \psi(q', 0) \quad , \tag{1}$$

which connects the wave function $\psi(q'', T)$ at time T with the wave function $\psi(q', 0)$ at an earlier time 0. The kernel $K(q'', T\,; q', 0)$ is given by

$$K(q'', T\,; q', 0) = N \sum_{pq} \exp i S_{pq} \quad , \tag{2}$$

where

$$S_{pq} = \int_0^T \{p\dot{q} - H(p, q)\}dt \quad . \tag{3}$$

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In (3), H is the classical Hamiltonian of the system while the subscripts p, q denote any history of the system specified by two arbitrary functions of time $q(t)$ and $p(t)$ subject to the restrictions

$$q(0) = q', \quad q(T) = q'' . \tag{4}$$

Thus S_{pq} is the classical action for a history p, q. In (2), $\sum_{pq}$ means a sum over all histories which satisfy the end conditions (4) and not only over the history which is the actual classical path between the end-points. The normalization factor N is chosen so that

$$K(q'',0\,;q',0) = \delta(q'-q'') , \tag{5}$$

where δ is the Dirac delta function.

The equivalence of this approach to the usual one based on the Schrödinger equation

$$H\psi = i\frac{\partial\psi}{\partial t} \tag{6}$$

is readily shown by means of setting up operators in a function space and defining an appropriate inner product. This will now be discussed.

3. The operators p and q

We suppose that the elements of the function space are $f(q)$, $g(q)$, etc. Then, following Davies [3], we define an inner product (f, g) as follows:

$$(f,g) = \int\int dq'\,dq''\,\bar{f}(q'')\,A(q'',T\,;q',0)\,g(q') , \tag{7}$$

where

$$A(q'',T\,;q',0) = N\sum_{pq} \exp i\int_0^T pdq , \tag{8}$$

and the $q-p$ histories to be summed over are those specified by giving $q(t)$, $p(t)$ arbitrary values over the range $0 \leq t \leq T$ subject to

$$q(0) = q', \quad q(T) = q'' . \tag{9}$$

By the method described in Section 5 we evaluate the summation in (8) and find that

$$A(q'', T ; q', 0) = \delta(q' - q'') , \tag{10}$$

a result independent of T. Hence the inner product (f, g) becomes

$$(f,g) = \int\int dq' dq'' \bar{f}(q'') \delta(q' - q'') g(q')$$

$$= \int dq' \bar{f}(q') g(q') , \tag{11}$$

which corresponds with the frequently used inner product of function space.

We now define operators corresponding to the variables q, p. First, we define the operator Q corresponding to q by

$$(f, Qg) = \int\int dq' dq'' \bar{f}(q'') B(q'', T ; q', 0) g(q') , \tag{12}$$

where

$$B(q'', T ; q', 0) = N \sum_{pq} q_{pq}(t) \exp i \int_0^T pdq ., \tag{13}$$

where a time t has been associated with Q such that $0 < t < T$, $q_{pq}(t)$ denotes the value of $q(t)$ for a particular $q - p$ history, and once again the summation has to carried out over all $q - p$ histories subject to the restrictions (9). The evaluation of (13) is carried out by the method described in Section 5 and we find that

$$B(q'', T ; q', 0) = q' \delta(q' - q'') . \tag{14}$$

Hence equation (12) becomes

$$(f, Qg) = \int\int dq' dq'' \bar{f}(q'') q' \delta(q' - q'') g(q')$$

$$= \int dq' \bar{f}(q') q' g(q') , \tag{15}$$

which is identical to the usual representation of the operator corresponding to q.

In a similar way, the operator corresponding to p is defined by

$$(f, Fg) = \iint dq'dq'' \bar{f}(q'') \, C(q'', T; q', 0) \, g(q') \quad , \tag{16}$$

where

$$C(q'', T; q', 0) = N \sum_{pq} p_{pq}(t) \exp i \int_0^T p dq \quad , \tag{17}$$

where $p_{pq}(t)$ denotes the value of $p(t)$ for a particular $q-p$ history. It can readily be shown that

$$C(q'', T; q', 0) = -i \, \delta'(q''-q') \quad , \tag{18}$$

where δ' is the first derivative of the Dirac delta function. Hence equation (16) becomes

$$\begin{aligned} (f, Pg) &= \iint dq'dq'' \bar{f}(q'') \, (-i) \, \delta'(q''-q') \, g(q') \\ &= \int dq'' \bar{f}(q'') \int dq' \{ i \frac{d}{dq'} \delta(q''-q') \} \, g(q') \\ &= \int dq'' \bar{f}(q'') \int dq' \, \delta(q''-q') \, (-i \frac{dg}{dq'}) \\ &= \int dq' \, \bar{f}(q') (-i \frac{d}{dq'}) \, g(q') \quad , \end{aligned} \tag{19}$$

where (19) has been obtained by integration by parts. Equation (19) shows the usual quantum mechanical correspondence of the variable p with the operator $-i \, d/dq$.

4. Equivalence of the Feynman and Schrödinger approaches

Since we have now defined the operators Q, P corresponding to the variables q, p, we can define similarly operators corresponding to q^2 and p^2 and indeed to a function $F(q, p)$. Thus, we write

$$(f, F(Q,P) g) = \iint dq'dq'' \bar{f}(q'') \, I(q'', T; q', 0) \, g(q') \quad , \tag{20}$$

where

$$I(q'', T; q', 0) = N \sum_{pq} F(q_{pq}(t), \, p_{pq}(t)) \exp i \int_0^T p dq \, . \tag{21}$$

In the same way we can define the operator

$$\exp\{-i \int_0^T F(Q,P,t)\,dt\}$$

by

$$(f, \exp\{-i \int_0^T F(Q,P,t)\,dt\} g) = \iint dq'\,dq''\,\bar{f}(q'')\,J(q'',T;q',0)\,g(q') \,, \quad (22)$$

where

$$J(q'',T;q',0) = N \sum_{pq} \exp i \int_0^T (p\,dq - F(q,p,t)\,dt) \quad . \quad (23)$$

We now choose $F(q,p,t)$ to be equal to $H(q,p,t)$, the Hamiltonian of a system. But now, for this particular choice of $F(q,p,t)$, the kernel J of equation (23) is identical to the kernel K of equations (1), (2) and (3) which determined a function $\psi(q'',T)$ from a function $\psi(q',0)$. So if we write $g(q) = \psi(q,0)$, we have

$$\begin{aligned}(f, \exp\{-i \int_0^T H(Q,P,t)\,dt\}\,\psi(q,0)) &= \int dq''\,\bar{f}(q'')\,\psi(q'',T) \\ &= (f, \psi(q,T)) \,, \quad (24)\end{aligned}$$

and therefore we have

$$\psi(q,T) = \exp\{-i \int_0^T H\,dt\}\,\psi(q,0) \,, \quad (25)$$

which is just the integral form of the Schrödinger equation

$$H\psi = i \frac{\partial\psi}{\partial t} \quad .$$

The equivalence of the Feynman approach with the Schrödinger approach is therefore established.

5. Classical wave motion

We take the classical wave equation in one space dimension

$$\frac{\partial^2 \varphi}{\partial q^2} = \frac{\partial^2 \varphi}{\partial t^2} \tag{26}$$

and write it in the two-component form

$$M\psi = i\,\frac{\partial \psi}{\partial t} \quad , \tag{27}$$

where

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \; , \quad u = \frac{\partial \varphi}{\partial q} \; , \quad v = \frac{\partial \varphi}{\partial t} \; , \tag{28}$$

and

$$M = -\sigma P \quad , \tag{29}$$

with

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad P = -i\,\frac{\partial}{\partial q} \quad . \tag{30}$$

Equation (27) is of Schrödinger type and the equivalence established in Section 4 enables us therefore to reformulate (27) as

$$\psi(q'', T) = \int dq' \; K(q'', T ; q', 0)\; \psi(q', 0) \quad , \tag{31}$$

with

$$K(q'', T ; q', 0) = N \sum_{pq} \exp i\, S_{pq} \quad , \tag{32}$$

and

$$S_{pq} = \int_0^T \{ p dq - M\, dt \} \; . \tag{33}$$

Following Davies [3] we use a Riemann definition of integral and write (33) as

$$S_{pq} = \sum_{r=1}^{n} \{ p_r (q_r - q_{r-1}) + \sigma p_r (t_r - t_{r-1}) \} \quad , \tag{34}$$

where a partition $t_0 = 0$, t_1, t_2, ..., $t_n = T$ has been made of the interval and where $q_r = q(t_r)$ and $p_r = p(\tau_r)$, with $t_{r-1} \le \tau_r < t_r$, and

$$q_0 = q' , \quad q_n = q'' . \tag{35}$$

Then

$$\begin{aligned} \exp i\, S_{pq} &= \prod_{r=1}^{n} \exp i \,\{p_r(q_r - q_{r-1}) + \sigma\, p_r(t_r - t_{r-1})\} \\ &= \prod_{r=1}^{n} \exp\{i\, p_r(q_r - q_{r-1})\}\{I \cos p_r(t_r - t_{r-1}) + i\,\sigma \sin p_r(t_r - t_{r-1})\} , \end{aligned} \tag{36}$$

where I is the unit 2×2 matrix.

The p-summation in (32) is now obtained by integrating over the variables $p_1, p_2, \ldots, p_n$, and we have

$$\begin{aligned} \int_{-\infty}^{\infty} dp_1 \ldots \int_{-\infty}^{\infty} dp_n \exp i\, S_{pq} &= \prod_{r=1}^{n} \int_{-\infty}^{\infty} dp_r \exp\{i\, p_r(q_r - q_{r-1})\} \\ &\quad \times \{I \cos p_r(t_r - t_{r-1}) + i\,\sigma \sin p_r(t_r - t_{r-1})\} \\ &= \prod_{r=1}^{n} \{\pi(I + \sigma)\, \delta(q_r - q_{r-1} + t_r - t_{r-1}) \\ &\quad + \pi(I - \sigma)\, \delta(q_r - q_{r-1} - t_r + t_{r-1})\} , \end{aligned} \tag{37}$$

where we have used the result

$$\int_{-\infty}^{\infty} dp \exp i\, pq = 2\pi\, \delta(q) . \tag{38}$$

The summation over histories is now completed by integrating (37) over $q_1, q_2, \ldots, q_{n-1}$, so that

$$\sum_{pq} \exp i S_{pq} = \int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_{n-1} \prod_{r=1}^{n} \{\pi(I+\sigma)\,\delta(q_r - q_{r-1} + t_r - t_{r-1})$$

$$+ \pi(I-\sigma)\,\delta(q_r - q_{r-1} - t_r + t_{r-1})\} \quad . \tag{39}$$

Performing the integration over q_1 we get

$$\int_{-\infty}^{\infty} dq_1 \{\pi(I+\sigma)\,\delta(q_1 - q_0 + t_1 - t_0) + \pi(I-\sigma)\,\delta(q_1 - q_0 - t_1 + t_0)\}$$

$$\times \{\pi(I+\sigma)\delta(q_2 - q_1 + t_2 - t_1) + \pi(I-\sigma)\,\delta(q_2 - q_1 - t_2 + t_1)\}$$

$$= \pi^2 (I+\sigma)^2\; \delta(q_2 - q_0 + t_2 - t_0) + \pi^2 (I-\sigma)^2\; \delta(q_2 - q_0 - t_2 + t_0) \quad , \tag{40}$$

since

$$(I+\sigma)(I-\sigma) = 0 \quad , \tag{41}$$

and

$$\int_{-\infty}^{\infty} dq_1\; \delta(q_1 + a)\,\delta(b - q_1) = \delta(a+b) \quad . \tag{42}$$

The integrals over $q_2, q_3, \ldots, q_{n-1}$ can be evaluated in the same way and we obtain

$$\sum_{pq} \exp i S_{pq} = \pi^n (I+\sigma)^n\; \delta(q'' - q' + T)$$

$$+ \pi^n (I-\sigma)^n\; \delta(q'' - q' - T) \quad . \tag{43}$$

Using the relations

$$(I \pm \sigma)^n = 2^{n-1} (I \pm \sigma) \quad , \tag{44}$$

and introducing the normalization factor $N = (2\pi)^{-n}$, we have

$$K(q'',T;q',0) = \frac{1}{2}(I+\sigma)\,\delta(q''-q'+T) + \frac{1}{2}(I-\sigma)\,\delta(q''-q-T) \quad . \tag{45}$$

This completes the determination of the Feynman integral for classical waves. With given initial conditions, equation (31) has a sclution which agrees of course with the standard D'Alembert solution.

6. Unified formulation of dynamics

We now consider the work of Corson [1] on the question of finding a single postulate that would cover both classical and quantum dynamics. This postulate therefore must lead to the Schrödinger equation in the quantum case and to Lagrange's equations (or equivalents) in the classical case.

The classical case is given by

$$\delta S = 0 \quad , \tag{46}$$

where S is the action defined in equation (3). Equation (46) means that S is stationary with respect to a small variation in the path between the end-points $(q',0)$ and (q'',T). Now Hamilton's principle (46) is really just the simplest form of a stationary condition involving S. It could be replaced by

$$\delta F(S) = 0 \tag{47}$$

with F some reasonable function of S. Corson's choice of F is essentially

$$F(S) = \exp i S \quad , \tag{48}$$

which is suggested by the form of the Feynman integral K in equation (2).

Corson [1] then postulates

$$\delta \sum_{pq} \exp i S_{pq} = 0 \tag{49}$$

as the fundamental equation of dynamics. There are two cases, namely, (i) the definite path case, and (ii) the indefinite path case.

(i) Definite path. If there is only <u>one</u> path p, q, then equation (49) reduces to

$$\delta \exp i S = 0 ,$$

or

$$\delta S = 0 , \tag{50}$$

and equation (50) leads, of course, to Lagrange's equations. This then covers the classical case.

(ii) Indefinite path. If there are many paths, $\sum_{pq} \exp i S_{pq}$ is a function of the end-points only, and since the end-points remain <u>fixed</u> under the δ-variation, it follows that

$$\delta \sum_{pq} \exp i S_{pq} = 0 \tag{51}$$

trivially. This condition therefore does not appear to lead to anything. The classical case $\delta S = 0$ leads to Lagrange's equations, but the many path case does not tell us what function of the end-points $\sum_{pq} \exp i S_{pq}$ actually is. This requires an additional postulate — one involving the notion of state.

Thus the conclusion would seem to be that Corson's single postulate is not enough. To formulate quantum dynamics from classical action expressions one must postulate the time evolution of the wave function as Feynman did.

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