TECHNICAL MEMORANDUM NO. 44

THE AVERAGE NUMBER OF ZERO CROSSINGS PER SECOND OF A SINUSOIDAL SIGNAL PLUS NOISE

Gerald Weiss

January 1965

Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND
CONTRACT NAS 5- 9809

LABORATORY FOR ELECTROSCIENCE RESEARCH

DEPARTMENT OF ELECTRICAL ENGINEERING

SCHOOL OF ENGINEERING AND SCIENCE

NEW YORK UNIVERSITY New York 53, New York

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ABSTRACT

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The measurement of the average frequency of a sinusoidal voltage source may be implemented by counting the average number of zero crossings which have accrued in a given counting period. If the source is present in a noiseless background, then the uncertainty in the measure of the average frequency is limited by the uncertainty of measuring the number of zero crossings and the observation time. If, in addition, the sinusoidal voltage is associated with random noise, then the uncertainty in the measure of the source frequency depends upon the noise characteristics. This report considers two models which show the effect of noise on the average number of zero crossings per second. One model utilizes as a signal source a quasi-harmonic sinusoidal voltage which is characteristic of a propagated signal subject to fading. The other model assumes a fixed sinusoidal voltage which may be the output of a signal generator. In both of these models, the signal source voltage is added to a random voltage which is Gaussian distributed, and the number of zero crossings is averaged over an infinite period of time. These results, then, establish an upper limit on the error in measuring frequency by a zero count technique.

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THE AVERAGE NUMBER OF ZERO CROSSINGS PER SECOND
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I. THE QUASI-HARMONIC SINUSOIDAL SOURCE

The average number of zero crossings per second for a narrow band sine wave process in random noise has been studied by many investigators.[1,2,3] In this model, a sine wave of fixed frequency $\omega_0 (=2\pi f_0)$ and random amplitude and phase is considered. The bandwidth of the signal process is assumed to be narrow compared to the frequency ω_0. A random noise process with a normal amplitude distribution is added to the random signal process resulting in perturbations in the average number of zero crossings compared to $2f_0$. The noise is represented by its autocorrelation function:

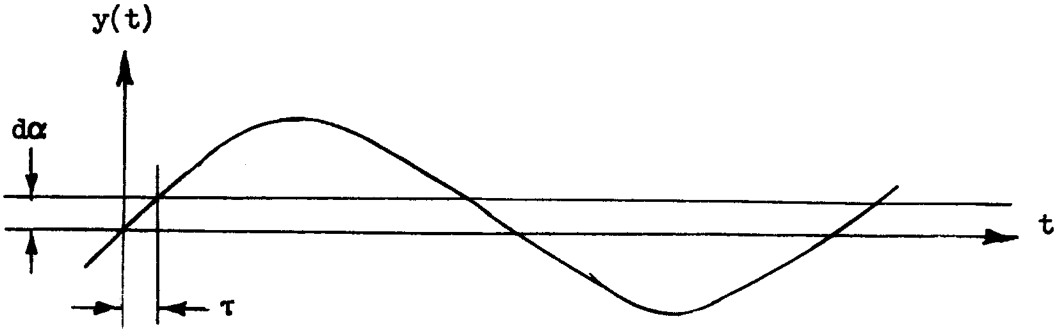
$$R(\tau) = \int_0^\infty G(\omega) \cos \omega\tau \, d\omega \quad , \tag{1}$$

where $G(\omega)$ is the power spectrum of the noise. The quasi-harmonic sine wave process and random noise process are assumed mutually independent, and the composite process given by:

$$y(t) = Q \sin (\omega_0 t + \phi_k) + n(t) \quad , \tag{2}$$

where Q and ϕ_k are the random amplitude and phase functions associated with the random signal process.

In one development[2] an incremental length $d\alpha$ is considered for a representative sine wave in the ensemble.



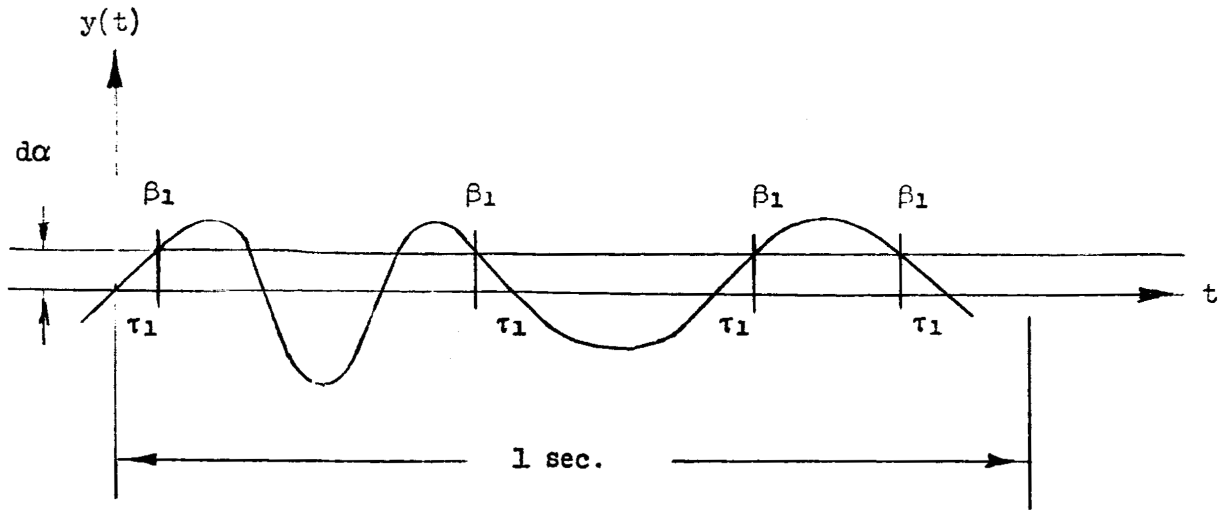
The time to cross the interval $d\alpha$ is related to the slope of the waveform:

$$\tau = \frac{d\alpha}{|\beta|} \quad . \tag{3}$$

The probability over the ensemble that y(t) is in the interval $(\alpha, \alpha+d\alpha)$ while its derivative $\dot{y}(t)$ is between $(\beta, \beta+d\beta)$ is:

$$f(\alpha, \beta)\, d\alpha d\beta \quad . \tag{4}$$

This may also be interpreted as the amount of the time per unit time that y(t) spends in the interval $(\alpha, \alpha+d\alpha)$ with velocity $(\beta, \beta+d\beta) \approx \beta$ as illustrated below:



The number of crossings per unit time through a level α with velocity β is

$$\frac{f(\alpha,\ \beta)\,d\alpha d\beta}{\tau} = |\beta| f(\alpha,\ \beta)\ d\beta \quad . \tag{5}$$

The average number of zero crossings for all β

$$\overline{N_{s+n}} = \int_0^\infty |\beta| f(0,\ \beta)\ d\beta \quad . \tag{6}$$

If $y(t)$ and $\dot{y}(t)$ are statistically independent:

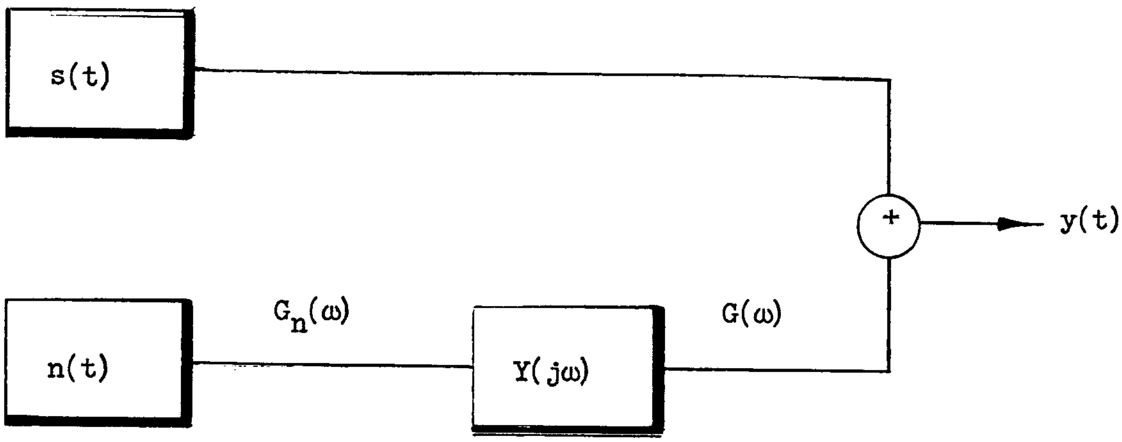
$$\overline{N_{s+n}} = \frac{1}{\pi}\left[\frac{\omega_0^2\ \frac{\overline{Q^2}}{2} + D_o}{\frac{\overline{Q^2}}{2} + R_o}\right]^{\frac{1}{2}} \tag{7}$$

where

$$D_o = \int_0^\infty \omega^2 G(\omega)\ d\omega \quad , \tag{8}$$

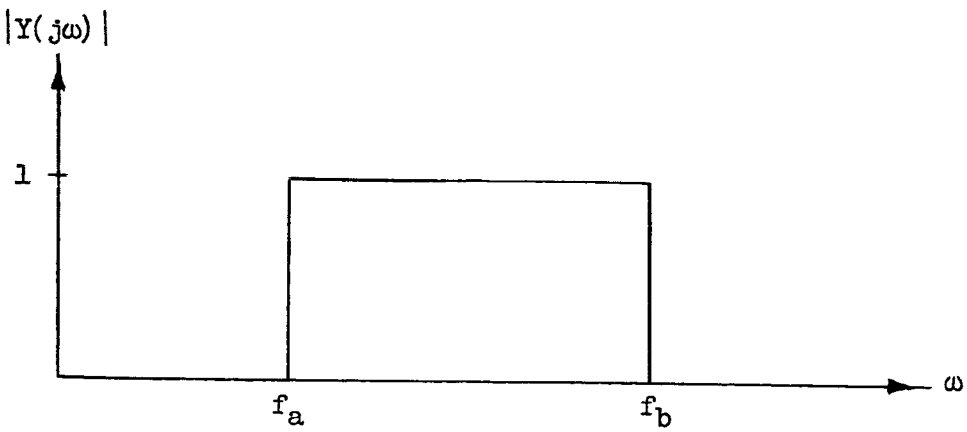
$$R_o = \int_0^\infty G(\omega)\, d\omega \quad . \qquad (9)$$

Consider the application of these results to the case where the noise is derived from the output of an ideal rectangular filter with a response which includes the sine wave frequency ω_o.



$$s(t) = Q \sin(\omega_o t + \phi_k)$$

The power spectrum $G(\omega)$ is obtained for the rectangular filter:



$$G_n(\omega) = K/2\pi \text{ volts}^2/\text{cps.} \quad , \tag{10}$$

$$G(\omega) = K/2\pi \; |T(j\omega)|^2 \tag{11}$$

$$G(\omega) = \begin{bmatrix} K/2\pi & , \; \omega_a \leq \omega \leq \omega_b \\ 0 & , \; \text{elsewhere} \end{bmatrix} . \tag{12}$$

$$R_o = \int_{\omega_a}^{\omega_b} K/2\pi \; d\omega = K(f_b - f_a) \quad , \tag{13}$$

$$D_o = \int_{\omega_a}^{\omega_b} K/2\pi \; \omega^2 \; d\omega = k(2\pi)^2 \left(\frac{f_a^3 - f_b^3}{3} \right) . \tag{14}$$

$$\overline{N}_n = \frac{1}{\pi} \left(\frac{D_o}{R_o} \right)^{\frac{1}{2}} = \frac{1}{\pi} \left[\frac{\int_0^\infty \omega^2 \; G(\omega) \; d\omega}{\int_0^\infty G(\omega) \; d\omega} \right]^{\frac{1}{2}} , \tag{15}$$

$$\overline{N}_n = 2 \left[\frac{\int_0^\infty f^2 \; G(f) \; df}{\int_0^\infty G(f) \; df} \right]^{\frac{1}{2}} , \tag{16}$$

$$\overline{N}_n = 2 \left[\frac{f_b^3 - f_a^3}{3(f_b - f_a)} \right]^{\frac{1}{2}} . \tag{17}$$

$$\overline{N}_{s+n} = \frac{1}{\pi}\left[\frac{\omega_o^2\rho + (\pi N_n)^2}{\rho + 1}\right]^{\frac{1}{2}} , \tag{18}$$

where ρ is the signal-to-noise power ratio:

$$\rho = \frac{\overline{Q^2}}{2R_o} = \frac{\overline{Q^2}}{2K(f_b-f_a)} . \tag{19}$$

$$\overline{N}_{s+n} = \left[\frac{\rho(2f_o)^2 + (\overline{N_n})^2}{1 + \rho}\right]^{\frac{1}{2}} . \tag{20}$$

As the quantity ρ increases, the average number of zero crossings approaches $2f_o$. For $\rho = 0$, the average number of zero crossings is $\overline{N_n}$, the expected number for noise alone. Both of these results are what would be expected. Expression (20) may be used to determine the fractional deviation in $2f_o$ when f_o lies between f_a and f_b:

$$\frac{\overline{N}_{s+n}}{2f_o} = \left[\frac{\rho + r^2}{\rho + 1}\right]^{\frac{1}{2}} , \tag{21}$$

where:

$$r = \frac{\overline{N}_n}{2f_o} .$$

The above results may also be applied to noise shaped by an RLC bandpass filter (see appendix) centered on ω_c by defining $r = \omega_c/\omega_o$. Figure 1 is a plot of (21) for $r < 1$ and Figure 2 for $r > 1$.

II. THE FIXED SINUSOIDAL SOURCE

The average number of zero crossings per second for a sinusoidal source of fixed amplitude, frequency and phase plus normal noise has been studied by S.O. Rice.[3] For the composite process $y(t) = Q \sin \omega_o t + n(t)$, the resulting expression is:

$$\bar{N}_{s+n} = \bar{N}_n\left[\epsilon^{-\alpha} I_o(\beta) + \frac{b^2}{2\alpha} I_e\left(\frac{\beta}{\alpha}, \alpha\right)\right] , \qquad (22)$$

where

$$\bar{N}_n = \frac{1}{\pi}\left(\frac{D_o}{R_o}\right)^{\frac{1}{2}} \quad \text{[See (15)]}$$

$$\alpha = \frac{a^2 + b^2}{4} , \quad \beta = \frac{a^2 - b^2}{4}$$

$$a^2 = \frac{Q^2}{R_o} , \quad b^2 = \left[\frac{2af_o}{N_n}\right]^2 .$$

Defining the following quentities

$$\rho = \frac{Q^2}{2R_o} \equiv \text{signal-to-noise power ratio, and}$$

$$\delta = \frac{2f_o}{N_n} \equiv \text{signal frequency offset relative to the no-signal case,}$$

then the parameters of (22) may be written as:

$$\alpha = \rho\left(\frac{1 + \delta^2}{2}\right) ,$$

$$\beta = \rho\left(\frac{1 - \delta^2}{2}\right) \quad ,$$

$$\frac{b^2}{2\alpha} = \frac{2\delta^2}{1 + \delta^2} \quad .$$

In addition to the above quantities

$I_o(\beta) \equiv$ Bessel function of order zero and imaginary argument,

$$I_e(k, x) = - \int_0^x e^{-u} I_o(ku) \, du \quad .$$

As an example of the above, consider the following cases:

<u>For the no-signal case ($Q = 0$)</u>:

$$\rho = 0 \quad , \quad \alpha = \beta = 0:$$

$$I_o(C) = 1 \quad ,$$

$$I_e(k, 0) = 0 \quad ,$$

$$\overline{N}_{s+n} = \overline{N}_n \quad .$$

<u>For the no-noise case ($R_o = 0$)</u>:

$$\rho = \infty \quad ; \quad \alpha = \beta = \infty \quad :$$

$$\lim_{\rho \to \infty} \frac{I_o(k_1\rho)}{\epsilon^{k_2\rho}} = 0$$

$$I_e(k,\ \infty) = (1-k^2)^{-\frac{1}{2}} = \left[1 - \left(\frac{1+\delta^2}{1-\delta^2}\right)^2\right]^{\frac{1}{2}}$$

then

$$\left(\frac{2\delta^2}{1+\delta^2}\right)\left[1 - \left(\frac{1+\delta^2}{1-\delta^2}\right)^2\right]^{\frac{1}{2}} = \delta$$

and

$$\overline{N}_{s+n}\Big|_{\rho = \infty} = \overline{N}_n(0+\delta) = 2f_o$$

<u>For the case where f_o is set equal to $N_n/2$ for any ρ:</u>

$$\delta = \frac{2f_o}{N_n} = 1 \quad ,$$

$$\beta = 0 \quad , \quad \alpha = \rho \quad :$$

$$I_e(0,\ x) = 1 - \epsilon^{-x} = 1 - \epsilon^{-\alpha} = 1 - \epsilon^{-\rho}$$

then,

$$\overline{N}_{s+n} = \overline{N}_n(\epsilon^{-\rho}+1-\epsilon^{-\rho}) = \overline{N}_n = 2f_o$$

The above results satisfy what would be intuitively expected in the extreme cases. For a rectangular filter with lower limit f_a and upper limit f_b shaping the noise:

$$\overline{N}_n = 2\left[\frac{f_b^3 - f_a^3}{3(f_b-f_a)}\right]^{\frac{1}{2}} \quad \text{[See (17)]} \quad . \tag{23}$$

For an RLC bandpass filter centered on f_c:

$$\overline{N}_n = 2f_c \quad \text{[See (A17)]} \quad . \tag{24}$$

Using an IBM 1620 computer, the average number of zero crossings per second for a rectangular noise filter was computed. A copy of the computer program is shown in Figure 25. The parameters chosen were:

$$f_a = 5\text{KC} \quad , \quad f_b = 15\text{KC}$$

$$f_a \leq f_o \leq f_b$$

$$\overline{N}_n = 10408.333 \text{ crossings per second}$$

$$1 \leq \rho \leq 10$$

The center frequency of the sinusoid f_o is varied between 5 and 15KC in 1KC steps. The average number of zero crossings per second with ρ and f_o as parameters is tabulated in Table 1 and the data plotted in Figures 3 through 13. From this data, curves of constant error in cps were plotted with ρ and f_o as parameters. Curves corresponding to errors in frequency of 5, 10, 20, 50, and 100 cps are plotted in Figures 14 through 18. These cyclic errors correspond to percent errors in measuring frequency of 0.05%, 0.10%, 0.20%, 0.50%, and 1.00% respectively.

It is seen from these curves that for a given error, the requisite signal-to-noise ratio decreases sharply as the sinusoidal frequency f_o decreases from f_a. As the frequency $\overline{N}_n/2 = f_o$ is approached from the left, the signal-to-noise ratio approaches zero. For frequencies greater than $f_o = \overline{N}_n/2$, the requisite signal-to-noise ratio increases again. The required signal-to-noise ratio is maximum at f_a and represents a worst case design point. Since the signal-to-noise ratio falls so sharply with frequency, then a departure from the worst case design may be made with a resulting compromise in the error in a small fraction of the data points. As an example, from Figure 15, it is seen that if $\rho = 10$, then all the data points between 6-15KC are within 0.1%. A reduction of ρ to 7.9 (9db) or a decrement of 1db results in a 10% loss of data points that are within 0.1%.

The effect of a change in bandwidth is shown in Figures 19 through 23. In Figures 19 and 20, the lower frequency f_a is reduced to 3KC and in Figures 21 through 23 to d.c.

The increase in error due to a change in bandwidth is minimal at high signal-to-noise ratio. For example, for $\rho = 6$, $f_a = 5$KC and $f_o = 15$KC an error of -6 cps results in the measured frequency based on the average number of zero crossings. If f_a is reduced to 3KC, the error in frequency is essentially unchanged. However, when f_a is reduced to zero frequency the error is -7 cps which represents a minor increase. Consequently, the effect of "roll-off" characteristics of the noise shaping filter on the error in measuring any one particular frequency is negligible.

The effect of heterodyning on the error in frequency may be determined from Figure 23. In this case, the 10KC bandwidth centered on a frequency of 10KC is translated to a center frequency of 40KC. In the curve, the cyclic error versus input frequency is plotted for a signal-to-noise ratio of 6 and 8. As an example, at the 10KC center frequency with $f_o = 6.9$KC and $\rho = 8$, the error in measuring the frequency is 10 cps. Heterodyning this data bandwidth to 40KC with

$f_o = 36.9$KC and $\rho = 8$ results in an error of 0.7 cps in the measured frequency. This decrease in frequency error is due to the narrow band effect of heterodyning. Since the noise bandwidth upon frequency translation represents a smaller fraction of the center frequency, the effect of noise perturbing this frequency is reduced.

III. CONCLUSION

Two models of a signal source have been considered for the effect of noise on the measurement of the frequency of the source. One model assumes a quasi-harmonic source representing a propagated signal, the other a fixed source representing a signal generator.

In either model, when the frequency of the source is measured by counting the average number of zero crossings per second, the signal-to-noise ratio over the entire data spectrum for a fixed error is not constant. The worst case (largest signal-to-noise ratio) occurs at the lower band edge, decreases to zero at the approximate arithmetic mean of the data band and then increases to a threshold value at the upper band edge.

When the data band is heterodyned up in frequency, the narrow-band effect of the noise on the frequency of the signal results in a reduced error in measuring frequency. The measured frequency, however, assumes an infinite averaging time. Sampling over finite time intervals will result in a spread of the measured frequency about the average value. If the process of heterodyning does not result in an increased spread in measured frequency, then it would appear that frequency translation will reduce measurement errors.

APPENDIX: Average Number of Zero Crossings of Signal Plus Noise Shaped by a Bandpass Filter

The average number of zero crossings is given by expression (7):

$$\overline{N_{s+n}} = \frac{1}{\pi}\left[\frac{\omega_o^2 \frac{\overline{Q^2}}{2} + D_o}{\frac{\overline{Q^2}}{2} + R_o}\right]^{\frac{1}{2}} ,$$

where:

$$D_o = \int_0^{\infty} \omega^2 \, G(\omega) \, d\omega \quad ,$$

$$R_o = \int_0^{\infty} G(\omega) \, d\omega \quad .$$

Consider the voltage transfer function of an RLC bandpass filter which shapes the noise power spectrum.

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = Y(s) \quad , \qquad \text{(A-1)}$$

$$Y(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}\, s + \frac{1}{LC}} \qquad \text{(A-2)}$$

$$Y(j\omega) = \frac{\omega_c^2}{-\omega^2 + 2\beta j\omega + \omega_c^2} \quad , \quad 2\beta = \frac{R}{L} \quad , \quad \omega_c = \frac{1}{LC} \qquad \text{(A-3)}$$

$$|Y(j\omega)|^2 = \frac{\omega_c^4}{(\omega_c^2-\omega^2)^2 + 4\beta\omega^2} \quad , \tag{A-4}$$

$$\begin{aligned} G(\omega) &= |Y(j\omega)|^2 \, G_n(\omega) \quad ; \\ G_n(\omega) &= K \text{ volts}^2/\text{rps} \end{aligned} \tag{A-5}$$

$$D_o = \int_0^\infty \omega^2 \, G(\omega) \, d\omega \quad , \tag{A-6}$$

$$D_o = K\omega \int_0^\infty \frac{\omega^2 \, d\omega}{(\omega_c^2-\omega^2)^2 + 4\beta\omega^2} \quad , \tag{A-7}$$

See Reference 4, Table 19, No. 7.

$$\int_0^\infty \frac{x^2 \, dx}{(p^2+q^2)^2 + 2(p^2-q^2)\, x^2 + x^4} = \frac{\pi}{4p} \quad . \tag{A-8}$$

$$\begin{aligned} (\omega_c^2-\omega^2)^2 + 4\beta\omega^2 &= \omega^4 + (4\beta-2\omega_c^2)\,\omega^2 + \omega_c^4 \\ (p^2+q^2) &= \omega_c^4 \\ 2(p^2-q^2) &= 4\beta - 2\omega_c^2 \\ p^2 + q^2 &= \omega_c^2 \\ p^2 - q^2 &= \beta/2 - \omega_c^2 \end{aligned}$$

$$p = \frac{\beta^{\frac{1}{2}}}{2}$$

$$q^2 = \omega_c^2 - p^2 = \omega_c^2 - \beta/4 \quad , \tag{A-9}$$

$$D_o = \frac{\pi K \omega_c^4}{2\beta^{\frac{1}{2}}} \quad , \tag{A-10}$$

$$R_o = \int_0^\infty G(\omega)\, d\omega \quad , \tag{A-11}$$

$$R_o = K\omega_c^4 \int_0^\infty \frac{d\omega}{(\omega_c^2-\omega^2)^2 + 4\beta\omega^2} \quad , \tag{A-12}$$

See Reference 4, Table 19, No. 6.

$$\int_0^\infty \frac{dx}{(p^2+q^2)^2 + 2(p^2-q^2)\, x^2 + x^4} = \frac{1}{4p}\, \frac{\pi}{p^2 + q^2} \quad , \tag{A-13}$$

$$p^2 = \beta/4 \quad ;$$

$$p^2 + q^2 = \omega_c^2$$

$$R_o = \frac{\pi K \omega_c^2}{2\beta^{\frac{1}{2}}} \quad , \tag{A-14}$$

Defining the signal-to-noise power ratio as:

$$\rho = \frac{\frac{\overline{Q^2}}{2}}{R_o} \quad ,$$

and from the previous results:

$$\frac{D_o}{R_o} = \omega_c^2 \quad ,$$

$$\overline{N_o} = \frac{1}{\pi}\left(\frac{\rho\omega_o^2 + \omega_c^2}{\rho + 1}\right)^{\frac{1}{2}} \quad , \tag{A-16}$$

$$\overline{N_o} = 2f_o\left(\frac{\rho + r^2}{\rho + 1}\right)^{\frac{1}{2}} \quad ;$$

$$r = \frac{\omega_c}{\omega_o} \quad . \tag{A-17}$$

The above results indicate that the average number of crossings of the axis of signal plus noise is independent of the shape characteristics of the spectrum of the noise, but does depend upon the relative displacement of the center of the noise spectrum and signal.

REFERENCES

1. H. Steinberg, P.M. Schultheiss, C.A. Wagrin, and F. Zweiz. "Short-Time Frequency Measurements of Narrow-Band Random Signals by Means of Zero Counting Process," J. Appl. Phys., vol. 26 (February 1955), pp. 195-201.

2. J.S. Bendat. "Principles and Applications of Random Noise Theory."

3. S.O. Rice. "Statistical Properties of a Sine Wave Plus Random Noise," Bell System Technical Journal (1948) 27, 1, 109.

4. Bierens DeHaan. Nouvelles Tables, Hafner Publishing Co. (1957).

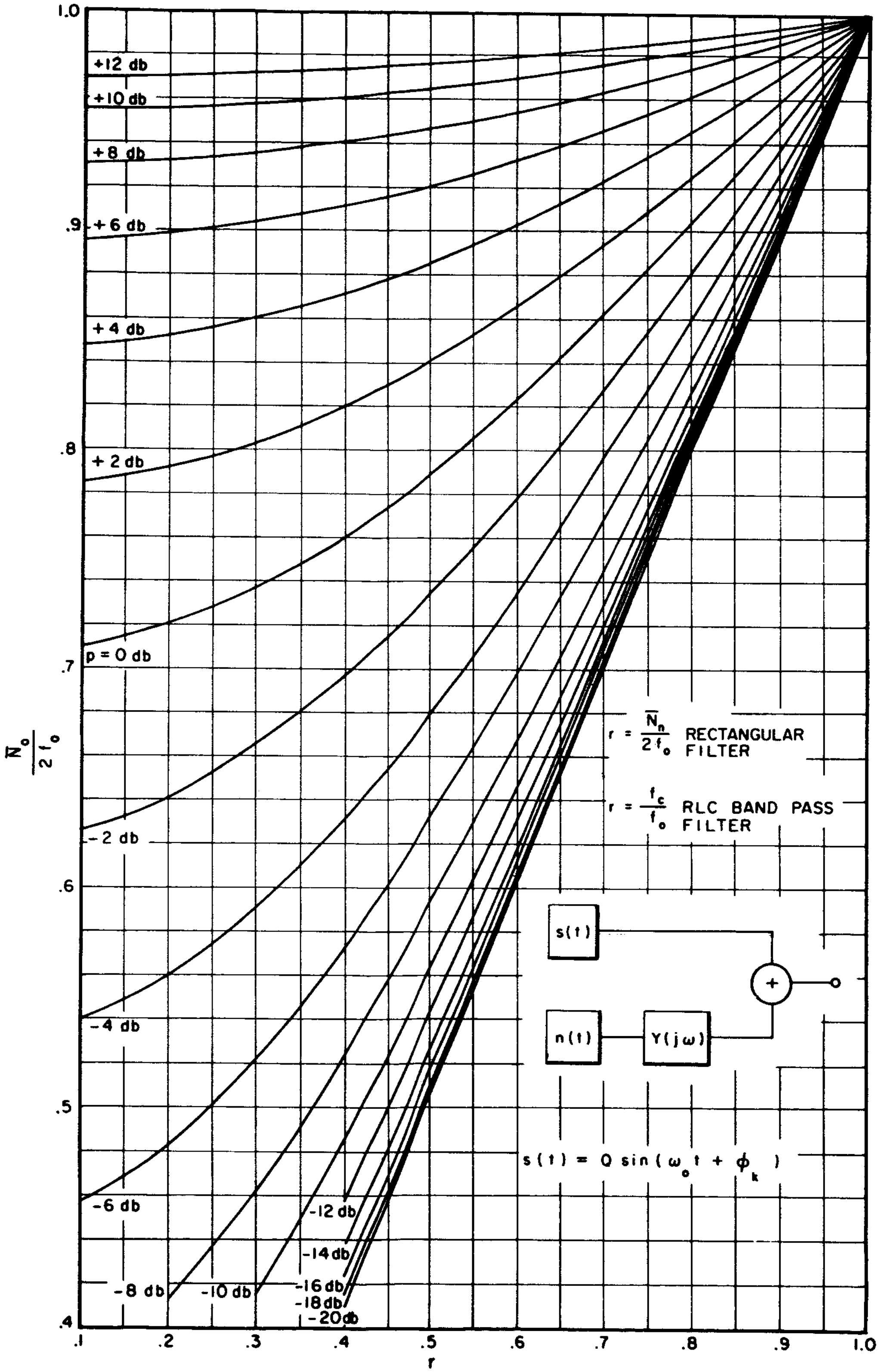


FIG. 1

PLOT OF $\frac{\bar{N}_o}{2f_o} = \left(\frac{\rho + r^2}{\rho + 1} \right)^{\frac{1}{2}}$

ρ = SIGNAL-TO-NOISE POWER RATIO; $r < 1$

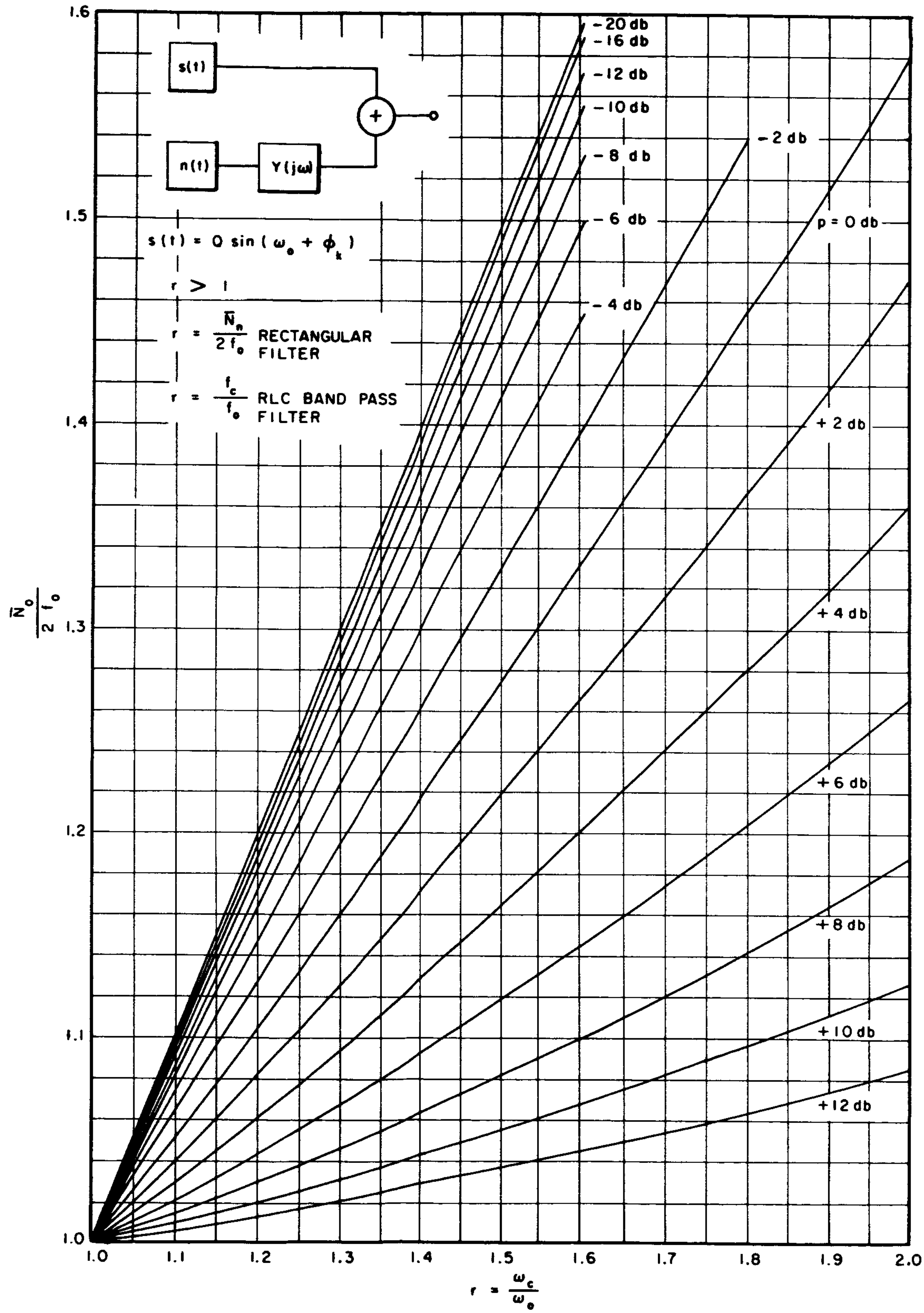


FIG. 2

PLOT OF $\frac{\bar{N}_0}{2 f_0} = \left(\frac{\rho + r^2}{\rho + 1} \right)^{\frac{1}{2}}$

ρ = SIGNAL-TO-NOISE POWER RATIO $r < 1$

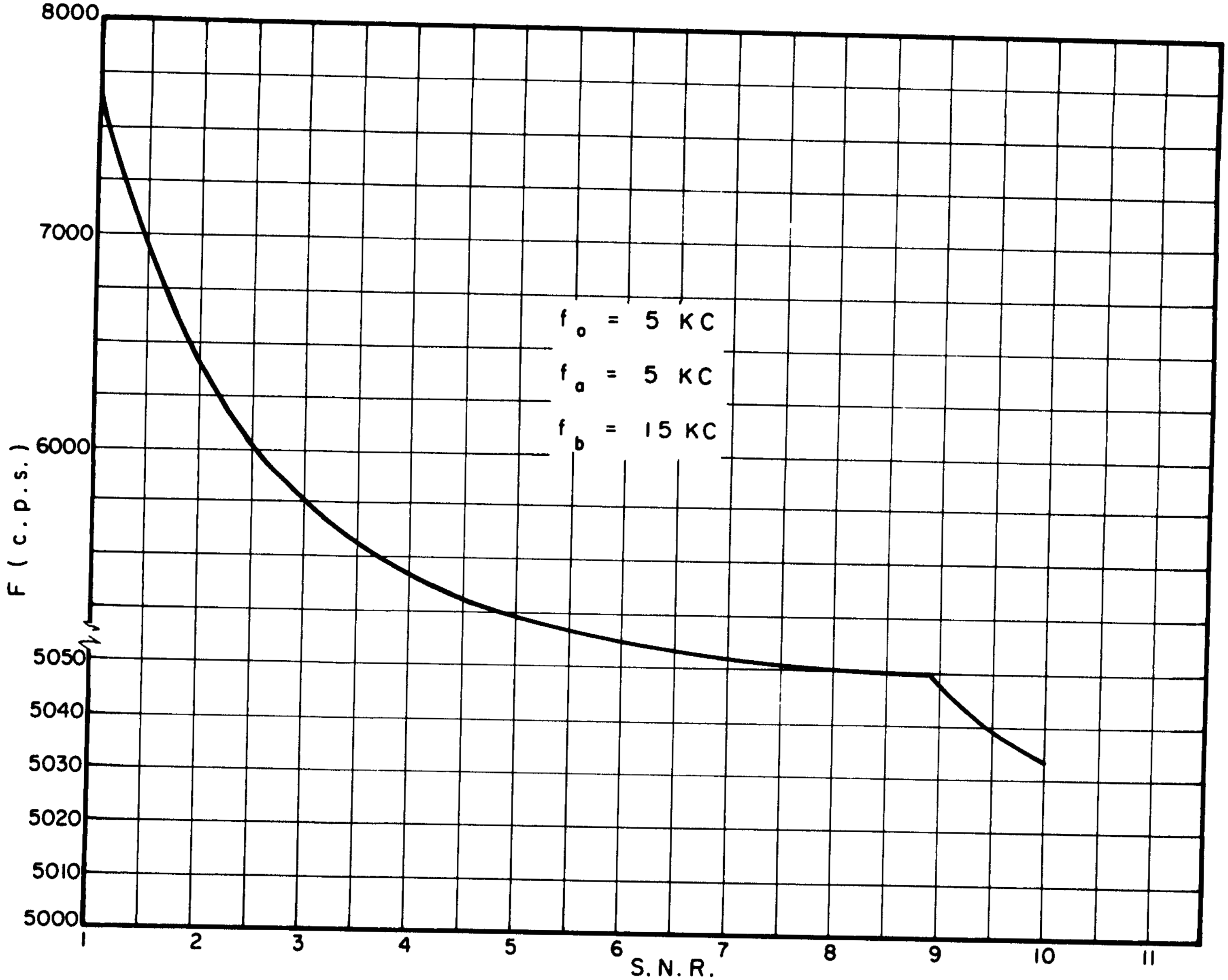


FIG. 3

AVERAGE FREQUENCY MEASURED VS S.N.R.

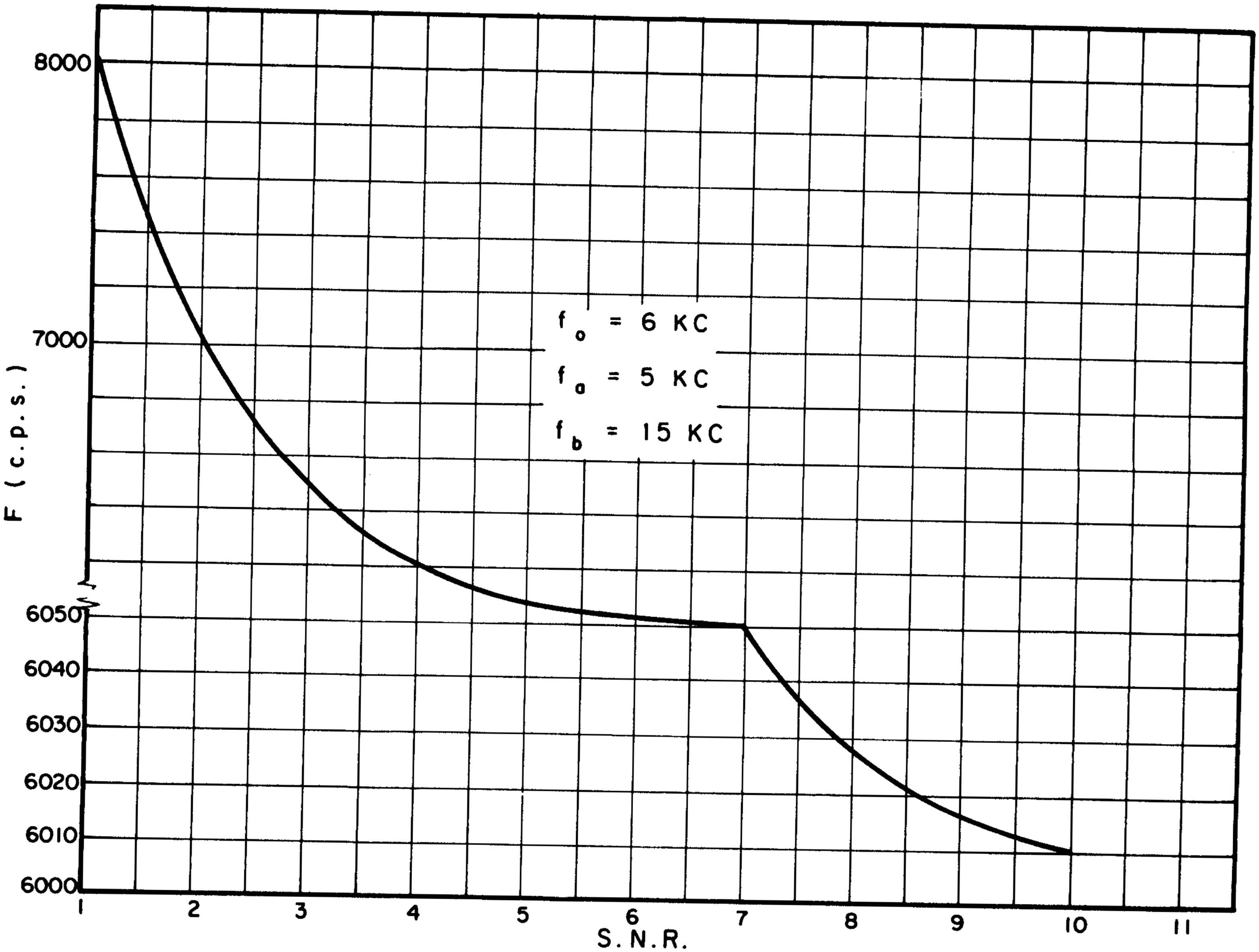


FIG. 4

AVERAGE FREQUENCY MEASURED VS S.N.R.

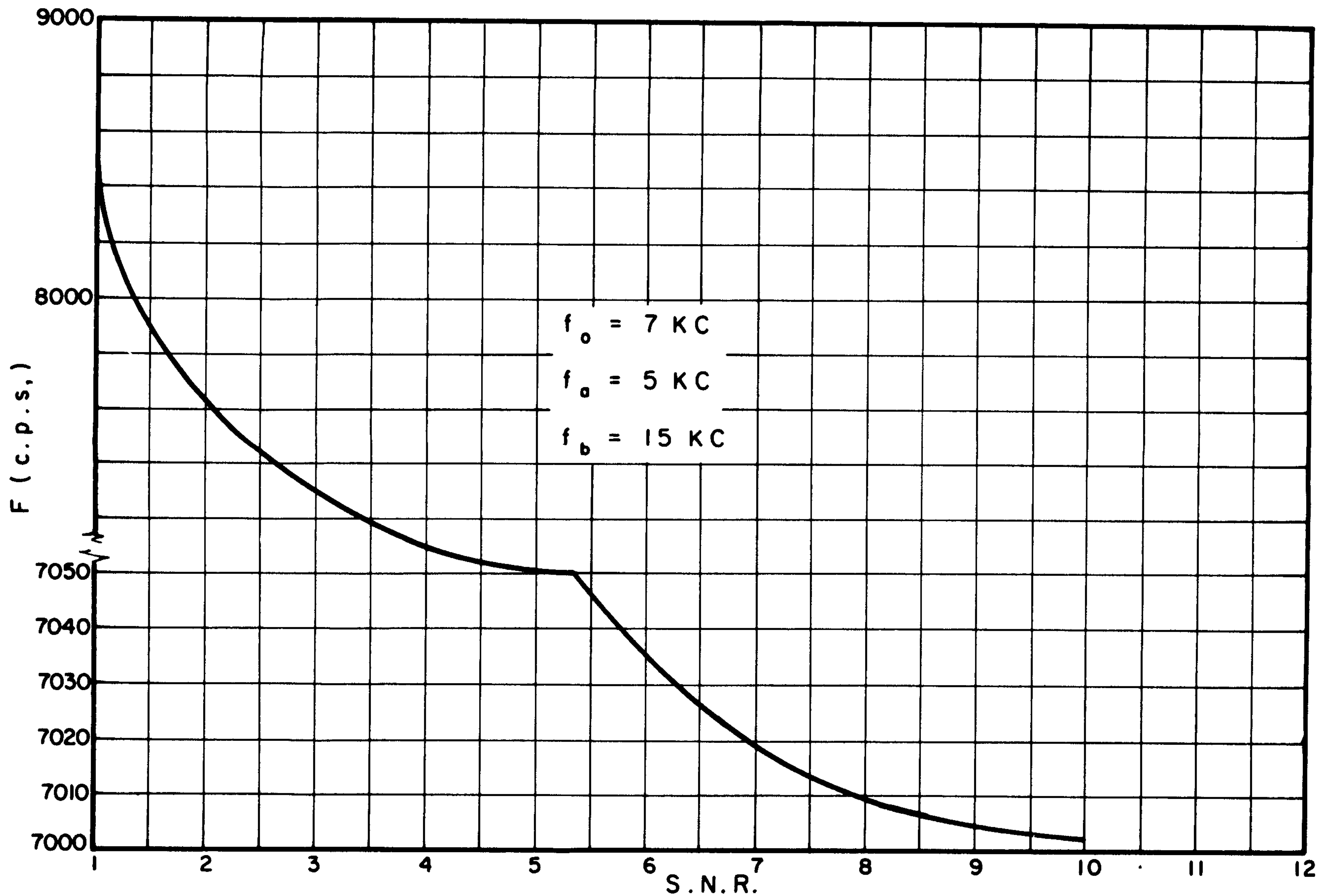
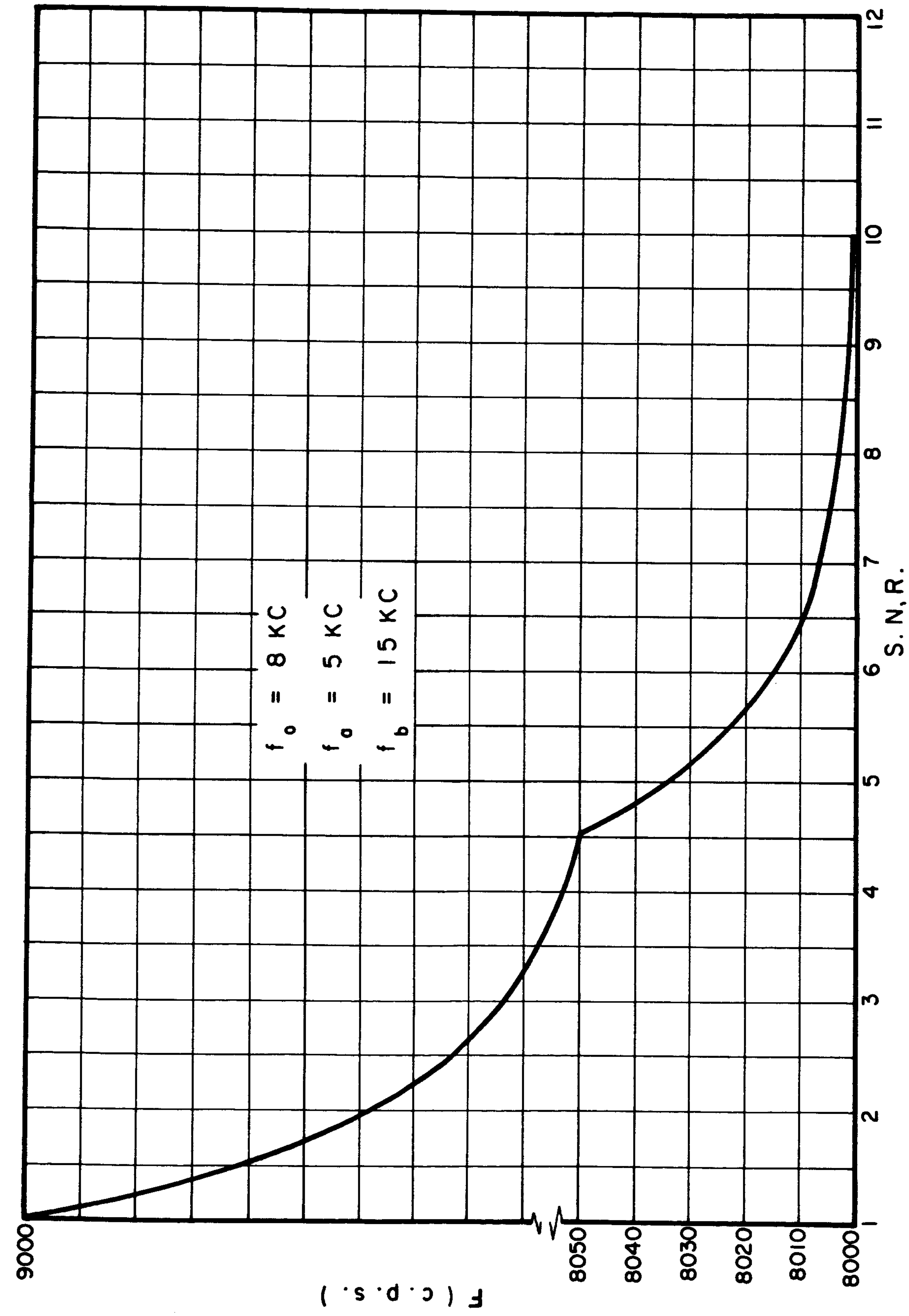


FIG. 5

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 6

AVERAGE FREQUENCY MEASURED VS S.N.R.

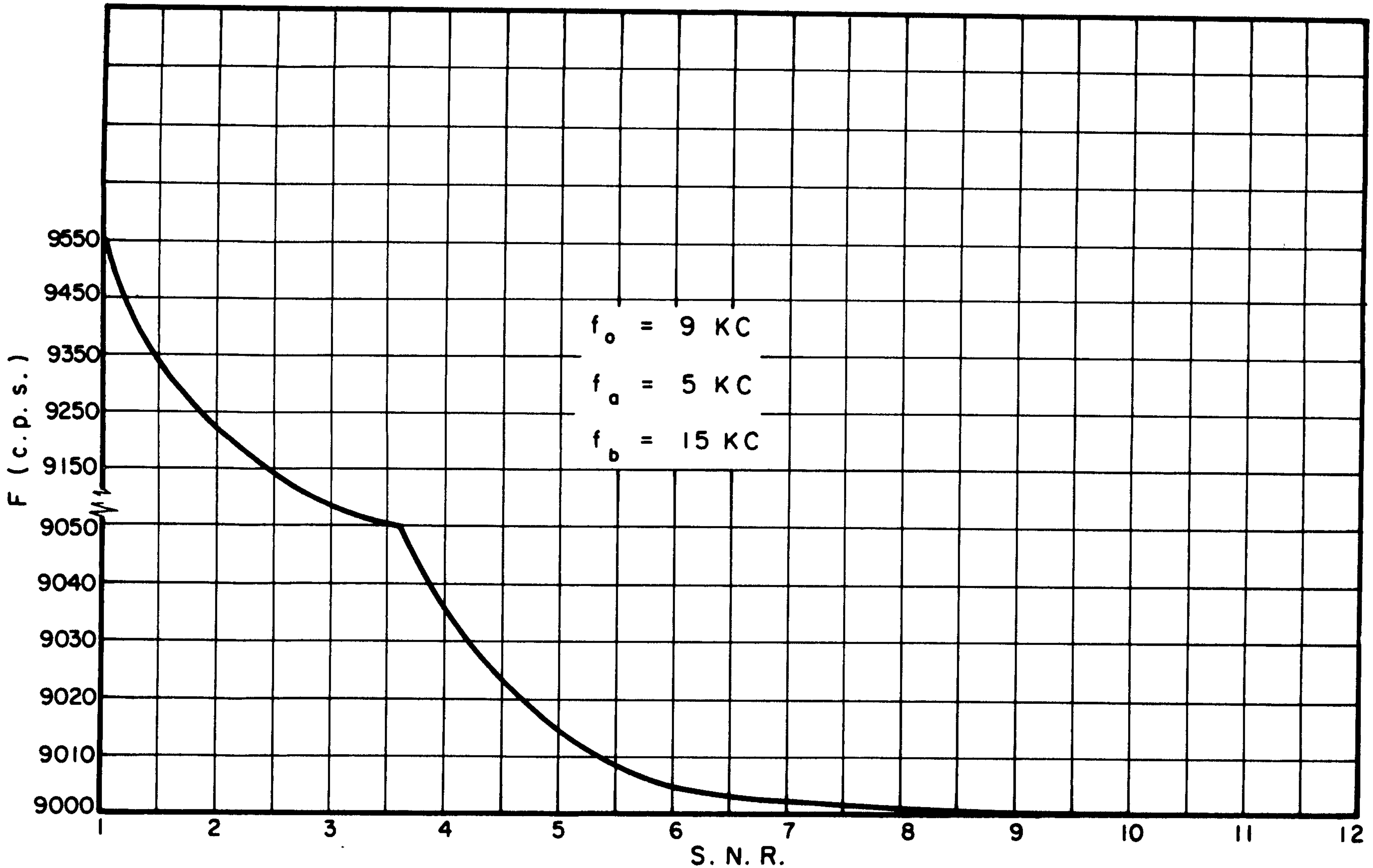


FIG. 7

AVERAGE FREQUENCY MEASURE VS S.N.R.

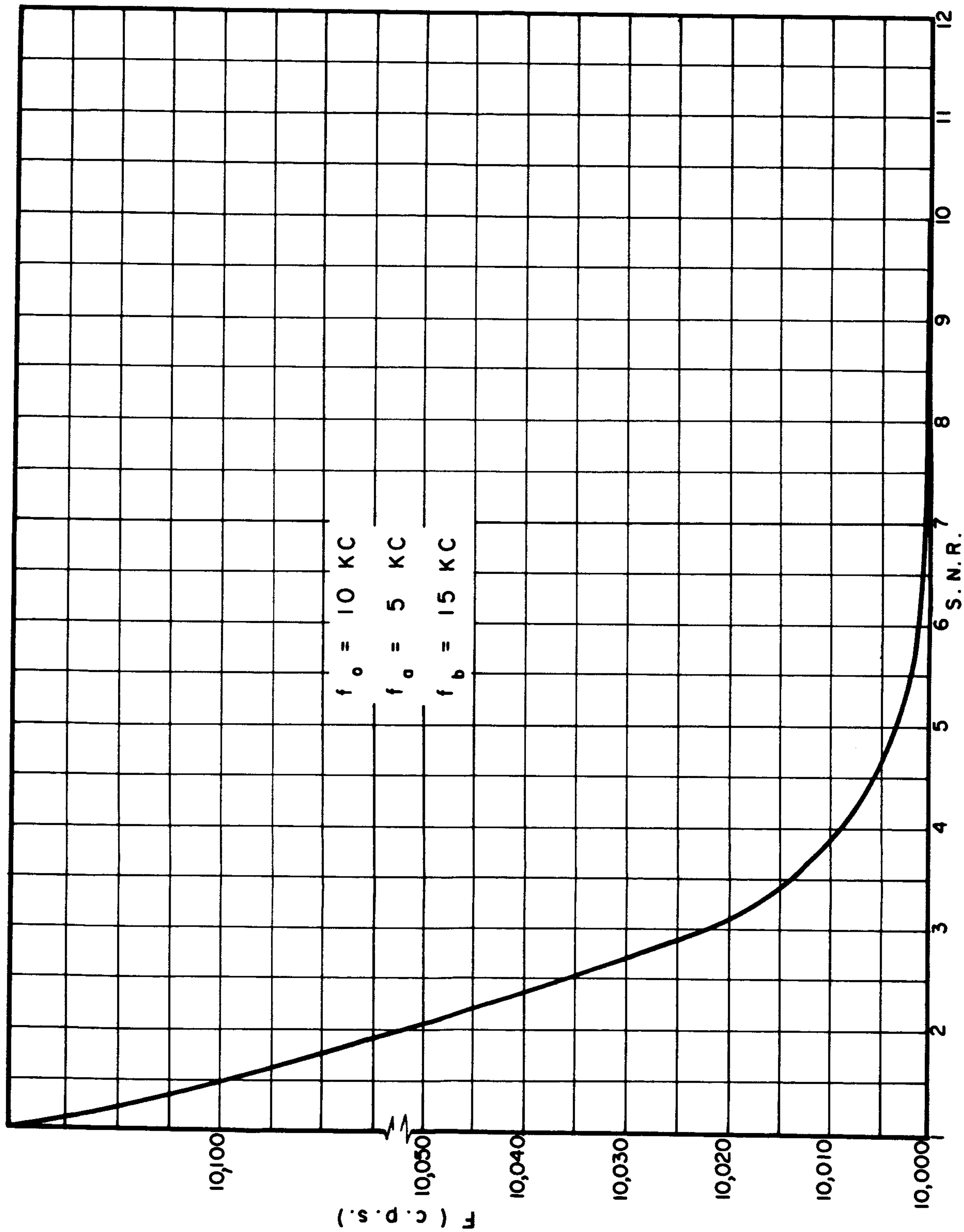


FIG. 8

AVERAGE FREQUENCY MEASURED VS S.N.R.

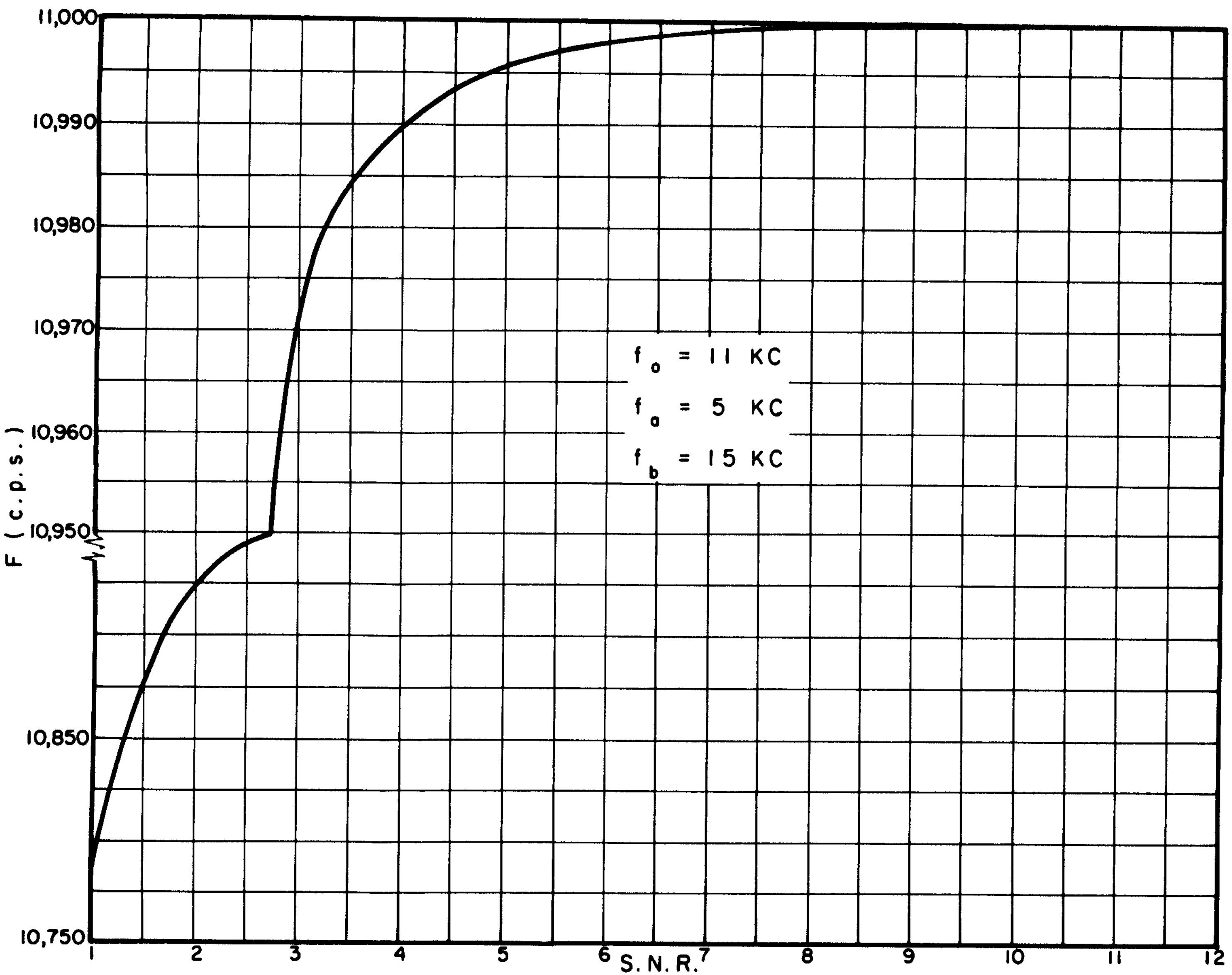


FIG. 9

AVERAGE FREQUENCY MEASURED VS S.N.R.

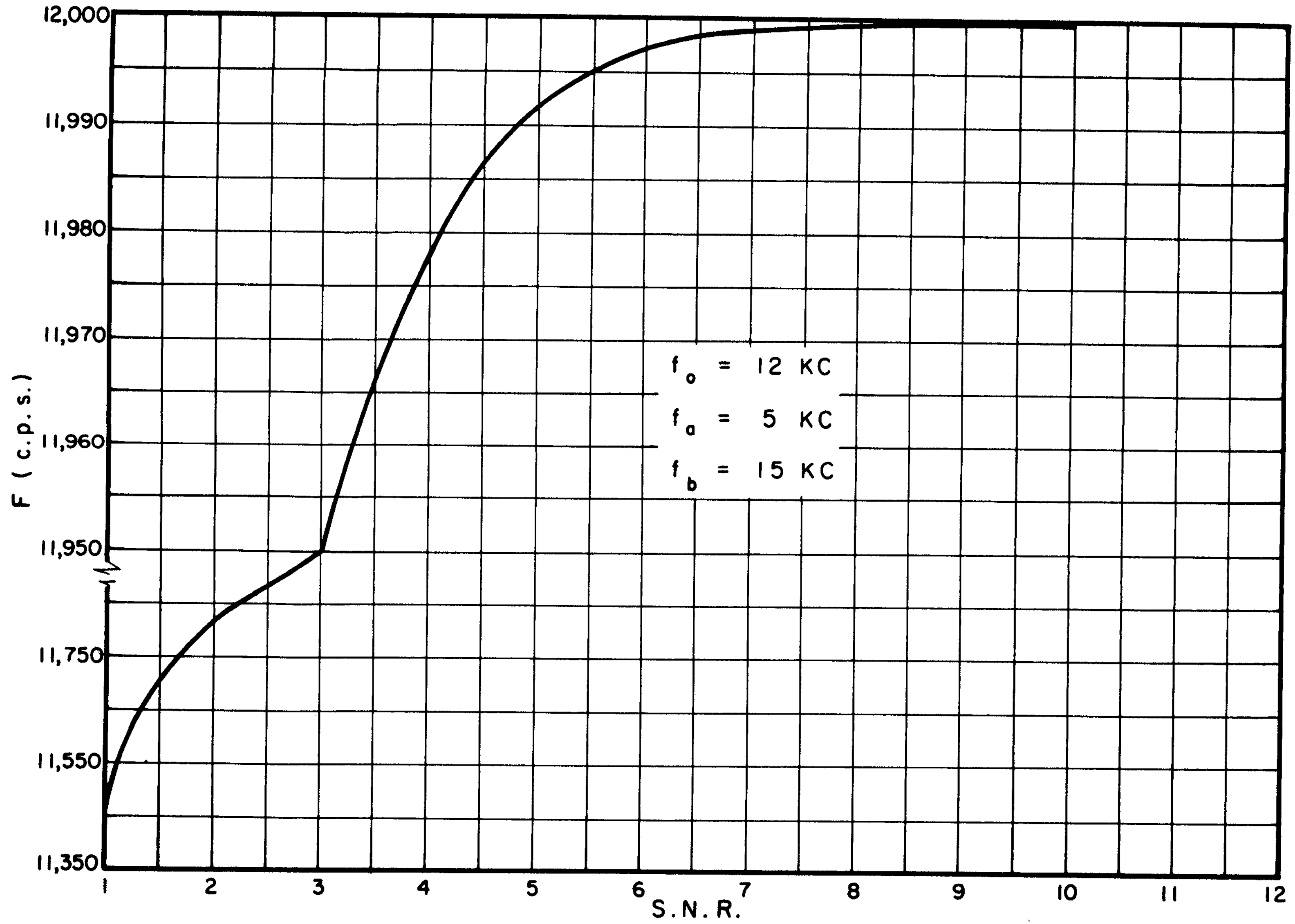
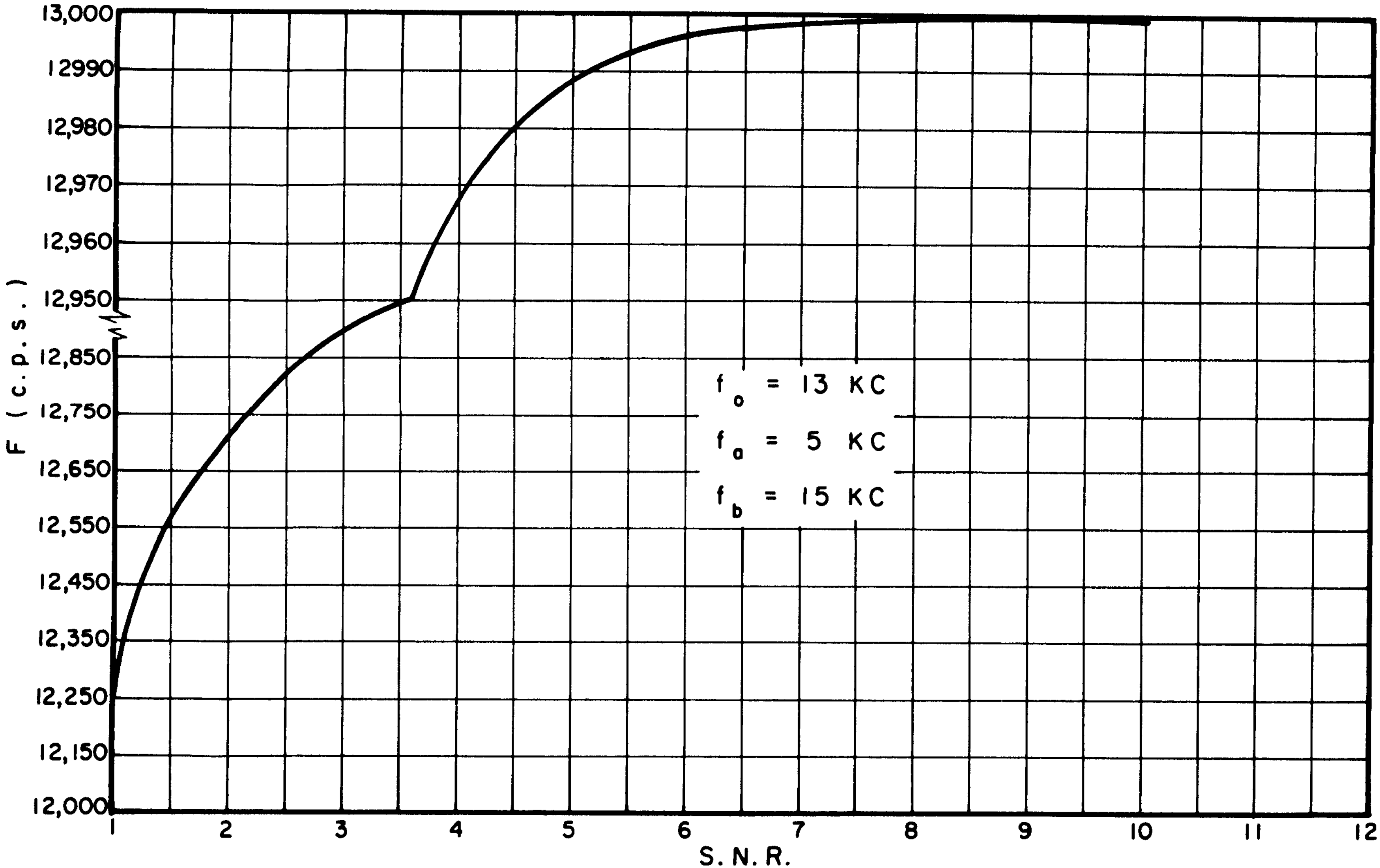


FIG. 10

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 11

AVERAGE FREQUENCY MEASURED VS S.N.R.

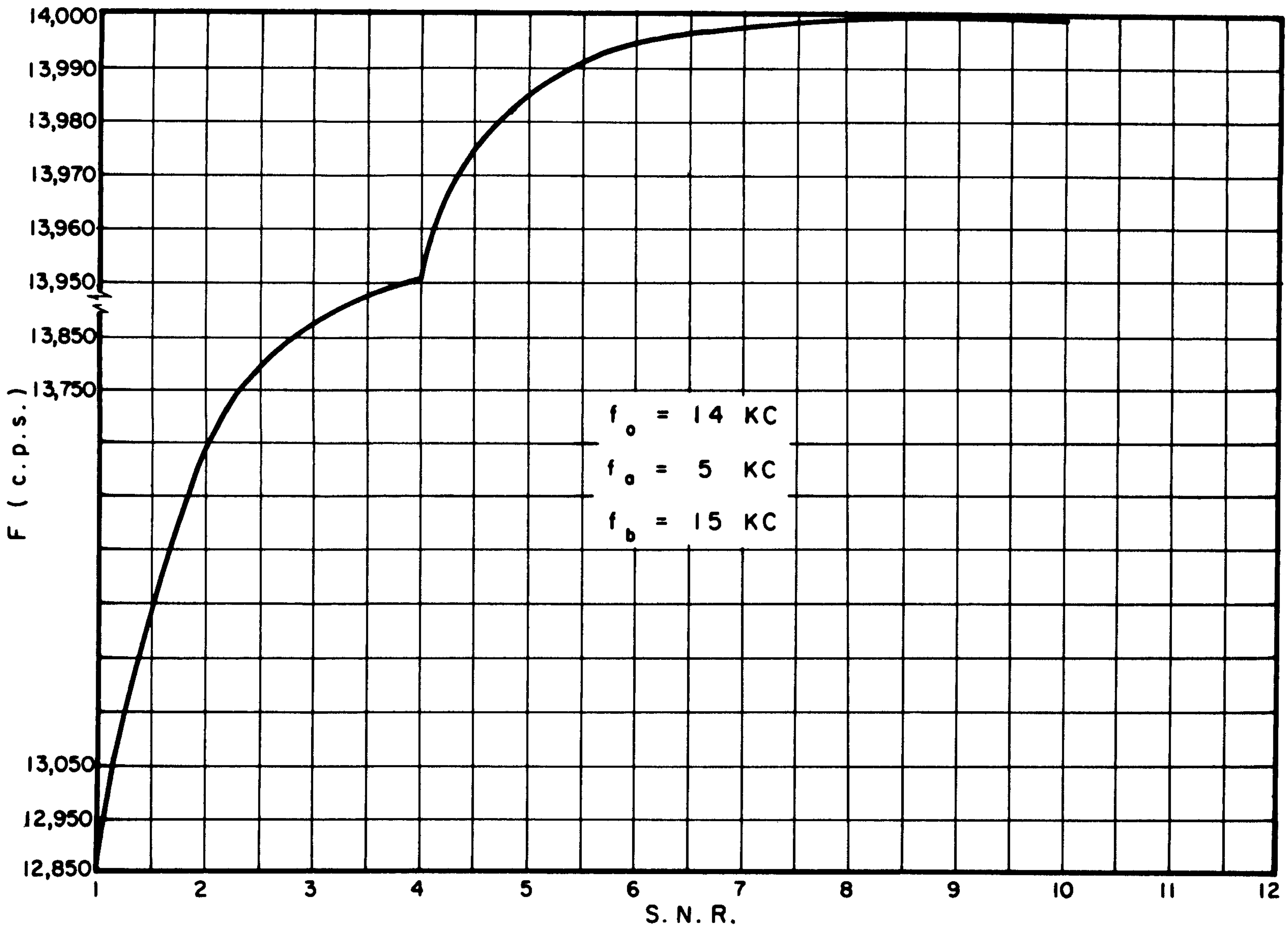


FIG. 12
AVERAGE FREQENCY MEASURED VS S.N.R.

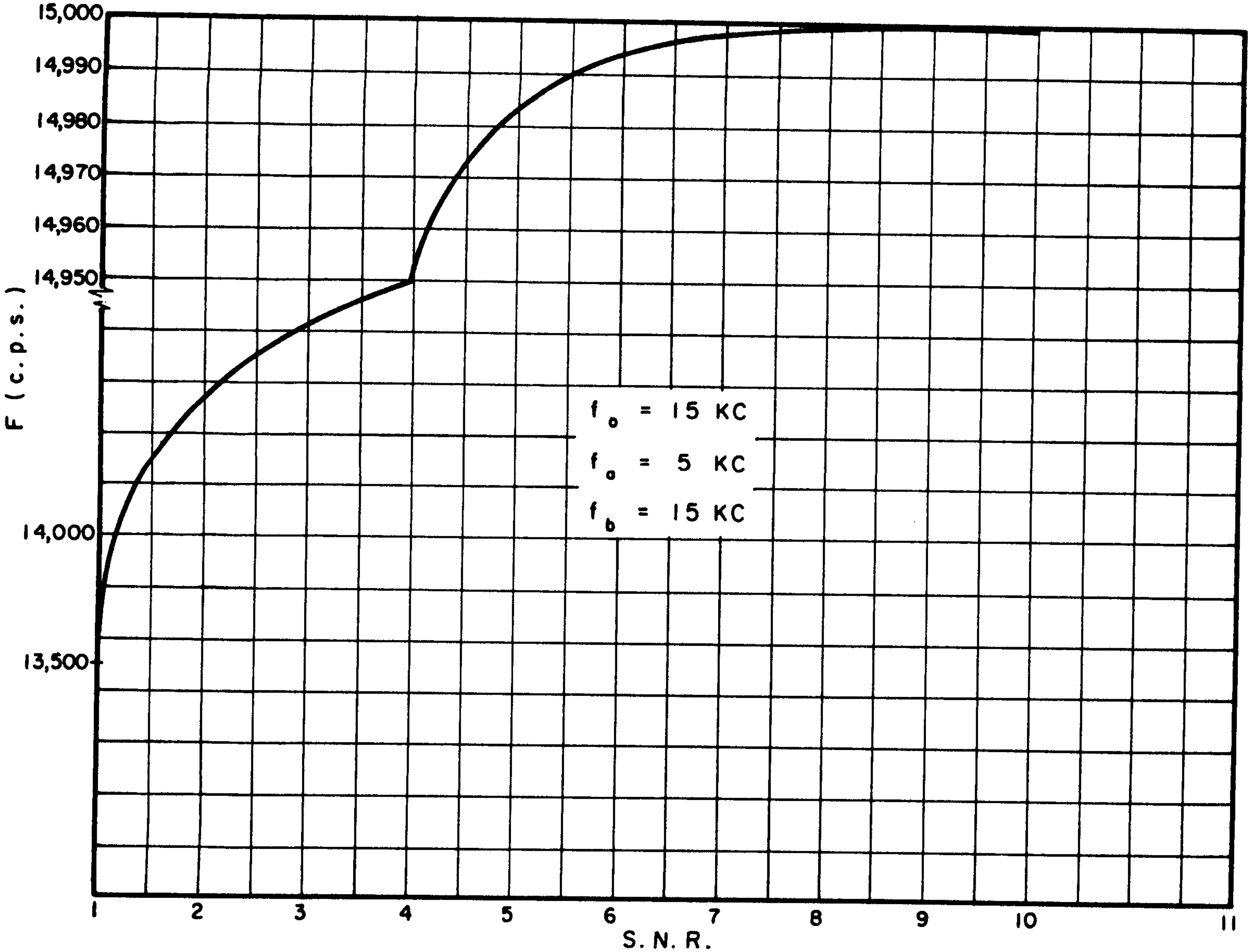
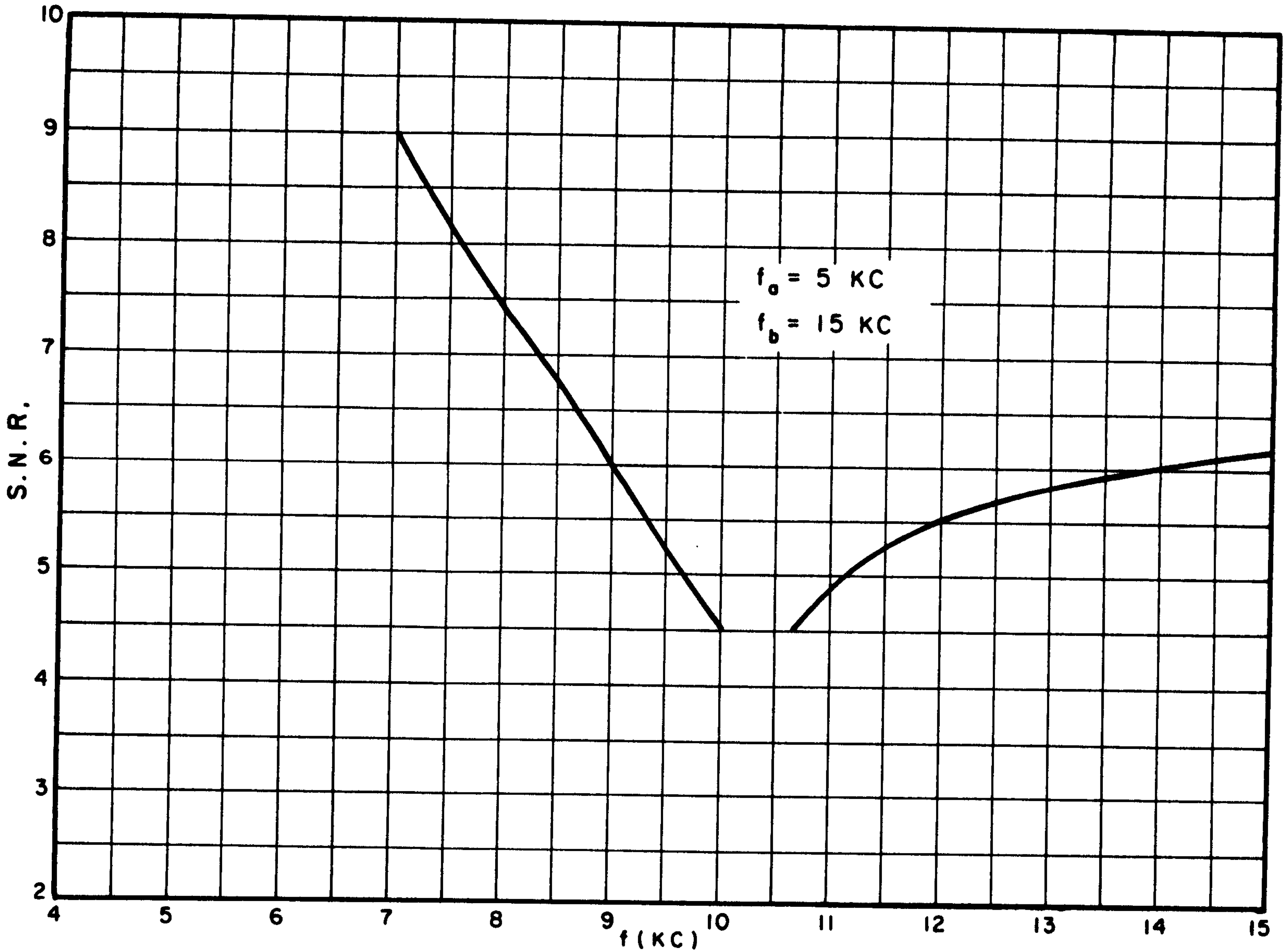


FIG. 13

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 14

S.N.R. VS FREQUENCY FOR CONSTANT ERROR.
AVE. MEASURED FREQUENCY DEVIATION 5 c/s

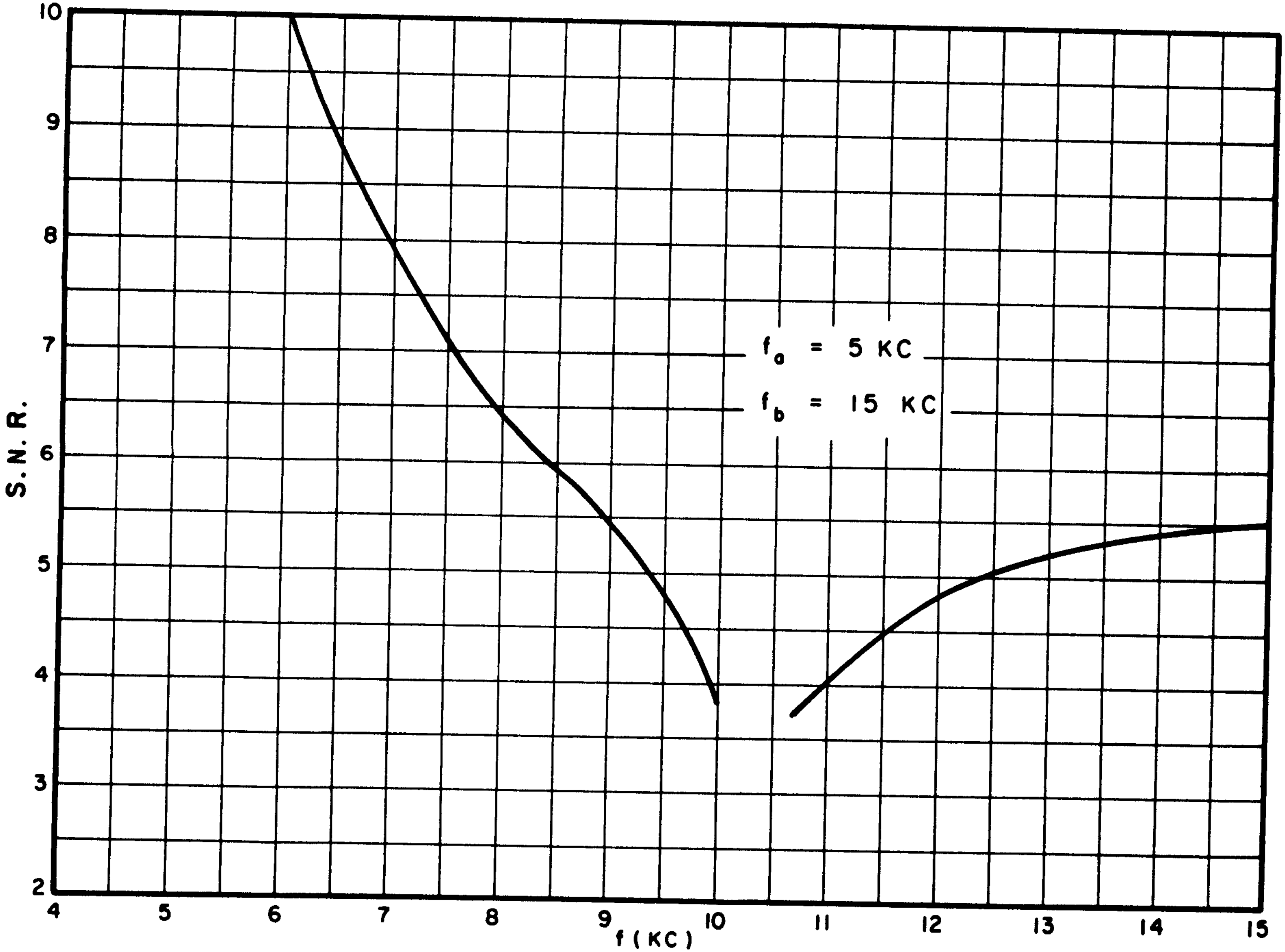


FIG. 15

S.N.R. VS FREQUENCY FOR CONSTANT ERROR.
AVE. MEASURED FREQUENCY DEVIATION 10 c/s

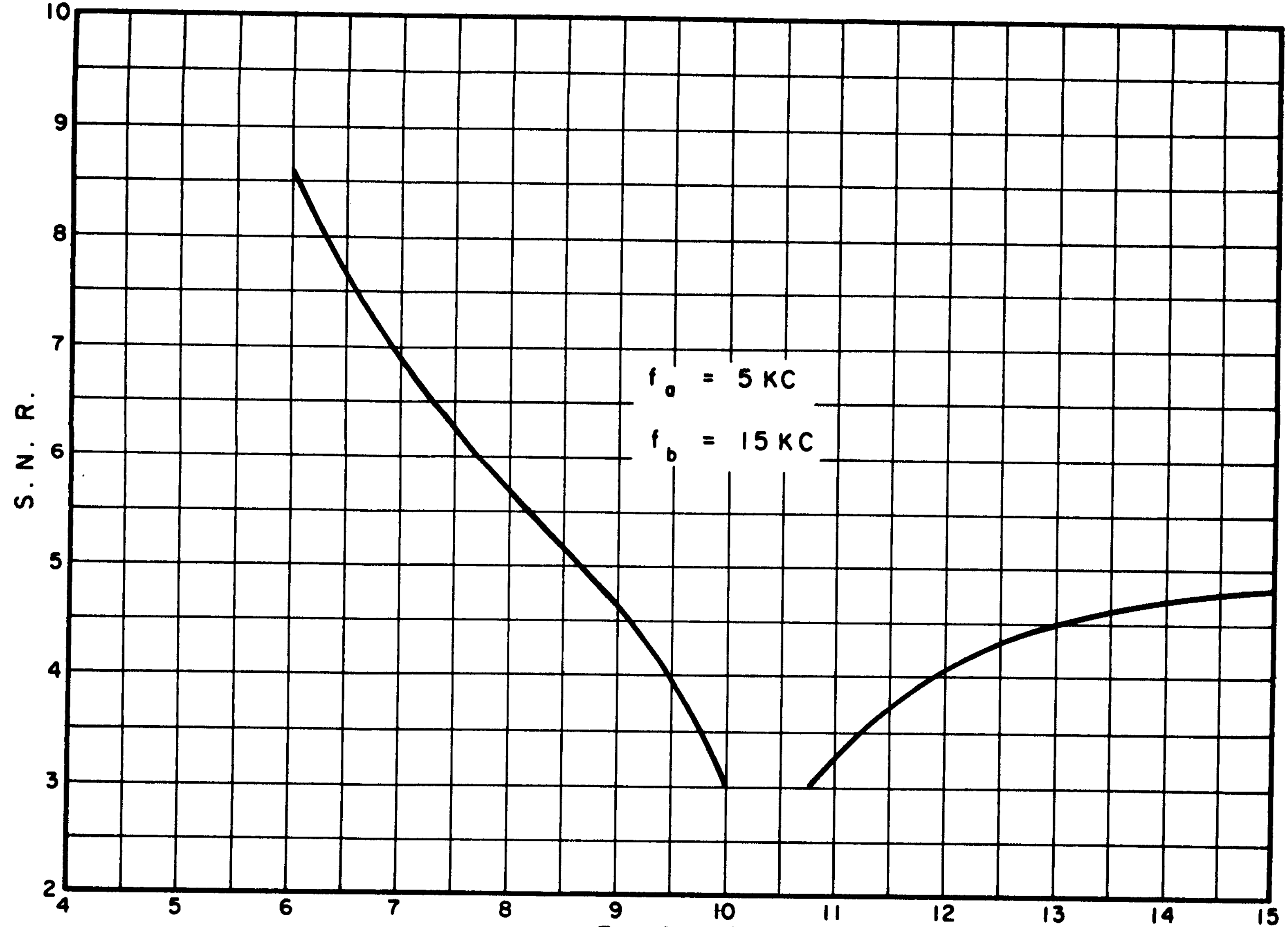


FIG. 16

S.N.R. VS FREQUENCY FOR CONSTANT ERROR
AVE. MEASURED FREQENCY DEVIATION 20 c/s

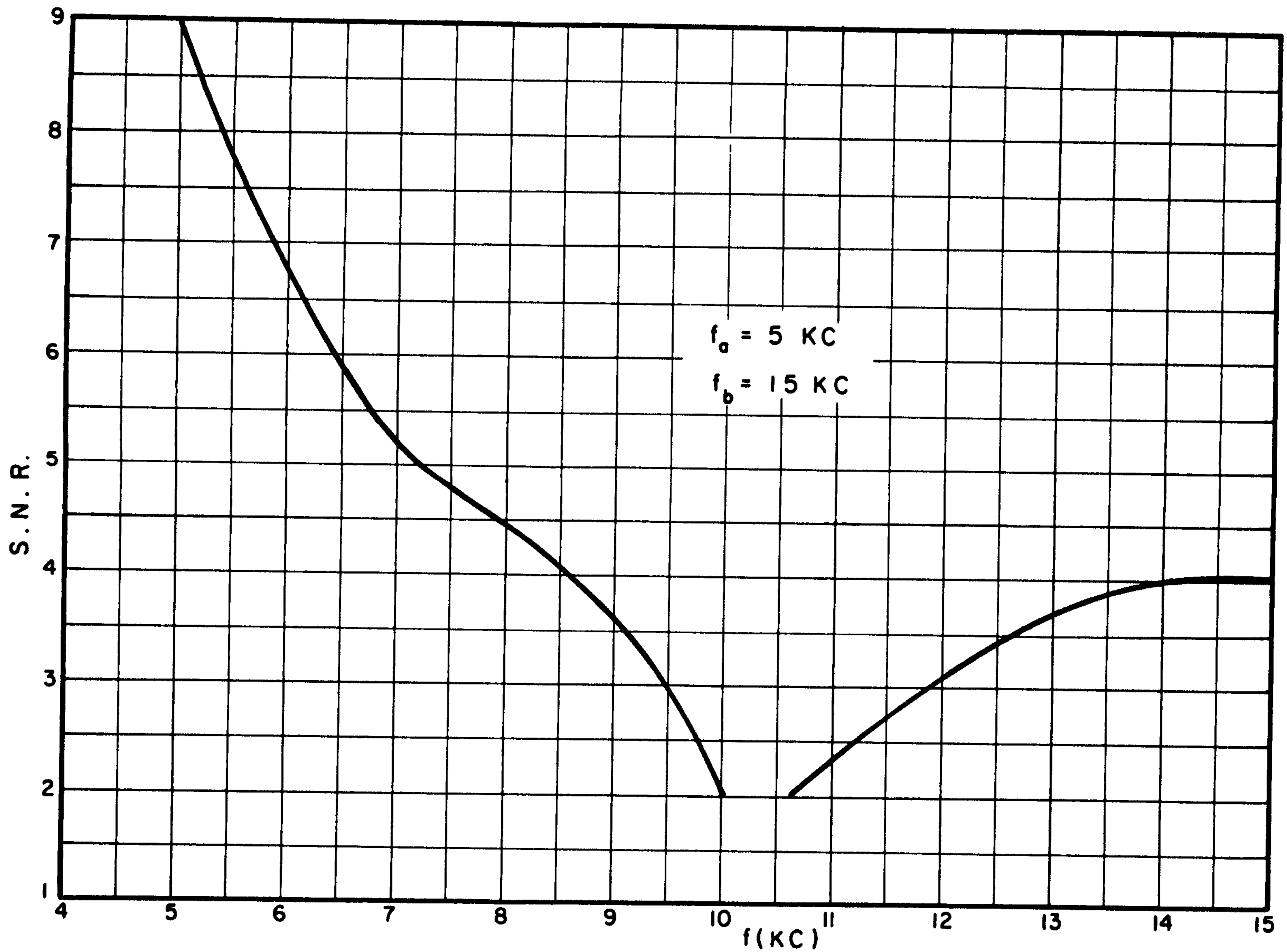


FIG. 17

S.N.R. VS FREQUENCY FOR CONSTANT ERROR.
AVE. MEASURED FREQUENCY DEVIATION 50 c/s

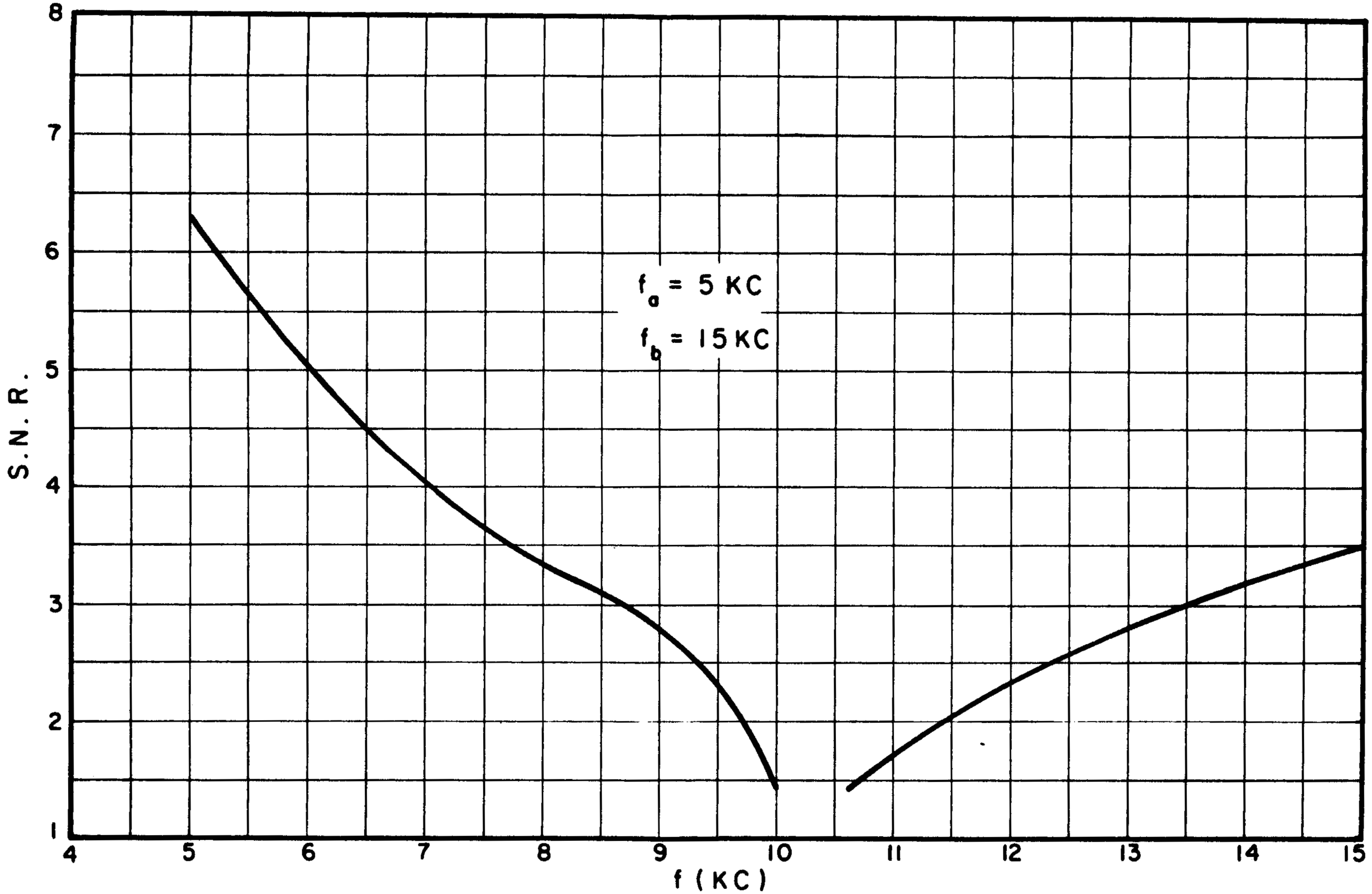


FIG. 18

S.N.R. VS FREQUENCY FOR CONSTANT ERROR.
AVE. MEASURED FREQENCY DEVIATION 100 c/s

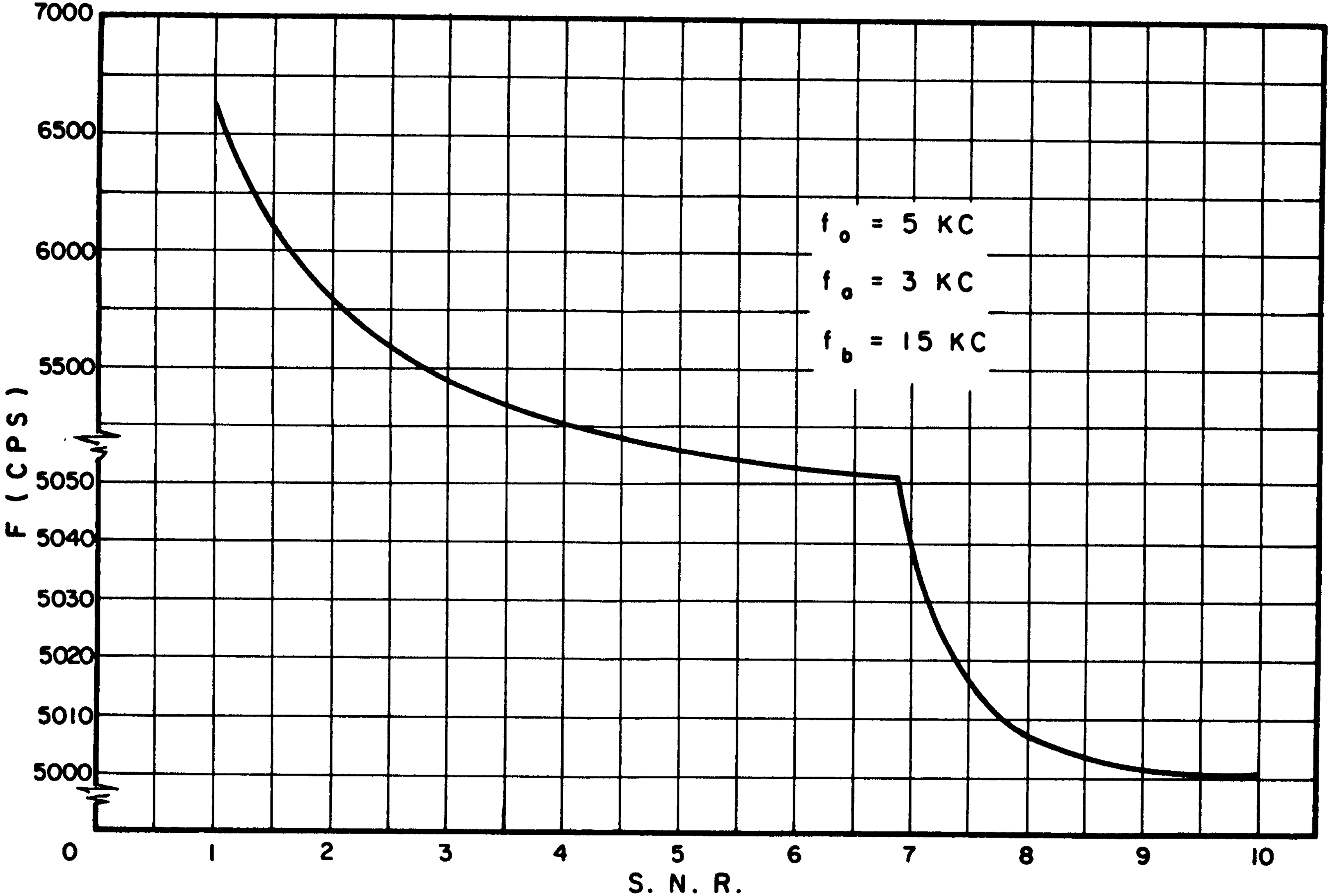
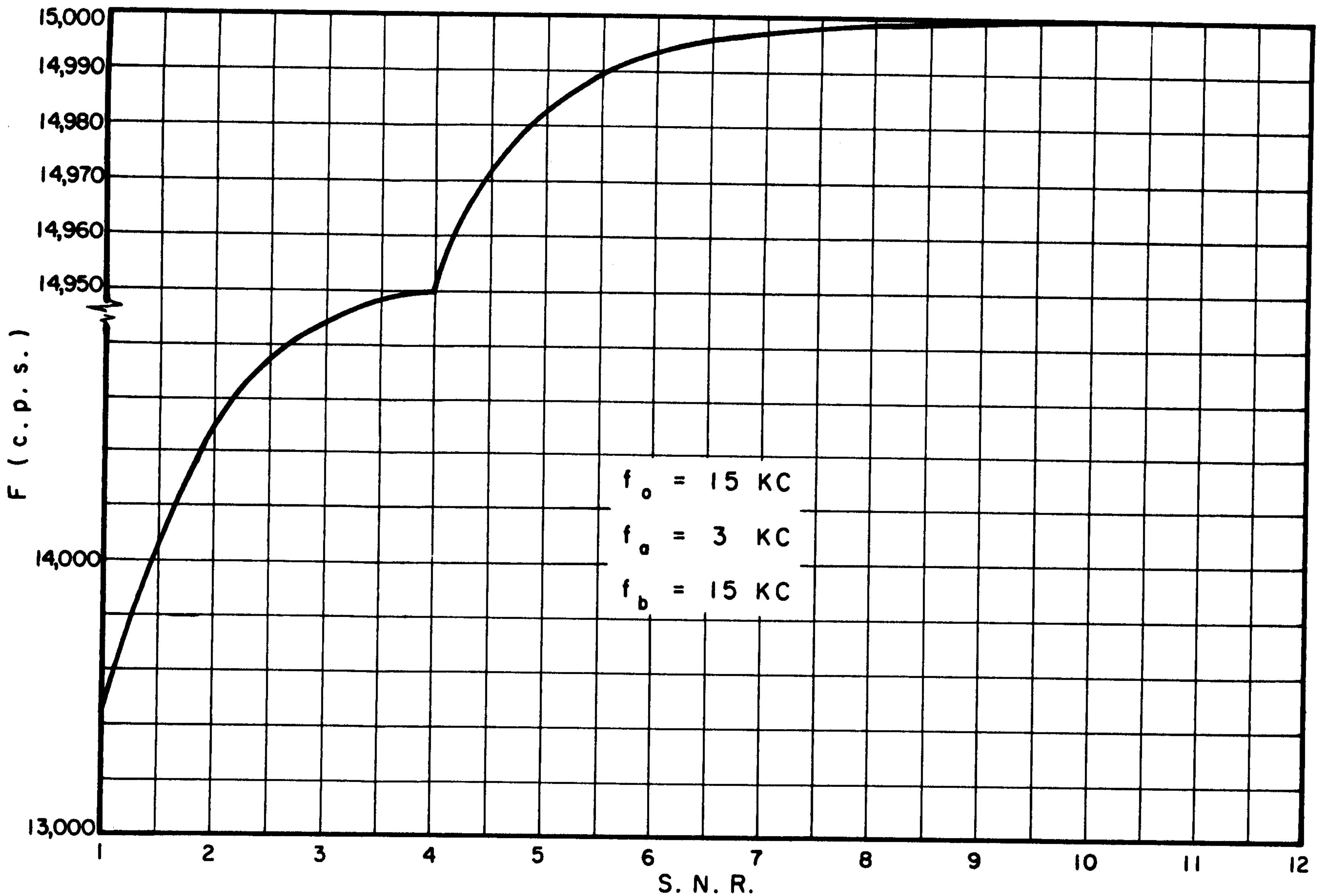


FIG. 19

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 20

AVERAGE FREQUENCY MEASURED VS S.N.R

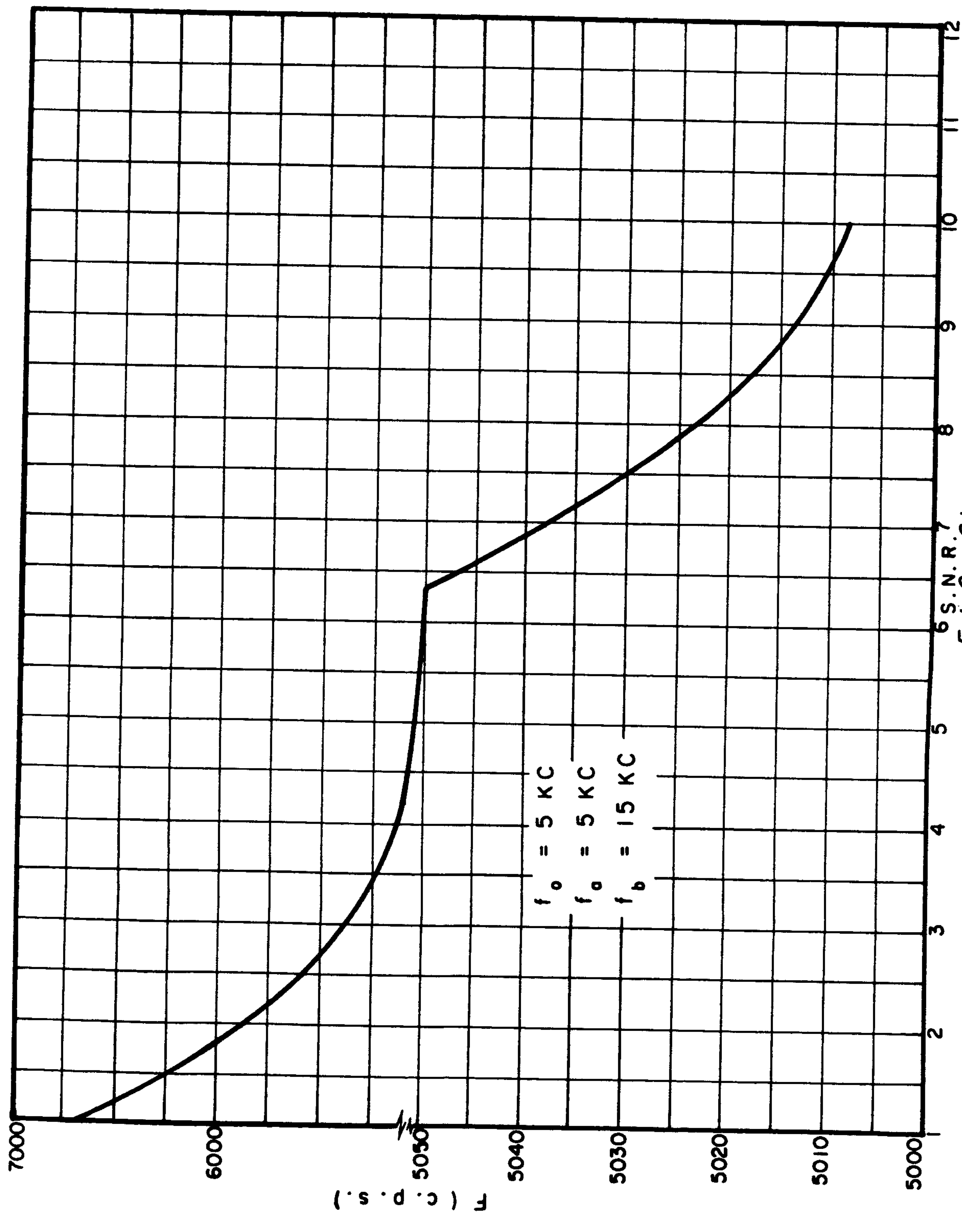
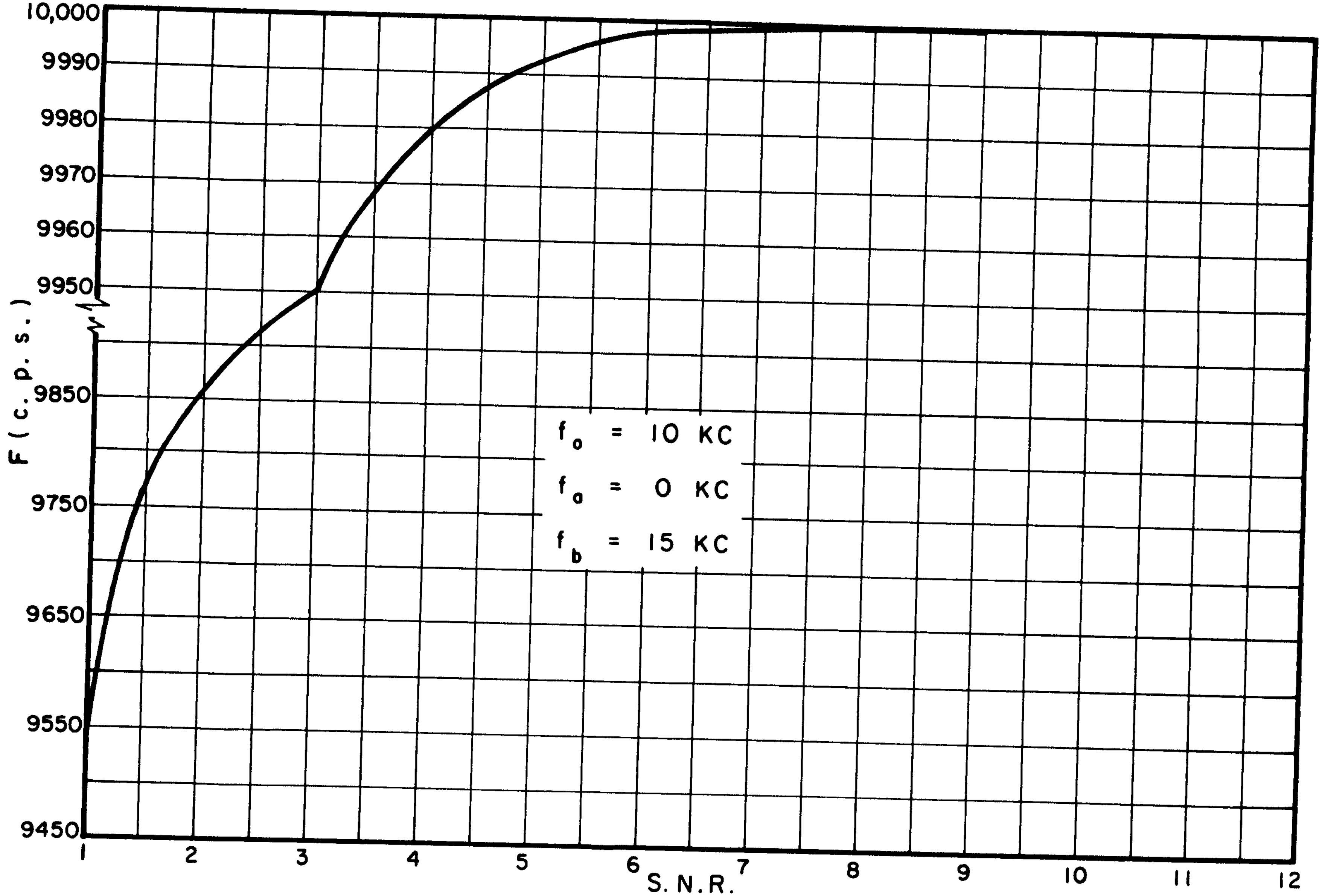


FIG. 21

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 22
AVERAGE FREQUENCY MEASURED VS S.N.R.

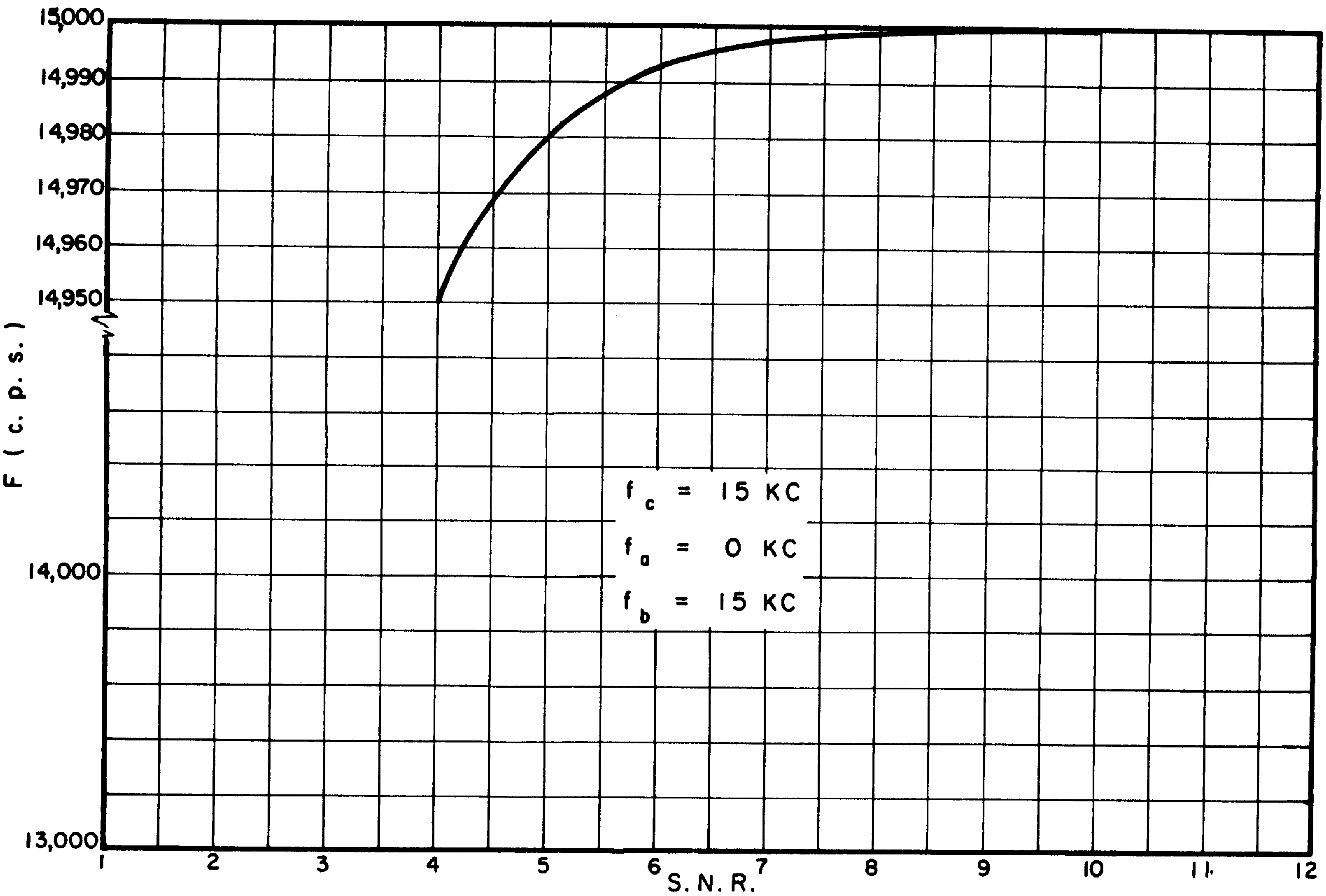
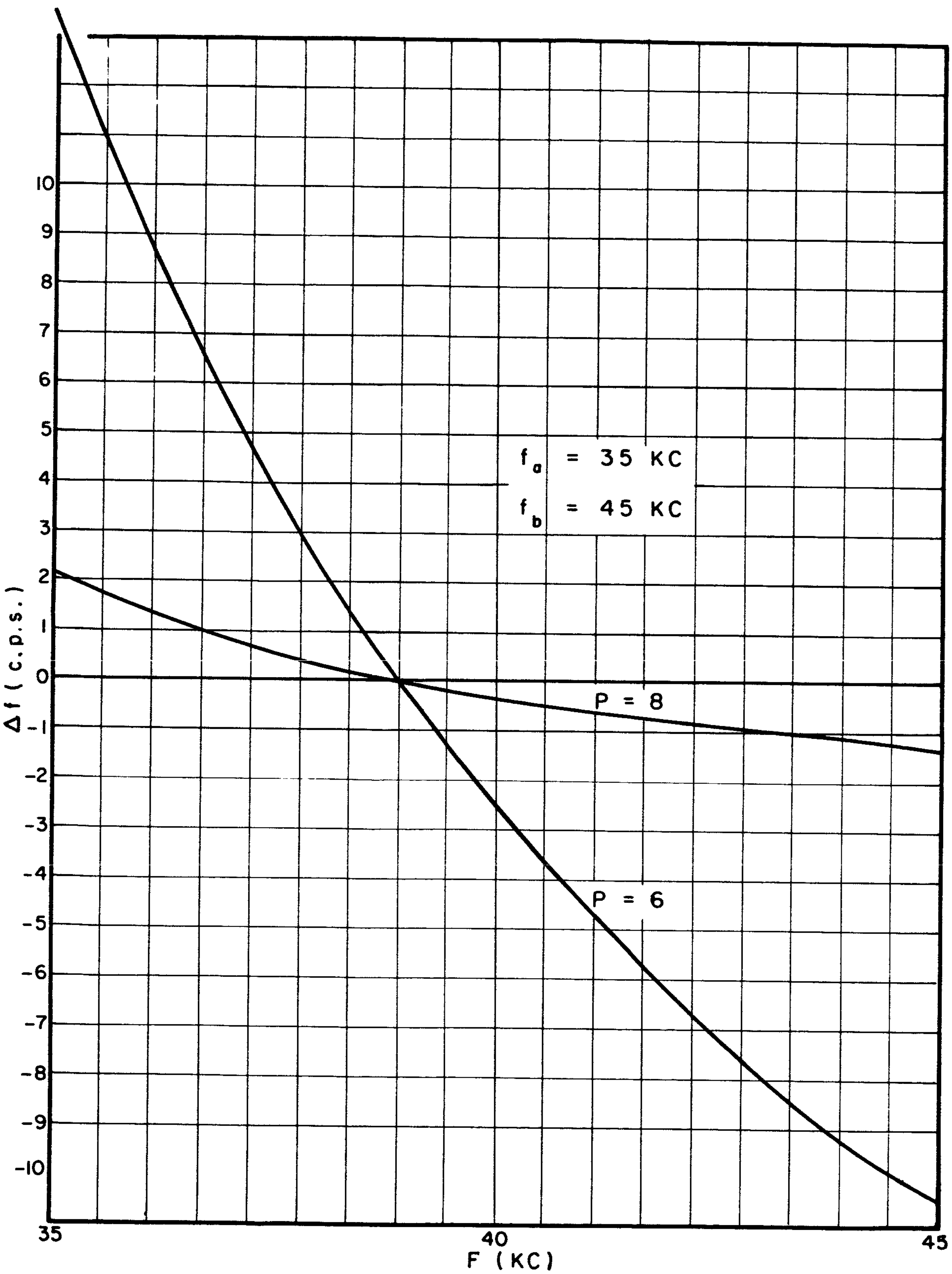


FIG. 23

AVERAGE FREQUENCY MEASURED VS S.N.R.



F I G. 24

FREQUENCY ERROR VS FREQUENCY WITH CONSTANT S.N.R.

```
C     THE EVALUATION OF THE NUMBER OF ZERO CROSSINGS
  110 READ 100, AFO,SNR
  100 FORMAT(F10.4,F10.4)
      XN=10408.333
      DELTA=2.*AFO/XN
      DELSQ=DELTA*DELTA
      ALPHA=SNR*(1.+DELSQ)/2.
      BETA=SNR*(1.-DELSQ)/2.
      ALAMB=2.*DELSQ/(1.+DELSQ)
      SFO=1.
      IO=1
      AK=(1.-DELSQ)/(1.+DELSQ)
      TEX=EXPF(-ALPHA)
      XO=BETA/2.
      TPO=10.
      SUMO=1.
   30 FIO=IO
      XPO=XO**(2*IO)
      SFO=SFO*FIO
      DIO=SFO*SFO
      TIO=XPO/DIO
      SUMO=SUMO+TIO
      DIIO=ABSF(TPO-TIO)
      IF(DIIO-.000001)32,31,31
   31 TPO=TIO
      IO=IO+1
      GO TO 30
   32 DELX=ALPHA/99.
      B=AK
      X=ALPHA
      U=0.
      RES=0.
      DO 18 M=1, 33
    4 DO16 K=1,4
      TP=10.
      XD=B*U/2.
      SF=1.
      SUM=1.
      I=1
    2 FI=I
      XP=XD**(2*I)
      SF=SF*FI
      D=SF*SF
      T=XP/D
      SUM=SUM+T
      DI=ABSF(TP-T)
      IF(DI-.000005)9,1,1
    1 TP=T
      I=I+1
      GO TO 2
    9 BES=SUM
      EX=EXPF(-U)
      ARG= BES*EX
      IF(K-2)14,13,13
   13 IF(K-3)15,15,14
   14 RES=RES+ARG
      GO TO 16
   15 RES=RES+3.*ARG
   16 U=U+DELX
      PUNCH 400,U,RES,ALPHA
  400 FORMAT(2HU=F8.3,4HRES=F12.6,6HALPHA=F10.6)
   18 U=U-DELX
   20 RINT=3.*DELX/8.*RES
      ZC=.5*XN*(TEX*SUMO+ALAMB*RINT)
      PUNCH120,AFO,SNR,RINT,ZC
  120 FORMAT(4HAFO=F12.4,4HSNR=F10.4,5HRINT=F9.5,3HZC=F11.4)
      GO TO 110
      END
```

FIG. 25

COMPUTER PROGRAM FOR COMPUTATION OF ZERO CROSSINGS

TABLE 1

AVERAGE FREQUENCY MEASURED VS. S.N.R.

$f_a = 5KC$, $f_b = 15KC$

f_o(KC)	SNR	f(cps)	f_o(KC)	SNR	f(cps)	f_o(KC)	SNR	f(cps)
5	1	7647.6042	9	1	9555.5548	13	1	12151.171
	2	6368.6252		2	9220.1368		2	12717.844
	3	5747.8672		3	9087.6552		3	12905.020
	4	5430.7860		4	9035.0678		4	12967.684
	5	5260.0673		5	9014.0492		5	12988.898
	6	5163.3565		6	9005.6964		6	12996.149
	7	5105.9838		7	9002.3056		7	12998.645
	8	5070.5673		8	9000.9364		8	12999.514
	9	5047.9578		9	9000.3184		9	12999.832
	10	5033.1329		10	9000.1491		10	12999.944
6	1	8041.3441	10	1	10153.2070	14	1	12869.963
	2	6982.7636		2	10057.9980		2	13634.571
	3	6492.8553		3	10021.5940		3	13879.285
	4	6257.4367		4	10008.1060		4	13959.448
	5	6139.7461		5	10003.0410		5	13986.197
	6	6078.5476		6	10001.1360		6	13995.247
	7	6095.5067		7	10000.4910		7	13998.340
	8	6027.0453		8	10000.1490		8	13999.420
	9	6016.4204		9	10000.0510		9	13999.808
	10	6010.1464		10	10000.0120		10	13999.955
7	1	8494.1444	11	1	10788.3290	15	1	13608.491
	2	7671.3354		2	10924.2100		2	14559.845
	3	7309.6753		3	10972.8470		3	14856.480
	4	7146.7817		4	10990.2530		4	14952.173
	5	7071.4766		5	10996.4940		5	14983.803
	6	7035.7246		6	10988.730		6	14994.442
	7	7018.2936		7	10999.5320		7	14998.076
	8	7009.5776		8	10999.822		8	14999.335
	9	7005.1115		9	10999.925		9	14999.792
	10	7002.7739		10	10999.962		10	14999.980
8	1	9000.7068	12	1	11455.876			
	2	8421.1804		2	11812.921			
	3	8179.7198		3	11935.338			
	4	8077.7991		4	11977.537			
	5	8034.1791		5	11992.185			
	6	8015.2413		6	11997.242			
	7	8006.8941		7	11999.019			
	8	8003.1623		8	11999.642			
	9	8001.4712		9	11999.864			
	10	8000.6876		10	11999.952			

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