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VOLUME I OF II

A STATISTICAL ANALYSIS OF GYRO DRIFT TEST DATA

by

John Richard Cooper, Lieutenant, Royal Navy

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EXPERIMENTAL ASTRONOMY LABORATORY

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VOLUME I

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M.A. (Cantab)

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Submitted to the Department of Aeronautics and Astronautics in September 1965, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

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The drift rate data derived from a manufacturer's testing of 50 inertial grade gyroscopes, is analysed to derive a statistical model for use as a predictor.

The initial hypothesis of a simple Random Walk is shown to be a reasonable approximation to the actual data, and an explanation of the discrepancies suggests a more complex model, but basically of the same type.

The simpler Model is then used to generate "synthetic" gyro drift rate data as an input to a computer simulation of an inertial navigator, and the resulting system output errors are derived.

Thesis Supervisor: James E. Potter
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The availability of a computer simulation of an inertial system, used in another project, provided a very instructive application of the main results of the thesis, and the help given by the authors of this report is greatly appreciated – Mr. Bernard E. Blood (who suggested the thesis topic), and Mr. Joel B. Searcy. Also, in this application, thanks are due to Dr. David C. Cooper of the Carnegie Institute of Technology, for suggesting the program for generating the Model data.

The author further expresses his appreciation to the Royal Navy for the opportunity to do this graduate work; and to the U.S. Government Defense Department and the Manufacturer concerned, who made the gyro test data available.

Finally, the author wishes to record his gratitude to his family, without whose forbearance this work could not have been completed.

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NOMENCLATURE

Time

Δt_o	discrete data time interval = 1 hour = unit of time
$t_i = i \cdot \Delta t_o$	$i = 0, 1, 2 \cdots n$

Drift Rate

DR	Drift Rate
MSDR	Mean Squared Drift Rate
$DR(t_i)$	Drift Rate at the end of the i^{th} interval from the start of the test run.

Incremental Drift Rate

IDR	Incremental Drift Rate, defined as the difference between the Drift Rates at one hour intervals.
MSIDR	Mean Squared Incremental Drift Rate

$$IDR(t_i) = DR(t_i) - DR(t_{i-1})$$

Drift Acceleration

DA	Drift Acceleration, defined as $IDR/\Delta t_o$
MSDA	Mean Squared Drift Acceleration

$$A_i = IDR(t_i)/\Delta t_o$$

$E(x)$	Statistical Expectation of the random variable x
$Est(x)$	Estimate of $E(x)$
S^2	Estimate of the Variance V
S	Estimate of the Standard Deviation σ

A random process is defined as being Stationary, in this thesis, if the ensemble Mean and Mean Squared values are time invariant.

Only terms and equations not found in standard statistical texts will be explained in detail.

CHAPTER I

INTRODUCTION

1-1 Background

The output errors of modern inertial navigators are usually very good indicators of the quality of the gyroscopes in the system. Constant errors in the gyro output can be handled by a variety of techniques, but the problem of random drift errors is far more complex. A knowledge of the expected random errors is highly desirable for two reasons:

1) If the random errors can be "modelled", some insight may be gained as to the causes thereof, and their subsequent elimination by improved design.

2) If the future random errors can be predicted in a statistical manner, then valuable information is available to help answer such questions as:

- i) the optimum time for external reset in moving base systems.
- ii) the frequency of gyro calibration in fixed base systems, such as a missile in standby operation.

One such way of representing the random errors is to derive a statistical model, based on the analysis of a large quantity of gyro data.

1-2 Purpose of the Thesis

The purpose of this thesis is to analyse a set of gyro drift test data, taken from one particular type of single degree of freedom gyros, for one particular orientation – Input axis along the local vertical. This is a very useful orientation as it is a difficult one to apply self-compensation techniques to, and consequently the gyro output may drift (randomly) over long time intervals between corrections.

The interpretation of the developed models, as to the possible causes of the errors, is left as an open question, but the use of a model as a predictor is considered by its application as an input to a system simulation.

1-3 Summary of Conclusions

The initial hypothesis of the existence of a simple Random Walk model is found to be a reasonable assumption, when compared with the test data. A development of the simple model is shown to explain certain discrepancies with the test data, but the precise specification of this more complex model is not derived, because the data was not a sufficiently large representative sample of the statistical population.

1-4 Organization of the Thesis

The thesis is divided into two Volumes, Volume 1 develops the theory and tests the Random Walk hypothesis – all the results are presented in normalized form or with the classified scales deleted*. Volume 2 applies the simple model as a system input and presents the results of the simulation. Chapter 10 identifies the gyro type, and the specific units from which the data was derived, together with the "missing scales" from Volume 1.

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*Where, in Vol. 1, a scale has not been shown, a mark with the letter S will be found – this is the identification for the value given in Chapter 10 of Vol. 2.

CHAPTER II

STATISTICAL MODELS IN GYRO APPLICATIONS

2-1 Ideal Sequence of Events

To investigate fully and develop a statistical model for a particular type of gyro it would be necessary to:

1) Generate several data records of the drift rate by test repetition on one gyro, always maintaining the same axes orientation. Subsequently, one would analyse the data and derive a statistical model.

2) Test the model developed in (1) with data taken from several gyros (of the same type), whilst still retaining the same orientation.

3) Repeat (1) on data from a different orientation (maybe a varying one, e.g., from a "tumbling" type of servo test).

4) Repeat (2) on data taken with the orientation of (3).

The procedure above could be continued as long as the data is forthcoming, but even then, one would very clearly have to state the conditions under which the developed model(s) is valid. For example, if the ultimate system application subjects the gyros to some form of base motion not present in the test conditions, then one cannot be sure of the value of the model in predicting system errors.

2-2 Practical Analysis of Gyro Drift Data

Unlike a statistical analysis of (say) radar noise, where a wealth of data can be made available in a short time, the limitation of the outline given in Section 2-1 is, directly, time, and indirectly, cost. Gyro drift is a very slowly varying quantity and, in order to have sufficient data for analysis in one run alone, the recording time is likely to be of the order of days. Therefore the situation of – many runs – many gyros – many orientations, is, to say the least, idealistic.

Consequently, one must specialize the analysis, and in considering high precision, costly gyros, the analyst is usually in the position of having to select data from what is currently available. This inevitably means data generated for other purposes, (i.e., not for the specific purpose of analysis) in particular, the only likely source of any quantity is from manufacturers' test data.

This raises the question of the validity of models produced from such data. A statistical model is not a precise mathematical formula, and therefore it is necessary to produce tolerances, confidence intervals, etc., in order to express the accuracy of the model. To do this, the data to be analysed must be a true representative sample of the data that could be generated, but with manufacturers' data this is unlikely to be the situation. To the manufacturer, naturally enough, "time is money" and to continue a test over the full specified time when he knows, by observation of trends, etc., and from previous experience, that it will fail, is an unlikely occurrence. Consequently, the analyst is usually only presented with data from gyros that have passed the particular test - and this is not a true sample. However, this undesirable situation is somewhat alleviated by the fact that it is only these gyros which will see operational use, so that the derived model is still basically valid for prediction, provided that one states the condition applicable, i.e., that the model is valid for those gyros which successfully passed a particular test specification. If the model is to be used to give an insight into the possible error causes, the false boundaries introduced by the non-random sample, may lead to incorrect deductions. A possible example of the effect of rejection of gyros is suggested in the analysis of the Mean Squared Drift Rate in Chapter 6, para. 2-2.

Test data may be broadly categorised into two classifications:

1) Gyro axes in a fixed orientation with respect to the gravity vector.
2) Gyro axes in a varying orientation with respect to the gravity vector.

The data to be analysed in this thesis falls into group (1), and it can be stated at the outset, that any model developed is unlikely to be applicable should the gyro be used in an application of group (2). At first sight this might seem to be a considerable limiting factor, but this is not the case, as many inertial navigation systems are of the type which instrument a navigation frame, wherein the gyros <u>are</u> maintained at a fixed orientation with respect to the gravity field.

CHAPTER III

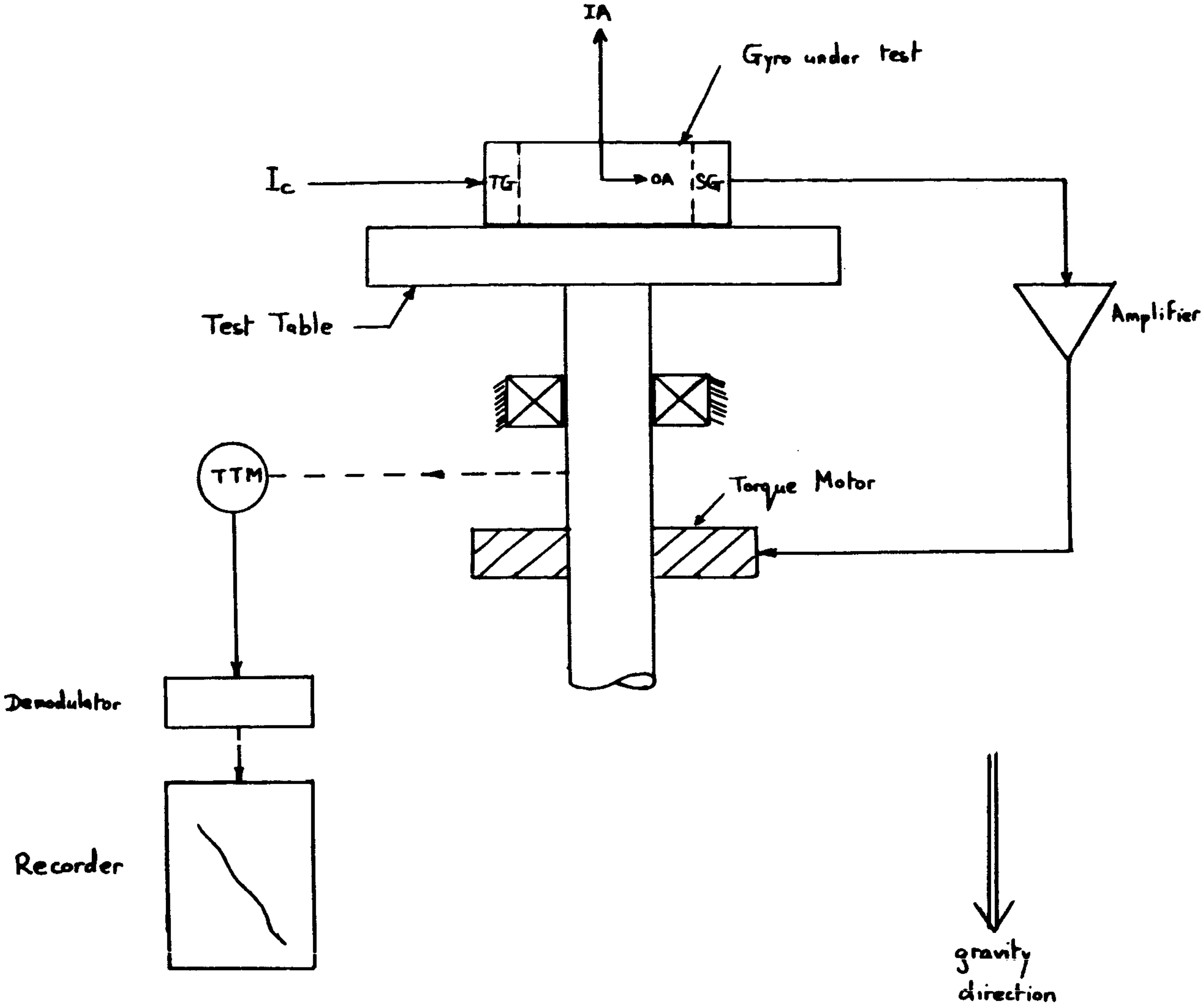
DESCRIPTION OF THE DATA TO BE ANALYSED

3-1 Test Configuration

The data to be analysed consists of a number of gyro runs performed by one manufacturer in meeting a certain test specification. Each drift run is conducted in a vertical earth-reference servo connection, with the gyro Input axis orientated along the gravity vector, as shown in Fig. 3-1. The drawing is simplified to show the basic features – for more detail on typical test installations, and construction of single degree of freedom gyros References 1 and 2 respectively, are suggested.

With no input current to the Torque generator, the gyro senses the vertical component of earth's angular rate with respect to inertial space, thereby producing an output from the signal generator, to cause the test table to rotate accordingly. The table will also rotate due to miscellaneous torques acting on the gyro, both constant and random, caused by a variety of sources (e.g. mass unbalances, lack of rigidity, etc.) The particular advantage of the test orientation is that gravity induced torques remain constant and can therefore, be virtually eliminated by calibration procedures, by applying a constant compensation current to the gyro torque generator. The effect of earth's rotation is removed in a similar manner and therefore, after this compensation, any motion of the test table will be due to small fluctuating torques acting on the gyro (which can usually be considered random in nature).

The measurement of the drift error to a high enough degree of accuracy can only be achieved by an angular measurement; in this case a signal from the Table Tracking Microsyn, proportional to angle, is fed to a Recorder to give a continuous presentation of the Drift angle, relative to some arbitrary datum position. However, in analysing a system performance, the vector quantity of drift rate is always considered as the input error source, consequently it must be computed from the gyro test output. The procedure used is the very simple one of differencing the Drift angle over hourly intervals.



IA	Input Axis	of Single-degree-of-freedom gyro
OA	Output Axis (nominally North)	
TG	Torque Generator	
SG	Signal Generator	
I_c	Compensating Current	
TTM	Table Tracking Microsyn	

Fig. 3-1 Simplified drawing of Test Configuration

This is the form of the data to be analysed and one can see at the outset, a possible source of error; namely, the data strictly represents the Incremental Drift Angle over one hour steps, and this can only approximate the average Drift Rate over one hour intervals. The actual pen records were not observed, but assurance was given that the Drift Angle was smoothly and slowly varying, so that it is presumed that the errors in the approximation are, in general, small. Attention will be drawn later to certain specific points on some gyro runs, where this assumption is probably not a good one – these points are few in number.

3-2 Test Method

The total test time for each run is 70 hours, of which the first 10 hours is a calibration period. Initially, a "best estimate" of the required compensation current is applied to the Torque Generator, based on a knowledge of earth's rotation and gyro drifts due to known constant error torques. A correction is made to this estimate after 10 hours running, based on observations of the data record to date; and, with the assumption of only short term transients, the time of this "cut" is considered as time zero for the drift test proper, over the remaining 60 hours. The calibration correction is obtained by computing the average of the drift rates (incremental drift angle) over 10, one hour intervals.

The test parameter (see Chapter 2, para. 1) during this phase of the manufacturer's evaluation of the gyro, is the Root Mean Square (RMS) of the Drift rates as computed over the 60, one hour intervals. This calculated value is referred to as the "Figure of Merit" (FOM) for the gyro.

i.e. $$\text{FOM} = \sqrt{\frac{1}{60} \sum_{i=1}^{60} (DR_i)^2} \qquad (3.1)$$

where DR_i = computed average drift rate over the "i" th one hour interval.

3-3 Initial Observations of the Data

The data as supplied, consisted of hourly tabulations of the computed drift rate for 50 runs, each run being for a different gyro. The 50 gyros were all of the same type of single degree of freedom gyros and will be simply indexed by the numbers 1 to 50 in the remainder of Vol. 1 – identification of the type of gyro, and the manufacturer's unit number is given in Chapter 10 of Vol. 2. In addition to the tabulated data, a simple graphical plot was provided, allowing a quick visual observation to be made of any obvious irregularities, etc. Thus, excluding the calibration data, a total of 50 x 60 data points was available for the analysis. This might appear to be a large sample, but as will become evident in later chapters, this is not the case, as a relatively small number of "irregular" points can have a marked effect on the calculation of certain statistics.

The first general impressions of the data were:

1) Many of the gyro runs show a steady rate of change of the drift rate (a ramp), with perturbations about this mean slope. Thus, in these cases, the data gives the appearance of being somewhat less random, but more deterministic, than one would have wished for at the start of a statistical analysis. However, comparison of these ramps in the runs where they occur, shows that the ramps themselves, appear to be random in nature.

2) Differencing of adjacent drift rate values (i.e., the incremental drift rate over one hour intervals) does show a very random pattern.

3) Nine, out of the 50 gyros, have a Figure of Merit equal to the specification tolerance. This seems a rather large proportion and one wonders what figures would have been achieved if the test runs had been repeated on those same nine gyros. This further supports the doubt expressed in Chapter 2, para. 2, of incomplete samples, should an attempt be made to link any model developed to the possible origins of the drift errors.

4) Gyros, Nos. 6, 45, and 47, exhibit particular features which are not readily observable in the other gyros.

i) Gyro No. 6 – considering the full 60 hour run and the 10 hour calibration period, this gyro shows far greater fluctuations than any of the other gyros.

ii) Gyro No. 45 – from time 11 to 15 (5 points) the drift rate shows a marked change in magnitude, from the values on either side of this range.

iii) Gyro No. 47 – at time 12 and 13 (2 points) the magnitude of the drift rate is approximately three times that of any other points in the run.

In the case of gyro 47, and to a lesser extent gyros 6 and 45, if the values have been obtained by differencing the test table angles at one hour intervals, they are likely to be considerably in error from the true drift rate at the specified times. It is further hypothesised, for gyros 45 and 47 only, that these values may well represent some sudden change, either in the test conditions, or due to some external influence.

All the data has been used in the analysis, as supplied – i. e., no modifications have been made on the basis of unlikely values in small samples. Consequently, attention will be drawn to calculations which are heavily dependent on these 3 gyros, and alternative values are given with certain points eliminated.

CHAPTER IV

THE MODEL HYPOTHESIS

4-1 General Considerations

Two different approaches can be considered in the analysis of test data and the subsequent derivation of a statistical model.

1) First select a simple model to be tested – the hypothesis – the choice of a likely model would obviously be based on the experience gained from other work in the same field, and on one's own initial interpretation of the data. Then promulgate as many tests as are considered necessary to evaluate this model, conduct the tests on the available data and assess the results.

If there is sufficient evidence to support the initial hypothesis, the job is complete; however, more likely is the situation where some test results are not adequately explained by the hypothesis. In this event, one has to consider whether to start again with a different type of model, or whether to follow up the initial hypothesis with a somewhat more complex model, but basically of the same type.

2) The alternative approach is to fully analyse the data, without any preconceived ideas, then to test the feasibility of the results to support different types of statistical models.

The second method is clearly the more unbiased (and time consuming) approach, which is a very strong point in its favour when one considers that the science of statistics is more of an art, and that preconceived ideas can usually be supported by statistical "evidence". Nevertheless, in this thesis the first approach has been adopted because there is evidence to support a likely initial hypothesis. In order to avoid the pitfalls stated previously, a full presentation of the test results is given, good and bad, in order that the line of inductive reasoning, subsequently developed, can be contested in open court!

4-2 Fundamental Assumptions

Three basic assumptions are implied in the analysis that follows:

1) The data represents Rate errors due to the gyros alone, i.e., it is not contaminated by errors introduced either by the test methods or the test equipment.

2) Gyro drift can be considered as a random process.

3) The discrete data to be analysed represent samples from the same statistical population.

The only justification for assumption (1) is that since the test equipment was designed specifically for this type of gyro, one should not have to consider it as a source of error at the start of an analysis.

Assumption (2) can only be supported in retrospect. Should a deterministic trend (e.g. a steadily increasing mean drift rate, due perhaps to the ramping noted in Chapter 3, para. 3 (1)) be observed, then the assumption would only be valid after mathematically removing the trend from the data.

Assumption (3) implies that the gyros are very uniform in construction, so that a sample consisting of one run on each of 50 gyros, is equivalent to a sample of 50 runs on one gyro. In less precise gyros one would be unwise to make this assumption, and even with these high tolerance instruments it must be applied with caution (Chapter 7, para. 3 investigates one possible breakdown of this assumption). To clarify the importance of this assumption further, if it is not valid, then one effectively has a sample of one run only for each of 50 statistical populations, differing to various degrees on which to base any conclusions.

4-3 Motivation for the Model Specification

From a review of the somewhat limited amount of literature available in this field, there was evidence to support a "Random Walk" type of model (e.g. Refs. (3), (4) and (5)). It was therefore decided to test the validity of this model to the particular gyro data under investigation here.

4-4 A Simple Random Walk

It is not intended in this thesis to develop fully the theory applicable to simplified Random Walks; only relevant steps to support the tests actually selected will be given in detail. For more detail in the application of this type of model to gyro drift analysis, reference could be made to Ref. (6). In order to illustrate the concept of the random walk the description given in the introductory remarks of this reference is reproduced here, with the permission of the author.

"Simplified Random Walk Process: This model consists of a man who has been placed in a very long corridor. The man has been instructed that at the end of each minute he must take one step forward, or one step backward, or remain in whatever position he is in. It is further demanded that his decision be completely random with an equal likelihood assigned to each of the three alternatives. Thus, the average rate of the man during any of the one minute intervals is: plus one step per minute (forward), minus one step per minute (backward), or zero steps per minute (remain). For purposes of simplification, it is further assumed that the man's rate is constant over any one minute interval."

4-5 <u>The Model Hypothesis</u> (see Nomenclature and Definitions on Page vii.)

The model to be tested is similar, but not as simple, as that outlined in 4-4. From the first inspection of the data (Chapter 3, para. 3) the Drift Acceleration (defined as the Incremental Drift Rate/step interval, and not the true acceleration) exhibited a very random pattern, whereas the Drift Rate was less random in nature. A rough check of the Ensemble average of the Mean Squared Drift Rate at various times during the test runs, suggested that this statistic was likely to be time-varying. It was therefore hypothesised that the Drift Acceleration was a stationary random process and the Drift Rate a non-stationary random process.

The model, so far, conforms to that of Chapter 4, para. 4, when the derivatives are considered, i.e., the Drift Acceleration is compared to the rate at which the man walks, and the Drift Rate to the position of the man corresponding to an integral number of step intervals from the commencement of the walk, where the unit of time is now one hour. However, it would indeed be fortuitous if the Drift Acceleration could be modelled by steps of equal, but opposite magnitude, or zero; therefore the model condition will be that the Expectation of the Mean of the Drift Acceleration (E(A)) is zero, and that the acceleration step amplitudes are statistically independent.

This model is illustrated by a typical picture in Fig. 4-1, together with the Drift Rate resulting from this process, and is now summarized.

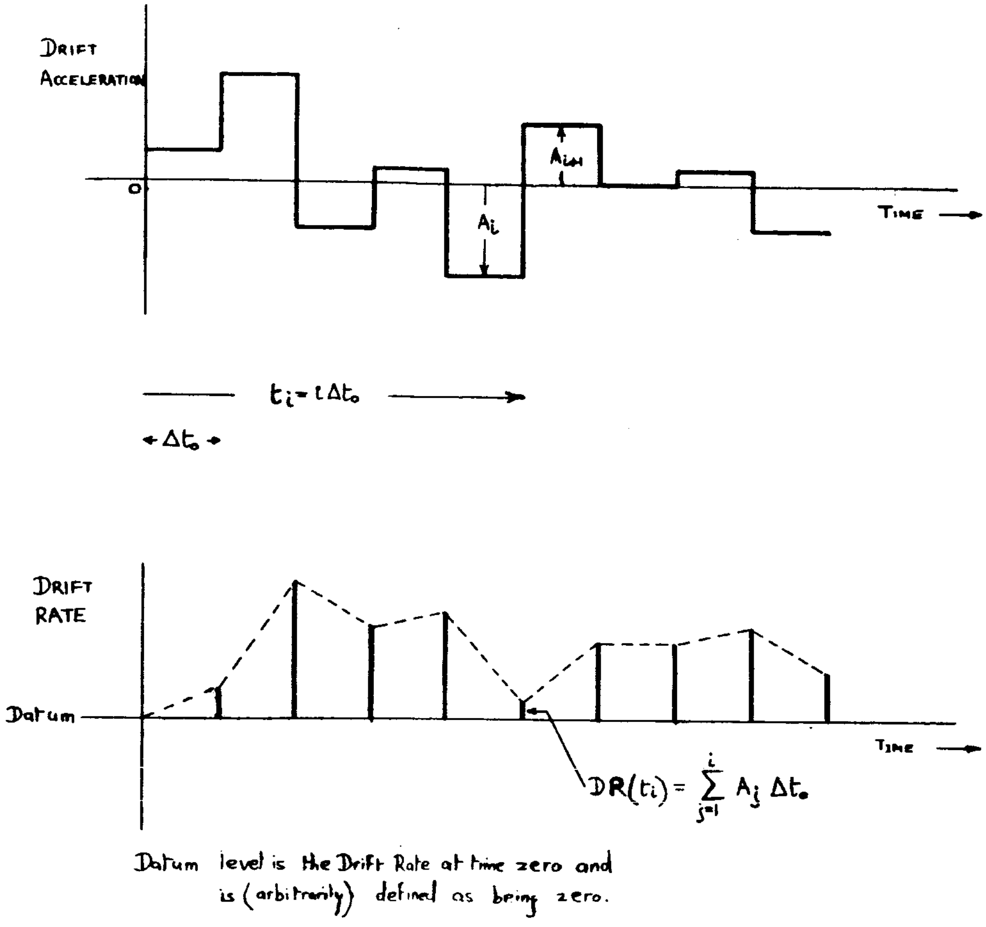


Fig. 4-1 A TYPICAL SAMPLE OF THE RANDOM WALK MODEL HYPOTHESIS

1) Drift Acceleration, as typified by A_i in Fig. 4-1, is a stationary random process, with:

$$E(A) = 0 \tag{4.1}$$

E(A) is obtained from the ensemble average over an infinite number of gyro runs at any time, t. Also, by applying the ergodic hypothesis because the process is stationary,

$$E(A) = \underset{n \to \infty}{\text{Limit}} \quad \frac{1}{n} \sum_{i=1}^{n} A_i = 0 \tag{4.2}$$

for each individual gyro.

2) The discrete values of the Drift Acceleration are statistically independent,

Therefore $$E(A_i \cdot A_j) = E(A^2) \cdot \delta_{ij} \tag{4.3}$$

where $$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

4-6 General Comments on the Model

It should be noted that the drift acceleration is defined as the incremental drift rate (IDR) divided by the time interval (Δt_o), and that

$$A_i \cdot \Delta t_o = DR(i \cdot \Delta t_o) - DR((i-1) \cdot \Delta t_o)$$

The step interval Δt_o = 1 hour, so using this as the basic time unit, the Drift Acceleration is numerically equal to the IDR over the "i" th time interval.

The Drift Rate is shown as discrete values in Fig. 4-1. One might infer from the drift acceleration that the continuous drift rate is being modelled, but this is not the case as the data to be analysed is in discrete form, hence the model is only applicable to the Drift Rate at one hour intervals. The dotted lines shown on Fig. 4-1 represent the only estimate one can make of the drift rate between the discrete time intervals.

As the Drift Rate is generated by a summation process, its value will depend on an initial condition DR($t = 0$). Since this is arbitrary it will be specified for convenience as being zero.

4-7 Compatability of the Model with the Initial Inspection of the Data (Chapter 3-3)

In Chapter 3, para 3 (4), it was noted that certain gyro runs had a few "odd" points. One would like therefore, to have some knowledge of the type of statistical frequency distributions that might be encountered, in order that the significance of these points can be analysed.

The many possible sources of error present in such a complex instrument leads to the prediction that the Drift Rate values will be Normally distributed. This follows as a result of the Central Limit Theorem, since the Drift Rate can be considered as the mean of many small errors, and the distribution of the means approaches a Normal distribution. Now consider the Model, and although it is not specified it is nevertheless implied, that the model drift rate will tend to a Normal distribution. (Note:- the IDR will have the same distribution as the drift acceleration; there being only a constant factor Δt_o relating them).

Therefore one might consider at the outset, plotting the frequency distribution of the drift rate data, to investigate if any of the data points could reasonably be rejected. However, the model has specified that the drift rate is non-stationary, therefore only the ensemble distributions at a particular time can be considered (i.e., both the Mean and the Variance of the Normal distribution could be time varying). Thus a sample would consist of only 50 points, one from each gyro, and this is too small to consider rejection limits at the start of an analysis. As was stated previously, none of the original data points were modified.

The other feature, from the initial observations, was "ramping" of the Drift Rate. Under certain conditions this is not incompatible with the random walk model, as, although the Expectation of the drift acceleration has been specified as zero, there is no reason why the actual drift acceleration should not be a value of (say) one Standard deviation away from the Mean, for certain lengths of time. This is equivalent to a ramp, and therefore the drift acceleration (or the IDR) distribution must be obtained and the distribution statistics compared with the data drift acceleration, before the data could be rejected on this basis. Nevertheless, one must still be mindful of the possibility that the ramping may be due, in part, to some deterministic non-random process.

CHAPTER V

TESTING THE MODEL - THEORETICAL DEVELOPMENT

5-1 The model, as specified by equations (4.1) to (4.3), implies certain statistical properties which can be used to test the degree of support of the data to the model. The theory in this Chapter and the results in Chapter 6, are presented in the order that the tests were conducted (not necessarily the best order - in retrospect).

5-2 Drift Rate - Stationarity Test

5-2-1 Expectation of the (Mean) Drift Rate

From Fig. 4-1:

$$DR(t_n) = \sum_{i=1}^{n} A_i \cdot \Delta t_o \tag{5.1}$$

$$E\left[DR(t_n)\right] = E\left[\sum_{i=1}^{n} A_i \cdot \Delta t_o\right]$$

but, Expectation of a Sum = Sum of the Expectations

$$\text{i.e.} \quad E(a + b + \cdot\cdot) = E(a) + E(b) + \cdots\cdot \tag{5.2}$$

therefore, $$E\left[DR(t_n)\right] = \sum_{i=1}^{n} E\left(A_i \cdot \Delta t_o\right)$$

but, Δt_o is a constant, and from equation (4.1), $E(A_i) = 0$
therefore

$$E\left[DR(t_n)\right] = 0$$

but this is true for any value of n

hence $$E\left[DR\right] = 0 \tag{5.3}$$

5-2-2 Expectation of the (Mean) Squared Drift Rate

From equation (5.1)

$$\left[DR(t_n)\right]^2 = \sum_{i=1}^{n} A_i \Delta t_o \sum_{j=1}^{n} A_j \Delta t_o$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j (\Delta t_o)^2 \qquad (5.4)$$

therefore $$E\left[DR(t_n)\right]^2 = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j (\Delta t_o)^2\right]$$

Using equation (4.3) and equation (5.2), and with Δt_o constant

$$E\left[DR(t_n)\right]^2 = n \cdot E(A^2 \Delta t_o^2$$

but $$t_n = n \cdot \Delta t_o$$

and $$E\left[DR(t_n)\right]^2 = E\left[MSDR(t_n)\right]$$

and $E(A^2) = E(MSDA)$ ... time invariant for a stationary process

therefore $$E\left[MSDR(t)\right] = E(MSDA) \cdot t \cdot \Delta t_o \qquad (5.5)$$

or, since the Drift Acceleration has been defined as the rate of change of the Incremental Drift Rate,

$$E\left[MSDR(t)\right] = E(MSIDR) \cdot \frac{t}{\Delta t_o} \qquad (5.6)$$

i.e. $$E\left[MSDR(t)\right] = k \cdot t \qquad (5.7)$$

where k is a constant and, with unit time as the step interval of one hour, is numerically equal to E(MSIDR); and t is an integer number of step intervals.

Thus, as stated earlier, the Model drift rate is a non-stationary process as, although the Mean is zero, the Mean Squared value increases linearly with time.

The Est(k) will be defined in the tests as the estimate of the slope of the MSDR/Time plot, and it is clearly the basic clue as to the validity of the Random Walk hypothesis. For simplicity, the Expectation of Est(k) will just be written as k.

5-2-3 Estimates of the Drift Rate Statistics

Since the model predicts a non-stationary process, Ensemble averages, and not Time averages, must be considered.

$$\mathrm{Est}\left[DR(t_n)\right] = \frac{1}{R} \cdot \sum_{r=1}^{R} DR(t_n)_r \tag{5.8}$$

$$\mathrm{Est}\left[MSDR(t_n)\right] = \frac{1}{R} \cdot \sum_{r=1}^{R} \left[DR(t_n)_r\right]^2 \tag{5.9}$$

where r refers to Gyro number r, drift run

and R = total number of drift runs = 50

As a computer was used for all calculations the statistics of Drift Rate Variance and Standard Deviation were also evaluated (see Nomenclature).

$$S^2(t_n) = \frac{\sum_{r=1}^{R} \left[DR(t_n)_r - \mathrm{Est}\left[DR(t_n)\right]\right]^2}{R-1}$$

$$= \left[\mathrm{Est}\left[MSDR(t_n)\right] - \left\{\mathrm{Est}\left[DR(t_n)\right]\right\}^2\right] \cdot \frac{R}{R-1} \tag{5.10}$$

$$S(t_n) = \sqrt{S^2(t_n)} \tag{5.11}$$

(Reference can be made to any standard statistical text for the proof of equations (5.10) and (5.11).)

It should be noted that since,

$$E(DR) = 0 \tag{5.3}$$

Population Variance $$V = E\left[S^2(t_n)\right] = E\left[MSDR(t_n)\right] \tag{5.12}$$

Population St. Dev. $$\sigma = \sqrt{E\left[MSDR(t_n)\right]} \tag{5.13}$$

5-2-4 Acceptable Limits Based on the Model Hypothesis

Consider that $E\left[MSDR(t)\right]$ is known.

This implies from equation (5.7) that k is also known, and from equation (5.13)

$$\sigma(t) = \sqrt{k \cdot t} \tag{5.14}$$

But, from Chapter 4, para. 7, it was shown that the model predicts that the Drift Rate at any time t_n will tend to be Normally distributed, and from equation (5.3) the expectation of the distribution Mean is zero. Therefore the probabilities associated with the Normal distribution can be considered, viz.

$$\left.\begin{array}{llllll} 68\% & \text{probability that} & DR(t) & \text{lies within} & \pm\sigma(t) & \text{of zero} \\ 95.5\% & \text{"} \quad \text{"} & DR(t) & \text{"} \quad \text{"} & \pm 2\sigma(t) & \text{"} \quad \text{"} \\ 99.74\% & \text{"} \quad \text{"} & DR(t) & \text{"} \quad \text{"} & \pm 3\sigma(t) & \text{"} \quad \text{"} \end{array}\right\} \tag{5.15}$$

$$\left.\begin{array}{llllll} 68\% & \text{probability that} & \text{Est } DR(t) & \text{is within} & \pm \frac{\sigma(t)}{\sqrt{R}} & \text{of zero} \\ 95.5\% & \text{"} \quad \text{"} & \text{Est } DR(t) & \text{is within} & \pm 2.\frac{\sigma(t)}{\sqrt{R}} & \text{"} \quad \text{"} \\ 99.74\% & \text{"} \quad \text{"} & \text{Est } DR(t) & \text{"} \quad \text{"} & \pm 3.\frac{\sigma(t)}{\sqrt{R}} & \text{"} \quad \text{"} \end{array}\right\} \tag{5.16}$$

For large sample sizes, i.e., large R, it can be shown that the Distribution of Est MSDR(t) also approaches a Normal distribution,

$$\text{with a Variance} \quad = \quad \frac{2.\sigma^4(t)}{R}$$

$$\text{and a Standard Deviation} \quad = \quad \sigma^2(t)\cdot\sqrt{\frac{2}{R}} \tag{5.17a}$$

Therefore:

68% probability that Est $\left[\text{MSDR}(t)\right]$ is within $\pm\ \sigma^2(t)\cdot\sqrt{\frac{2}{R}}$ of E $\left[\text{MSDR}(t)\right]$ etc. (5.17)

5-2-5 <u>Acceptable Limits Based on Estimated Statistics</u>

In general $\sigma(t)$ will not be known, only the Est $\sigma(t)$, i.e. S(t), will be available from calculations. For large samples equations (5.16) and (5.17) can be modified to the following statements, to a reasonable approximation.

68% probability that E$\left[\text{DR}(t)\right]$ is within $\pm\frac{S(t)}{\sqrt{R}}$ of Est$\left[\text{DR}(t)\right]$ etc. (5.18)

68% probability that E$\left[\text{MSDR}(t)\right]$ is within $\pm S^2(t)\sqrt{\frac{2}{R}}$ of Est$\left[\text{MSDR}(t)\right]$ (5.19)

Terminology:

(i) $\frac{S(t)}{\sqrt{R}}$ is called the Standard Error of the Mean.

(ii) $S^2(t)\cdot\sqrt{\frac{2}{R}}$ is called the Standard Error of the Variance, and since in this situation E(DR) = 0, it could also be called the Standard Error of the MSDR.

(iii) 2 Standard Errors of the Mean = $2\cdot\frac{s(t)}{\sqrt{R}}$ etc.

5-2-6 Sample Size

The equations given in 5-2-4 and 5-2-5 are only valid for large sample sizes. For small samples, it would have been necessary to consider Student's "t" distribution and Chi-Squared distribution, instead of the Normal distribution in establishing the probability limits for the Estimated Mean and Mean Squared values, respectively. In this analysis the Sample size, for ensemble calculations (R), is 50. This number might be described as the "in between" sample size, neither large nor small. For no other reason than simplicity, it will be categorized as a large sample in the subsequent analysis.

5-3 Drift Rate – Figure of Merit Test

In Chapter 3-2 the test parameter of the RMS value of the Drift Rate after 60 hours, was defined as the Figure of Merit. If this parameter is now generalized to any discrete interval time t_n, equation (3.1) can be modified, for the Random Walk model, to:

$$FOM(t_n) = \sqrt{\frac{1}{n+1} \sum_{i=0}^{n} \left[DR(t_i) \right]^2} \qquad (5.20)$$

From equation (5.5) it follows that,

$$E\left[DR(t_i)\right]^2 = E(MSDA) \cdot t_i \cdot \Delta t_o$$

$$= E(MSDA) \cdot (\Delta t_o^2) \cdot i \qquad (5.21)$$

Therefore, combining equations (5.21) and (5.2) with equation (5.20)

$$E\left[FOM(t_n)\right]^2 = \frac{1}{n+1}\sum_{i=0}^{n} E\left[MSDA\right] \cdot \Delta t_o^2 \cdot i$$

$$= E\,(MSDA) \cdot (\Delta t_o^2) \cdot \sum_{i=0}^{n}\left(\frac{i}{n+1}\right)$$

$$= E(MSDA) \cdot (\Delta t_o^2) \cdot \frac{n}{2}$$

$$= \frac{1}{2} \cdot E(MSDA) \cdot t_n \cdot \Delta t_o \qquad (5.21)$$

In the terminology of equation (5.7)

$$E\left[FOM(t)\right]^2 = \frac{1}{2}k \cdot t \qquad (5.23)$$

i.e. the slope of the $(FOM)^2$/Time plot is a half the slope of the (MSDR)/Time plot.

The $E\left[FOM(t)\right]$ is a more complex expectation to evaluate, but to a first approximation, it will be proportional to $\sqrt{t}$. The ensemble estimate of $(FOM)^2$ is given by:

$$Est\left[FOM(t_n)\right]^2 = \frac{1}{(n+1)\cdot R} \cdot \sum_{r=1}^{R}\sum_{i=0}^{n}\left[DR(t_i)\right]_r^2 \qquad (5.24a)$$

and Est(FOM) by,

$$Est\left[FOM(t_n)\right] = \frac{1}{R}\sum_{r=1}^{R}\sqrt{\frac{1}{n+1}\sum_{i=0}^{n}\left[DR(t_i)\right]_r^2} \qquad (5.24b)$$

5-4 Incremental Drift Rate – Stationarity Test

From equation (4.1)

$$E(IDR) = \Delta t_o \cdot E(A) = 0 \tag{5.25}$$

From equation (4.3)

$$E\left[(IDR)^2\right] = E(MSIDR) = (\Delta t_o^2) \cdot E(A^2) \tag{5.26}$$

i.e., Incremental Drift Rate is a stationary process.

Estimates are similar to equations (5.8) and (5.9) for Drift Rate, viz.

$$Est\left[IDR(t_n)\right] = \frac{1}{R} \cdot \sum_{r=1}^{R} IDR(t_n)_r \tag{5.27}$$

$$Est\left[MSIDR(t_n)\right] = \frac{1}{R} \cdot \sum_{r=1}^{R} \left[IDR(t_n)_r\right]^2 \tag{5.28}$$

The ensemble estimates when plotted against time should be a straight line with zero slope if the model is valid, but, since a stationary process is being tested, time averages can also be used for additional confirmation if necessary.

5-5 Incremental Drift Rate – Correlation Test

Since the model specified that the Drift Acceleration in any given step interval, was to be independent of that during any other interval, the same must be applicable to the IDR.

The model IDR is a stationary process, therefore a measure of the independance can be derived by calculating the Autocorrelation Function (ACF) over time, where,

$$\begin{aligned} ACF(t_i, t_j) &= E\left[IDR(t_i) \cdot IDR(t_j)\right] \\ &= (\Delta t_o^2) \cdot E\left[A(t_i) \cdot A(t_j)\right] \end{aligned}$$

Therefore, from equation (4.3)

$$ACF(t_i, t_j) = (\Delta t_o^2) \cdot E(A^2) \cdot \delta_{ij}$$

$$= E(MSIDR) \cdot \delta_{ij}$$

For a stationary, ergodic process the ACF is not dependent on the absolute values of t_i and t_j, but only on the the time difference τ (= $t_j - t_i$).

i.e.
$$ACF(\tau) = \begin{cases} E(MSIDR) & \text{when } \tau = 0 \\ 0 & \text{when } \tau \neq 0 \end{cases} \tag{5.29}$$

For discrete data, with zero mean, the estimate is given by,

$$Est\left[ACF(n)\right] = \frac{1}{N-n} \cdot \sum_{i=1}^{N-n} \left\{ IDR(t_i) \cdot IDR(t_i + n \cdot \Delta t_o) \right\} \tag{5.30}$$

Equation (5.30) can be evaluated for each gyro drift run, where N is the total number of points in the run (N = 59, as 60 Drift Rate points can only give 59 IDR values). It can be seen from equation (5.30), that the larger the value of n, the fewer are the number of terms in the summation and consequently the poorer the estimate of the ACF. Even with n = 0, the estimate is still only based on 59 terms, therefore, in an effort to improve this accuracy, the Ensemble estimate can be considered, where

$$\text{Ensemble } Est\left[ACF(n)\right] = \frac{1}{R} \sum_{r=1}^{R} Est\left[ACF(n)\right]_r \tag{5.31}$$

where r signifies the ACF of the r^{th} data run

and $R = 50$

Notes:-

i) As the samples, both in time, and number of runs, are neither very large, it would be wise to compare the general trend of equation (5.31), over the range of n selected, with each of the 50 separate trends obtained from equation (5.30), before deciding that equation (5.31) is this best estimate.

ii) The "pre-requisite" for the ACF equations given above, is that the IDR has been proven to be a stationary process (Section 5-4).

5-6 Frequency Distributions

5-6-1 Drift Rate

In Chapter 4, para. 7, the expected ensemble Drift Rate distribution was shown to be a Normal distribution, but, because the parameters specifying the distribution are time varying, the estimate of the data distribution can only contain one point from each gyro. A total of 50 points is too small for any accuracy in a histogram presentation.

5-6-2 Incremental Drift Rate Distribution

If the tests of Sections 5-4 and 5-5 are satisfactory, i.e. the IDR is a stationary, independent over one hour intervals, process – then there are 59 independent points in each of the 50 gyro runs. Each gyro run is clearly independent of any other run, and, since a fundamental assumption was made that all data is from the same population, there will be 2,950 independent data points. Consequently, a very useful histogram can be drawn to indicate the expected IDR distribution.

Should the IDR be shown to be stationary but (say) correlated over one hour intervals, then one would have to use every other point in estimating the sample, as the total number of independent points would have been reduced by a half. Thus, the longer the correlation in the test of Section 5-5, the poorer will be the estimate of the IDR frequency distribution.

Although, in the Model specification, a knowledge of the IDR distribution was not required, it was nevertheless considered not unlikely that it would be Normally distributed. As it will transpire (Chapter 6, para. 7), this may well not be the case.

5-7 Drift Rate Correlation

5-7-1 Expectation and Estimation

If for the Model, the Drift Rate Correlation (COR) is defined as,

$$COR(t_m, t_n) = E\left[DR(t_m) \cdot DR(t_n)\right] \quad \text{where } n \geq m$$

then

$$COR(t_m, t_n) = E\left[DR(t_m)\left\{DR(t_m) + \sum_{j=m+1}^{n} A_j \Delta t_o\right\}\right]$$

$$= E\left[DR(t_m)^2\right] + E\left[DR(t_m) \sum_{j=m+1}^{n} A_j \Delta t_o\right] \tag{5.32}$$

but

$$DR(t_m) = \sum_{i=1}^{m} A_i \Delta t_o$$

therefore, from equations (4.3) and (5.2)

$$E\left[\sum_{i=1}^{m} A_i \Delta t_o \sum_{j=m+1}^{n} A_j \Delta t_o\right] = 0 \quad (\text{since } i \neq j) \tag{5.33}$$

therefore, from the analysis in Section 5-2-2,

$$COR(t_m, t_n) = E\left[MSDR(t_m)\right] = k \cdot t_m \tag{5.34}$$

$$\text{for all } n \geq m$$

i.e. For a given t_m, $COR(t_m, t_n)$ is a constant for all $n \geq m$ (5.35)

To obtain an estimate of the Drift Rate Correlation, ensemble averages must be used (non-stationary).

Therefore

$$\text{Est COR}(t_m, t_n) = \frac{1}{R} \cdot \sum_{r=1}^{R} \left\{ DR(t_m)_r \cdot DR(t_n)_r \right\} \quad n \geq m \tag{5.36}$$

where, as before, r refers to the Drift Rate during the r^{th} run
and R = total number of runs = 50.

5-7-2 Acceptable Limits

To evaluate the Drift Correlation of the data, the results obtained in equation (5.36) can be plotted as a family of curves, each representing a selected t_m and varying t_n ($\geq t_m$). Theses curves should approximate to a straight line with zero slope, and the height of the line being equal to $k \cdot t_m$ (equation (5.34)), where k is numerically equal to the expectation of the slope of the MSDR/Time plot. However, the sample size is only 50, therefore it is necessary to examine what would be acceptable as support for the Random Walk hypothesis.

If limits are based on the model hypothesis and the model population parameters are assumed to be known, then for the random variable x, where

$$x = DR(t_m) \cdot DR(t_n) \qquad n \geq m \tag{5.37}$$

$$E(x) = COR(t_m, t_n) = k \cdot t_m \tag{5.38}$$

$$\begin{aligned} E(x^2) &= E\left[DR(t_m)^2 \left\{DR(t_m) + \sum_{j=m+1}^{n} A_j \Delta t_o\right\}^2\right] \\ &= E\left[DR(t_m)^4\right] + 2E\left[DR(t_m)^3 \cdot \sum_{j=m+1}^{n} A_j \Delta t_o\right] \\ &\quad + E\left[DR(t_m)^2 \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} A_i A_j (\Delta t_o)^2\right] \end{aligned} \tag{5.39}$$

Now consider the 3 terms in equation (5.39)

If the $DR(t_m)$ is Normally distributed (Chapter 4, para. 7 with Variance σ^2 and Mean, zero, then

$$E\left[DR(t_m)^4\right] = \int_{-\infty}^{\infty} DR(t_m)^4 \cdot \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{DR(t_m)^2}{2\sigma^2}} \cdot d\left[DR(t_m)\right]$$

$$= 3\sigma^4$$

but, from equation (5.12)

$$\sigma^2 = E\left[MSDR(t_m)\right] = k \cdot t_m$$

therefore $\quad E\left[DR(t_m)^4\right] = 3\, k^2\, t_m^2 \quad .$ (5.40)

By a similar analysis used in equation (5.33) it follows that

$$E\left[DR(t_m)^3 \cdot \sum_{j=m+1}^{n} A_j \Delta t_o\right] = 0 \qquad (5.41)$$

Also,

$$E\left[DR(t_m)^2 \cdot \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} A_i A_j \Delta t_o^2\right] = E\left[DR(t_m)^2\right] \cdot E\left[\sum_{i=m+1}^{n} \sum_{j=m+1}^{n} A_i A_j \Delta t_o^2\right]$$

$$= k\, t_m \left[(n-m) \cdot E(A_i^2)\, \Delta t_o^2\right]$$

$$= k\, t_m (t_n - t_m) \cdot E(MSDA) \cdot \Delta t_o$$

which, from equations (5.5) and and (5.7) $\quad = k^2\, t_m\, (t_n - t_m)$ (5.42)

Combining equations (5.40), (5.41) and (5.42) with (5.39)

$$E(x^2) = 2k^2 t_m^{\ 2} + k^2 t_m t_n \tag{5.43}$$

Therefore $\quad$ $\text{Variance}(x) = E(x^2) - \left[E(x)\right]^2$

which from
equations (5.38) and (5.43) $= k^2 t_m (t_m + t_n)$ $\quad$ (5.44)

But, $\text{Est}\left[\text{COR}(t_m, t_n)\right]$ as given in equation (5.36) is a summation of 50 values of x,

therefore,

$$\text{Variance of Est}\left[\text{COR}(t_m, t_n)\right] = \frac{\text{Variance}(x)}{50}$$

and $\quad$ $$\text{Standard Deviation of Est}\left[\text{COR}(t_m, t_n)\right] = \sqrt{\frac{k^2 \cdot t_m(t_m + t_n)}{50}} \tag{5.45}$$

CHAPTER VI

TESTING THE MODEL - ANALYSIS OF RESULTS

6-1 The interpretation of the results outlined in the preceding Chapter is presented here with the emphasis on those results which basically support the Model hypothesis of Chapter 4; where obvious discrepancies with the Model are present, a fuller investigation is delayed until Chapter 7. All computer programs were run on an IBM 1620, and are presented in Appendix B.

The scales (except time) have been intentionally omitted from graphs for classification purposes, and are listed in Chapter 10 of Vol. 2. On most of the graphs the computed points have been joined together by straight lines only for ease of visual interpretation, and are not intended to portray the statistical quantity between the discrete time intervals.

6-2 Drift Rate - Stationarity Test

Test theory - Chapter 5, para. 2
Program numbers, 1, 2, 3
Figs. 6-1 to 6-4

6-2-1 First Check

The ensemble statistics were computed (Program 1) of the original Drift Rate data, i.e. the Mean, Mean Square, Variance and Standard Deviation of the drift rate at the discrete time intervals (equations (5.8 to (5.11)); however, difficulty was found in relating the results to the Random Walk model, where it was specified (arbitrarily) that the drift rate was to start from zero. This is illustrated in Fig. 6-1, where the MSDR/Time curve is plotted (note: the first data point available is at $t = 1$). From Chapter 5, para. 2-2, the model predicts that the MSDR should be linearly increasing with time. Inspection of Fig. 6-1 suggests that this straight line may be present, but with a slight curve upwards with time. The questions to be answered are: What does the non-zero value at $t = 0$ signify (assuming a linear projection back in time, from the first plotted point)? Is this indicative of a stationary drift rate component as well as a non-stationary one

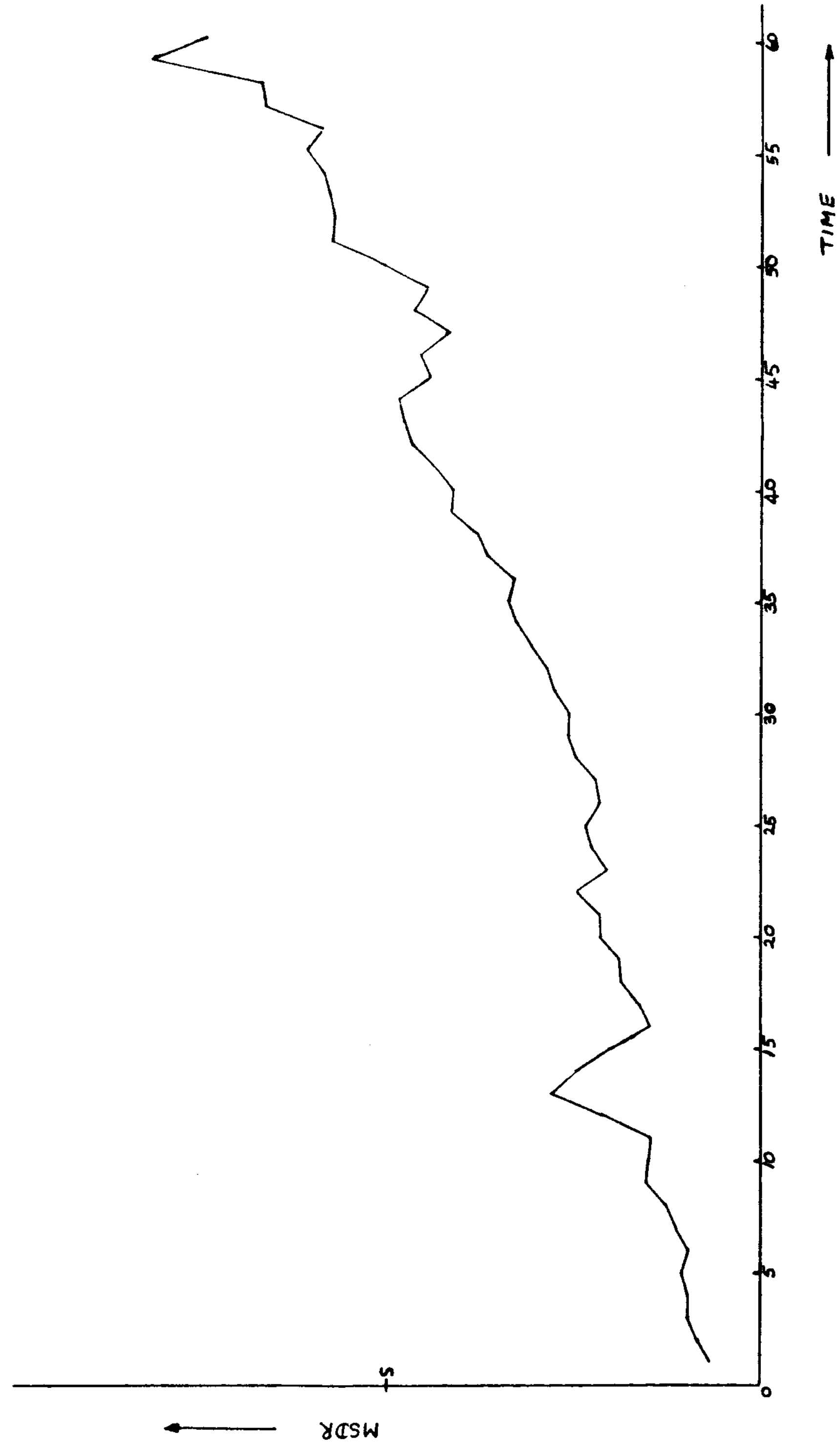


Fig. 6-1 MEAN SQUARED DRIFT RATE - ORIGINAL DATA

(possibly a Random Walk)? The questions cannot be satisfactorily answered at this stage and will be delayed until Section 6-2-2. It was theorized that the results reflected that calibration methods (Chapter 3, para. 2), and that a 10 hour calibration period is insufficient to remove any constant drift rate component from the subsequent data run proper, particularly if the drift rate is a non-stationary process. Therefore, in order to test the hypothesis that a Random Walk is representative of the major component of drift rate variations, it was decided to force the data to zero at the start of each run by subtracting the first drift rate data point from all other values in the same run. Unless otherwise stated, all remaining tests were applied to the data, modified in this manner. Furthermore, for consistency in the time scale, since the $DR(t_1)$ have been forced to zero, this point will be considered as time zero for all subsequent tests (e.g. $DR(t_{10})$ of the original data now becomes part of the data at t = 9, where new $DR(t_9)$ = original $DR(t_{10})$ – original $DR(t_1)$, etc.).

The results of recalculating the ensemble statistics – zero corrected (Program 2), are shown in Figs. 6-2 to 6-4.

6-2-2 Estimate of the Mean Squared Drift Rate

The MSDR/Time plot of Fig. 6-2 shows, for the most part, a linearly increasing trend consistent with the Random Walk hypothesis. To bring out the discrepancies more clearly, a straight line was fitted to the plot, passing through the origin, using the least squares technique (Program 3). The errors in the fit of this line can be considered in three separate ranges.

i) Times 11 to 14 deviations above
ii) Times 25 to 49 deviations below
iii) Times 56 to 59 deviations above

An attempt was then made to relate these discrepancies back to the original data with the following conclusions:

i) The values at times 11 and 12 (12 and 13 of the original data) can be directly attributed to the two "wild" points on Gyro No. 47 (Chapter 3, para. 3(4) (iii)), and the peak is considerably reduced if these values are eliminated from the calculations. Deviations at times 11, 12, 13 and 14 are still further reduced if Gyro No. 45 is also eliminated (Chapter 3, para. 3(4) (ii)).

ii) The other two ranges cannot be attributed directly to a few individual points. If the Model is correct, one possible explanation (and it is stressed that this is only a theory) is that the data shows the effect of incomplete samples (Chapter 2, para. 3). If the rejection zone based on past experience were to cover the range of times 25 to 49, and the rejected gyros had been included in the population from which the test data was sampled, then the deviation in this range might have been reduced. Now, suppose a gyro under test was showing a tendency to drift off in a more pronounced manner than normally expected, towards the end of the test run; then one would be loathe to abort the run, in the event that it might still remain inside tolerance. This latter situation could possibly account for the deviations in the range of times 56 to 59!

Conclusions

1) The MSDR/Time plot supports the Model hypothesis, the discrepancies could only be investigated further by larger samples and longer runs.

2) If the slope of the least squares straight line is designated Est(k) then limits can be considered. For ease of presentation assume $k = Est(k)$, then from equations (5.14) and (5.17a), with $R = 50$,

$$SD(t) = \text{Standard Deviation}(t) = \frac{k \cdot t}{5}$$

The following observations were made (reference equation (5.17)). All Est$\left[MSDR(t)\right]$ points fall inside the limits given below, except at the tabulated times.

LIMITS	Time at which Est$\left[MSDR(t)\right]$ outside limit (max. possible = 59 points)
± 1 SD	1, 2, 3, 11, 12, 13, 14, 58
± 2 SD	1, 2, 11, 12, 13,
± 3 SD	12, 13

The ± 2SD limits are shown on Fig. 6-2.

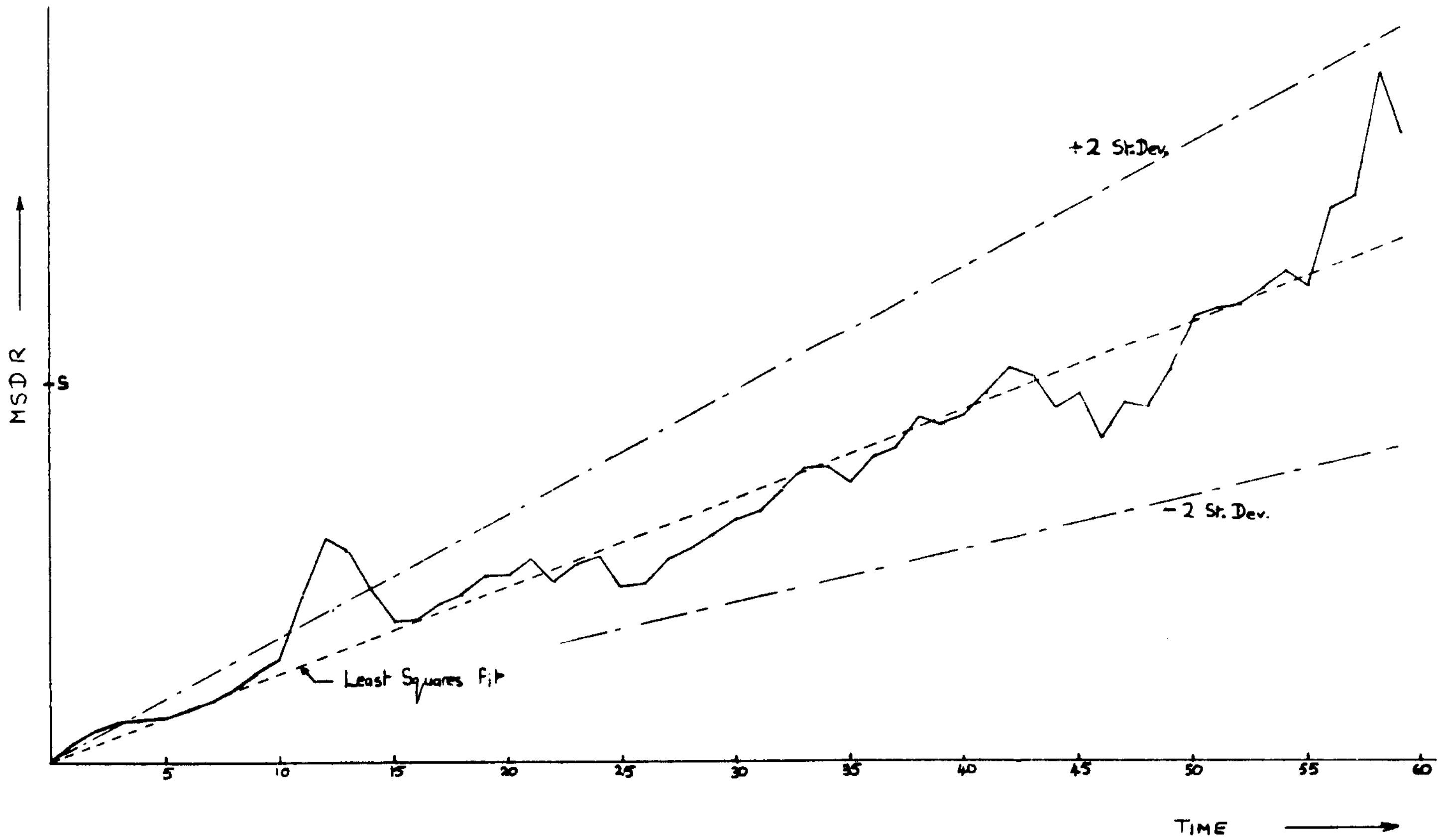


Fig. 6-2. MEAN SQUARED DRIFT RATE - ZERO CORRECTION

3) The slight curvature from t = 0 to 3 may indicate a stationary drift rate component or some correlation in the Random Walk drift acceleration, but it was not considered significant at this stage to investigate further (Chapter 7).

6-2-3 Estimate of the Drift Rate Mean

The first check applied to the Estimated Mean Drift Rate / Time plot was to test the Model hypothesis

$$E(DR) = 0 \tag{5.3}$$

This check was combined with the Est(k) derived in Section 6-2-2 and, as before, it was decided to use the expected value of k as being equal to Est(k). From equations (5.14) and (5.16), with R = 50, it follows that the Standard Deviation of the population of the Est DR(t) is given by $\sqrt{\frac{k \cdot t}{50}}$, and the Mean of this population is zero. As can be seen from the plot in Fig. 6-3, all the Estimates of the Mean Drift Rate fall within the ± 2 Standard Deviation limits as drawn.

Another feature observable in Fig. 6-3, is the possibility that some deterministic trend may be present as illustrated by the non-randomness of the Est$\left[DR(t)\right]$ about the E(DR) of zero. This could be explained, in part, by a time correlation of the Drift Rate (Section 6-8). It is also interesting to note that, for the Random Walk model, should the drift rate (for any reason) be off from zero by (say) "a" degrees /hour, then the most likely value of the drift rate at any later time is still "a" degrees / hour, even though E(DR) = 0 (cf. from t = 38 to 59).

Conclusions

1) The data does not contradict the Random Walk hypothesis when combined with the Slope of the MSDR/Time plot, the test being applied at the "2 Standard Deviation of the Mean" level.

2) The possibility of some deterministic trend, e.g., a non-zero mean "ramp", cannot be eliminated unless more data is included in the sample.

6-2-4 Estimate of the Drift Rate Frequency Distribution

It was shown in Chapter 4, para. 7, that the Model hypothesis implies that the drift rate will tend to a Normal distribution, but with a time varying Standard Deviation. To test this statement, the Ensemble drift rate distributions were evaluated at 5 different times and the results are plotted in histogram form in Fig. 6-4.

Conclusions

1) Because of the small sample for each distribution (50 points), the only reasonable comment in support of the hypothesis of a Normal Distribution, is that it has not been disproven.

2) The distributions do become "flatter", the larger the value of t, in accordance with an increasing standard deviation, and if a Normal distribution is present, then, with this small sample, it would be unlikely that any points would fall outside the 3 standard deviation value – only one point did, gyro 45 at time 14 (15 on the original data), as shown in Fig. 6-4.

Note: On Fig. 6-4, the estimated Mean has been shown dotted, with the "± 2 Standard Error of the Mean" range shown at the top of each distribution. This range has been calculated in accordance with equation (5.18), viz.,

$$2 \text{ St. error of the mean} = \frac{2 \cdot s(t)}{\sqrt{R}}$$

where s(t) is the computed estimate of the ensemble standard deviation at time t (equation (5.11) and Program 2). In all cases the $\pm \frac{2 \cdot s(t)}{\sqrt{R}}$ range encompasses the zero drift rate level, which gives additional support to Section 6-2-3 (where the population statistics were assumed to be known).

6-3 Drift Rate – Figure of Merit Test

Test theory – Chapter 5, para. 3.
Figs. 6-5 and 6-6

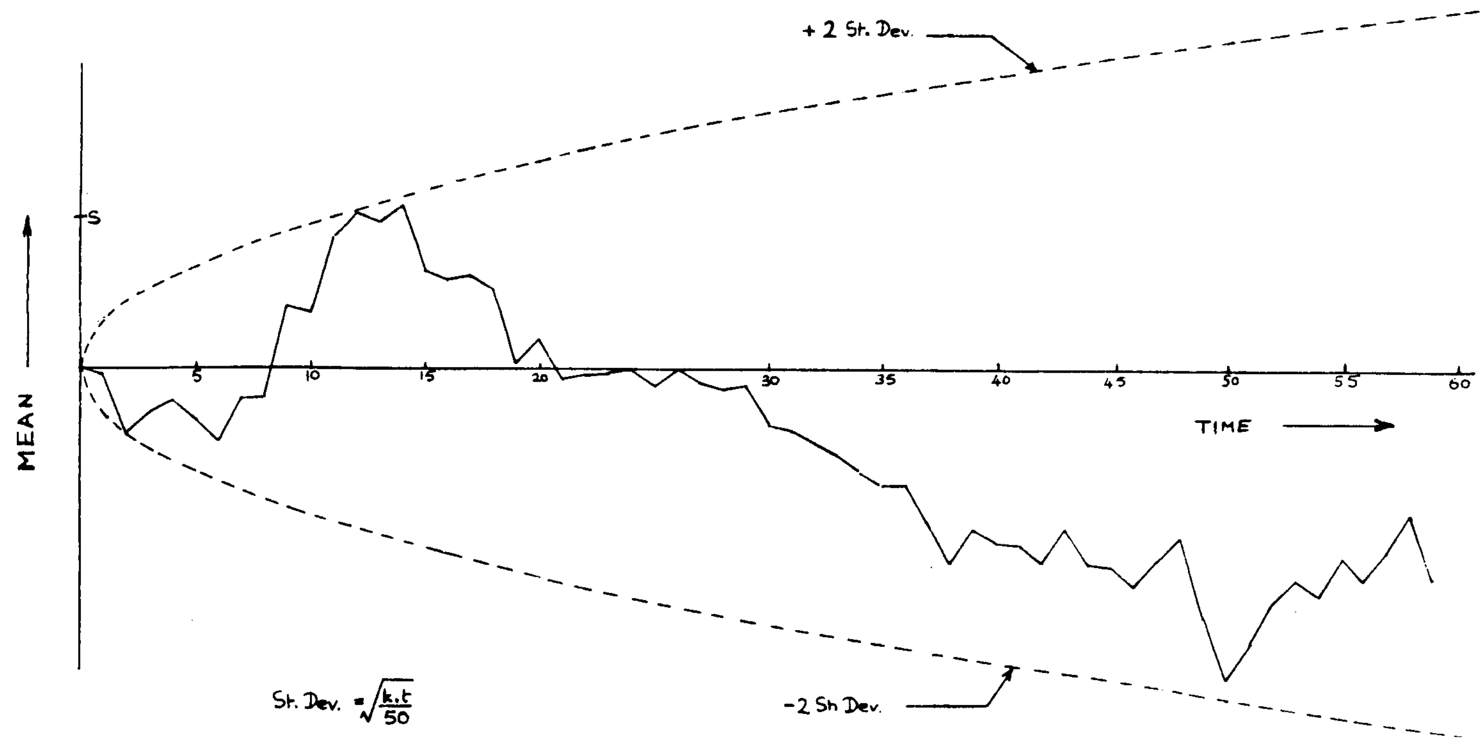
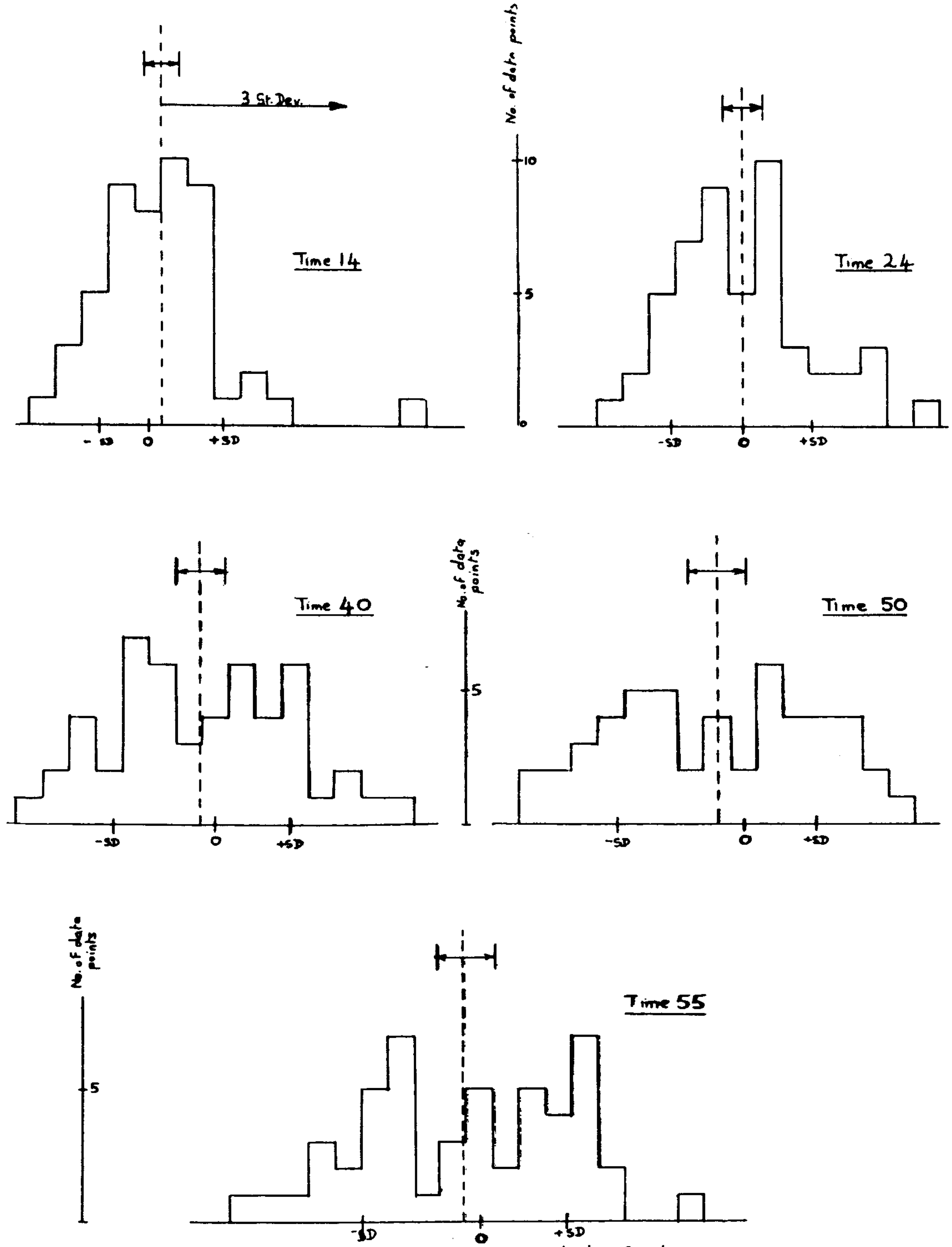


Fig 6-3 MEAN DRIFT RATE - ZERO CORRECTION



1) The range of ± twice the Standard error of the Mean is shown at the top of each Histogram.

2) All points fall within ± 3 Standard Deviations of the Estimated Mean (shown dotted), except for Time 14

3) The range of ± 1 Standard Deviation (from the Est. Mean) is shown on the horizontal axis of each histogram.

Fig. 6-4 ENSEMBLE DRIFT RATE DISTRIBUTIONS - ZERO CORRN.

The Est(FOM)^2 is clearly an easier parameter to investigate, as the Random Walk E(FOM)^2 is a straight line (equation (5.23)). Equation (5.24a) can be modified by reversal of the summations to:

$$\text{Est}\left[\text{FOM}(t_n)\right]^2 = \frac{1}{n+1} \cdot \sum_{i=0}^{n} \frac{1}{R} \sum_{r=1}^{R} \left[\text{DR}(t_i)\right]_r^2$$

which, from equation (5.9) $$= \frac{1}{n+1} \sum_{i=0}^{n} \text{Est}\left[\text{MSDR}(t_i)\right] \qquad (6.1)$$

Therefore, this estimate can be directly computed from the results in Sections 6-2-1 and 6-2-2. The stimates of the squared Figure of Merit are shown in Fig. 6-5 for the original data, and in Fig. 6-6 for the zero corrected data. To test the Random Walk hypothesis,

$$E\left[\text{FOM}(t)\right]^2 = \frac{k \cdot t}{2} \qquad (5.23a)$$

a straight line must be fitted to the data and passing through the origin of the graph in Fig. 6-6; however, implicit in the usual methods of least squares fitting is that the deviations of the data from the fitted line be independent from each other. But, from equation (6-1) it can be seen that a cumulative error can be present, thereby invalidating the least squares fit to the Est(FOM)^2 data.

Consider, in equation (6-1) that

$$\text{Est}\left[\text{MSDR}(t_i)\right] = E\left[\text{MSDR}(t_i)\right] + e_i$$

where the e_is are independent errors.

then $$\text{Est}\left[\text{FOM}(t_n)\right]^2 = \frac{1}{n+1} \sum_{i=0}^{n} E\left[\text{MSDR}(t_i)\right] + \frac{1}{n+1} \sum_{i=1}^{n} e_i$$

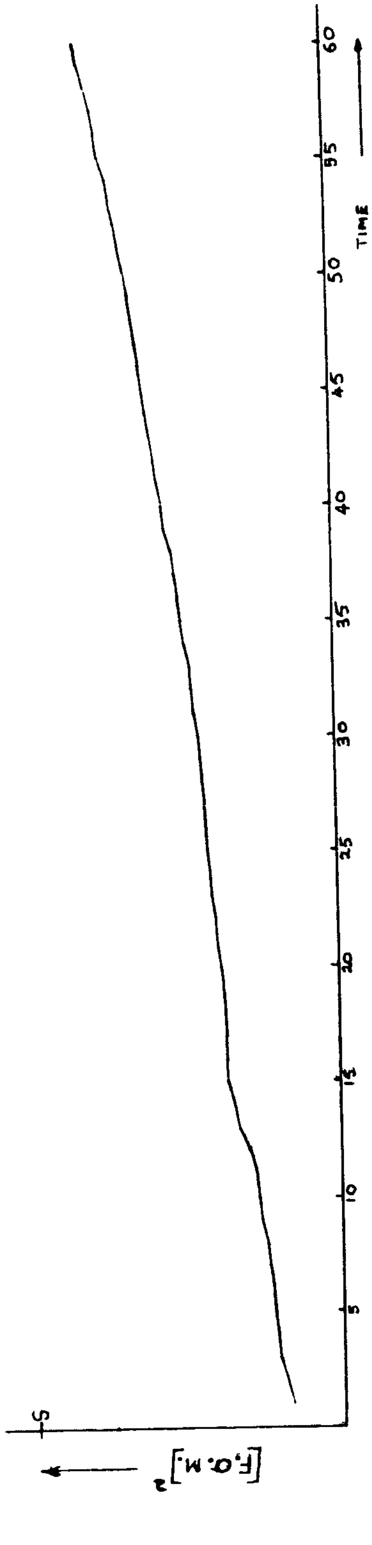


Fig. 6-5 SQUARED FIGURE OF MERIT – ORIGINAL DATA

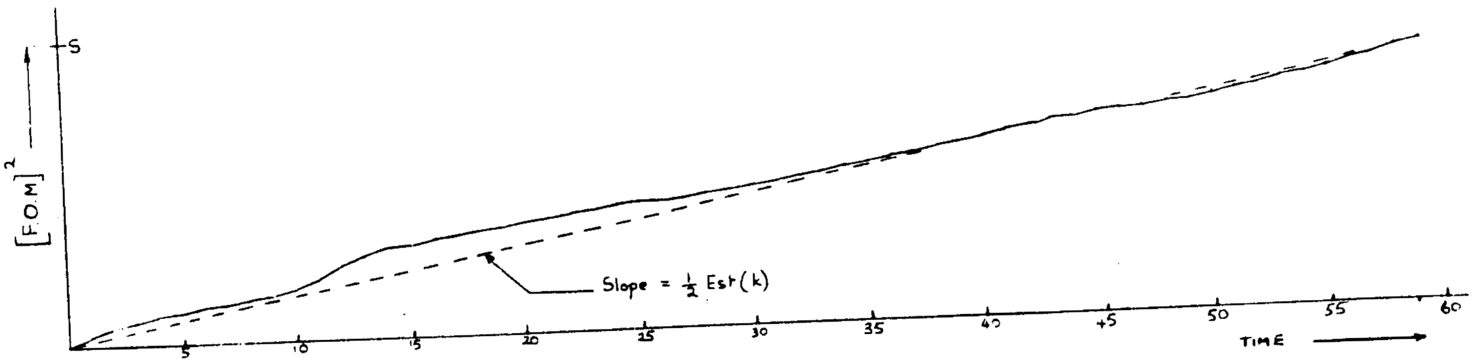


Fig. 6-6 SQUARED FIGURE OF MERIT - ZERO CORRECTION

Figs. 6-5 and 6-6 are drawn to the same scale as Figs 6-1 and 6-2 if $$f_n = \frac{1}{n+1} \sum_{i=1}^{n} e_i \tag{6.2}$$

then, even though the e_is are independent, it can be seen that the f_ns will not be independent.

The straight line through the origin shown in Fig. 6-6, is the value predicted if the Random Walk model is valid (equation (5.23a), where the value of k=Slope of the MSDR/Time plot). The poor fit for the smaller values of t is explained by equation (6-2), as should e_i be continuously positive over the early range of time, then e_n may become large. It can be seen from Fig. 6-2 that this is the situation, e_i is positive up to time 21. As t becomes large, it would be expected that the e_is from 0 to t, would have an average coming closer and closer to zero, and hence this explains the much better fit of Est$\left[FOM(t)\right]^2$ for large t.

<u>Conclusions</u>

The $(FOM)^2$ parameter was not calculated to provide additional proof as to the validity of the Model hypothesis, but rather to give a complete picture of the data, as this is the manufacturer's test parameter (square root thereof). It supports the model to the same degree as the MSDR does (this is not surprising as it can be directly derived from it – equation (6.1)).

6-4 Incremental Drift Rate – Stationarity Test

Test theory – Chapter 5, para. 4.
Program Nos. 4, 5.
Figs. 6-7 to 6-10.

6-4-1 Estimate of the Mean of the Incremental Drift Rate

The ensemble statistics of the IDR were calculated from equations (5.27) and (5.28), Program 4, and the Est(IDR)/Time graph is shown in Fig. 6-7. The Est$\left[IDR(t)\right]$ is very randomly distributed about the Model E(IDR) = 0 (equation (5.25)), and with the assumption that the estimates are uncorrelated (Section 6-6), the value of $\frac{1}{59}\sum_{i=1}^{59} IDR(t_i)$, will give a

E = one Standard error of the Mean = $\frac{S(IDR)}{\sqrt{50}}$

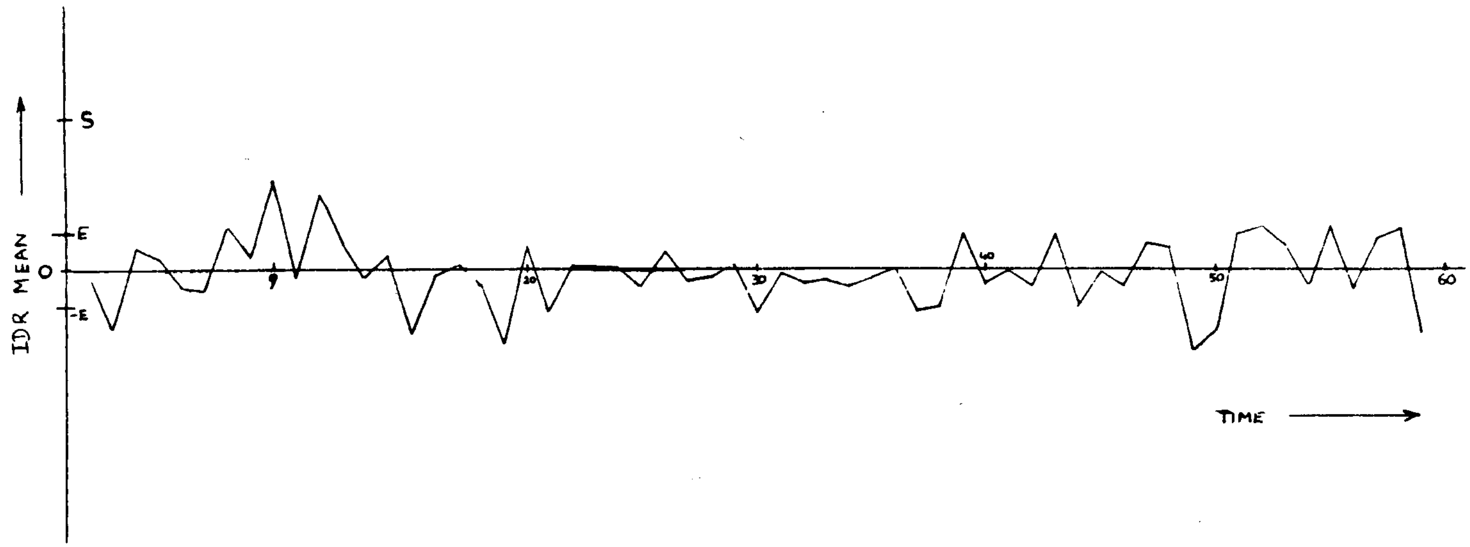


Fig 6-7 ENSEMBLE MEAN OF THE INCREMENTAL DRIFT RATE

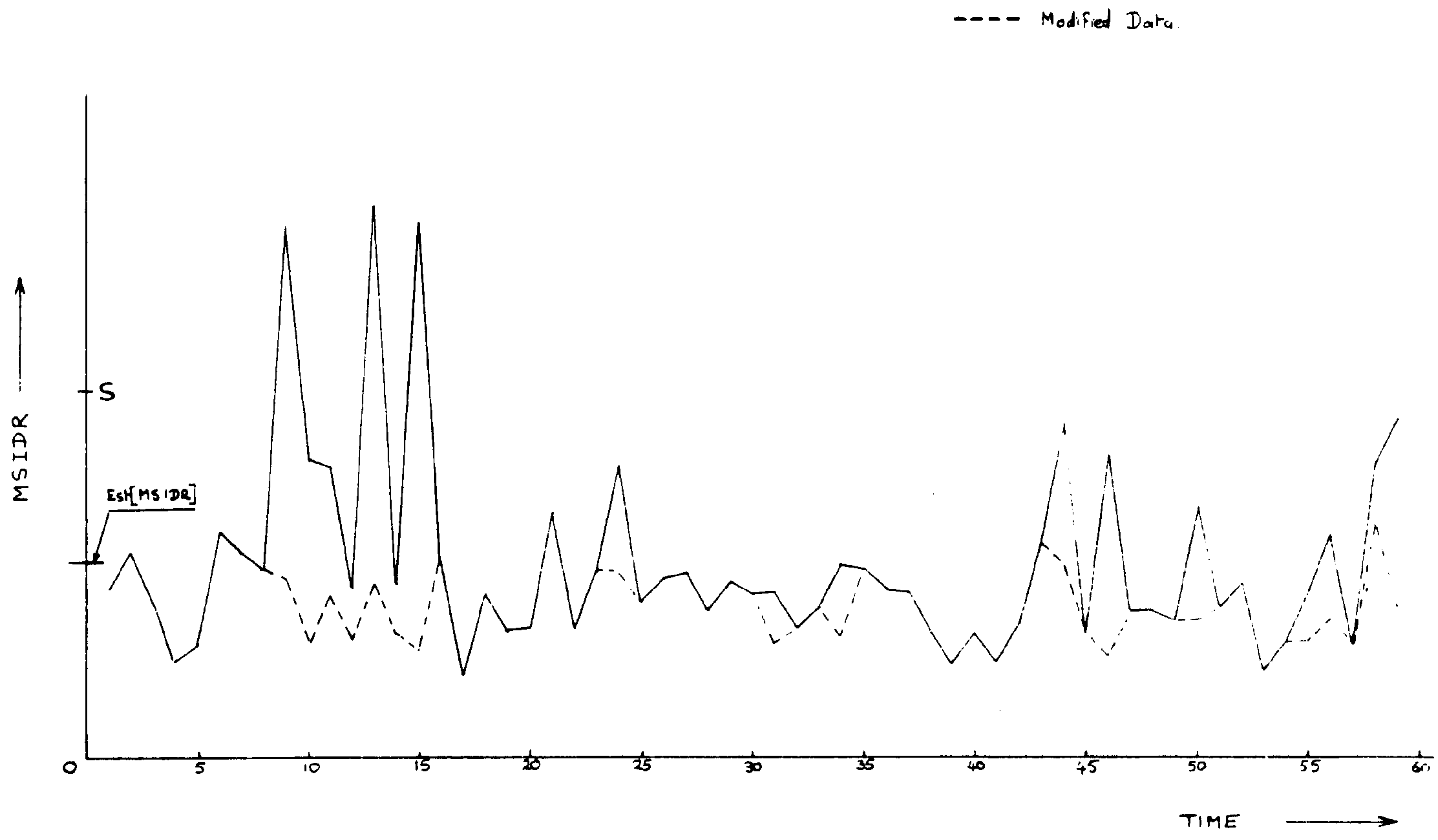


Fig. 6-8 ENSEMBLE MEAN SQUARED INCREMENTAL DRIFT RATE

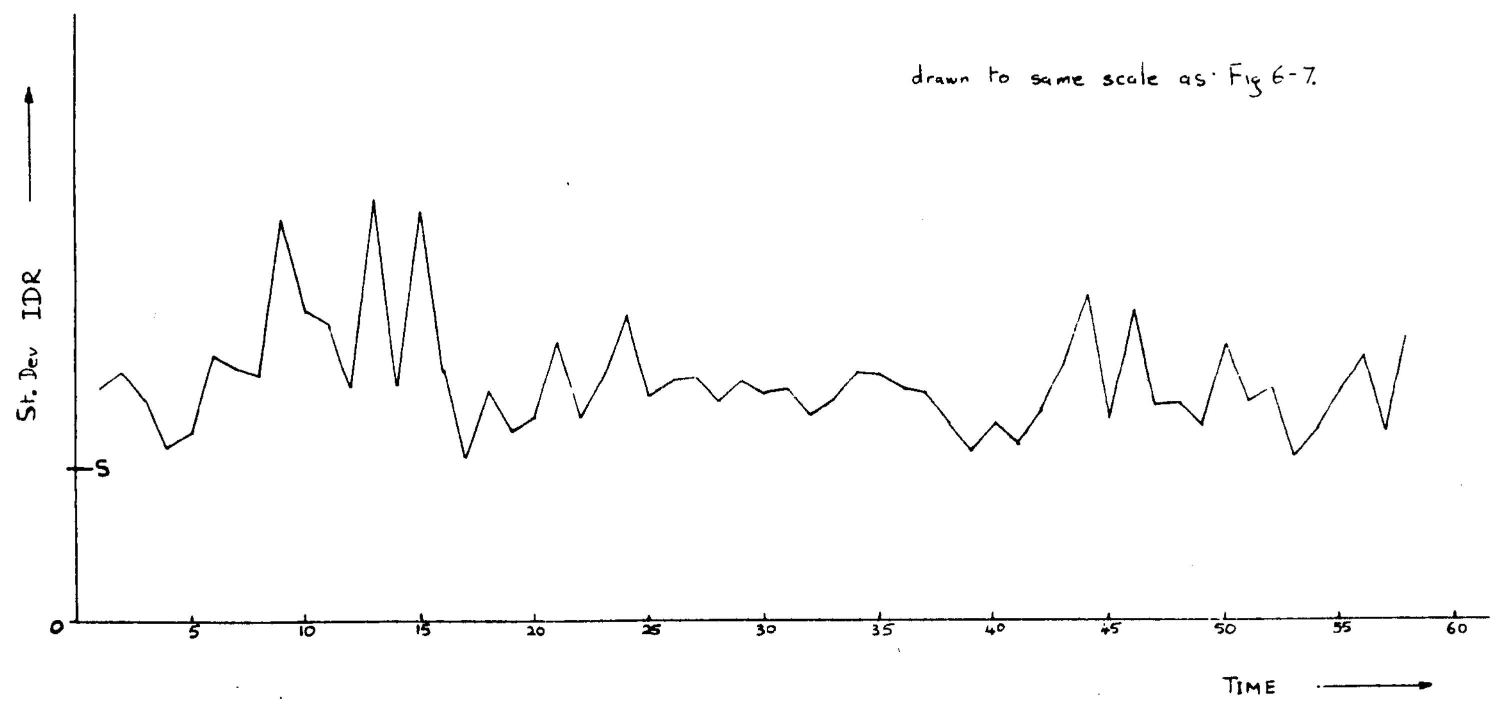
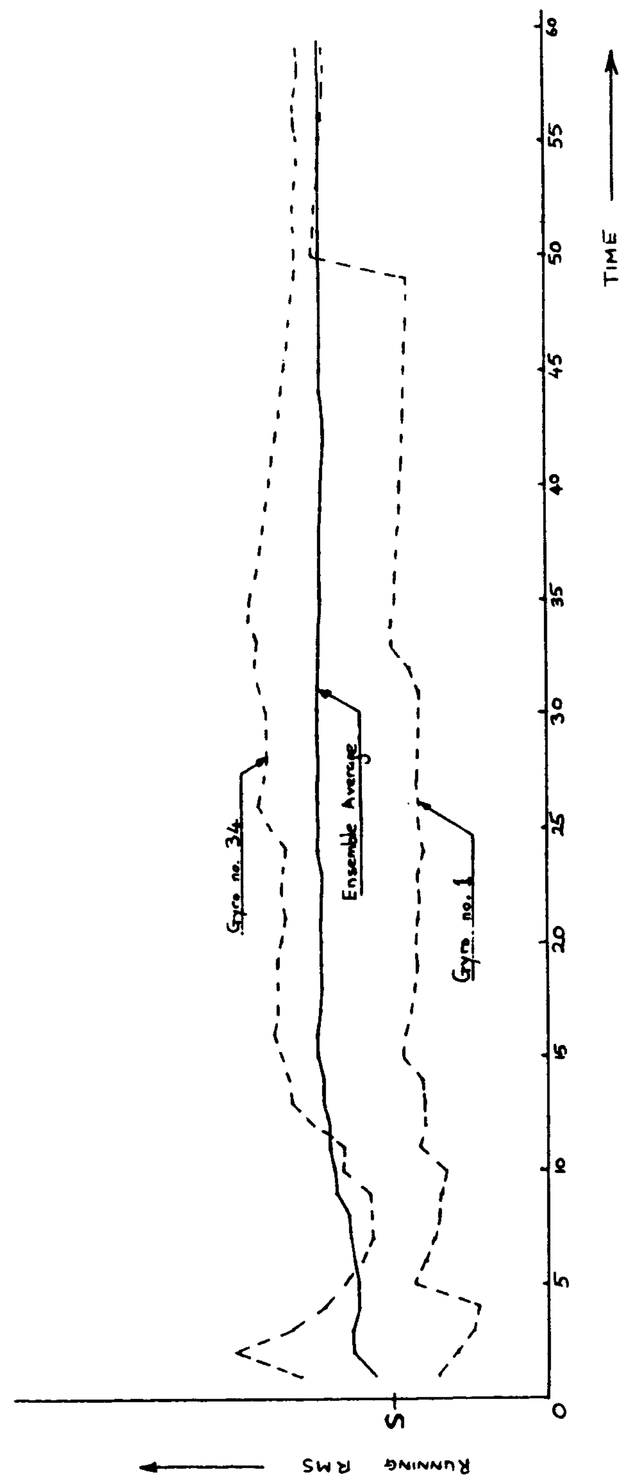


Fig 6-9 ENSEMBLE STANDARD DEVIATION OF THE INCREMENTAL DRIFT RATE



Drawn to the same scale as Fig 6-7

Fig. 6-10 RUNNING R.M.S. OF THE INCREMENTAL DRIFT RATE good estimate of the true mean – this value was very close to zero. The Standard Deviation of the Random variable EST IDR(t), (i.e., the Standard error of the Mean), is to a good approximation, $= \frac{S(IDR)}{\sqrt{R}}$ – where S(IDR) is calculated in equation (6.3). As can be seen from equation (5.16) most of the Estimates are within ± one Standard error.

Conclusions

The Estimate of the Mean IDR supports the Model hypothesis very closely.

6-4-2 Estimate of the Mean Squared Incremental Drift Rate

The model hypothesis requires that the IDR be a stationary process, consequently the MSIDR should be constant with time. The first impression of the plot in Fig. 6-8, is that this is not the case; however, the following series of steps is offered in mitigation for the Random Walk hypothesis.

1) Visual interpretation: If initially one ignores the three peaks at time 9, 13, 15, then there does not appear to be an obvious trend of the $\text{Est}\left[\text{MSIDR}(t)\right]$, then inclusion of the three peaks does not add a trend.

2) Statistical interpretation: It was thought possible that the peaks in the plot could have been caused by the sample size for each estimate (50) not being large enough. To test this, it was assumed that the IDR was Normally distributed (this is not implied in the model hypothesis (Chapter 5, para. 6-2)). One could now apply limits in a similar manner to those applied to the MSDR plot, but at this stage the results of the next test (IDR correlation) were available, indicating that IDR was reasonably uncorrelated, therefore a different procedure was adopted. If the 2,950 data points used to obtain Fig. 6-8, are independent and Normally distributed then it could be expected (on the average) that only 8 of these points would lie outside the bound of ± 3S from the Mean, where S is the Estimated Standard Deviation of the 2,950 points. But, from Fig. 6-7, the Mean is effectively zero, and S is calculated from the results of Program 4, viz.

$$S(IDR) = \sqrt{\frac{1}{59} \sum_{i=1}^{59} Var(t_i)} \tag{6.3}$$

(since all points are assumed independent)

where $Var(t_i)$ is the Estimated Variance calculated in a similar manner to equation (5.10);

however, it was found that 39 points (cf. 8), fell outside this range. The bound was then increased to ±3.5S at which level not more than one point would (on average) have been expected outside the range – there were 22. Therefore, either the IDR is not Normally distributed, or the distribution tails are being weighted more than the remainder of the population (which still retains the basic Normal distribution shape). At this stage it was decided not to abandon the Normal distribution theory, as many of the "outsiders" had their origin in the drift rate data points previously noted for gyros 45 and 47 (Chapter 3, para. 3(4)). The estimates of the MSIDR were recalculated with these 22, out the total 2,950 points, removed. The corresponding modifications to the MSIDR/Time plot are shown by the dashed lines in Fig. 6-8, and one would now conclude, that the modified data is indicative of a stationary process.

3) Returning to the unmodified data, it was realised that, being a "squared" statistic, perturbations would be emphasised. Therefore, since the estimate of the Mean IDR is so close to zero (Fig. 6-7), the "non-squared" statistic of the ensemble Standard Deviation can be considered as a measure of a stationary process in this situation. The results (from Program 4) are shown in Fig. 6-9, and the same type of perturbations, but less pronounced, can be seen when compared to Fig. 6-8. However, one can now see more clearly, that there is no obvious trend, other than a constant zero slope, present in the data.

4) Another technique, when testing for a stationary, ergodic process, uses statistic computed in time (cf. ensemble) as an indicator. By choice of the computed statistics, one can still further smooth out any perturbations caused by small samples. For instance, if the Running RMS

of the IDR is plotted, it should very quickly, as time increases, tend to a zero slope straight line – the time to achieve this, and the subsequent non-smoothness of the plot, reflect the degree of support given to a stationary hypothesis. In Fig. 6-10, the ensemble average of the Running RMS Incremental drift rate computed for each gyro separately (Program 5), is shown in Fig. 6-10, together with the Running RMS for 2 of the individual gyro runs selected at random (shown dashed). The ensemble average supports the stationary hypothesis and the Running RMS for each of the individual gyro runs tended to a constant value in a very short time, except where isolated large steps were present in the IDR data. This last point is illustrated by the plot for Gyro No. 1, shown in Fig. 6-10, where the sharp rise at time 50, is accounted for, by the IDR at that time having a value given by -5.5S(IDR); it was one of the 22 points referred to previously.

Conclusions

1) It is thought likely that a larger sample would have smoothed out the perturbations sufficiently to support the Model hypothesis, that the MSIDR is constant with time; and, combined with the Est(IDR) = 0 at any time (Chapter 6, para. 4-1), that the IDR is a random stationary process.

2) It will later be shown (Section 6-7-2), that the IDR may not be Normally distributed, therefore rejection of all 22 points in step (2) of the preceeding analysis, may not have been acceptable. Nevertheless, the three data points causing the peaks at times 9, 13, 15 in Fig. 6-8, will be seen to be a long way outside the probability zone covering 2,950 points, even with the better estimate fo the IDR frequency distribution. Consequently, it is considered reasonable to ignore these three peaks.

6-5 Comparison of the Estimates of the Mean Squared Incremental Drift Rate and the Slope of the Mean Squared Drift Rate /Time Plot.

Test theory – Chapter 5, para. 2-2.
Figs. 6-2 and 6-8.

From equations (5.6) and (5.7), with the unit of time defined as Δt_o (1 hour).

$$k = E(\text{MSIDR}) \qquad (6.4)$$

where equation (6.4) is a numerical equality

and k = Expectation of the slope of the MSDR /Time plot.

The least squares Est(k) was derived in Section 6-2-2, and to derive a best estimate of the MSIDR, the average was computed of all the $\mathrm{Est}\left[\mathrm{MSIDR(t)}\right]$.

$$\text{i.e.} \qquad \mathrm{Est(MSIDR)} = \frac{1}{59} \sum_{i=1}^{59} \mathrm{Est}\left[\mathrm{MSIDR}(t_i)\right] \qquad (6.5)$$

(this value is shown on Fig. 6-8)

Equation (6.5) requires that the Incremental Drift Rate be a stationary process, and equation (6.4) implies that the IDRs are statistically independent. The results of these calculations can be expressed, relatively:

$$\mathrm{Est(MSIDR)} = 2.24\mathrm{Est(k)} \qquad (6.6)$$

This proportionality factor of 2.24 is obviously too large to support the Model factor of 1.0. Elimination of the 22 points outside the range $\pm 3.5\mathrm{S(IDR)}$, (Section 6-4-2), would give a lower factor, the result being,

$$\mathrm{Est(MSIDR)} = 1.73\mathrm{Est(k)} \qquad (6.7)$$

Even with the data modified in this manner, the proportionality factor of 1.73 is still too large.

Conclusions

1) Although the IDR is probably a stationary process, it does not support the Model hypothesis that the Expected value of the MSIDR be numerically equal to the slope of the MSDR/Time plot.

2) Further investigation of this discrepancy is delayed until Chapter 7, where a more complex model is discussed. In the remaining tests of the validity of the simple model, where equations are used which imply the equality given in equation (6.4), the value of Est(k) will be used in calculations and not the value of Est(MSIDR). This choice is made because the Model is intended, ultimately, to reproduce the Drift Rate, and not the Incremental Drift Rate.

6-6 Incremental Drift Rate – Correlation Test

Test theory – Chapter 5, para. 5.
Program No. 6.
Fig. 6-11.

As a zero Mean of the IDR has been established in Section 6-4-1, the estimate of the Autocorrelation Function was computed in accordance with equation (5.31), (Program 6). The program was arranged so that the individual Autocorrelation Functions for each run (equation (5.30)) were also available for comparison with the final ensemble average. The ensemble estimate of the correlation functions is shown in normalized form (i.e. ACF(O) = 1), in Fig. 6.11. A check of the 50 individual autocorrelation functions showed that they were generally of the same form, so the ensemble average was considered to be the best estimate. There was some tendency for greater negative correlation to be present, but it was not considered at this stage (Chapter 7).

The Model ACF(τ) of the IDR is given in equation (5.29). In normalized form this becomes:

$$\text{Normalized ACF}(\tau) = \begin{matrix} 1 & \text{when } \tau = 0 \\ 0 & \text{when } \tau \neq 0 \end{matrix} \qquad (6.8)$$

Conclusions

1) It would have been surprising if no correlation (when $\tau \neq 0$) had existed in any of the estimated ACFs, and therefore one concludes from Fig. 6.11, that the Random Walk hypothesis is not disproven by this test. To this statement, the qualification must be added that a normalized ACF of 0.06 has not been considered significant.

2) The above conclusion will be re-examined in more detail in Chapter 7.

6-7 Frequency Distributions

Test Theory – Chapter 5, para. 6.
Program No. 7.
Fig. 6-12.

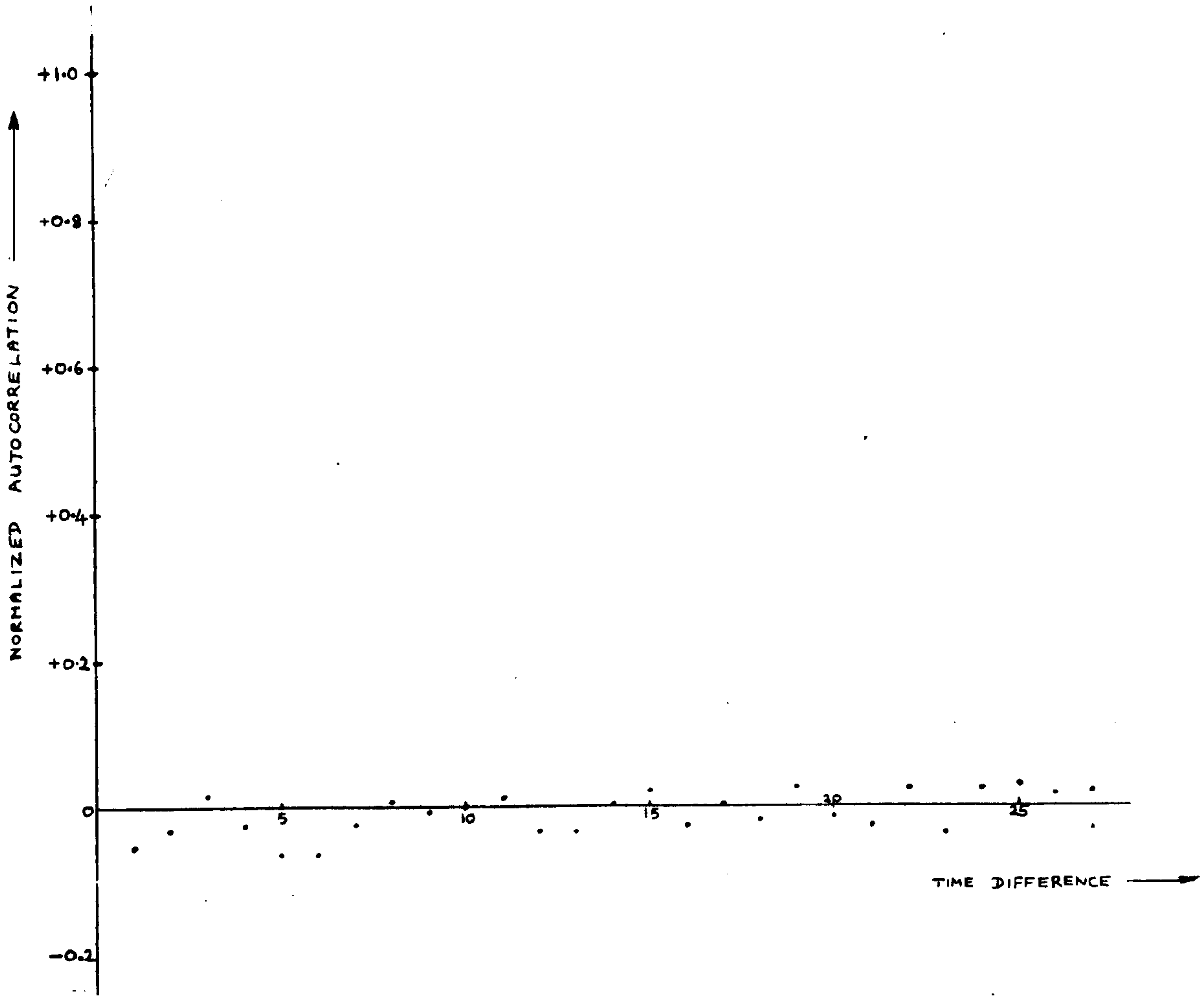


Fig. 6-11 ENSEMBLE AVERAGE OF THE AUTO CORRELATION (normalized) OF IDR

6-7-1 Drift Rate Distributions

Previously discussed in Section 6-2-4.

6-7-2 Incremental Drift Rate Distribution

Combining the theory of Chapter 5, para. 6-2 with the conclusion in Chapter 6, para. 6, the IDR distribution can be estimated from 2,950 independent (to a good approximation) data points. This was done by grouping the data points (Program 7) and the results are presented as a histogram in Fig. 6-12. The vertical scale gives the number of points falling in each rectangular block of the histogram, and all blocks are of equal width. To give a better presentation of the major part of the distribuion, the "tails" have been drawn separately; with their vertical scale magnified by 10 and width scale diminished by 4.

The first impression was that the data might be Normally distributed except for the long tails. However, a closer look suggested that there was no point of inflexion (the one standard deviation point in the Normal distribution). To confirm this, a plot of a Normal distribution, using standard tables, with Mean, zero, and Standard Deviation given by S(IDR) from equation (6.3), was superimposed on the histogram (Curve A), and as can be seen, the fit is not very good.

An interesting point was then noticed. For a better fit to the major part of the distribution, using a Normal curve, the Standard Deviation needed to be smaller. Therefore a new Normal distribution curve was superimposed (Curve B), with Mean, zero, and a standard deviation given by $\sqrt{Est(k)}$, where k has been previously defined as the expectation of the slope of the MSDR/Time plot. The fit would appear to be better than Curve A over the majority of the Histogram, except for the tails (as was to be expected). The significance of this result is that, based on the Model hypothesis, the Variance of the IDR distribution (not necessarily Normal) is k. This can be evaluated from equations (5.25), (5.26) and (6.4). The improved fit is a very useful result and will be referred to later (Chapter 8, Table 8-1).

Returning to the correct estimate of the IDR standard deviation it was thought that a better mathematical fit of the histogram would be achieved by considering some other type of Distribution. The obvious choice of a simple distribution is an exponential one, symmetric about zero;

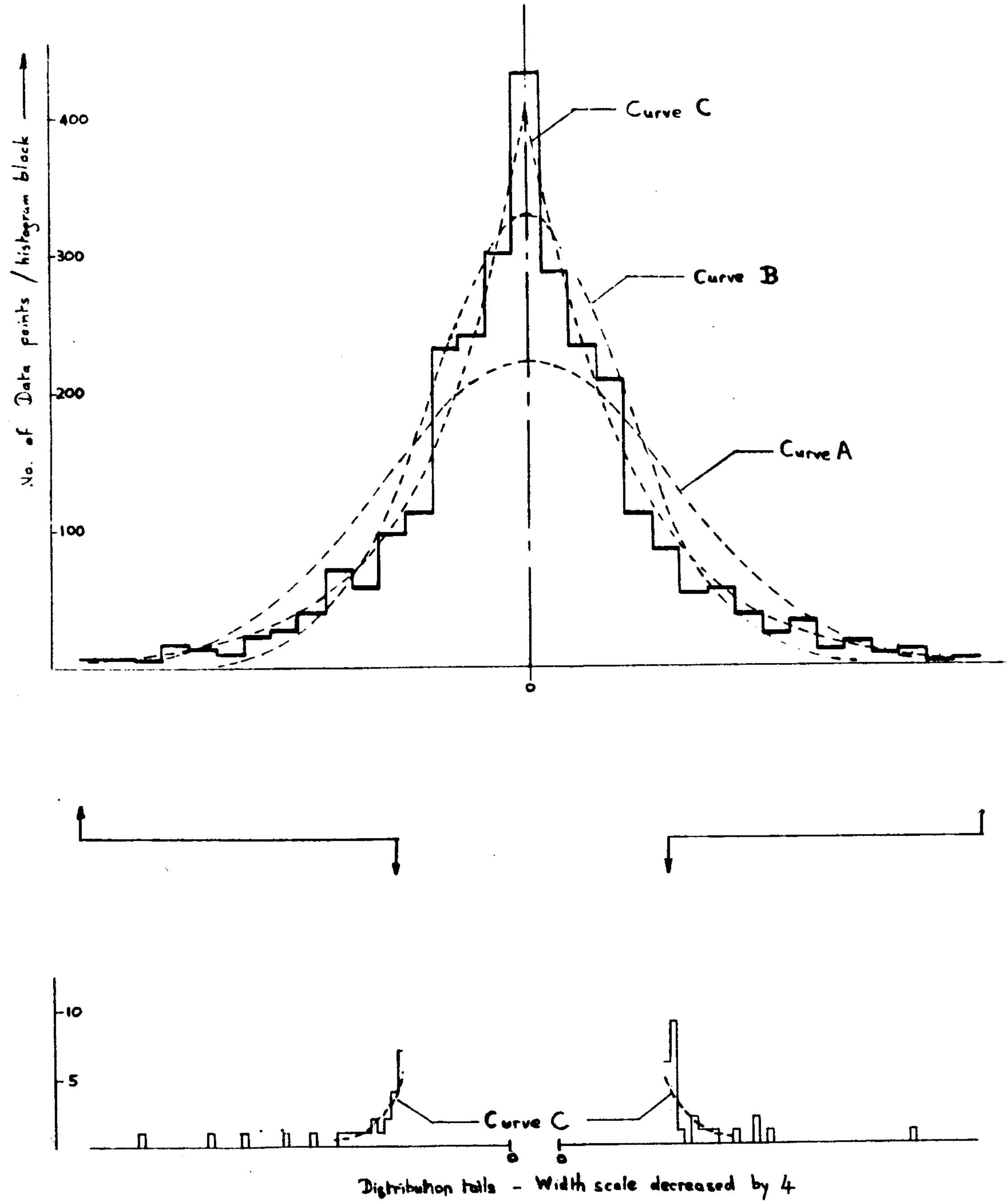


Fig 6-12 INCREMENTAL DRIFT RATE DISTRIBUTION

was given added support by replotting the histogram with a vertical logarithmic scale — ignoring the scatter of the tails, a very reasonable tri-angular shape resulted. As this distribution is not usually given in standard statistical texts, the theory is developed in Appendix A, where the probability distribution is shown to be

$$f(x) = \sqrt{\frac{1}{2\sigma^2}} \cdot e^{-\sqrt{\frac{2}{\sigma^2}} \cdot x} \qquad (A.6)$$

and the boundary containing, on average, all but one of the 2, 950 IDR points, is given by:

$$x = \pm 5.65\,\sigma \qquad (A.9)$$

where σ is the standard deviation of the distribution.

Now, if S(IDR) from equation (6.3) is substituted for σ in equation (A. 9), it is found that only 6 data points lie outside this boundary, which is a much more logical answer than the 22 points for a Normal Distribution with a boundary of ± 3. 5S(IDR), as used in Section 6-4-2. These 6 values are listed in Table 6-1, as functions of S(IDR).

Gyro Run No.	Time	IDR As A Function of S(IDR)
6	44	- 6.2S
6	46	- 7.3S
45	9	+ 9.7S
45	15	- 8.3S
47	11	+ 5.9S
47	13	-10.2S

TABLE 6-1

Analysing these 6 points, it can be seen that the last 4 points can be directly attributed to the visually observed "bad" drift rate points, and the first 2 points to the Gyro with most fluctuations, as listed in Chapter 3, para. 3(4). Also, the values at times 9, 13, 15 are so large, that removal of these points eliminates the 3 peaks at the corresponding times on the MSIDR/Time plot of Fig. 6-8, which is a much sounder reasoning than that previously used in Section 6-4-2. A recalculation of

S(IDR) with these 6 points removed (equation (6.3)) gave,

$$S(IDR)_{modified} = 0.94.S(IDR)_{original} \tag{6.9}$$

or in the terminology of equation (6.6)

$$Est(MSIDR) = 1.98.Est(k) \tag{6.10}$$

(since the Variance of the IDR differed only in the last figure, from the MSIDR)

The value of $S(IDR)_{modified}$ was considered to be the best estimate of the standard deviation of the Incremental Drift Rate, and consequently, this value was substituted for σ in equations (A.6) and (A.9) to derive the estimated "mathematical" distribution of the Incremental Drift Rate. The result (Curve C) has been superimposed on the histogram in Fig. 6-12 and the fit to the actual data is remarkably close.

Conclusions

1) Although the IDR (or Drift Acceleration) distribution was not specified as part of the model hypothesis (Chapter 4, para. 7 and Chapter 5, para. 6-2), a knowledge of this distribution is required if ultimately it is intended to generate synthetic data, representative of actual gyro data.

2) The prior assumption of the IDR being Normally distributed, is not supported by the actual data distribution (Curve A). A better Normal distribution fit is obtained if a standard deviation, given (numerically) by the square root of the slope of the MSDR/Time plot – i.e. $\sqrt{Est(k)}$, is used (Curve B). The expectation of this last statistic is equal to the expectation of the standard deviation of the specified Model IDR. (Note: it has been shown previously, that this value was not supported by the data (equation (6.6)).

3) The Normal distribution hypothesis was maintained in previous tests, with the thought that the large tails might have been contributed by "noise", introduced by the method of obtaining the IDR from the original Drift Rate data. (Differencing is one form of differentiating; and differentiating is, inherently, a "noise" accentuating source.) However, from the

histogram in Fig. 6-12, it is not possible to extend this reasoning to account for the large peak surrounding the Mean, zero, of the IDR. Therefore an exponential distribution, of the form given by equation (A.6), was fitted with a S(IDR) modified by the elimination of 6 (out of 2, 950) unlikely data points. The closeness of the resulting fit was such that the conclusion is:

If the incremental drift rate, as calculated, is a true representation of the actual difference of the gyro drift rate over one hour intervals, then this parameter has a probability distribution, to a very good approximation, given by

$$f(x) = \sqrt{\frac{1}{2\sigma^2}} \cdot e^{-\sqrt{\frac{2}{\sigma^2}} \cdot |x|} \tag{A.6}$$

where the best estimate fromthe data, for the variance σ^2, is, from equation (6.10),

$$Est(\sigma^2) = 1.98 Est(k) \tag{6.11}$$

(Note: equation (6.11) has been expressed in this form for classification purposes only – Est(k) is given in Vol. 2. The Est(k) was not used, in any way, to obtain the Est(σ^2).)

This conclusion was unexpected at the outset, and the closeness of fit makes this a somewhat curious result, leading to the question as to why this should be so? A possible answer is discussed in Chapter 7, para. 3.

4) The value of $S(IDR)_{modified}$ will be used for the St. Dev. of the Incr. Drift Rate in all subsequent calculations.

6-8 Drift Rate Correlation

Test theory – Chapter 5, para. 7.
Program No. 8.
Figs. 6-13 and 6-14.

It was stated in Chapter 5, para. 7, that a family of curves is probably the most useful presentation of the Drift Rate Correlation test results.

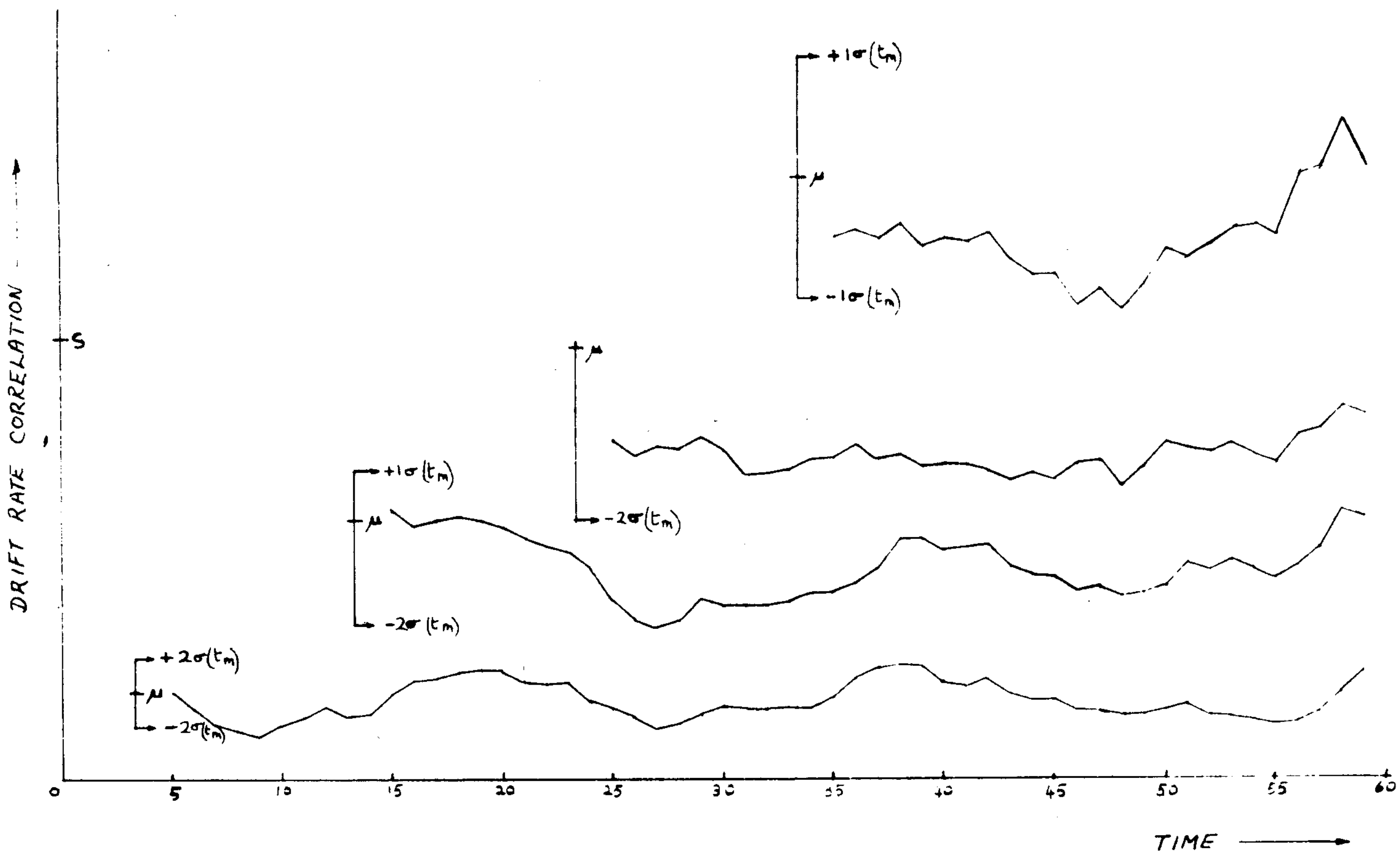


Fig. 6-13 DRIFT RATE CORRELATIONS For t_m = 5, 15, 25 and 35

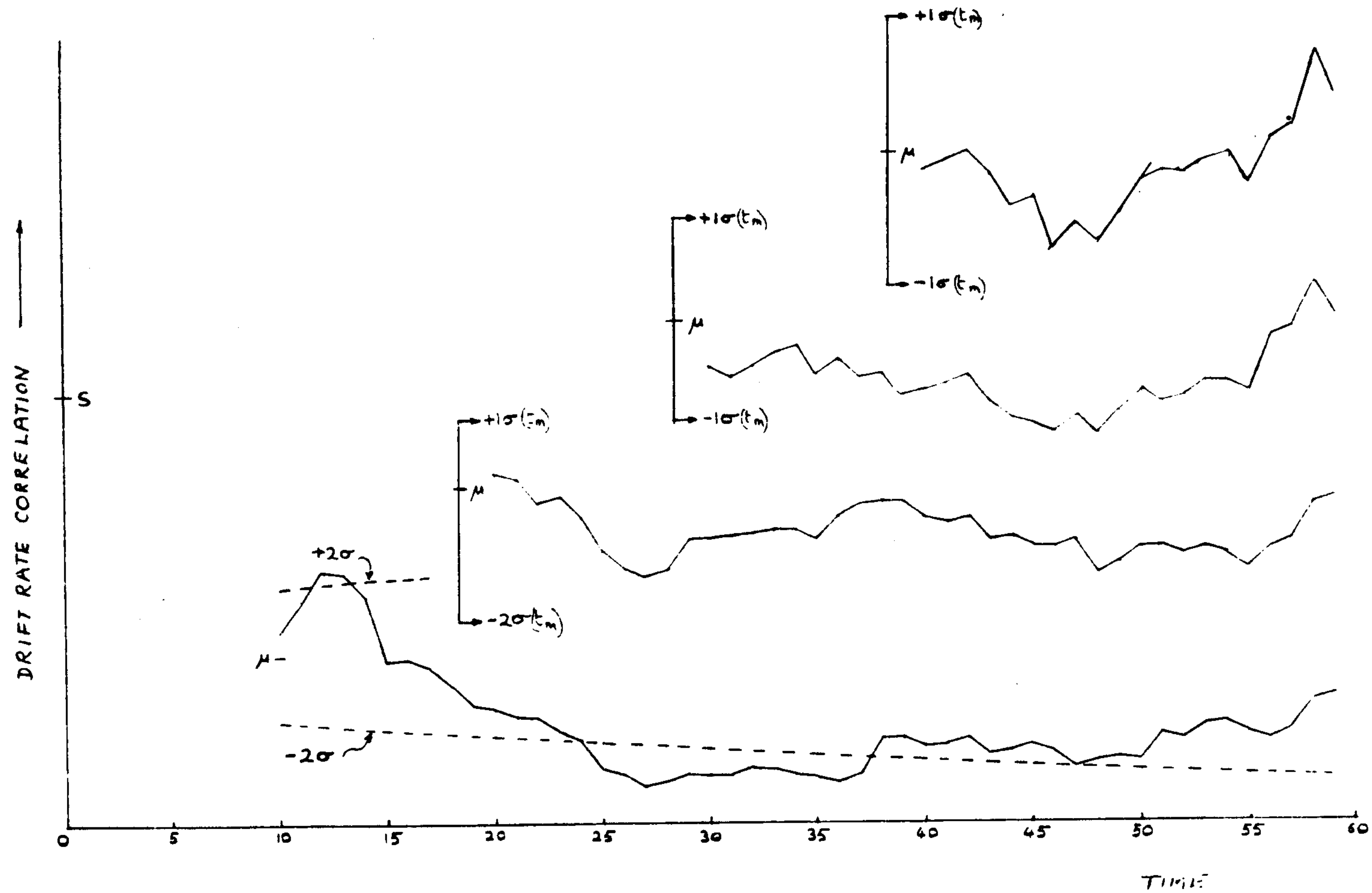


Fig. 6-14 DRIFT RATE CORRELATIONS for t_m = 10, 20, 30 and 40

Eight correlation curves were computed (Program 8) from equation (5.36), for values of t_m – 5, 10, 15, 20, 25, 30, 35, and 40 (the data was zero-corrected at time 1 of the original data, i.e. time 0 on the graphs). The results for odd values of t_m are shown in Fig. 6-13, and for even values in Fig. 6-14.

The Model prediction is a straight line (equation (5.35)), of magnitude $k \cdot t_m$ (equation (5.34)), for any individual plot. Inspection of the plots does not confirm or refute the straight line hypothesis, and therefore, the distributions were considered (Chapter 5, para. 7-2). The expected Normal distribution cannot be tested, as the points on any one curve are all a function of the data at the start of the run (t_m) – this also explains the lack of randomness in the calculated points. Tolerances have been shown on Figs. 6-13, and 6-14, derived from equations (5.38) and (5.45), where the Est(k) has been substituted for k.

i.e. $$\text{graph symbol } \mu = \mathrm{Est}(k) \cdot t_m \qquad (6.12)$$

$$\text{graph symbol } \sigma = \mathrm{Est}(k) \cdot \sqrt{\frac{t_m(t_m + t_n)}{50}} \qquad (6.13)$$

For ease of presentation, integer multiples of σ calculated for $t_n = t_m$ (designated $\sigma(t_m)$), which enclose all of the points are shown at the start of each curve, except for $t_m = 10$. As can be seen from equation (6.13), this is the smallest value of σ (σ increases with t_n and $t_n \geq t_m$). From the plots it is observed that all 7 curves fall inside $\pm 2\,\sigma(t_m)$, and consequently, all points will fall inside the true bound of $\pm 2\,\sigma\,(t_n)$.

For the plot starting at $t_m = 10$, the correct bound of $\pm\ 2\,\sigma(t_n)$ is shown dashed. The points which fall outside this bound are all within $\pm\ 3\,\sigma(t_n)$.

The final observation from the plots, is that there is a general tendency for the results to be lower than the Model predicted value (μ). This is explained, in part, by many of the curves having an Est$\left[\mathrm{MSDR}(t_m)\right]$ below the predicted value, but a further explanation is offered in Chapter 7.

Conclusions

On the assumption that the Drift Rate (and the Est COR(t_m, t_n) are Normally distributed the results are in reasonable agreement with the Model hypothesis.

Footnote to Chapter 6:-

It may appear that this Chapter has been written to prove the Model hypothesis, rather than to test it! However, all the facts that were available have been presented, and it is left to the reader's own judgement as to whether or not, he agrees with the author's reasoning.

CHAPTER VII

FURTHER ANALYSIS OF THE DISCREPANCIES WITH THE MODEL HYPOTHESIS

7-1 The theory of Chapter 5 and the analysis of the results in Chapter 6, were both concerned with testing the specified Model; and the Conclusions at the end of each section in Chapter 6 were primarily orientated to give the answers "Yes", "No" or "Inconclusive", to the question of whether or not, that particular test supported the Model. In this Chapter, the emphasis is changed from "testing a Model", to "fitting the data", and the theory and results presented here, are done so, in retrospect – i.e. they gave the best answers of the many methods tried (cf. the chronological presentation in Chapter 6).

7-2 <u>IDR Correlation</u>

The major discrepancy between the specified Model and the data was in the disagreement between the estimated slope of the MSDR/Time plot and the estimated mean squared IDR (Chapter 6, para. 5). With the motivation provided by Reference 6, the IDR autocorrelation functions were reconsidered, and for ease of reference, the previous results of the computation (Fig. 6-11) of the normalized ensemble autocorrelations (equation (5.31)) are given in Table 7-1.

IDR Ensemble Autocorrelations (normalized)

Time Difference	ACF	Time Difference	ACF
0	1.000	14	.002
1	-.054	15	.021
2	-.032	16	-.026
3	.015	17	.002
4	-.025	18	-.020
5	-.068	19	.027
6	-.067	20	-.011
7	-.023	21	-.024
8	.008	22	.024
9	-.008	23	-.035
10	.000	24	.026
11	.011	25	.030
12	-.031	26	.020
13	-.032	27	.022

TABLE 7-1

In the Conclusions of Chapter 6, para. 6, the level of correlation in Table 7-1 was not considered significant, and IDR was treated as being statistically independent over one hour intervals, thereby supporting the Model hypothesis that

$$E(A_i \cdot A_j) = E(A^2) \cdot \delta_{ij} \qquad (4.3)$$

Now, consider the situation where equation (4.3) is not assumed to be proven for the data. In consequence the Model hypothesis must be modified to:

$$E(A_i \cdot A_j) = E(A^2) \cdot C_{|j-i|} \qquad (7.1)$$

where all the properties of the original Model are retained (in particular that Drift Acceleration (and IDR) are stationary processes), except those dependent on equation (7.1)

and $C_{|j-i|}$ is the normalized autocorrelation function for a time difference $|j-i|$ (i.e. $C_o = 1$, etc.).

From equations (5.2), (5.4) and (7.1) it follows that

$$E\left[DR(t_n)\right]^2 = E\left[(A_1 \; A_2 \; \cdots \; A_n)^2\right] \cdot (\Delta t_o)^2$$

$$= E\left[\sum_{i=1}^{n} A_i^2 + 2\sum_{i=1}^{n-1} A_i \cdot A_{i+1} + \cdots + 2\sum_{i=1}^{n-(n-1)} A_i \cdot A_{i+n-1}\right] \cdot (\Delta t_o)^2$$

$$= E(A^2) \cdot \left[nC_o + 2(n-1)C_1 + \cdots + 2(n-(n-1))C_{n-1}\right] \cdot (\Delta t_o)^2 \tag{7.2}$$

Since $t_n = n \cdot \Delta t_o$, and $IDR_i = A_i \cdot \Delta t_o$

equation (7.2) can be rewritten as,

$$E\left[MSDR(t_n)\right] = E(MSIDR)\left\{1 + 2\sum_{i=1}^{n-1} C_i \left(1 - \frac{i}{n}\right)\right\} \cdot \frac{t_n}{\Delta t_o} \tag{7.3}$$

If the IDR is corrected over "a" intervals (i.e. C_i for $(i > a) = 0$)

then $$E\left[MSDR(t_n)\right] = E(MSIDR) \cdot \left\{1 + 2 \cdot \sum_{i=1}^{a} C_i \left(1 - \frac{i}{n}\right)\right\} \cdot \frac{t_n}{\Delta t_o} \tag{7.4}$$

For values of $n \geq a + 1$, equation (7.4) represents a straight line (with a non-zero slope, and not passing through the origin on the MSDR/Time plot).

Inspection of equation (7.4) shows that $(1-\frac{i}{n})$ is always positive, and therefore, a large negative C_i (possibly more than one), would have the effect of reducing the gradient which, as can be seen from equations (6.6) and (6.10), is the required result for a better comparison of the Model to the data.

7-3 Check on Gyro Uniformity

A large negative C_i, to support the analysis in Section 7-2, is not present (Table 7-1), therefore the derivation of the ensemble average of the Autocorrelation Functions was questioned. The results in Table 7-1, were normalized on completion of the evaluation in equation (5.31). Suppose, however, that the gyros were not uniform; more specifically, that the MSIDR as computed for each gyro separately (i.e. a time average), did not come from the same population – this was a fundamental assumption in Chapter 4, para. 2(3). Then a better estimate of the Autocorrelation Function would be to take an ensemble average of the normalized ACFs for each gyro. Program 6 printed out each individual gyro MSIDR and normalized ACF so this was a comparatively easy point to check. A histogram of the MSIDRs for each gyro is shown in Fig. 7-1a and it was noted that the 3 points remote from the other 47, were the MSIDR. For gyro Nos. 6, 45 and 47, each one of which contains 2 points of the outsiders given in Table 6-1. Elimination of these points and the recalculation of the MSIDR for the 3 gyros, based on 57 (cf. 59) IDR data points, gave the histogram shown in Fig. 7-1b.

To evaluate Fig. 7-1b it was assumed that, regardless of the IDR distribution, the MSIDR distribution should be approximately Normal (there being 59 points in the determination of each MSIDR). The 15 gyros having an MSIDR in the histogram block nearest to zero are contrary to a Normal Distribution hypothesis, but it should be noted that all 15 points fell in the upper half of the block. The Est$\left[\text{MSIDR}\right]$ for the modified data (6, out of the 2,950 points, eliminated) is shown dashed on Fig. 7-1b, and the range of $\pm 3\sigma_s$ is shown at the top of the histogram where σ_s is derived in Appendix A, assuming the IDR to be exponentially distributed.

$$\text{Standard deviation (y)} = \sigma^2 \sqrt{\frac{5}{N}} \qquad \text{(A.15)}$$

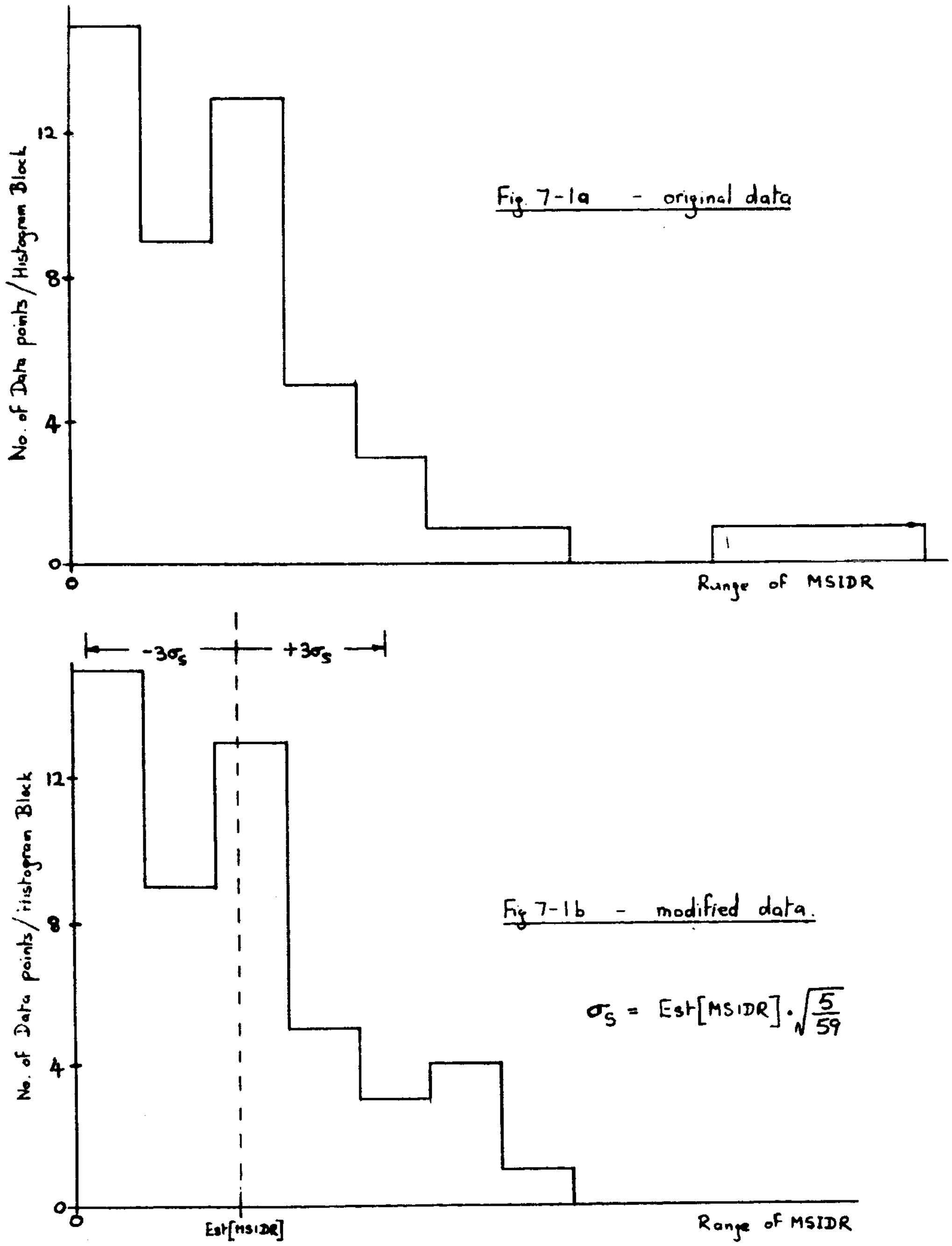


Fig. 7-1 DISTRIBUTION OF MSIDR COMPUTED FOR EACH GYRO (time average)

Therefore, if $\text{Est}\left[\text{MSIDR}\right]$ is substituted for σ^2 and N = 59, σ_s is given approximately by

$$\sigma_s = \text{Est MSIDR} \cdot \sqrt{\frac{5}{59}} \tag{7.5}$$

6 gyros, out of the total of 50, had a MSIDR outside the $\pm 3\sigma_s$ range (all 6 were above $+3\sigma_s$), and therefore, it is possible that the gyros are not uniform – this was a basic assumption in Chapter 4, para. 2 (Assumption 3). Futhermore, if the gyros are non-uniform, the exponential distribution of the IDR could be attributed to a summation of different Normal distributions for each gyro (i.e. different Variances for each gyro). In an attempt to investigate further, the uniformity of the gyros, the distributions of the individual gyro IDRS were considered. 6 gyros were selected on the basis that that MSIDRs for these gyros were spread throughout the range of the histogram in Fig. 7-1b. The IDR distribution histograms for these gyros are shown in Figs. 7-2 and 7-3; with the $\pm\ \sigma$ points ($\sigma = \sqrt{\text{Est(MSIDR)}}$) marked on the horizontal axes. Also shown is the ratio of the MSIDR for the particular gyro, to the $\text{Est}\left[\text{MSIDR}\right]$.

There are 2 possible alternatives to be considered:

1) The histograms all represent samples from an exponential distribution.
2) The histograms each represent a sample from different statistical populations (the populations being, probably, Normally distributed).

Because of the basic similarity of the Normal and Exponential distributions, and the small number of data points (59) in each histogram, no positive conclusions can be made from the inspection of Figs. 7-2 and 7-3. Noting that an Exponential distribution would have more points around the mean (zero), more points in the tails and no point of inflexion, as compared with a Normal distribution; it is suggested that better support is given by,

i) the 3 gyros in Fig. 7-3 to alternative (1)
ii) the 3 gyros in Fig. 7-4 to alternative (2).

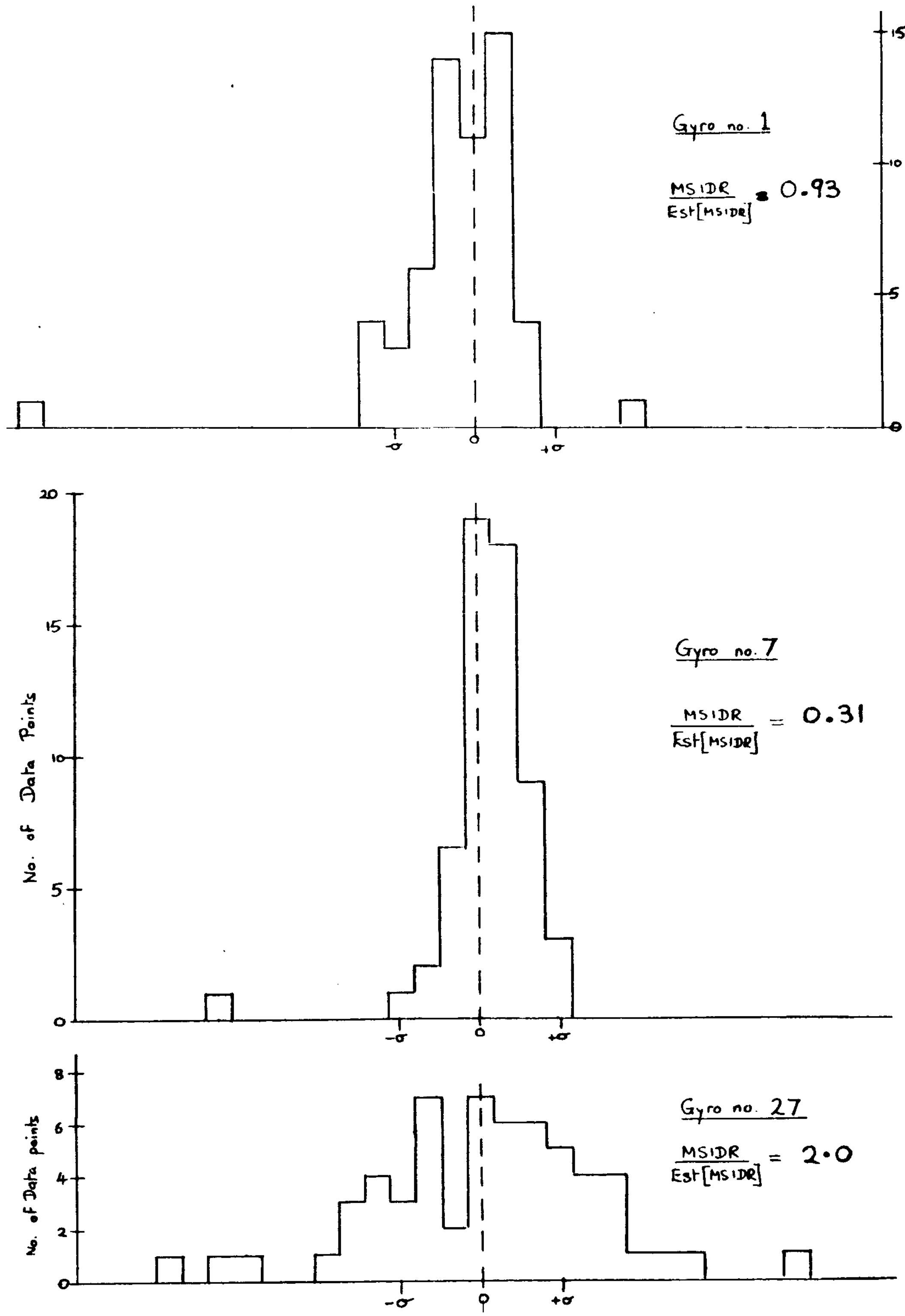


Fig. 7-2 IDR DISTRIBUTION FOR GYRO NOS. 1, 7 and 27

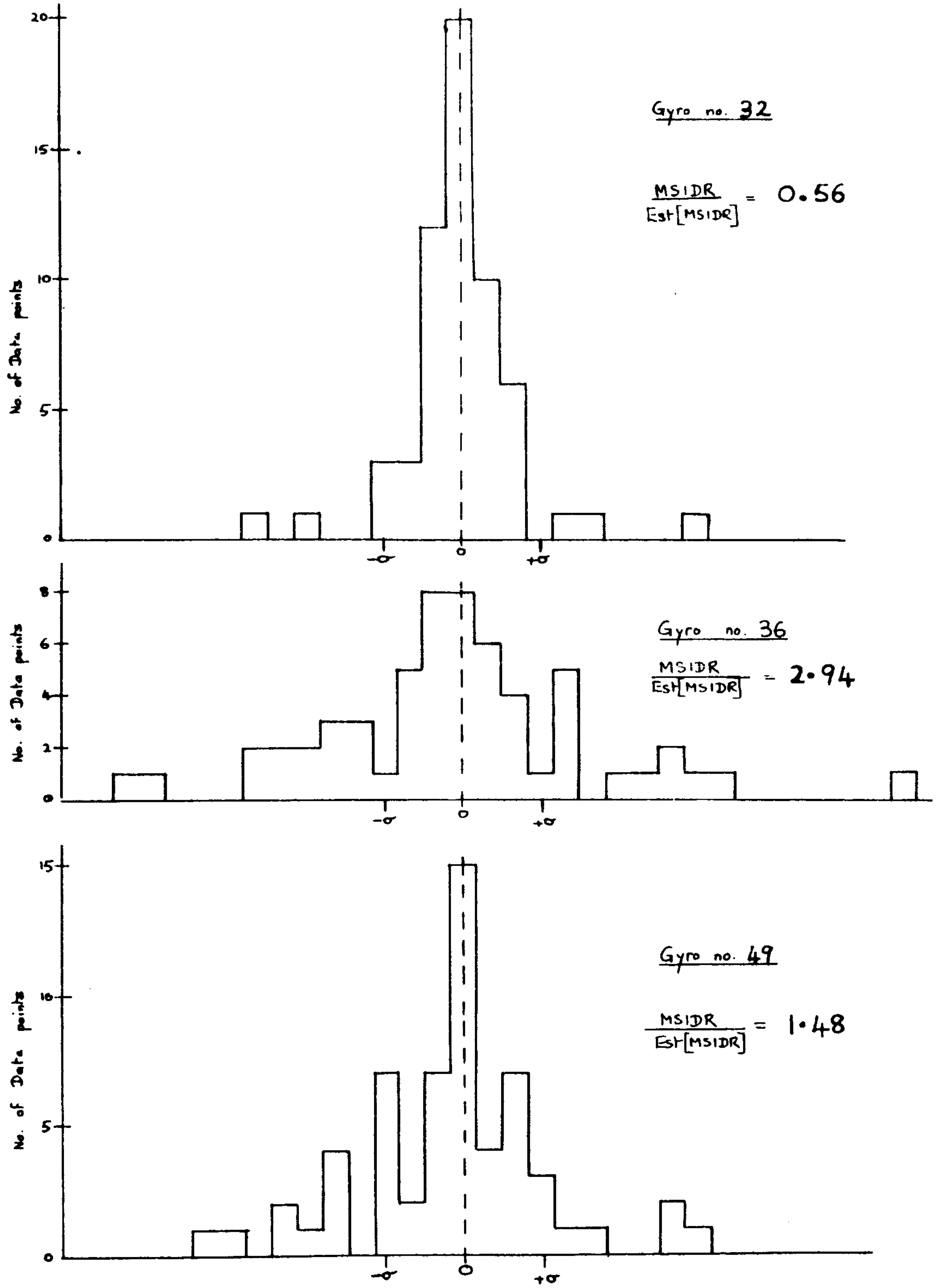


Fig 7-3 IDR DISTRIBUTION FOR GYRO NOS. 32, 36 and 49

Conclusions

1) Doubt has been raised on the fundamental assumption of much of the preceding analysis, that the gyros are uniform (i. e. all the data analysed is a sample from the same statistical population). Because of the small number of data points in Figs. 7-1 to 7-3, the results are inconclusive, and further investigation is beyond the scope of this thesis.

2) If the gyros are non-uniform, then it might be possible to "normalize" the data for each gyro to achieve uniformity in any subsequent analysis; however, a check on the ensemble average of the normalized ACFs for each gyro, gave different, but no more predominant values of correlation (C_i) than those in Table 7-1.

7-4 Reconsideration of the Autocorrelations of Table 7-1

As no large negative C_i is present in Table 7-1, the effect of several small values was next investigated. A maximum correlation time difference of 7 was selected (a = 7 in equation (7.4)), based on the following retrospective arguments:

1) From time difference 1 to 7, there are six negative values of C_i and only one small positive value.
2) From time difference 8 on, the correlations are much more randomly distributed about zero, thereby tending to cancel out in equation (7.4).
3) The values for large time differences are based on a fewer number of terms in the estimate (equation (5.30)), and are therefore less reliable than the small time difference correlations.

Substitution of C_i for i = 1 to 7, in equation (7.4) reduces it to:

$$E\left[MSDR(t_n)\right] = E(MSIDR) \cdot (0.492n + 2.15) \qquad (7.6)$$

$$\text{for } n \geq 8$$

Now substituting Est(MSIDR) from equation (6.10), (the estimate based on 2,944 out of the 2,950 IDR points – see Conclusions (3) at the end of this Section), for E(MSIDR) in equation (7.6),

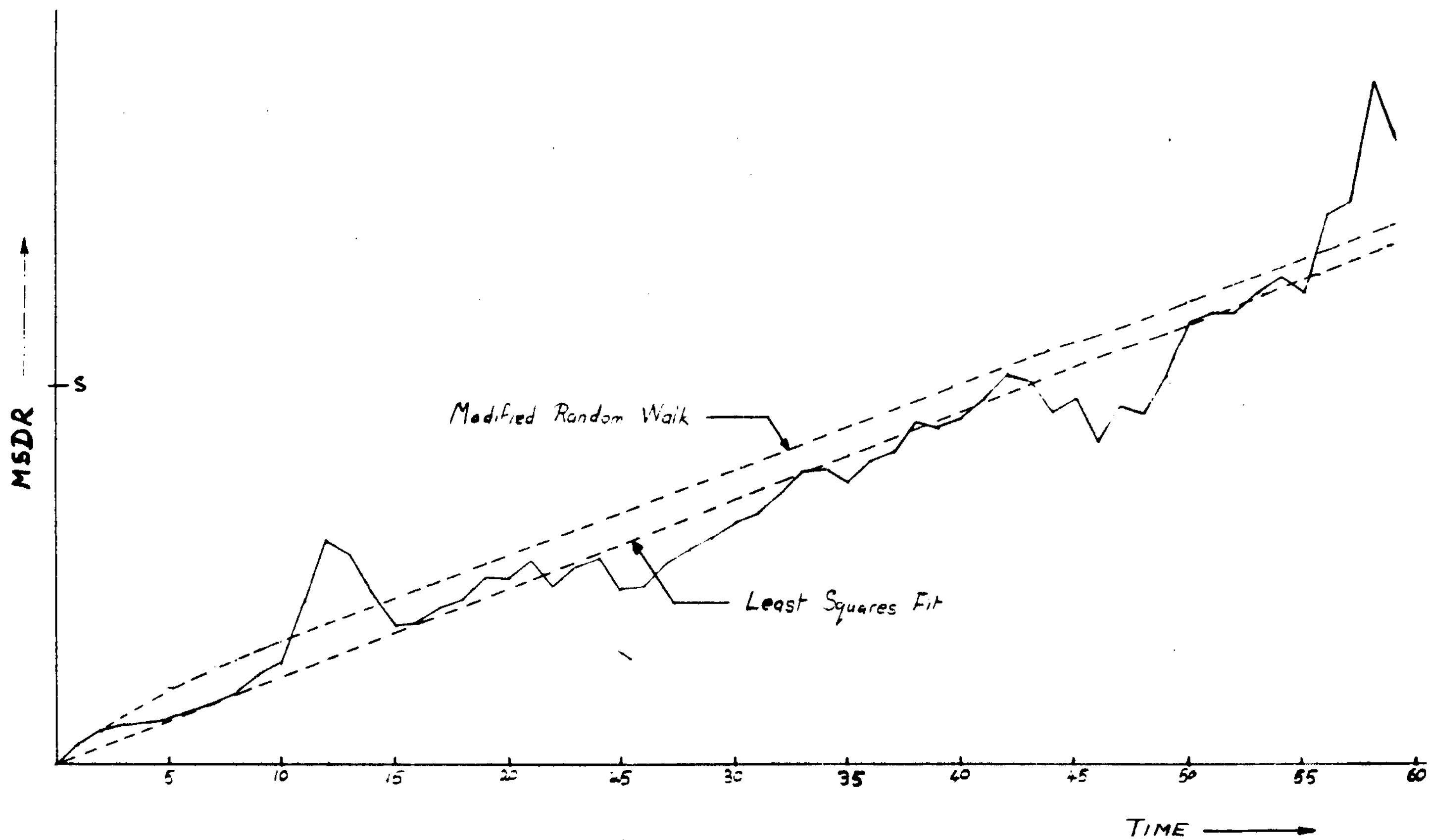


Fig 7-4 MEAN SQUARED DRIFT RATE - ZERO CORRECTION

where Est(k) is the least squares fit for the slope of the MSDR/Time plot.

Thus a close fit of the modified "correlated" Random Walk model to the actual data will result. The MSDR/Time plot of Fig. 6-2 is redrawn in Fig. 7-4, with equation (7.6)(points for $n < 8$ are calculated from equation (7.3)), and the least squares fit of Fig. 6-2, superimposed.

7-5 Conclusions (assuming Gyro Uniformity, Section 7-3)

1) The modified model hypothesis illustrates the probable cause of the major discrepancy between the previous Model and the test data – the change in Drift Rate (IDR) is not statistically independent over one hour intervals.

2) The correlation is small but extends over several hours (7 was suggested by the analysis). Because of these low values, the estimates of the ACFs are probably in error, but the final result, expressed in equation (7.6), is a good approximation to the data.

3) Two questions are raised by this analysis. One, can a better estimate of the ACF be obtained, either by elimination of the "bad" data or, preferably, by a bigger sample? Secondly, is the IDR frequency distribution, and its mathematical fit (Fig. 6-12), still valid? Neither question was pursued further, but it is thought unlikely that the distribution would be varied much, through having considered the 2,950 points to be statistically independent, if the correlation levels in Table 7-1 are approximately correct.

4) The Drift Rate correlation test results are also supported by the modified Model, as revision of the theory in Chapter 5, para. 7, shows that the expectation of the Drift Rate correlation is $E\left[\mathrm{MSDR}(t_m)\right]$ at t_m, as before, but falling to a somewhat lower constant value at t_n, for all $n \geq a + 1$, where a = number of steps over which the IDR is correlated. The change in the expected level at t_n, to that at t_m, for $n \geq a + 1$, is a constant for all t_m (i.e. for all 8 curves in Figs. 6-13 and 6-14); but the value calculated for $a = 7$, and using Table 7-1, was numerically equal to 1.1S(IDR), (or $1.55\sqrt{\mathrm{Est}(k)}$), which was too small to add any additional remarks to those of Chapter 6, para. 8.

5) If the IDR is correlated, the calculation of σ_s in Fig. 7-1 would underestimate the true value. A larger value of σ_s would give support to the basic assumption of gyro uniformity as fewer gyros would have an MSIDR outside the $\pm\ 3\,\sigma_s$ boundary.

CHAPTER VIII

SUMMARY OF RESULTS, AND VALIDITY OF THE PROPOSED STATISTICAL MODELS

8-1 Properties of the Proposed Statistical Models

8-1-1 Model I

A Random Walk type of model, having a representative sample as shown in Fig. 4-1, The Model is completely specified by the "Drift Accelerations", which are defined as the difference in the Drift Rates at one hour intervals, divided by the time interval (one hour). The properties of the "Drift Acceleration" (A_i) are:

1) The (A_i)s constitute a stationary, ergodic, random process, and are statistically independent of each other.
2) The Drift Acceleration is Normally distributed, with Mean, zero, and a best estimate of the Standard Deviation given by $\sqrt{\frac{Est(k)}{\Delta t_o}}$; where Est(k) is the slope of the least squares straight line fit, through the origin, for the Mean Squared Drift Rate /Time plot of the analysed data. This data is the original drift rate data as tabulated at one hour intervals, corrected to zero by subtracting the first data point from all other points in the same run.

The other properties of the Model can all be developed from these two conditions.

8-1-2 Model II

The basic specification of the Model, as shown in Fig. 4-1, remains similar to Model I; however, the conditions (1) and (2) of Section 8-1-2 are now modified, so that:

1) The (A_i)s constitute a stationary, ergodic, random process, but they are not statistically independent quantities. The level of correlation is low, but extends over several hours, the best estimate available is given by the correlation figures from time difference 0 to 7, in Table 7-1, resulting in an expression for the $E\left[MSDR(t_n)\right]$, given by equation (7.6).

2) The Drift Acceleration is "exponentially" distributed, having a Probability distribution given by:

$$f(x) = \sqrt{\frac{1}{2\sigma^2}} \cdot e^{-\sqrt{\frac{2}{\sigma^2}} \cdot |x|} \tag{A.6}$$

where the best estimate of σ^2 is given by $S^2(DA)_{modified}$, this quantity being numerically equal to $S^2(IDR)_{modified}$ as derived in Chapter 6, para. 7-2. This latter quantity is the Estimated Variance of the Incremental Drift Rate, with 6, out of the total 2,950 data points, eliminated (Table 6-1).

8-2 Comparison of the Models With the Test Data

The various tests, and the comparison of the results with the Model predictions, are summarized in Table 8-1.

8-3 Use of Models

As was stated in the Introduction, the use of the Models, to indicate the possible causes of gyro drift errors, will not be pursued (obviously Model II provides more information in this respect).

For prediction, the Models serve a two-fold purpose:

1) For prediction of the Distribution of the gyro drift rate at some future time, after correction of the instantaneous gyro drift rate to zero, a diagram, such as Fig. 8-1, can be used. The interpretation of the diagram

Comparison of the Test Results With the Model Predictions

Data Test			Comparison: Test Results /Model Prediction	
No.	Description	Relevant Figures	MODEL I	MODEL II
1	MSDR /Time	6-2	Good *	Expectation is Good
2	Mean DR /Time	6-3	Acceptable	Acceptable
3	DR Distribution /Time	6-4	Acceptable	Acceptable
4	Mean IDR /Time	6-7	Good	Good
5	MSIDR /Time	6-8 6-9 6-10	Acceptable	Acceptable
6	IDR Autocorrelation	6-11	Fair (in retrospect)	Good *
7	IDR Distribution	6-12	Approximation (Curve B)	Good (Curve C)
8	DR Correlation	6-13 6-14	Acceptable	Acceptable

Translation (English /Statistics)

* . . . by definition of the Model

Good . . . not statistically disproven

Acceptable)
Fair) . . . more data require, i. e. sample too small

Approximation . . . known error

TABLE 8-1

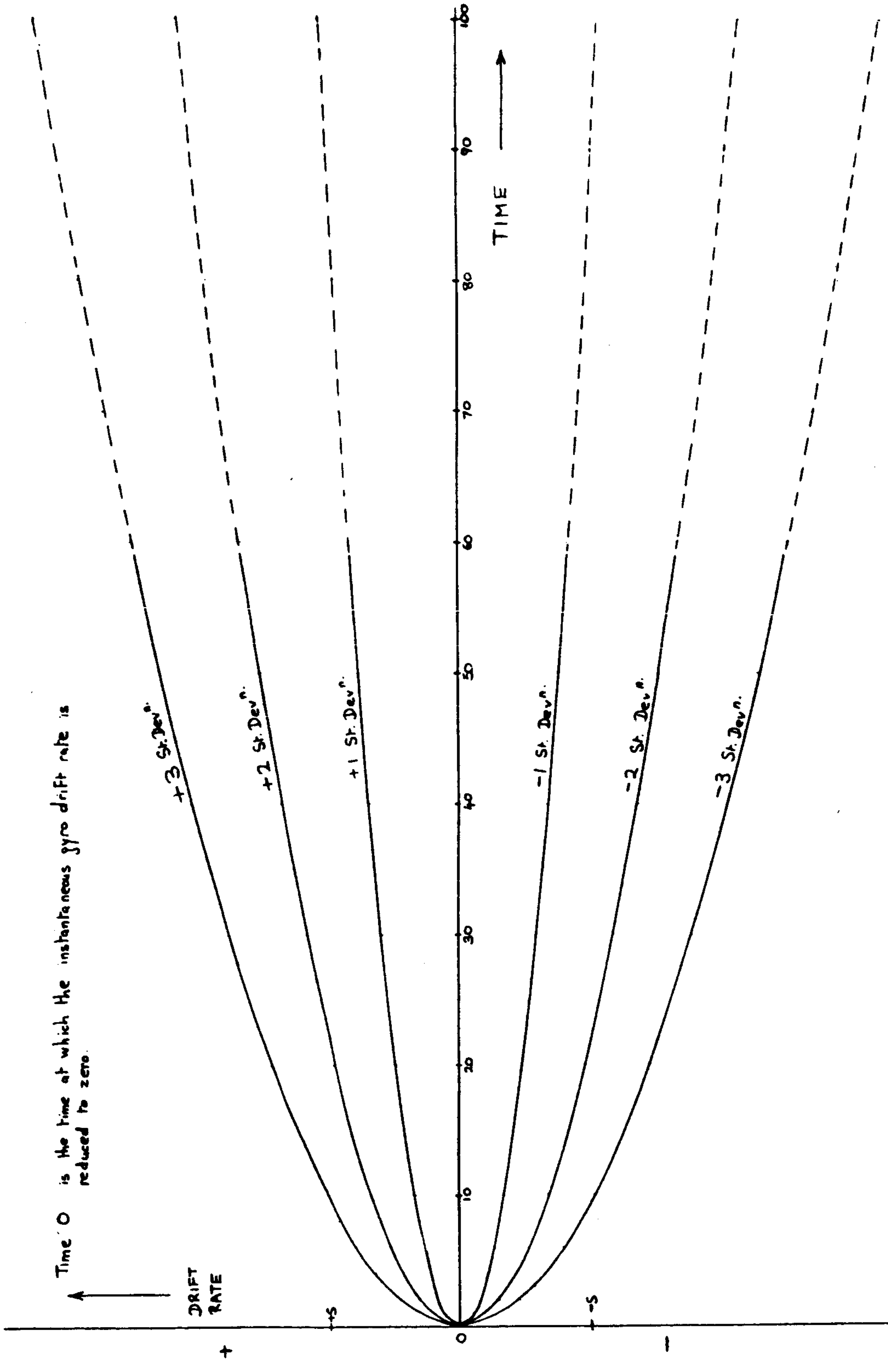


Fig. 8-1 STATISTICAL PREDICTION OF DRIFT RATE

is based on equation (5.15), i.e. "At time T, there is a 68% probability that the drift rate will be within ±1 Standard Deviation from zero." The SD curves have been computed on the basis of,

$$SD(t) = \sqrt{Est(k) \cdot t} \tag{8.1}$$

(cf. equation (5.14))

which is a reasonable prediction based on the test data, and confirmed by both of the Models. The curves are shown dashed after t = 59, as they represent Model I prediction, and not the test data results (see Section 8-4).

2) The vital link missing in Fig. 8-1, is the relationship between the gyro drift errors and the system output errors. This can be investigated by using the Model to generate synthetic data as the input to a computer simulation of the system (Chapter 9, Vol. 2). Model II is the better model to use, but it has not been well defined by this analysis, and clearly it is considerably harder to use as a data generator (with correlation, and exponential distributions, to be included), than Model I. The use of Model I would ignore, completely, the correlation effects, and would approximate the estimated IDR distribution by a Normal distribution. This latter approximation is a reasonable one as, from Curve B in Fig. 6-12, it can be seen that, although it eliminates the tails, this is counteracted by the generation of fewer points around the mean.

8-4 Validity of Models

The validity of a statistical model must be very clearly stated. In addition to the discrepancies noted in Chapters 6 and 7, the following conditions must be applied to the use of the Models developed in this thesis.

Models are only applicable,

1) To gyros of the specific type analysed.
2) To the conditions of the test configuration; shown simplied in Fig. 3-1.
3) Over the time of the data runs (effectively 59 hours, from zero correction).

Extension of the Models to applications outside these conditions (e.g. when base motion present, or for longer times), may possibly be inferred from the analysis, but certainly has not been proven.

APPENDIX A

THE STATISTICS OF A SYMMETRIC EXPONENTIAL DISTRIBUTION

Consider

$$f(x) = a \cdot e^{-b \cdot |x|} \tag{A. 1}$$

This will be a Probability Distribution

$$\text{if} \qquad \int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{A. 2}$$

$$\text{i.e. if} \qquad 2 \int_{0}^{\infty} a \cdot e^{-b \cdot x}\, dx = 1$$

$$\text{or} \qquad b = 2a \tag{A. 3}$$

Now, since the Mean = 0, the Standard Deviation (σ), is given by,

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx \tag{A. 4}$$

$$= 2 \int_{0}^{\infty} x^2 \cdot a \cdot e^{-b \cdot x}\, dx$$

$$= \frac{4a}{b^3} \tag{A. 5}$$

Combining equations (A. 1), (A. 3) and (A. 5), the exponential probability distribution is given by,

$$f(x) = \sqrt{\frac{1}{2\sigma^2}} \cdot e^{-\sqrt{\frac{2}{\sigma^2}} \cdot |x|} \qquad \text{(A. 6)}$$

and the area under this "normalized" exponential distribution is

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= 1 - \frac{1}{2} e^{-\sqrt{\frac{2}{\sigma^2}}\, x} \qquad \text{for } x \geq 0 \qquad \text{(A. 7)}$$

Now, applying equation (A. 7) to the IDR data (Chapter 6, para. 7-2), consiting of 2, 950 points, what is the boundary ($\pm$ B σ) at which there will be, on the average, only one point present in the tails?

Since the distribution is symmetric, it is only necessary to consider the "upper end" tail, containing $\frac{1}{2}$ data point, i.e. that the upper tail encloses an area = $\frac{1}{59}$ % of the total area.

From equation (A. 7) it follows that,

$$1 - F(x) = \frac{1}{5,900} \qquad \text{(A. 8)}$$

i.e. $$e^{-\sqrt{\frac{2}{\sigma^2}} \cdot x} = \frac{1}{2,950}$$

giving $$x = 5.65\,\sigma \qquad \text{(A. 9)}$$

$$\text{or} \qquad B = 5.65 \tag{A.10}$$

Consider the random variable x^2 where x has the exponential distribution of equation (A.6)

since
$$E(x) = 0$$

$$E(x^2) = \sigma^2 \tag{A.11}$$

also
$$E(x^4) = \int_{-\infty}^{\infty} x^4 f(x) \; dx$$

$$= 2 \int_{0}^{\infty} x^4 \cdot \frac{1}{\sqrt{2\sigma^2}} \cdot e^{-\sqrt{\frac{2}{\sigma^2}} x} \; dx$$

$$= 6 \sigma^4 \tag{A.12}$$

therefore
$$\text{Variance } (x^2) = E(x^4) - \left[E(x^2) \right]^2$$

$$= 5 \sigma^4 \tag{A.13}$$

and
$$\text{Standard Deviation } (x^2) = \sigma^2 \sqrt{5} \tag{A.14}$$

If the random variable y is defined as the average of N independent values of x^2

then
$$\text{Standard Deviation } (y) = \sigma^2 \sqrt{\frac{5}{N}} = E(x^2) \sqrt{\frac{5}{N}} \tag{A.15}$$

and for large values of N the distribution of y will approach the Normal Distribution.

APPENDIX B

FORTRAN PROGRAMS USED ON IBM 1620 COMPUTER

```
C    PROGRAM 3     MSDR/TIME - LEAST SQUARES SLOPE
      SUM = 0.
      DEN = 0.
      DO 10 N=1,60
      S=N-1
      READ 100,Y
      SUM = SUM + S*Y
   10 DEN = DEN + S*S
      SUM = SUM/DEN
      PUNCH 101, SUM
      CALL EXIT
  100 FORMAT (27X,E10.3)
  101 FORMAT (5X,17HMSDR/TIME SLOPE =F6.3)
      END
```

```
C    PROGRAM 4     INCR. DRIFT RATE - ENSEMBLE STATISTICS
*FANDK0504
      DIMENSION D(50,60)
    1 READ 101, ((D(I,J), J=1,60), I=1,50)
C     COMPUTE INCREMENTAL DRIFT RATE
      DO 6 I =1,50
      DO 5 J =1,59
      JJ=61-J
    5 D(I,JJ) = D(I,JJ) - D(I,JJ-1)
    6 PUNCH 102, I,(D(I,J),J=2,60)
C     COMPUTE IDR ENSEMBLE STATISTICS
      PUNCH 104
      PUNCH 105
      DO 20 J=2,60
      EMEAN =0.
      EMSQ=0.
      DO 10 I=1,50
      EMEAN = EMEAN + D(I,J)
   10 EMSQ = EMSQ + D(I,J)**2
      EMEAN = EMEAN/50.
      EMSQ = EMSQ/50.
      VAR = ((EMSQ-EMEAN**2)*50.)/49.
      SD = VAR**.5
   20 PUNCH 106, J,EMEAN,EMSQ,VAR,SD
      GO TO 1
  101 FORMAT (10F7.1)
  102 FORMAT (1X,I3/(10F7.1))
  104 FORMAT (10X,42HINCREMENTAL DRIFT RATE ENSEMBLE STATISTICS///)
  105 FORMAT (4X,4HTIME6X,4HMEAN12X,3HMSQ12X,3HVAR13X,2HSD//)
  106 FORMAT (4X,I3,4(6X,E9.2))
      END
```

```
C    PROGRAM 1      DRIFT RATE ENSEMBLE STATISTICS

*FANDK0504
      DIMENSION D(50,60)
    1 READ 101, ((D(I,J),J=1,60),I=1,50)
      PUNCH 102
      PUNCH 103
      DO 20 J=1,60
      EMEAN = 0.
      EMSQ = 0.
      DO 10 I = 1,50
      EMEAN = EMEAN + D(I,J)
   10 EMSQ = EMSQ + D(I,J)**2
      EMEAN = EMEAN/50.
      EMSQ = EMSQ/50.
      VAR = ((EMSQ - EMEAN**2)*50.)/49.
      SD = VAR**.5
   20 PUNCH 104,J,EMEAN,EMSQ,VAR,SD
      GO TO 1
  101 FORMAT (10F7.1)
  102 FORMAT (15X,30HDRIFT RATE ENSEMBLE STATISTICS///)
  103 FORMAT (4X,4HTIME6X,4HMEAN12X,3HMSQ12X,3HVAR13X,2HSD//)
  104 FORMAT (4X,I3,4(5X,E10.3))
      END
```

```
C    PROGRAM 2      DRIFT RATE ENSEMBLE STATISTICS - ZERO CORRECTION

*FANDK0504
      DIMENSION D(50,60)
    1 READ 101, ((D(I,J),J=1,60),I=1,50)
      PUNCH 102
      PUNCH 103
      DO 5 I = 1,50
      DO 5 J = 1,60
      JJ = 61-J
    5 D(I,JJ) = D(I,JJ) - D(I,1)
      DO 20 J=1,60
      EMEAN = 0.
      EMSQ = 0.
      DO 10 I = 1,50
      EMEAN = EMEAN + D(I,J)
   10 EMSQ = EMSQ + D(I,J)**2
      EMEAN = EMEAN/50.
      EMSQ = EMSQ/50.
      VAR = ((EMSQ - EMEAN**2)*50.)/49.
      SD = VAR**.5
   20 PUNCH 104,J,EMEAN,EMSQ,VAR,SD
      GO TO 1
  101 FORMAT (10F7.1)
  102 FORMAT(12X,46HDRIFT RATE ENSEMBLE STATISTICS,ZERO CORRECTION///)
  103 FORMAT (4X,4HTIME6X,4HMEAN12X,3HMSQ12X,3HVAR13X,2HSD//)
  104 FORMAT (4X,I3,4(5X,E10.3))
      END
```

```
C    PROGRAM 5      INCR. DRIFT RATE - RUNNING RMS (TIME)
      DIMENSION D(59), RMS(59), TRMS(59)
      DO 10 J=1,59
   10 TRMS(J) = 0.
      PUNCH 102
      DO 30 N=1,50
      READ 101, I, (D(J), J=1,59)
      EL = 0.
      DO 20 J=1,59
      EL = EL + D(J)*D(J)
      DEN = J
      RMS(J) = SQRTF(EL/DEN)
   20 TRMS(J) = TRMS(J) + RMS(J)
   30 PUNCH 103, I, (RMS(J), J=1,59)
C     COMPUTE ENSEMBLE RUNNING RMS
      PUNCH 104
      PUNCH 105
      DO 40 J=1,59
      TRMS(J) = TRMS(J)/50.
   40 PUNCH 106, J, TRMS(J)
      CALL EXIT
  101 FORMAT (1X,I3/(10F7.1))
  102 FORMAT (14X,17HIDR - RUNNING RMS///)
  103 FORMAT (1X,4HGYROI3/(10F7.2))
  104 FORMAT (////13X,26HIDR - ENSEMBLE RUNNING RMS///)
  105 FORMAT (16X,4HTIME5X,3HRMS/)
  106 FORMAT (16X,I3,3X,F7.3)
      END
```

```
C     PROGRAM 6      INCR. DRIFT RATE - AUTOCORRELATION
      DIMENSION D(59), SUM(28), ESUM(28)
      DO 10 K=1,28
   10 ESUM(K) = 0.
      PUNCH 102
      DO 60 N=1,50
      READ 101, I, (D(J), J=1,59)
      DO 30 K=1,28
      JJ=60-K
      DEN=JJ
      SUM(K)=0.
      DO 20 J=1,JJ
      JK=J+K-1
      EL=D(J)*D(JK)
   20 SUM(K) = SUM(K) + EL
      SUM(K) = SUM(K)/DEN
   30 ESUM(K) = ESUM(K) + SUM(K)
      PUNCH 103, I, SUM(1)
      EMSQ = SUM(1)
      DO 40 K=1,28
   40 SUM(K) = SUM(K)/EMSQ
   60 PUNCH 104, (SUM(K), K=1,28)
C     COMPUTE ENSEMBLE AUTOCORRELATIONS
      PUNCH 105
      EMSQ = ESUM(1)/50.
      PUNCH 106, EMSQ
      PUNCH 107
      EMSQ = ESUM(1)
      DO 50 K=1,28
      ESUM(K) = ESUM(K)/EMSQ
      KK=K-1
   50 PUNCH 108, KK, ESUM(K)
      CALL EXIT
  101 FORMAT (1X,I3/(10F7.1))
  102 FORMAT (14X,25HNORMALIZED IDR AUTOCORRS.///)
  103 FORMAT (1X,4HGYROI3,6X,7HMSIDR =F6.2)
  104 FORMAT (10F7.2)
  105 FORMAT (////13X,34HNORMALIZED ENSEMBLE IDR AUTOCORRS.///)
  106 FORMAT (4X,7HMSIDR =F6.2//)
  107 FORMAT (16X,10HTIME DIFF.5X,15HAUTOCORRELATION/)
  108 FORMAT (19X,I3,12X,F7.3)
      END
```

```
C     PROGRAM 7      INCR. DRIFT RATE - FREQUENCY DISTRIBUTION
          DIMENSION D(59), KSUM(33)
          DO 1 K=1,33
        1 KSUM(K) = 0.
          PUNCH 102
          DO 8  L=1,50
        2 READ 101, I, (D(J), J=1,59)
          DO 8  J=1,59
          IF (D(J)) 3,4,4
        3 M=1
          N=0
          GO TO 5
        4 M=0
          N=1
        5 DO 6  K=1,17
          KM=18-K
          KN=16+K
          BB=K-1
          B=.15 + .3*BB
          IF (ABSF(D(J)) - B) 7,6,6
        6 CONTINUE
          PUNCH 103, I, D(J), J
          GO TO 8
        7 KSUM(KM) = KSUM(KM) + M
          KSUM(KN) = KSUM(KN) + N
        8 CONTINUE
          PUNCH 104, (KSUM(K), K=1,33)
          CALL EXIT
      101 FORMAT (1X,I3/(10F7.1))
      102 FORMAT (5X,29HINCR. DRIFT RATE FREQ. DISTR.///)
      103 FORMAT (10X,I3,4X,F7.1,6X,I3)
      104 FORMAT (//(20X,I4))
          END
```

```
C    PROGRAM 8      DRIFT RATE CORRELATION

*FANDK0504
      DIMENSION D(50,60)
    1 READ 101, ((D(I,J),J=1,60),I=1,50)
      DO 10 I=1,50
      DO 10 J=1,60
      JJ = 61-J
   10 D(I,JJ) = D(I,JJ)-D(I,1)
      PUNCH 102
      DO 30 J=6,41,5
      N = J-1
      PUNCH 103, N
      PUNCH 104
      K = 61-J
      DO 30 JJ=1,K
      SUM=0.
      JK = J+JJ-2
      DO 20 I=1,50
      EL = D(I,J) * D(I,JK+1)
   20 SUM = SUM+EL
      SUM = SUM/50.
   30 PUNCH 105, JK,SUM
      GO TO 1
  101 FORMAT (10F7.1)
  102 FORMAT (10X,36HDRIFT RATE ENSEMBLE AUTOCORRELATIONS///)
  103 FORMAT (///3X,15HACFS WITH TIME I3//)
  104 FORMAT (7X,4HTIME11X,3HACF//)
  105 FORMAT (7X,I3,9X,F7.2)
      END
```

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