T. 41411

X-641-66-103

NASA TM X- 55477

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GPO PRICE $

CFSTI PRICE(S) $

Hard copy (HC) 3.00

Microfiche (MF) .75

ff 653 July 65

MARCH 1966

NASA

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-641-66-103

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ABSTRACT

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The aim is to review and extend our knowledge of the time dependence of fluctuations in density in homogeneous and isotropic models of the universe. Perturbations are assumed to be small and a linearized, normal mode analysis is used. Only gravitational interactions and irrotational motions are considered. Using first Newtonian and then general relativity theory it is shown that the amplitude of various modes increases in expanding and contracting models of the universe. The origin and growth of celestial structure require that uniform models of the universe are unstable against arbitrarily small perturbations. However, the rate of growth of the various modes, particularly in an expanding universe, is sufficiently slow to cast doubt on the instability of the models. A linearized gravitational theory appears inadequate to account for the origin of structure in the universe; hence alternative explanations must be found. Author

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VIBRATIONS OF THE UNIVERSE

1. INTRODUCTION

Nobody believes that the universe was created in its present form, complete in all its array of celestial detail. If the inchoate universe is featureless, then we must show how differentiation and structure contrive to evolve. The physics and evolution of stellar and galactic structure is the concern of astrophysics. But the nature and development of an environment favorable to the formation of astrophysical objects is the concern of cosmology.

The study of the vibrations of the universe is a fascinating subject, apart from any other consideration. However, in an attempt to account for the actual universe, as distinct from the smooth idealized models of cosmology, we might hopefully expect that the universe is unstable. In this way, various modes of vibration of arbitrarily small amplitude grow relatively rapidly and eventually, in a time short compared with the age of the universe, the foundations of astrophysical structure are laid down. However, it turns out that in an expanding universe the growth rates of the various modes are too small to account for appreciable irregularity.

We assume that all modes have small amplitudes and use linearized equations. Only gravitational interactions are considered; the motions are assumed to be irrotational; and a simple equation of state is used. The behaviour of the modes is then studied, first with Newtonian and then general relativity theory. Some general comments are made in Section 6.

2. ORIGIN OF STRUCTURE IN THE UNIVERSE

2.1 Initial Conditions

Cosmology is the science of the universe; at present it seeks only to account for the macroscopic nature of the universe. Instead of the physical universe, cosmology deals with a featureless, idealized universe that is everywhere isotropic and homogeneous and contains a uniform fluid of rudimentary properties.[1,2] Out of such generalizations emerge several simple models, and one's choice is largely an act of faith. Cosmology, in spite of its antiquity, is in an immature state; ultimately it must bridge the gap between its present idealizations and the physical universe with realistic models.

The provision of background and initial conditions for the origin and formation of structure is a cosmological problem. The subsequent evolution of structure into its detailed manifestations lies in the provinces of cosmogony, astrophysics, and every other science. The physical universe, as distinct from the current cosmological models, is complex and diverse. From the simplest of all possible points of view the diversity consists of variations in the density and motion of matter. Given a cosmological model that is a valid description of the universe in the large, it should be possible to show, with refinements of the cosmic fluid when necessary, that perturbations are capable of evolving in time into configurations of density and motion which resemble the grosser features of the physical world. The first step therefore is to inject some realism into the cosmological models by perturbing their fluid density and motion. The behaviour of the perturbations might then provide a means of judging whether a model is acceptable or not.

Two hypotheses are possible concerning the initial conditions in an evolving universe:*

A. Structure is created with the universe, or imprinted in it from the earliest moment of its expansion.

B. Structure develops in the universe from initially small random disturbances.

Hypothesis A is as old as cosmology. All current models (in which the cosmological term is zero) evolve either from or through a singular state. The universe originates before, at the time of, or after the singular state. It is impossible to imagine any structure surviving the singular state and therefore hypothesis A demands that the origin occurs after the singular state. Alternatively, by modifying the laws of physics,[4] the singular state can be avoided and the universe contracts to and expands from a finite size. This theory has the advantage that some structure might survive passage through the 'bounce' and act as dense nuclei around which further condensations subsequently form. It has the further advantage that in the linear theory, as we shall show, the growth rate of disturbances is usually larger in a contracting universe than in an expanding universe.

The concept of condensations growing in a uniformly dispersed medium (hypothesis B) is as old as the idea of gravitation. Jeans points out that in some correspondence Newton remarks:[5] 'But if the matter were evenly disposed throughout an infinite space, it could never convene into one mass; but some of

*These hypotheses do not cover the case of the steady state model which is outside the range of this discussion.[3]

it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered great distances from one to another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing that matter were of a lucid nature.' In his own work on gravitational instability Jeans writes:[6] 'We have found that, as Newton first conjectured, a chaotic mass of gas of approximately uniform density and of very great extent would be dynamically unstable; nuclei would tend to form in it, around which the whole of the matter would ultimately condense.' These comments were made with the idea in mind of a static universe. If we consider a universe already fragmented into widely separated 'islands,' each having a density large compared with the average density, such that the contents of each island do not partake in the expansion between the islands, then Jeans' comments are acceptable. But these are the initial conditions of cosmogony; the problem for cosmology is to explain how it is possible in the first place for islands to form.

A principal difference between the two hypotheses is that in B perturbations of the universe can, at least initially, be studied with a linear theory, whereas in A it is doubtful whether a linear theory is valid at any stage in an expanding universe.

In this discussion we shall discount ad hoc modifications of gravitational theory and hence A reduces to the proposition that the universe began with much of its detailed design predetermined. The obvious comment is that such ideas run counter to the main trend of science. By developing and employing the laws of physics, with their properties of symmetry and invariance, the aim is to rationalize our observations in terms of given initial conditions.[7] With the advance of science the laws of physics account for more and more of the regularity of nature at the expense of the need to assume regularity in the initial conditions. One might hope that ultimately the laws of physics, operating in a universe containing random and irregular fluctuations of small amplitude, will be adequate to account for the observed macroscopic structure of the universe.

The choice between hypotheses A and B is a matter of preference; if A, then there is scarcely any problem for cosmology; but if B, then it is necessary to show at least that in a linear theory certain fluctuations can grow in an expanding universe. In the following we adopt hypothesis B.

2.2 Linear Stability Theory

It is assumed that all disturbances are small and consist of a superposition of normal modes of a complete set; only gravitational interactions are considered using a linearized theory.

Within a comoving system of coordinates the uniform fluid of the cosmological models is in a hydrodynamic stationary state. A system in a stationary state is unstable when a small disturbance grows in time and leads to a changed configuration of the system. Thus, if one or more modes is time-growing, and the characteristic growth time is short compared with the lifetime of the system, the system is unstable.[8] According to hypothesis B the universe is an unstable system.

Planets, stars, stellar associations and clusters, galaxies, galactic clusters, ..., form a hierarchy of perturbations in which the amplitude diminishes as its spatial extent increases. It follows that a linearized treatment is most valid for fluctuations extending over extreme distances; that is for the lower vibration modes of the universe. As we go back in time the hierarchy of celestial objects dissolves into the increasing mean density of the universe, and the amplitude of all perturbations relative to the mean density diminishes. At a sufficiently early epoch, according to hypothesis B, the contrast density (ratio of the perturbation in density and unperturbed density) is small enough for a linear treatment. It follows therefore that a linear gravitational theory is cosmological in the sense that it is limited to small contrast densities, either remote in time, or extending over cosmic distances.

If the universe is unstable only for long wavelength perturbations we might conjecture with Jeans that there is a process of fragmentation[9] 'of nebulae out of chaos, of stars out of nebulae, ...,' and so on. Protogalaxies, or larger masses, first form because the universe is unstable. Regions of enhanced density, in which matter no longer expands with the universe, provide an environment more favourable for the formation of smaller condensations. Inhomogeneity, anisotropy, and complex properties of the fluid develop. No longer limited to a linearized treatment, we are free to invoke all the cosmogonic paraphenalia of turbulence, magnetic fields, radiation, dust, and so forth.[10]

Alternatively, if the universe is unstable for short wavelengths we might conjecture with Layzer that there is a process of clustering,[11] whereby small condensations first form and subsequently interact to create larger and larger self-gravitating systems. The arguments in favour of this process, however, have not progressed very far nor have they gained wide acceptance.

From the cosmological point of view the concepts of fragmentation and clustering are by no means mutually exclusive. The universe could be unstable for a large class of modes, or a wide spectrum of wavelengths, and the rate of growth of the different wavelengths determines whether elementary structure evolves by fragmentation or clustering, or by both processes acting simultaneously. Conceivably, early population II stars evolve out of inhomogeneities laid down either prior to or at the same time as those leading to galactic structure. All this, however, is speculation and we must wait for cosmology to give a clear and reliable account of the origin of structure in the universe.

3. NEWTONIAN COSMOLOGY

3.1 Newtonian Models

In 1934 McCrea and Milne[12,13] used Newtonian theory to derive the equations of a universe obeying the cosmological principal.[2] It is assumed, as in general relativity, that in the unperturbed state there is everywhere a perfect fluid of uniform mass density ρ and isotropic pressure p. The equations are identical with those derived using general relativity theory, provided the pressure is negligible in comparison with the energy density ρc^2 (c is the speed of light). Several authors[14-18] have discussed the validity and limitations of Newtonian cosmology.

As we shall show, the equations of Newtonian and general relativity theory for the perturbed state are similar when the pressure is negligible. The advantage of the Newtonian treatment is its simplicity; furthermore, it provides physical insight which helps to reduce the general relativity equations to their simplest form. Before proceeding to the Newtonian equations of a universe in a perturbed state, the treatment for the unperturbed state is presented briefly.

Let $\mathbf{r}$ be the position at time t_0 of any element of fluid. At time t, let the position of the same fluid element be $(S/S_0)\,\mathbf{r}$, where $S(t)$ is a universal function of time and $S_0 = S(t_0)$. This condition ensures that the fluid density ρ remains uniform and is a function only of time. Thus $\mathbf{r}$ is a comoving position vector and r, θ, ϕ are comoving spherical coordinates. The velocity and acceleration of a fluid element are

$$\mathbf{u} = \frac{\dot{S}}{S_0}\,\mathbf{r}, \quad \frac{d\mathbf{u}}{dt} = \frac{\ddot{S}}{S_0}\,\mathbf{r}, \tag{1}$$

where dots denote time differentiation. Within the comoving coordinate system the ordinary gradient operator ∇' transforms to $(S_0/S)\nabla$.

The equations of motion, continuity, and Poissions equation, are

$$\frac{S}{S_0}\,\frac{d\mathbf{u}}{dt} = -\nabla\psi^* - \frac{1}{\rho}\,\nabla p, \tag{2}$$

$$\frac{S}{S_0}\,\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\,\mathbf{u}), \tag{3}$$

$$(S_0/S)^2\,\nabla^2\psi^* = 4\pi G\rho - \Lambda, \tag{4}$$

where p is the pressure, G the gravitational constant, and ψ^* the gravitational potential. The cosmological term Λ is included for historical interest. In a uniform universe the ∇p term in (2) vanishes. Using (1), and taking the divergence of (2), it follows that

$$3\ddot{S} + (4\pi G\rho - \Lambda) S = 0. \tag{5}$$

From the equation of continuity (3),

$$\rho S^3 = \text{constant}, \tag{6}$$

and hence (5) can be integrated and becomes

$$\dot{S}^2 = \frac{1}{3}(8\pi G\rho + \Lambda)S^2 - \kappa \tag{7}$$

A universal constant of integration C is absorbed by the transformation: $S \rightarrow S\,|C|^{1/2}$; and S has now the dimension of time and κ has the value of 1, 0, or -1.

Equations (6) and (7) are the Newtonian equations of an isotropic and homogeneous universe. Furthermore, if

$$S \rightarrow c^{-1} R,$$

the equations are

$$\dot{R}^2 = \frac{1}{3}(8\pi G\rho - \Lambda) R^2 - \kappa c^2 \tag{8}$$

$$\rho R^3 = \text{constant}, \tag{9}$$

which are identical with those usually derived with general relativity, when the pressure is small compared with the energy density ρc^2. (First derived by

Friedmann for $p = 0$, and[19] $\kappa = 1$ and[20] $\kappa = -1$.) The velocity of light is an alien intruder in Newtonian hydrodynamics and therefore we keep to (6) and (7) for the present.

Using the constant

$$\beta = 8\pi G \rho S^3 / 3, \tag{10}$$

the solutions of (6) and (7) are

$$\begin{aligned} \kappa = 0: &\quad S = \beta\chi^2, \\ \kappa = 1: &\quad S = \beta \sin^2 \chi, \\ \kappa = -1: &\quad S = \beta \sinh^2 \chi, \end{aligned} \tag{11}$$

and $t(\chi)$ is found by integrating

$$dt = 2 S \, d\chi. \tag{12}$$

Within a sphere of radius r the potential energy is

$$\Omega = -\frac{16}{15}\left(\frac{S}{S_0}\right)^5 \pi^2 G \rho^2 r^5 = \frac{\beta E}{\kappa S}, \tag{13}$$

and the kinetic energy is

$$T = \frac{2}{5}\,\frac{\dot{S}^2 S^3}{S_0^5}\,\pi \rho r^5 = -\frac{\dot{S}^2 E}{\kappa}, \tag{14}$$

where $E = \Omega + T$ is the total energy and $\Lambda = 0$. From (14): $E = -\kappa T / \dot{S}^2$; and the total energy is zero when $\kappa = 0$.

The advantage of the Newtonian equations is the ease with which they can be physically interpreted. Thus, assuming $\Lambda = 0$, when $\kappa = 0$, $E = 0$, the fluid elements have velocities equal to their escape velocity and their trajectories are of the parabolic class; and for $\kappa = 1$ ($\kappa = -1$), $E < 0$ ($E > 0$), the fluid elements have velocities less (more) than their escape velocity and their trajectories are of the elliptical (hyperbolic) class. These interpretations are less obvious when (8) and (9) are derived from general relativity. In that case κ is the curvature constant and space is flat ($\kappa = 0$), spherical or elliptical ($\kappa = 1$), and hyperbolic ($\kappa = -1$).

3.2 Perturbed Newtonian Models

Various authors[21-29] have used Newtonian gravitational theory to study the time dependence of density fluctuations in a uniform fluid of finite or infinite extension. From the cosmological point of view a Newtonian treatment is scarcely adequate; not only is it limited to low density, but also the long wavelength modes of Euclidean space are inapplicable in curved space. Fluctuations at high density and large-scale fluctuations at low density are the conditions for which the linear theory is most valid but the Newtonian approach is least valid. Nevertheless, the simplicity of the Newtonian approach serves as a valuable guide in the subsequent treatment, and so far as the author is aware, it has not been previously developed in comoving coordinates.

For a collisionless fluid, such as a supergas of stars or galaxies, the formal approach is by way of the Vlasov equations,[30] as in plasma physics. This has been used by Gilbert,[29] and by Sweet[31] for counter-streaming fluids. Particles travelling an appreciable fraction of a wavelength in an oscillation period cause Landau damping. Since we are concerned with an initially structureless fluid we shall use the fluid approximation; this will tend to overestimate the rate of growth of perturbations. In this discussion the velocity components are everywhere single-valued, and only gravitational interactions are considered.

Let the disturbed velocity, density, pressure, and gravitational potential be

$$\mathbf{u} \to \mathbf{u} + \mathbf{v}, \qquad \rho \to \rho + \delta\rho,$$

$$\mathrm{p} \to \mathrm{p} + \delta\mathrm{p}, \qquad \psi^* \to \psi^* + \psi,$$

where the small quantities are functions of $\mathbf{r}$ and t. In the usual coordinates t, $\mathbf{r}\, S/S_0$ the linearized equations of motion and continuity are

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla'\right) \mathbf{v} + \mathbf{v}\cdot\nabla' \mathbf{u} + \nabla' \psi + \frac{1}{\rho}\nabla' \delta\mathrm{p} = 0, \tag{15}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla' + \nabla'\cdot\mathbf{u}\right)\delta\rho + \rho\nabla'\cdot\mathbf{v} = 0, \tag{16}$$

and terms quadratic in small quantities are neglected. The perturbation of Poission's equation is

$$\nabla'^2 \psi - 4\pi G \delta\rho = 0. \tag{17}$$

The transformation to comoving coordinates is

$$\nabla' \to \frac{S_0}{S} \nabla = \frac{S_0}{S} \left(\mathbf{i}_r \frac{\partial}{\partial r} + \mathbf{i}_\theta \frac{\partial}{r\,\partial\theta} + \mathbf{i}_\phi \frac{\partial}{r \sin\theta\,\partial\phi} \right),$$

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla' = \frac{\partial}{\partial t} + r \frac{\dot{S}}{S} \frac{\partial}{\partial r} \to \frac{d}{dt},$$

where the time derivative now follows the unperturbed motion of the fluid. In ordinary coordinates $\partial/\partial t$ commutes with ∇', whereas in comoving coordinates:

$$\frac{d}{dt} \nabla = \nabla \frac{d}{dt}.$$

With the use of comoving coordinates, (15) – (17) become

$$\frac{d}{dt}\left(\frac{S}{S_0} \mathbf{v}\right) + \nabla\psi + \frac{1}{\rho} \nabla \delta p = 0, \tag{18}$$

$$\frac{d}{dt}\left(\frac{\delta\rho}{\rho}\right) + \frac{S_0}{S} \nabla \cdot \mathbf{v} = 0, \tag{19}$$

$$S_0^2 \nabla^2 \psi - 4\pi G S^2 \delta\rho = 0, \tag{20}$$

where $\nabla \cdot \mathbf{u} = 3\dot{S}/S_0$, $\mathbf{v} \cdot \nabla\mathbf{u} = \mathbf{v}\,\dot{S}/S_0$, and $\rho S^3 = \text{constant}$.

Taking the curl of (18), we have

$$\frac{d}{dt}(S^2 \zeta) = 0, \tag{21}$$

where $\nabla' \wedge \mathbf{v} = \zeta$ is the vorticity. Hence $S^2\zeta$ is conserved. In the absence of pressure and density gradients both $\mathbf{v}$ and ζ are zero. The vorticity is therefore permanently zero and the motion is irrotational:

$$\mathbf{v} = (S_0/S)\,\nabla\,\varphi. \tag{22}$$

For the four unknowns $\delta p, \delta\rho, \psi$, and φ, there are the three equations (18) - (20). An equation of state is therefore necessary. Let $p \propto \rho^\gamma$, where γ is the ratio of specific heats, then

$$\delta p = \gamma(p/\rho)\,\delta\rho = c_s^2\,\delta\rho, \tag{23}$$

and c_s is the speed of sound. Taking the divergence of (18) and using (19), we find

$$\left[\left(\frac{d}{dt} + 3\frac{\dot{S}}{S}\right)\left(\frac{d}{dt} + \frac{\dot{S}}{S}\right) - 4\pi G\rho - \frac{S_0^2}{S^2}\,c_s^2\,\nabla^2\right]\nabla^2\psi = 0. \tag{24}$$

It is now possible to write $\psi = \psi(t)\,\Pi(r,\theta,\phi)$, and

$$\nabla^2\Pi + k^2\Pi = 0, \tag{25}$$

where k^2 is the separation constant. With (5) and (7), Equation (24) becomes

$$\ddot{\psi} + 4 \frac{\dot{S}}{S} \dot{\psi} + \left(\frac{S_0^2}{S^2} c_s^2 k^2 - 2 \frac{\kappa}{S^2} + \Lambda \right) \psi = 0, \tag{26}$$

and dots denote time derivatives. The remaining two equations are

$$\frac{1}{S} \frac{d}{dt} (S\psi) = 4\pi G \rho \varphi, \tag{27}$$

$$S_0^2 k^2 \psi = - 4\pi G S^2 \delta \rho. \tag{28}$$

The last three equations determine the fluctuations of density, velocity, and gravitational potential for an inviscid, irrotational fluid. They are identical with those obtained in Section 5.5 from the theory of general relativity for $c_s^2 \ll c^2$.

3.3 Normal Modes of Vibration

An arbitrary disturbance in a scalar quantity consists of a superposition of normal modes given by (25). These modes can be constructed from plane waves

$$\Pi_k \propto \exp i \mathbf{k} \cdot \mathbf{r}$$

The modes in spherical coordinates are given for comparison with those in Section 5.7 for curved space.

Let $\Pi = \Psi(r)\Theta(\theta)\Phi(\phi)$; by separating the variables, Equation (25) in flat space becomes

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0, \tag{29}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0, \tag{30}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + \left(k^2 - \frac{n(n+1)}{r^2}\right)\Psi = 0. \tag{31}$$

The tesseral harmonics of n th degree and nth order are

$$\Theta\,\Phi = Y_n^m(\theta, \phi) = (a_{nm}\,e^{im\phi} + b_{nm}\,e^{-im\phi})\,P_n^m(\cos\theta),$$

for integral values of n and m, and the spherical surface harmonics of degree n are

$$Y_n(\theta, \phi) = \sum_{m=0}^{n} Y_n^m(\theta, \phi). \tag{32}$$

The solutions for the radial function $\psi(r)$, for n integral, are the spherical Bessel functions

$$\Psi_n = \left(\frac{\pi}{2kr}\right)^{1/2} J_{n+\frac{1}{2}}(kr), \; \Psi_{-n} = \left(\frac{\pi}{2kr}\right)^{1/2} J_{n-\frac{1}{2}}(kr).$$

There are no boundary or periodic conditions to satisfy, and the only condition is that Ψ must be finite everywhere. For $kr \to 0$,

$$\Psi_n \to (kr)^n / [1.3.5.\ldots(2n+1)],$$

$$\Psi_{-n} \to [1.3.5.\ldots(2n-1)]/(kr)^{n+1},$$

and $kr \to \infty$,

$$\Psi_n \to (kr)^{-1} \cos\left[kr - \tfrac{1}{2}(n+1)\pi\right],$$

$$\Psi_{-n} \to (kr)^{-1} \sin\left[kr - \tfrac{1}{2}(n+1)\pi\right],$$

and hence Ψ_{-n} is rejected since it diverges as $kr \to 0$. The radial functions form a continuous set having all eigenvalues of $k > 0$ for each value of n. In particular, for $n = 1, 2$, and 3:

$$\Psi_0 = (kr)^{-1} \mathcal{S}$$

$$\Psi_1 = (kr)^{-1} [(kr)^{-1} \mathcal{S} - C], \tag{33}$$

$$\Psi_2 = (kr)^{-1} [\{3(kr)^{-2} - 1\} \mathcal{S} - 3(kr)^{-1} C],$$

$\mathcal{S} = \sin kr$, $\mathcal{C} = \cos kr$. The spatial disturbances are represented by summations and integrations over the complete set of wave functions

$$\Pi = \Psi_n (kr)\, Y_n^m (\theta, \phi). \tag{34}$$

4. GRAVITATIONAL INSTABILITY

4.1 General Criteria

Let

$$\alpha_m = S^m \, \delta\rho \propto \delta\rho / \rho^{m/3}. \tag{35}$$

The existence of time growing modes of an arbitrary quantity such as α_m does not necessarily imply instability in a nonstatic universe. For example, if $\alpha_0 = \delta\rho$ grows, an expanding universe is unstable, but a contracting universe is stable if ρ increases more rapidly than $\delta\rho$. The only time growing quantity that denotes unambiguously a changing configuration is the contrast density $\alpha_3 = \delta\rho/\rho$. Suppose that α_m grows in time; then in an expanding (contracting) universe $m = 3$ is necessary and $m \leqslant 3$ ($m \geqslant 3$) is sufficient for instability. Thus the growth of the gravitational potential (27) = $\psi \propto \alpha_2$, is sufficient only for instability in an expanding universe.

When the growth time of a mode is greater than the age of the universe it cannot contribute to a significant change in configuration. For instability we require

$$\dot{\alpha}_3/\alpha_3 \gg |\dot{S}/S|, \tag{36}$$

or, if $\alpha_3 \propto S^{\pm\ell}$ ($+\ell$ for $\dot{S} > 0$, $-\ell$ for $\dot{S} < 0$),

$$\ell >> 1.$$

Even then a clear case of instability requires an adequate amplitude of the initial disturbance. For arbitrarily small initial conditions a clear case of incipient instability requires $\alpha_3 \propto \exp \omega t$, $\omega >> |\dot{S}/S|$.

The Newtonian equations of the unperturbed cosmological models, and the linearized Newtonian equations of the perturbed models, are identical with the corresponding general relativity equations for small pressure. The Newtonian equations are limited to flat space and cannot be used to determine the behaviour of modes in curved space. This is not a severe limitation as the eigenvalues k in (26) are multiplied by the speed of sound c_s, and the time dependence of α_m can be discussed for a given value of kc_s.

From (26) the general equation for α_m is

$$\ddot{\alpha}_m + (8 - 2m)\frac{\dot{S}}{S}\dot{\alpha}_m + \left[K^2 + (6 - m)\frac{\ddot{S}}{S} + (4 - m)(3 - m)\frac{\dot{S}^2}{S^2} - \Lambda\right]\alpha_m = 0, \tag{38}$$

where $K = c_s k S_0/S$. Equation (26) is for m = 2; and for m = 3:

$$\ddot{\alpha}_3 + 2\frac{\dot{S}}{S}\dot{\alpha}_3 + (K^2 - 4\pi G\rho)\alpha_3 = 0. \tag{39}$$

This equation has been derived by Bonnor[24] and van Albada[25] for radial type perturbations and has been discussed more generally by Savedoff and Vila.[27] From (27) the velocity potential is given by

$$\dot{\alpha}_3 + \varphi k^2 S_0^2/S^2 = 0. \tag{40}$$

For m = 4,

$$\ddot{\alpha}_4 + \left(K^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda\right)\alpha_4 = 0. \tag{41}$$

Unless stated otherwise we shall assume that the cosmological term Λ is zero. When

$$K^2 > 8\pi G\rho/3, \tag{42}$$

α_4 is oscillatory, and hence α_m is also oscillatory for all m. For $\kappa = 0$ (zero energy) this condition is $c_s^2 > (\dot{S}\lambda\!\!\!^{-}/S_0)^2$, where $\lambda\!\!\!^{-} = k^{-1}$, and the velocity of sound exceeds the expansion or contraction velocity (1) at distance $\lambda\!\!\!^{-}$.

Since $p \propto \rho^\gamma$, it follows

$$c_s^2 = c_{s0}^2\ (S_0/S)^{3\gamma - 3} \tag{43}$$

or $K^2 = K_0^2(S_0/S)^{3\gamma - 1}$ where $K_0 = c_{s0}k$ at $S = S_0$. Equation (41) is now

$$\ddot{\alpha}_4 + [K_0^2(S_0/S)^{3\gamma - 4} - \beta/S_0^3](S_0/S)^3\alpha_4 = 0. \tag{44}$$

This eauation is useful for discussing qualitatively the behaviour of α_m as a function of time. So far S_0 is arbitrary; let $S = K_0^2$ at $S_0 = \beta/S_0^3$, or

$$c_{s0}^2 k^2 = 8\pi G\rho_0/3, \tag{45}$$

and (44) becomes

$$\ddot{\alpha}_4 + [(S_0/S)^{3\gamma-4} - 1](\beta/S)^3 \alpha_4 = 0. \tag{46}$$

Hence, α_4 has marginal stability at $S = S_0$. The results are shown in Table 1. In an expanding (contracting) universe α_m is a growing function of time for $m > 4$ ($m < 4$) for all γ. More precise conditions are given in Section 4.3 from the solutions of (38). At this stage we are unable to deduce any conditions concerning the growth of the contrast density ($m = 3$) in the important case of an expanding universe.

A marginal state for any value of m is obtained by using

$$\epsilon = \int (S/S_0)^{2m-8} dt,$$

in (38):

$$\frac{d^2\alpha_m}{d\epsilon^2} + \left[K^2 + (6-m)\frac{\ddot{S}}{S} + (m-4)(m-3)\frac{\dot{S}^2}{S^2}\right]\dot{\epsilon}^2 \alpha_m = 0.$$

For $m = 3$, this is van Albada's[25] equation

$$\frac{d^2\alpha_3}{d\epsilon^2} + (K^2 - 4\pi G\rho)\left(\frac{S_0}{S}\right)^4 \alpha_3 = 0, \tag{47}$$

TABLE 1

Time Dependence of α_m

		m < 4	m = 4	m > 4
$\gamma < \frac{4}{3}$	$\dot{S} > 0$	$\sim\downarrow$, (?)	$\sim$, ($\uparrow$)	$\sim\uparrow$, ($\uparrow$)
	$\dot{S} < 0$	$\sim\uparrow$, ($\uparrow$)	$\sim$, ($\uparrow$)	$\sim\downarrow$, (?)
$\gamma = \frac{4}{3}$	$\dot{S} > 0$	?, (?)	$\uparrow$, ($\uparrow$)	$\uparrow$, ($\uparrow$)
	$\dot{S} < 0$	$\uparrow$, ($\uparrow$)	$\uparrow$, ($\uparrow$)	?, (?)
$\gamma > \frac{4}{3}$	$\dot{S} > 0$	?, ($\sim\uparrow$)	$\uparrow$, ($\sim$)	$\uparrow$, ($\sim\uparrow$)
	$\dot{S} < 0$	$\uparrow$, ($\sim\uparrow$)	$\uparrow$, ($\sim$)	?, ($\sim\downarrow$)

The symbols denote: $\dot{S} > 0$, expansion; $\dot{S} < 0$, contraction; $\uparrow$, growth; $\downarrow$, decay; $\sim$, oscillating; without brackets, $S > S_0$; with brackets, $S < S_0$. For example: ($\sim\downarrow$) means decaying oscillation for $S < S_0$.

and therefore, Jeans' criterion of marginal 'stability': $K^2 = 4\pi G\rho$, holds for $\alpha_3(\epsilon)$. However, ϵ is not a linear function of time, and in general this marginal state does not give an unequivocal stability criterion.

4.2 Jeans' Criterion

Jeans' treatment[21] of gravitational instability resembles Lord Rayleigh's formulation of the problem of oscillations in a fluid of positive and negative charges.[32] Jeans assumes that a uniform, gravitating fluid is in a stationary state (as for a neutral plasma) and $\dot{S} = 0$. Hence (39) becomes

$$\ddot{\alpha}_3 + (c_s^2 k^2 - 4\pi G\rho)\,\alpha_3 = 0, \tag{48}$$

and $S = S_0$. For $\alpha_3 \propto \exp i\omega t$ this equation gives the dispersion relation

$$\omega^2 = c_s^2 k^2 + \omega_p^2. \tag{49}$$

A similar relation holds for electrostatic oscillations in a plasma, and $\omega_p = (4\pi n e^2/m)^{1/2}$ is the plasma frequency for electrons of number density n and charge to mass ratio e/m. In Jeans' dispersion relation the 'gravitational frequency' is imaginary: $\omega_p = (-4\pi G\rho)^{1/2}$. There is thus a marginal state

$$k_J = (4\pi G\rho/c_s^2)^{1/2}, \tag{50}$$

and for $k > k_J$, ω is real and the disturbance oscillates at constant amplitude; and for $k < k_J$, ω is imaginary and the disturbance grows exponentially in time. Jeans' stability criterion is therefore $k > k_J$, or $\lambda < \lambda_J$, where $\lambda = k^{-1}$, $\lambda_J = k_J^{-1}$. The Debye length $\lambda_D = c_s/\omega_p$ plays a similar role in plasma physics and disturbances of $\bar{\lambda} < \lambda_D$ in many cases tend to be stable. In a Jeans' sphere (as compared with a Debye sphere) of radius $\bar{\lambda}_J$ the sum of the thermal and potential energies is of the order $c_s^2 \rho \bar{\lambda}_J^3 - 4\pi G \rho^2 \bar{\lambda}_J^5 = 0$. In a sphere of radius $\bar{\lambda} < \lambda_J$, the thermal energy predominates and collective interactions are of little consequence.

Jeans' analysis suffers from the defect that in general there is no initial stationary state in a uniform nonrotating fluid. When $\bar{\lambda} << \bar{\lambda}_J$, then $\omega^2 >> \dot{S}^2/S^2$, and the dispersion relation (49) is an acceptable approximation. But as $\bar{\lambda}$ increases, the oscillation period also increases and is infinite at $\bar{\lambda} = \lambda_J$; and when $\bar{\lambda} > \bar{\lambda}_J$, the e-folding time $i\omega_p^{-1}$ of a disturbance is comparable with the collapse time $i(2/3)^{1/2}\omega_p^{-1}$ of the system. Thus, in the range of interest: $\bar{\lambda} \gtrsim \bar{\lambda}_J$, the dispersion relation (49) fails and we must fall back on solving (39). It is seen that Jeans' instability criterion: $\bar{\lambda} > \lambda_J$, is necessary for expansion ($\dot{S} > 0$) and is sufficient for contraction ($\dot{S} < 0$), but in neither case is it both necessary and sufficient.

In Einstein's static universe (which has its analogy in Newtonian theory,[2] provided $p << \rho c^2$): $\dot{S} = 0$, $\ddot{S} = 0$, at $S = S_0$, and according to (5) and (7) this is possible for $\kappa = 1$ and

$$\Lambda = S_0^{-2} = 4\pi G \rho_0 . \tag{51}$$

Equation (38) for α_m becomes

$$\ddot{\mu} + (c_s^2 k^2 - \Lambda)\mu = 0 , \tag{52}$$

where μ stands for α_m, ψ, or φ. As Bonnor[24] has shown, Jeans' dispersion relation (49) holds true without modification in Einstein's static universe. The cosmological term neutralizes the gravitational field and there is a similarity with the neutral plasma state. If the pressure is zero, as in a fluid consisting of dust particles having no peculiar motion of their own, the velocity of sound is zero and all modes grow as $\exp \Lambda^{1/2} t$.

4.3 General Equations

To solve (38) we use the relations (11) and (12) and take χ as the independent variable. For $\kappa = 0$ (zero energy).

$$\frac{d^2 \alpha_m}{d\chi^2} + \frac{(14 - 4m)}{\chi} \frac{d\alpha_m}{d\chi} + 4\left[L(\chi) + \frac{(m-2)(m-9/2)}{\chi^2}\right] \alpha_m = 0,$$

$$L(\chi) = K_0^2 \beta^2 \chi_0^{6\gamma - 2} \Big/ \chi^{6\gamma - 6} \; ; \qquad (53)$$

$\kappa = 1$ (negative energy):

$$\frac{d^2 \alpha_m}{d\chi^2} + (14 - 4m)\cot \chi \frac{d\alpha_m}{d\chi} + 4\left[L(\sin \chi) - (m-4)(m-3) + \frac{(m-2)(m-9/2)}{\sin^2 \chi}\right] \alpha_m = 0,$$

$$L(\sin \chi) = K_0^2 \beta^2 (\sin \chi_0)^{6\gamma - 2} \Big/ (\sin \chi)^{6\gamma - 6} \; ; \qquad (54)$$

and for $\kappa = -1$ (positive energy):

$$\frac{d^2 \alpha_m}{d\chi^2} + (14 - 4m)\coth \chi \frac{d\alpha_m}{d\chi} + 4\left[L(\sinh \chi) + (m-4)(m-3) + \frac{(m-2)(m-9/2)}{\sinh^2 \chi}\right] \alpha_m = 0,$$

$$L(\sinh \chi) = K_0^2 \beta^2 (\sinh \chi_0)^{6\gamma - 2} \Big/ (\sinh \chi)^{6\gamma - 6} . \qquad (55)$$

These equations are used to consider the growth rates in (i) a cold universe, (ii) an isothermal universe ($\gamma = 1$), and in the cases when (iii) $\gamma = 4/3$, and (iv) $\gamma = 5/3$.

4.4 Cold Universe

If the pressure is zero, as in a universe containing dust particles having no peculiar motion of their own, all time growing perturbations have their maximum possible rate of growth. A cold universe is the most unstable of all, and is not only a very simple but is also a very interesting model to study. In this model $K_0 = 0$ in (53)-(55).

For $\kappa = 0$ (zero energy) the solution of (53) is

$$\alpha_m = A\, S^{m-9/2} + B\, S^{m-2}, \tag{56}$$

A, B are constants. Hence, during expansion ($\dot{S} > 0$), α_2 is either constant or decays, and during contraction ($\dot{S} < 0$), $\alpha_{9/2}$ is either constant or decays. For $\dot{S} > 0$ ($\dot{S} < 0$), α_m grows for $m > 2$ ($m < 9/2$). Using only the growing terms

$$\dot{S} > 0: \quad \delta\rho/\rho \propto S \quad \propto \rho^{-1/3},$$

$$\dot{S} < 0: \quad \delta\rho/\rho \propto S^{-3/2} \propto \rho^{1/2},$$

and these growths are the same for all modes.

The absence of exponential growth is typical of all non-static models. According to (36) neither $\ell = 1$ (expansion) nor $\ell = \frac{3}{2}$ (contraction) satisfy the condition (37): $\ell >> 1$, for a convincing case of instability. Because the universe is expanding the result $\ell = 1$ is of particular interest. It shows that during the life-span of the universe in which the Newtonian theory is valid small initial disturbances tend to remain small. For example, if initially $\rho_1 \sim 1$ g cm^{-3} and $\delta\rho_1/\rho_1 \sim 10^{-15}$, then at $\rho \sim 10^{-30}$ g cm^{-3} it follows that $\delta\rho/\rho \sim 10^{-5}$.

In the $\kappa = 1$ (negative energy) or oscillating model the solution of (54) is

$$\alpha_m = (\sin\chi)^{2m-7} \left[A P_2^1 (i \cot\chi) + B Q_2^1 (i \cot\chi)\right], \tag{57}$$

where $0 \leq \chi \leq \pi$, A and B are different constants, and P_ν^μ, Q_ν^μ are the associated Lengendre functions. Since $P_2^1(-ix) = -P_2^1(ix)$, $Q_2^1(-ix) = Q_2^1(ix)$, we need consider only the range $0 \leq \chi \leq \frac{1}{2}\pi$.. Using the relations[33]

$$P_2^1 (ix) = -3(1 + x^2)^{1/2} x,$$

$$Q_2^1 (ix) = i(1 + x^2)^{1/2} \left[3x \cot^{-1} x - (3x^2 + 2)/(1 + x^2)\right],$$

(57) becomes

$$\alpha_m = \mathcal{S}^{2m-9} \left[A C + B(3\mathcal{S} - \mathcal{S}^3 - 3\chi C)\right], \tag{58}$$

$\mathcal{S} = \sin\chi$, $C = \cos\chi$. As $\chi \to 0$, this equation is identical with (56).

In the $\kappa = -1$ (positive energy) model the solution of (55) is

$$\alpha_m = (\sinh\chi)^{2m-7} \left[A P_2^1 (\coth\chi) + B Q_2^1 (\coth\chi)\right]. \tag{59}$$

From the expressions $(x > 1)$

$$P_2^1(x) = 3(x^2 - 1)^{1/2}\, x,$$

$$Q_2^1(x) = (x^2 - 1)^{1/2}\left[\frac{3}{2}\, x \,\ell n\, \frac{x+1}{x-1} - \frac{3x^2 - 2}{x^2 - 1}\right],$$

it is found

$$\alpha_m = S^{2m-9}\,[AC + B(3S + S^3 - 3\chi C)], \tag{60}$$

$S = \sinh\chi$, $C = \cosh\chi$. As $\chi \to 0$, (56) is recovered; however, as $\chi \to \infty$, $\alpha_m = A S^{2m-8} + B S^{2m-6}$, and in an expanding (contracting) universe α_m grows for $m > 3$ $(m < 4)$.

The solutions (56), (58), and (60), for $\alpha_3 = \delta\rho/\rho$, are shown in Figure 1 with the constants A and B equal to unity. In an expanding universe the growth is least, as one would expect, in the case of positive energy of $\kappa = -1$. The rate of growth is greatest for $\kappa = 1$, but the growth is limited because the universe oscillates.

4.5 Isothermal Universe

Under certain conditions it is possible for a fluid to expand and contract isothermally.[22] Furthermore, it may be possible for regions of increased density to be cooled by radiation, thus tending to make the fluctuations isothermal. When $\gamma = 1$, the L's in (53) - (55) equal $(c_s k S_0)^2$. At $S = S_0$, let

$$c_{s0}^2\, k^2 = 8\pi G \rho_0 / 3 \tag{61}$$

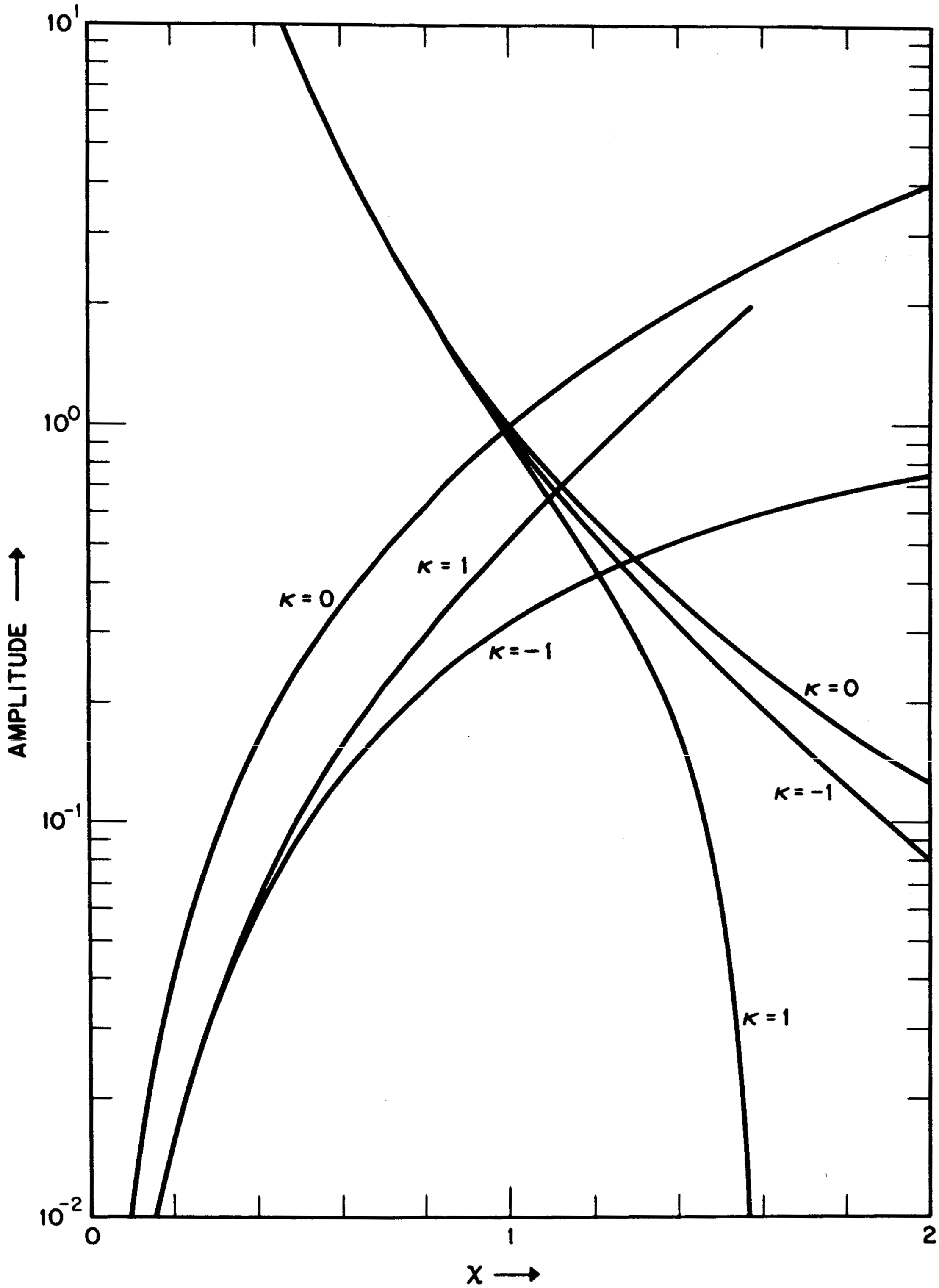

Figure 1—Curves increasing from left to right show the growth in amplitude of $\delta\rho/\rho$ for $\kappa = 0, \pm 1$ in an expanding cold universe. Curves increasing from right to left show the corresponding growth in amplitude in a contracting cold universe.

Then, for $\kappa = 0$, $(c_{s0}\ k\ S_0)^2 = \beta/S_0$, and

$$\alpha_m = S^{m-13/4} \left[A\, J_{\frac{5}{2}} (2S^{1/2} \big/ S_0^{1/2}) + B\, J_{-\frac{5}{2}} (2\, S^{1/2} \big/ S_0^{1/2}) \right] \qquad (62)$$

and therefore

$$S \ll S_0 \;:\; \alpha_m = A\, S^{m-9/2} + B\, S^{m-2}$$

$$S \gg S_0 \;:\; \alpha_m = S^{m-7/2}\ [A \sin 2(S/S_0)^{1/2} + B \cos 2(S/S_0)^{1/2}\,].$$

At high density or long wavelengths α_m behaves as in the cold universe, and at low density or short wavelengths $\alpha_{7/2}$ oscillates.

For $\kappa = 1$,

$$\alpha_m = (\sin \chi)^{2m-13/2}\ [A P_\lambda^{5/2}\ (\cos \chi) + B Q_\lambda^{5/2}\ (\cos \chi)] \qquad (63)$$

and $\kappa = -1$,

$$\alpha_m = (\sinh \chi)^{2m-13/2}\ [A P_\lambda^{5/2}\ (\cosh \chi) + B Q_\lambda^{5/2}\ (\cosh \chi)] \qquad (64)$$

where $\lambda = -1/2 \pm (1 + 4\ \kappa\, c_{s0}^2\ k^2\ S_0^2)^{1/2} = -1/2 \pm (5 + 4\ \kappa\ \dot{S}_0^2)^{1/2}$ and $\dot{S}_0$ occurs at S_0. When $2 c_{s0} k S_0 < 1$, these equations are similar to those of the cold model, and when $2 c_{s0} k S_0 > 1$ they resemble the zero curvature solution (62). Savedoff and Vila[27] have derived results similar to (63) and (64) in terms of hypergeometric functions.

4.6 Arbitrary γ

The general solution of (53) for $\kappa = 0$ and γ arbitrary is

$$\alpha_m \propto S^{m-13/4} J_{\pm p} (2 [S/S_0]^{2-3\gamma/2}), \tag{65}$$

$$P = 5/2(4 - 3\gamma),$$

where (61) holds at $S = S_0$; and for $\gamma = 4/3$,

$$\alpha_m \propto S^{m \pm \lambda - 13/4}, \tag{66}$$

$$\lambda = \left(\frac{25}{16} - \frac{3c_{s0}^2 k^2}{8\pi G \rho_0} \right)^{1/2}$$

and c_{s0} is the speed of sound at density ρ_0. From (65)

$$\left. \begin{array}{l} \gamma < \dfrac{4}{3}, \ S \ll S_0: \\ \\ \gamma > \dfrac{4}{3}, \ S \gg S_0: \end{array} \right\} \quad \alpha_m = A S^{m-9/2} + B S^{m-2}, \tag{67}$$

$$\gamma < \frac{4}{3}, \ S \gg S_0: \quad \alpha_m = S^{m-n-13/4} (A \sin x + B \cos x),$$

$$\gamma > \frac{4}{3}, \ S \ll S_0: \quad \alpha_m = S^{m+n-13/4} (A \sin x + B \cos x), \tag{68}$$

where $n = 1 - 3/4\gamma$, $x \simeq 2(S/S_0)^{2-3\gamma/2}$. The general solution has been given by Savedoff and Vila.[27]

A photon, neutrino, or relativistic gas has a ratio of specific heats of 4/3. When the pressure of such gases is dominant but their density is small compared with the total density of the fluid, the fluid as a whole has $\gamma = 4/3$ and can be treated with the Newtonian approximation. When ($\kappa = 0$)

$$c_{s0}^2 k^2 > \frac{25}{6} \pi G \rho_0, \tag{69}$$

$\alpha_{13/4}$ oscillates:

$$\alpha_m \propto S^{m-13/4} e^{\pm i \mu \ln S}, \tag{70}$$

where $i\mu = \lambda$ in (66). In the case of $\kappa = \pm 1$

$$\alpha_m = S^{m-7/2}[AP_\lambda^1(i \cot \kappa\chi) + BQ_\lambda^1(i \cot \kappa\chi)], \tag{71}$$

and

$$\lambda = -\frac{1}{2} + (25/4 - 3c_{s0}^2 k^2/2\pi G \rho_0)^{1/2}.$$

P_λ^1, Q_λ^1 are nonperiodic for condition (69).

When $\gamma = 5/3$, $\kappa = 0$, (65) is

$$\alpha_m \propto S^{m-13/4} J_{\pm 5/2}(2S_0^{1/2}/S^{1/2}) \tag{72}$$

as shown by van Albada.[25] There appears to be no simple analytic solutions for $\kappa = \pm 1$ for arbitrary γ, nor is there in the important case of $\gamma = 5/3$.

5. RELATIVISTIC COSMOLOGY

5.1 Unperturbed Models

In its unperturbed state we assume that the universe is homogeneous and isotropic, and the metric is given by the Robertson-Walker line element

$$ds^2 = dt^2 - \frac{R^2}{c^2\left(1 + \frac{1}{4}\kappa r^2\right)^2}(dr^2 + r^2\, d\Omega^2), \tag{73}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, r, θ, ϕ are comoving coordinates, $R(t)$ has the dimensions of length, and $\kappa = 0, \pm 1$ is the curvature constant. The energy-momentum tensor of a perfect fluid is

$$T^i_j = (\rho c^2 + p)\, g_{kj}\, u^k u^i - \delta^i_j\, p, \tag{74}$$

in which ρc^2 is the energy density, u^i the four velocity, and p is the isotropic pressure. For a fluid that is stationary in the comoving system, $u^0 = 1$, $u^1 = u^2 = u^3 = 0$, and the components of the energy-momentum tensor are

$$T^0_0 = \rho c^2, \; T^1_1 = T^2_2 = T^3_3 = -p. \tag{75}$$

All that now remains is to determine $R(t)$ in (73) with the Einstein equation

$$R^i_j - \frac{1}{2} \delta^i_j R^\ell_\ell + \delta^i_j \Lambda = -(8\pi G/c^2) T^i_j. \tag{76}$$

In this equation, R^i_j is the contracted Riemann-Christoffel or Ricci tensor (its further contraction R^ℓ_ℓ is shown explicitly to avoid confusion with $R(t)$), and the cosmological term Λ is included as in the Newtonian treatment for historical interest.

Equation (76) is readily solved using the line-element (73) and the components (75) of the energy-momentum tensor.[1,34] The following method however is particularly simple. Transforming to the coordinates:

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta,$$

$r^2 = x^2 + y^2 + z^2$, the line element (73) in the immediate neighborhood of $r = 0$ becomes

$$ds^2 = dt^2 - c^{-2} R^2 \left[1 - \frac{1}{2} \kappa (x^2 + y^2 + z^2)\right] (dx^2 + dy^2 + dz^2). \tag{77}$$

At any desired point r = 0 the coordinates are natural,[35] as in the Minkowski metric. For these coordinates the Christoffel symbols

$$\Gamma^{r}_{pq} = \frac{1}{2} g^{rs} \left(\frac{\partial g_{ps}}{\partial x^{q}} + \frac{\partial g_{qs}}{\partial x^{p}} - \frac{\partial g_{pq}}{\partial x^{s}} \right),$$

are*

$$\Gamma^{\mu}_{0\nu} = c^2 R^{-2} \Gamma^{0}_{\mu\nu} = \delta^{\mu}_{\nu} (\dot{R}/R),$$

$$\Gamma^{\alpha}_{\beta\beta} = -\Gamma^{\beta}_{\beta\alpha} = -\Gamma^{\alpha}_{\alpha\alpha} = \frac{1}{2} \kappa x^{\alpha}. \tag{78}$$

As $r \to 0$, the only surviving Christoffel symbols are $\Gamma^{\alpha}_{0\alpha}$ and $\Gamma^{0}_{\alpha\alpha}$, and the spatial derivatives of (78) are

$$\frac{\partial}{\partial x^{\alpha}} \Gamma^{\alpha}_{\beta\beta} = -\frac{\partial}{\partial x^{\alpha}} \Gamma^{\beta}_{\beta\alpha} = -\frac{\partial}{\partial x^{\alpha}} \Gamma^{\alpha}_{\alpha\alpha} = \frac{1}{2} \kappa. \tag{79}$$

Since

$$R_{kj} = \frac{\partial}{\partial x^{j}} \Gamma^{\ell}_{k\ell} - \frac{\partial}{\partial x^{\ell}} \Gamma^{\ell}_{kj} + \Gamma^{m}_{k\ell} \Gamma^{\ell}_{j\ell} - \Gamma^{\ell}_{kj} \Gamma^{m}_{\ell m}, \tag{80}$$

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*The convention adopted is that Latin indices assume all values 0,1,2,3: Greek indices λ, μ, ν assume only the values 1,2,3, and α, β, γ are used when there is no summation and $\alpha \neq \beta \neq \gamma$.

we find from $R^i_j = g^{ik} R_{kj}$ that

$$R^0_0 = 3\ddot{R}/R,$$

$$R^\alpha_\alpha = (\ddot{R}R + 2\dot{R}^2 + 2\kappa c^2)/R^2, \tag{81}$$

and all other components vanish. From (76) we now obtain the well-known equations

$$\dot{R}^2 = \frac{1}{3}(8\pi G\rho + \Lambda) R^2 - \kappa c^2, \tag{82}$$

$$\frac{d}{dt}(R^3\rho) + \frac{p}{c^2}\frac{dR^3}{dt} = 0. \tag{83}$$

Every observer can adjust the origin of his comoving coordinate system to give $r = 0$, and therefore (82) - (83) apply to all comoving observers. Equations (82)-(83) are identical with (6)-(7), provided $R = cS$ and p/c^2 is vanishingly small. In the relativistic treatment we shall use R rather than S.

5.2 Equation of State

We have two equations, (82) and (83), for R, ρ, and p, and therefore require an equation of state:

$$p = (\nu - 1)\rho c^2. \tag{84}$$

In any range of density in which ν is constant, (83) becomes

$$\rho R^{3\nu} = \text{constant}.$$

If, now

$$\beta_\nu = 8\pi G \rho R^{3\nu}/3, \tag{85}$$

(82) can be written as

$$\dot{R}^2 = \beta_\nu R^{2-3\nu} + \frac{1}{3} \Lambda R^2 - \kappa c^2. \tag{86}$$

For the three models $\kappa = 0, 1, -1$, and $\Lambda = 0$, it is found that

$$\begin{aligned}
\kappa = 0: \quad & R = (\beta_\nu c^{-2} \chi^2)^{1/(3\nu-2)}, \\
\kappa = 1: \quad & R = (\beta_\nu c^{-2} \sin^2 \chi)^{1/(3\nu-2)}, \\
\kappa = -1: \quad & R = (\beta_\nu c^{-2} \sinh^2 \chi)^{1/(3\nu-2)},
\end{aligned} \tag{87}$$

where

$$\frac{dt}{d\chi} = \frac{2}{3\nu - 2} \frac{R}{C}. \tag{88}$$

In the relativistic treatment the pressure contributes to the mass density of the fluid. If the kinetic and non-gravitational interaction energies of the constituent particles of the fluid are small compared with the particle rest masses, as in the universe in its present state, there is justification for assuming $\nu = 1$ as in the Newtonian theory. However, there is no justification for assuming that the pressure gradients in a perturbed universe of $p \ll \rho c^2$ are also of negligible effect. If, at low density, instead of (84) we adopt

$$\delta p = (\nu - 1)\, c^2\, \delta\rho \tag{89}$$

where $\delta p = c_s^2\, \delta\rho$, then

$$\nu = 1 + c_s^2 / c^2 = 1 + \gamma p / \rho c^2 , \tag{90}$$

and $c_s^2 \ll c^2$. In an isotropic photon or neutrino gas, or in an isotropic fluid in which the particles and their fields have energies large compared with the rest masses, ν attains its maximum value[36] of 4/3 and $p = \frac{1}{3} \rho c^2$. The range of ν is therefore $1 \leq \nu \leq 4/3$. (Zel'dovitch[37] has proposed increasing the upper limit to 2, but it is possible that this is unrealistic.[38] McCrea[39] suggests a lower limit of −2 as a physical explanation of a steady state universe, but this is outside the scope of our discussion.)

In the case of a gas containing degenerate electrons, the electrons are relativistic when their Fermi energy E_e exceeds $m_e c^2$ (m_e is the electron mass). But[40]

$$\frac{E_e}{m_e c^2} = \left(\frac{3^5}{4^4}\pi\right)^{1/3} \frac{\lambda_e}{a_e} \simeq \frac{\lambda_e}{a_e} \tag{91}$$

where a_e is the mean interelectron distance: $\frac{4}{3}\pi a_e^3 n_e = 1$, n_e the electron density, and $\lambda_e = h/m_e c$ the Compton wavelength. Hence, $E_e > m_e c^2$ occurs in hydrogen at $\rho > 10^6$ g cm^{-3}, approximately. It is reasonable to suppose as a first approximation[41] that the Fermi energy level is comparable with kT, where T is the temperature (in this section k is the Boltzmann constant). As the density rises, electromagnetic and weak interaction populate the photon and neutrino states,[42] thus ensuring that the Fermi and thermal energies (after allowance for rest masses) of the fermions and bosons are of comparable magnitude. The leptons and photons contribute most of the pressure and yet contribute only a small fraction of the total energy ρc^2 of the fluid. Therefore (90) is used with $\gamma = 4/3$. The lepton and photon energies become significant as the Fermi energy approaches $m_n c^2$ (m_n is the nucleon mass), and according to (91) this occurs for $a_e \sim \lambda_n$, where $\lambda_n = \hbar/m_n c$ (m_n is the nucleon mass), or when $\rho \sim 10^{16}$ g cm^{-3}. (The condition $E_e \sim kT$ limits electron pair production and the number of electrons does not greatly differ from the number of nucleons. Inverse β decay:[40,43] $p + e^- \to n + \nu$, which is of importance in stellar physics, plays only a minor role because of the dense neutrino-antineutrino background. The results of other studies,[44] in which it is assumed that the temperature is low, are also inapplicable.) The density of nuclear matter is $\rho_n \sim 10^{15}$ g cm^{-3}. For $\rho < \rho_n$ the pressure in general is small compared with the energy density and ν has a value close to unity. When $\rho \sim \rho_n$ the composition of the fluid is complex, and at higher densities progressively increases in complexity. However, whatever the nature of the fluid at supernuclear density, it seems likely[38] that ν will have a value close to 4/3. By these crude arguments it is possible to distinguish two states of the universe:

$$\text{subnuclear state:} \quad \rho < \rho_n, \quad \nu \simeq 1,$$

$$\text{supernuclear state:} \quad \rho > \rho_n, \quad \nu \simeq \frac{4}{3},$$

and in each of these states ν is roughly constant. The principal change in ν occurs in the transition through nuclear density.

Neglecting the transition region of ν there are now altogether six models:

Subnuclear-models. $\nu = 1$. These have been previously given in (11)-(12), but are now repeated using (87)-(88):

$$\begin{aligned} \kappa = 0: &\quad R = \beta_1 c^{-2} \chi^2, \\ \kappa = 1: &\quad R = \beta_1 c^{-2} \sin^2 \chi, \\ \kappa = -1: &\quad R = \beta_1 c^{-2} \sinh^2 \chi. \end{aligned} \tag{92}$$

and $dt = 2R\, d\chi / c$.

Supernuclear models. $\nu = 4/3$. Again, from (87)-(88):

$$\kappa = 0: \qquad R = \beta_{4/3}^{1/2}\, c^{-1}\, \chi,$$

$$\kappa = 1: \qquad R = \beta_{4/3}^{1/2}\, c^{-1} \sin \chi, \tag{93}$$

$$\kappa = -1: \qquad R = \beta_{4/3}^{1/2}\, c^{-1} \sinh \chi,$$

and $dt = R d\chi / c$.

The value of χ_n at the density ρ_n is estimated in the following way. At present in the universe $\beta_1 \sim R_0 c^2$, and hence $\beta_{4/3} \sim R_n \beta_1 \sim R_n R_0 c^2$. Also, $\chi_n \sim R_n c / \beta_{4/3}^{1/2}$, and therefore

$$\chi_n \sim (R_n / R_0)^{1/2} \sim (\rho_0 / \rho_n)^{1/6}. \tag{94}$$

From the present density of $\rho_0 \sim 10^{-30}\,\mathrm{g\,cm^{-3}}$ it follows that $\chi_n \sim 10^{-8}$. The supernuclear models (93), in which $\chi < \chi_n$, have therefore negligible difference and for simplicity we can assume $\kappa = 0$.

5.3 Linearized Equations of Perturbed Models

We consider small departures from the metric (73) as the result of displacements of the fluid. A perturbation treatment of the cosmological models, as distinct from a static and flat metric,[34, 45] encounters the slight complication of non-vanishing Christoffel symbols. The most general treatment has been given by Lifshitz[46] and includes rotational motions and gravitational waves. Owing to the conditions imposed on the metric tensor, simple irrotational motions were excluded. The Newtonian approach shows us that in the simplest treatment these are the motions of most inerest. A treatment of the problem, analogous to the Newtonian treatment of a uniform fluid, does not appear to have been previously attempted in a simple and straightfoward manner.

Small variations in the metric tensor are expressed as $g_{jk} + \delta g_{jk}$, where the g_{jk} are given by (73) and

$$\delta g_{jk} = h_{jk} \, . \tag{95}$$

We assume h_{jk} and its derivatives are everywhere small, and that quadratic and higher order terms in small quantities are negligible. Thus the unperturbed tensors g_{jk}, g^{ik} are used for lowering and raising the indices of h^{ik}, h_{jk} : $h^i_j = g^{ik} h_{jk} = g_{jk} h^{ik}$, and in effect h_{jk} is a tensor field in the unperturbed g_{jk} space. Since $g_{jk}\, g^{ik} = \delta^i_j$, to a first order

$$\delta (g_{jk} g^{ik}) = h^i_j + g_{jk}\, \delta g^{ik} = 0,$$

and therefore

$$\delta g^{ik} = - h^{ik} \, . \tag{96}$$

It is more convenient to use Einstein's equation (76) in the alternative form

$$R^i_j = - (8\pi G / c^2) \left(T^i_j - \frac{1}{2}\, \delta_j\, T \right) + \delta_j\, \Lambda \, , \tag{97}$$

and its perturbation is

$$\delta R^i_j = - (8\pi G / c^2)\, \delta \left(T^i_j - \frac{1}{2}\, \delta^i_j\, T \right) . \tag{98}$$

The perturbation δR^i_j is evaluated in terms of h^i_j as follows. From (80),

$$\delta R_{jk} = \left(\frac{\partial}{\partial x^j} \delta\Gamma^{\ell}_{k\ell} - \Gamma^{\ell}_{kj} \delta\Gamma^{m}_{\ell m} - \Gamma^{m}_{\ell j} \delta\Gamma^{\ell}_{km} + \Gamma^{\ell}_{jm} \delta\Gamma^{m}_{k\ell}\right)$$

$$- \left(\frac{\partial}{\partial x^{\ell}} \delta\Gamma^{\ell}_{kj} - \Gamma^{m}_{k\ell} \delta\Gamma^{\ell}_{jm} - \Gamma^{m}_{j\ell} \delta\Gamma^{\ell}_{km} + \Gamma^{m}_{\ell m} \delta\Gamma^{\ell}_{kj}\right),$$

thus giving the Palatini equation[47]

$$\delta R_{jk} = \delta\Gamma^{\ell}_{k\ell;j} - \delta\Gamma^{\ell}_{kj;\ell}, \tag{99}$$

where a semicolon denotes covariant differentiation. Also, to a first order,

$$\delta R^i_j = \delta(g^{ik} R_{jk}) = g^{ik}\delta R_{jk} - h^i_k R^k_j, \tag{100}$$

and with (99), we obtain

$$g^{ik} (\delta\Gamma^{\ell}_{k\ell;j} - \delta\Gamma^{\ell}_{kj;\ell}) = \delta R^i_j + h^i_k R^k_j. \tag{101}$$

The perturbed Christoffel symbols are

$$\delta\Gamma^{\ell}_{kj} = \delta(g^{\ell r} g_{rs} \Gamma^{s}_{kj}) = g^{\ell r} \delta(g_{rs}\Gamma^{s}_{kj}) - \Gamma^{s}_{kj} h^{\ell}_{s}$$

$$= \frac{1}{2} g^{\ell r} \left(\frac{\partial h_{kr}}{\partial x^j} + \frac{\partial h_{jr}}{\partial x^k} - \frac{\partial h_{kj}}{\partial x^r}\right) - \Gamma^{s}_{kj} h^{\ell}_{s},$$

thus giving Lifshitz's[46] equation

$$\delta\Gamma^{\ell}_{kj} = \frac{1}{2} g^{\ell r} (h_{kr;j} + h_{jr;k} - h_{kj;r}). \qquad (102)$$

In particular, $\delta\Gamma^{\ell}_{k\ell} = \frac{1}{2} h_{;k}$, where h is the trace h^{ℓ}_{ℓ}.

From (101)-(102) we have a linearized differential equation in h^i_j:

$$g^{ik}(h_{;kj} - h^{\ell}_{k;j\ell} - h^{\ell}_{j;k\ell}) + g^{\ell m} h^i_{j;\ell m} = 2(\delta R^i_j + h^i_k R^k_j), \qquad (103)$$

The g_{jk} and R^i_j are known, and δR^i_j is given by (98) in terms of the perturbed energy-momentum tensor.

The trace of T^i_j is $T = \rho c^2 - 3p$, and therefore

$$\delta T = c^2 \delta\rho - 3\delta p. \qquad (104)$$

Furthermore, $g_{kj} u^k u^j = 1$, and from $\delta(g_{kj} u^k u^j) = 0$,

$$h_{kj} u^k u^j + g_{kj} \delta u^k u^j + g_{kj} u^k \delta u^j = 0$$

where $u^0 = 1$, $u^\mu = 0$, and therefore

$$h^0_0 + 2 u_0 \delta u^0 = 0. \qquad (105)$$

Also, $g_{kj}u^k u^i = \delta^i_0 \delta^0_j$, and

$$\delta(g_{kj} u^k u^i) = \delta^i_0 h^0_j + \delta^0_j u_0 \delta u^i + g_{kj} \delta u^k \delta^i_0 u^0 .$$

We therefore find, from (74)

$$\delta T^i_j = \delta(\rho c^2 + p) g_{kj} u^k u^i + (\rho c^2 + p) \delta(g_{kj} u^k u^i) - \delta^i_j \delta p$$

$$= \delta^i_0 \delta^0_j \delta(\rho c^2 + p) - \delta^i_j \delta p + (\rho c^2 + p) (\delta^i_0 h^0_j + \delta^0_j u_0 \delta u^i + g_{kj} \delta u^k \delta^i_0 u^0). \quad (106)$$

From (105) and (106) it follows that the components of δT^i_j are

$$\begin{aligned} \delta T^0_0 &= c^2 \delta\rho, \\ \delta T^0_\alpha &= (\rho c^2 + p)(h^0_\alpha + g_{\alpha\alpha} \delta u^\alpha), \\ \delta T^\alpha_\alpha &= -\delta p, \\ \delta T^\alpha_\beta &= 0, \end{aligned} \quad (107)$$

where $\alpha, \beta = 1, 2,$ or 3, $\alpha \neq \beta$, and no summation. From (98), (104), and (107), we now have

$$\delta R^0_0 = -4\pi G\, \delta(\rho + 3p/c^2),$$

$$\delta R^0_\alpha = -8\pi G(\rho + p/c^2)\,(h^0_\alpha + g_{\alpha\alpha}\, \delta u^\alpha), \tag{108}$$

$$\delta R^\alpha_\alpha = 4\pi G\, \delta(\rho - p/c^2),$$

and $\delta R^\alpha_\beta = 0$. In addition, the components of R^i_j are

$$R^0_0 = -4\pi G\,(\rho + 3p/c^2) + \Lambda,$$

$$R^\alpha_\alpha = 4\pi G\,(\rho - p/c^2) + \Lambda. \tag{109}$$

Collecting together the equations (103), (108), and (109), we have, for $i = j = 0$:

$$h_{;00} - 2h^{\ell}_{0;0\ell} + g^{\ell m} h^0_{0;\ell m} = -8\pi G\,(h^0_0 + \delta)\,(\rho + 3p/c^2) + 2h^0_0\Lambda; \tag{110}$$

for $i = 0$, $j = \alpha$, ($\alpha = 1, 2,$ or 3; no summation over α):

$$h_{;0\alpha} - h^{\ell}_{0;\alpha\ell} - h^{\ell}_{\alpha;0\ell} + g^{\ell m} h^0_{\alpha;\ell m} =$$

$$-8\pi G\,[h^0_\alpha\,(\rho + 3p/c^2) + 2\,(\rho + p/c^2)\, g_{\alpha\alpha}\, \delta u^\alpha] + 2h^0_\alpha\,\Lambda; \tag{111}$$

for $i = \alpha, j = \alpha$:

$$g^{\alpha\alpha}\left(h_{;\alpha\alpha} - 2h^{\ell}_{\alpha;\alpha\ell}\right) + g^{\ell m} h^{\alpha}_{\alpha;\ell m} = 8\pi G(h^{\alpha}_{\alpha} + \delta)(\rho - p/c^2) + 2h^{\alpha}_{\alpha}\Lambda; \tag{112}$$

and for $i = \alpha,\ j = \beta,\ (\alpha \neq \beta)$:

$$g^{\alpha\alpha}\left(h_{;\alpha\beta} - h^{\ell}_{\alpha;\beta\ell} - h^{\ell}_{\beta;\alpha\ell}\right) + g^{\ell m} h^{\alpha}_{\beta;\ell m} = 0. \tag{113}$$

Altogether, (110)-(113) provide ten equations for the determination of the ten unknowns: h_{k_j} (four component's can be discarded by coordinate transformations), δu^{α} (since $\delta u^0 = -1/2\ h^0_0$), and $\delta\rho$ (δp is given by an equation of state).

5.4 Irrotational Motion

For irrotational motion it can be shown that (110)-(113) reduce to three equations determining ψ, φ, $\delta\rho$, as in the Newtonian treatment.

The simplest procedure is to transform to local coordinates where the Christoffel symbols are given by (78). In addition, we adopt a system of coordinates in which

$$h_{0\alpha} = 0, \tag{114}$$

and

$$\frac{\partial}{\partial x^i}\left(h^i_{\alpha} - \frac{1}{2}\delta^i_{\alpha} h\right) = \theta_{\alpha}. \tag{115}$$

The only equation involving the velocity δu^α is (111). This equation now becomes

$$\frac{\partial}{\partial x^\alpha}\left[\frac{1}{2}\frac{\partial h}{\partial t} - \frac{1}{R^2}\frac{\partial}{\partial t}(R^2 h_0^0)\right] - \frac{\partial \theta_\alpha}{\partial t} = -16\pi G(\rho + p/c^2)\, v_\alpha, \tag{116}$$

where $g_{\alpha\alpha}\,\delta u^\alpha = v_\alpha$. For irrotational motion $v_{\alpha;\beta} = v_{\beta;\alpha}$, or

$$v_\alpha = \partial\varphi/c^2\,\partial x^\alpha, \tag{117}$$

and this is possible when the left-hand side of (116) is a derivative with respect to x^α. Hence, $\theta_\alpha = \partial\theta/\partial x^\alpha$; and since θ can be absorbed into φ, we assume $\theta = 0$.

Equation (113) is now

$$\left[\frac{\partial}{\partial t}\left(2\dot{R}R^2 + R^3\frac{\partial}{\partial t}\right) + R c^2 (3\kappa - \nabla^2)\right] h_\beta^\alpha = \Delta_1^2\, h_\beta^\alpha = 0.$$

For (111) we take $i = 0$, $j = \alpha$, and subtract $i = 0$, $j = \beta$:

$$\left[8\pi G(\rho - p/c^2) + 2\Lambda + \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R}\frac{\partial}{\partial t} - \Box^2\right](h_\alpha^\alpha - h_\alpha^\beta) =$$

$$\Delta_2^2\,(h_\alpha^\alpha - h_\beta^\beta) = 0,$$

where $\square^2$ is the d'Alembertian operator. From $\Delta_1^2 h^\alpha_\beta = 0$, $\Delta_2^2 (h^\alpha_\alpha - h^\beta_\beta) = 0$, it is seen that h^α_β and $h^\alpha_\alpha - h^\beta_\beta$ are propogated independent of any fluid perturbation. Since h^α_β and $h^\alpha_\alpha - h^\beta_\beta$ are zero in the absence of perturbations, and we require only those h^i_j which result from fluid disturbances, we can choose coordinates in which $h^\alpha_\beta = 0$, $h^\alpha_\alpha = h^\beta_\beta$. Thus, all nondiagonal components of h^i_j are zero, and

$$h^0_0 = -h^1_1 = -h^2_2 = h^3_3 = 2\psi/c^2 . \tag{118}$$

The factor $2c^{-2}$ is chosen so that ψ is equivalent to the gravitational potential of Newtonian theory. For small irrotational fluid disturbances the line element is therefore

$$ds^2 = \left(1 + 2\frac{\psi}{c^2}\right)dt^2 - \frac{R^2\,(1 - 2\psi c^{-2})}{c^2\left(1 + \frac{1}{4}\kappa\, r^2\right)^2}\,(dr^2 + r^2\,d\Omega^2). \tag{119}$$

A transformation of the coordinates $x^i \to x'^i = x^i + \epsilon^i$, where ϵ^i are small quantities, leads to $g_{jk} + \gamma_{jk}$, where

$$\gamma_{jk} = h_{jk} - \epsilon_{j;k} - \epsilon_{k;j} .$$

The line element (119) is unchanged with the Killing equation[48]

$$\epsilon_{j;k} + \epsilon_{k;j} = 0 .$$

From

$$(g_{jk} + h_{jk}) \frac{dx^j}{ds} \frac{dx^k}{ds} = (g_{jk} + \gamma_{jk}) \frac{dx'^j}{ds} \frac{dx'^k}{ds} = 1$$

it is found $d\epsilon^0/ds = d\epsilon^0/dt = 0$. Also, for δu^α to remain unchanged, $d\epsilon^\alpha/dt = 0$. Clearly, these conditions are not limited to infinitesimal transformations, and any $x^i \to x'^i$ leading to $\psi(x^i) \to \psi'(x'^i)$ is admissable

5.5 Equations of Irrotational Motion

With the line element (119), the only surviving equations of (110)-(113) are

$$4\ddot{\psi} + 12\dot{R}R^{-1}\dot{\psi} - \Box^2\psi = 4\pi G(2\psi + c^2\delta)(\rho + 3p/c^2) - 2\psi\Lambda, \tag{120}$$

$$4\dot{R}R^{-1}\dot{\psi} + 4(\ddot{R}R^{-1} + 2\dot{R}^2R^{-2})\psi + \Box^2\psi = 4\pi G(2\psi - c^2\delta)(\rho - p/c^2) + 2\psi\Lambda, \tag{121}$$

$$\frac{1}{R}\frac{d}{dt}(R\psi) = 4\pi G(\rho + p/c^2)\varphi. \tag{122}$$

These equations are obtained either by working through the covariant differentiations, or more simply using local coordinates. The results have been checked with Dingle's[49] formulae for an orthogonal line element. With

$$\nabla^2\psi = -k^2\psi,$$

$$\Box^2 \psi = \left(\frac{\partial^2}{\partial t^2} + 3 \frac{\dot{R}}{R} \frac{\partial}{\partial t} + \frac{c^2}{R} k^2 \right) \psi ,$$

(120)-(121) become

$$4\pi G c^2 \delta\rho = -3 \dot{R} R^{-1} \dot{\psi} - (k^2 c^2 + 2\Lambda R^2 + 3\dot{R}^2 - 3\kappa c^2) R^{-2} \psi , \tag{123}$$

$$4\pi G \delta\rho = \ddot{\psi} + 4\dot{R} R^{-1} \dot{\psi} + (2\ddot{R} R + \dot{R}^2 - \kappa c^2) R^{-2} \psi . \tag{124}$$

The fluid motions are therefore parallel to the pressure and density gradients. With $p = (\nu - 1) \rho c^2$, for constant ν, we obtain from (123)-(124)

$$\ddot{\psi} + (1 + 3\nu) \dot{R} R^{-1} \dot{\psi} + [(\nu - 1)(k^2 c^2 + 2\Lambda R^2) + 2\ddot{R} R$$

$$+ (3\nu - 2)(\dot{R}^2 - \kappa c^2)] R^{-2} \psi = 0 . \tag{125}$$

At subnuclear density $\nu = 1 + c_s^2/c^2$, $c_s^2 \ll c^2$, and therefore

$$\ddot{\psi} + 4 \frac{\dot{R}}{R} \dot{\psi} + \left[\left(\frac{c_s k}{R} \right)^2 - 2 \frac{c^2 \kappa}{R^2} + \Lambda \right] \psi = 0 , \tag{126}$$

since $2\ddot{R} R + \dot{R}^2 = \Lambda R^2 - \kappa c^2$. This equation is identical with the Newtonian result (26) with the transformations $R \to cS$ and $k^2 \to k^2 S_0^2/c^2$. Because $\delta p = c_s^2 \delta\rho$, it follows from (124) and (126) that

$$k^2 \psi = -\nabla^2 \psi = -4\pi G R^2 \delta\rho, \tag{127}$$

which is Poisson's equation. Thus, (122) with $p = 0$, (126) and (127) are the Newtonian equations (26)-(27).

At supernuclear density $\nu = 4/3$, and from (125)

$$\ddot{\psi} + 5\frac{\dot{R}}{R}\dot{\psi} + \left(\frac{c^2 k^2}{3R^2} - 4\frac{c^2 \kappa}{R^2} + 2\Lambda\right)\psi = 0. \tag{128}$$

This equation is discussed in Section 5.8.

5.6 Previous Work

The problem of perturbed cosmological models has on the whole received scant attention in the theory of general relativity. The Newtonian approach to the problem is more popular in spite of its limitations. In the case of irrotational motion the two approaches lead however to identical results for relatively low pressure.

Lanczos[50] derives an equation similar to (103) and adopts a generalized de Donder[51] coordinate condition

$$\left(h^i_j - \frac{1}{2}\delta^i_j h\right)_{;j} = 0, \tag{128}$$

which is resembles (115) with $\theta_\alpha = 0$, $j \,\square\, \alpha$. However, a scalar wave equation of zero nondiagonal components h^i_j, and $h^0_0 = -h^\alpha_\alpha = -\frac{1}{2}h$, is impossible according to (128). The coordinate condition (128), commonly adopted in the study of gravitational waves,[34,45] yields in general tensor wave equations.

Irvine[52] uses Lanczos' equation and shows that for irregularities of scale length $\lambdabar$, such that $\lambdabar \ll R$, $\lambdabar \dot{R}/R \ll c$, an approximate form of (103) is

$$\square^2 h^i_j = 2\,\delta R^i_j = -16\pi G\,\delta\left(T^i_j - \frac{1}{2}\delta^i_j T\right). \tag{129}$$

Also, when the fluid velocity is small compared with the velocity of light the principle component of (129) is

$$\square^2 h^0_0 = -8\pi G\,\delta\,(\rho + 3p/c^2). \tag{130}$$

In this approximation $\delta\rho/\rho$, $\delta p/p$, $h^0_0/\lambdabar^2$, need not be small quantities. The results are equivalent to the Newtonian results for small scale irregularities and $p \ll \rho c^2$.

For $|\psi| \ll \frac{1}{2}c^2$, the Schwarzschild line element is

$$ds^2 = \left(1 + \frac{2\psi}{c^2}\right)dt^2 - \left(1 - \frac{2\psi}{c^2}\right)(dr^2 + r^2\,d\Omega^2). \tag{131}$$

With this as an analogy McVittie[53] assumes that the line element is orthogonal and of the form (119) for a perturbed Einstein universe, and considers a universe of discrete condensations. By supposing in a linear treatment that ψ depends upon the entire mass of the condensations it was found that the volume of a static universe depends on the number of condensations. Modifications[54] were subsequently made in the treatment which showed that the volume was independent of disturbances.

Bonnor[56] has considered the growth of single condensations in the linear approximation for a pressure free fluid and has concluded that in an expanding universe the nebulae cannot have originated from infinitesimally small disturbances.

The most general and elegant contribution to the subject has been made by Lifshitz.[46] His basic equations are similar to (110)-(113). In general, coordinate transformations are possible which allow four of the h_{kj} to be zero. Which of the h_{kj} are made zero determines to some extent the simplicity of the equations for a given physical problem. Lifshitz makes the choice $h_{0j} = 0$. This choice is appropriate for complex disturbances containing rotational motion and gravitational waves, but the simplest of all disturbances – irrotational motions – are concealed within cumbersome equations that require tensorial spherical harmonics for their analysis. Lifshitz finds that all perturbations either decay or grow very slowly in an expanding universe.

5.7 Normal Modes of Vibration in Curved Space

Using r, θ, ϕ coordinates in

$$\nabla^2 \psi + k^2 \psi = 0,$$

and separating the variables: $\psi = \psi(t)\, \Psi(r)\, Y_n^m(\theta, \phi)$, we find that only the radial function $\Psi(r)$ is different in the three cases $\kappa = 0, \pm 1$.

For spherical harmonics of degree n the radial equation is

$$\frac{1}{q^{3/2}\, r^2} \frac{d}{dr}\left(q^{1/2}\, r^2 \frac{d\Psi}{dr}\right) + \left[k^2 - \frac{n(n+1)}{qr^2}\right] \Psi = 0, \tag{132}$$

where $q = \left(1 + \frac{1}{4}\, \kappa\, r^2\right)^{-2}$. We consider briefly the wavefunctions and eigenvalues in a space of negative and positive curvature.[46, 57]

For $\kappa = 1$ let

$$\sin\alpha = \frac{r}{1 + \frac{1}{4} r^2}\; ; \quad \text{hence,} \quad d\alpha = \frac{dr}{1 + \frac{1}{4} r^2}, \tag{133}$$

and (132) becomes

$$\frac{1}{\sin^2\alpha} \frac{d}{d\alpha}\left(\sin^2\alpha \frac{d\Psi}{d\alpha}\right) + \left[k^2 - \frac{n(n+1)}{\sin^2\alpha}\right] \Psi = 0.$$

With $\Psi = \Pi \sin^{-1/2}\alpha$ as the new variable, this equation is

$$\frac{1}{\sin\alpha} \frac{d}{d\alpha}\left(\sin\alpha \frac{d\Pi}{d\alpha}\right) + \left[\lambda(\lambda + 1) - \frac{\left(n + \frac{1}{2}\right)^2}{\sin^2\alpha}\right] \Pi = 0, \tag{134}$$

and $\lambda(\lambda + 1) = k^2 + 3/4$.

For $\kappa = -1$ let

$$\sinh \alpha = \frac{r}{1 - \frac{1}{4} r^2}; \quad \text{hence,} \quad d\alpha = \frac{dr}{1 - \frac{1}{4} r^2}, \tag{135}$$

and $\Psi = \Pi \sinh^{-1/2} \alpha$; then (132) becomes

$$\frac{1}{\sinh \alpha} \frac{d}{d\alpha} \left(\sinh \alpha \frac{d\Pi}{d\alpha} \right) + \left[\lambda (\lambda + 1) - \frac{\left(n + \frac{1}{2}\right)^2}{\sinh^2 \alpha} \right] \Pi = 0, \tag{136}$$

and $\lambda (\lambda + 1) = k^2 - 3/4$.

Let $\xi = \kappa^{1/2} \alpha$, then both (134) and (136) can be expressed in the one equation

$$\frac{1}{\sin \xi} \frac{d}{d\xi} \left(\sin \xi \frac{d\Pi}{d\xi} \right) + \left[\lambda (\lambda + 1) - \frac{\left(n + \frac{1}{2}\right)^2}{\sin^2 \xi} \right] \Pi = 0, \tag{137}$$

where now $\lambda(\lambda + 1) = \kappa k^2 + 3/4$, or

$$\lambda_{1,2} = - \frac{1}{2} \pm (1 + \kappa k^2)^{1/2} \tag{138}$$

The solutions of (137) are the associated Legendre functions $P_\nu^\mu(\cos \xi)$, $Q_\nu^\mu(\cos \xi)$, and $\mu = \frac{1}{2} + n$, $\nu = \lambda$.

Positive curvature ($\kappa = 1$). In this case it is more convenient to set

$$k^2 = \gamma(\gamma + 1). \tag{139}$$

Hence, $\lambda_1 = \frac{1}{2} + \lambda$; and since $P^{\mu}_{\nu} = P^{\mu}_{-\nu-1}$, or $P^{\mu}_{\lambda_1} = P^{\mu}_{\lambda_2}$, we use $\nu = \lambda_1$. Because $\mu \pm \nu$ is an integer, but μ is not an integer, we can use P^{μ}_{ν} and $P^{-\mu}_{\nu}$ as linearly independent solutions.[58] These are:

$$\Psi^{n}_{\gamma} = (\pi/2 \sin \alpha)^{1/2} P^{\frac{1}{2}+n}_{\frac{1}{2}+\gamma}(\cos \alpha),$$

$$\Psi^{-n}_{\gamma} = (\pi/2 \sin \alpha)^{1/2} P^{-\frac{1}{2}-n}_{\frac{1}{2}+\gamma}(\cos \alpha). \tag{140}$$

From the definition

$$P^{\pm\left(\frac{1}{2}+n\right)}_{\frac{1}{2}+\gamma}(\cos \alpha) = \frac{\left(\cot \frac{1}{2}\alpha\right)^{\pm\left(\frac{1}{2}+n\right)}}{\Gamma\left(1 \mp \left(\frac{1}{2}+n\right)\right)} F\left(\frac{1}{2} - \gamma, \frac{1}{2} + \gamma; 1 \mp \left(\frac{1}{2}+n\right); \sin^2 \frac{1}{2}\alpha\right), \tag{141}$$

it is seen that as $\alpha \to 0$ (or $r \to 0$)

$$\Psi^{\pm n}_{\gamma} \to \sim \frac{\left(\cot \frac{1}{2}\alpha\right)^{\pm\left(\frac{1}{2}+n\right)}}{(\sin \alpha)^{1/2}\, \Gamma\left(1 \mp \left(\frac{1}{2}+n\right)\right)},$$

and as n is a positive integer, only Ψ_γ^{-n} is regular at the origin. Therefore

$$\Psi = a_{n\gamma}\, \Psi_\gamma^{-n} = a_{n\gamma}\, (\pi/2 \sin \alpha)^{1/2}\, P_{\frac{1}{2}+\gamma}^{-\frac{1}{2}-n} (\cos \alpha). \tag{142}$$

We have assumed that γ is integral; it can be shown that this is necessary in order that Ψ is periodic or single valued:

$$\Psi_\gamma^{-n} (- \cos \alpha) = \cos (\gamma - n)\pi\, \Psi_\gamma^{-n} (\cos \alpha). \tag{143}$$

Thus the wavefunction is symmetric (antisymmetric) about $\alpha = \frac{1}{2}\pi$, or $r = 2$, when $\gamma - n$ is an even (odd) integer. This has interesting consequences for elliptical space: $0 \le r \le 2$, and spherical space: $0 \le r \le \infty$. The transformation: $r \to 4/r$, leaves the metric (73) unchanged.* It is therefore said[59] that elliptical and spherical space are indistinguishable because $2 \le r \le \infty$ is merely a remapping of elliptical space. In a perturbed universe, however, elliptical space is not a mirror image of $2 \le r \le \infty$ for the antisymmetric wavefunctions, and therefore it would seem to be an inadequate model of the universe.

For $n = 0, 1, \ldots \gamma$,

$$\Psi_\gamma^{-n} = \frac{\sin^n \alpha}{M_n} \frac{d^{n+1} [\cos (1+\gamma)\alpha]}{d (\cos \alpha)^{n+1}},$$

$$M_n = (1+\gamma)^2 \,[(1+\gamma)^2 - 1] \cdots [(1+\gamma)^2 - n^2]. \tag{144}$$

*But with reversed parity.

It follows, for n = 0,

$$\Psi_{\gamma}^{0} = \frac{\sin (1 + \gamma) \alpha}{(1 + \gamma) \sin \alpha}, \tag{145}$$

and for $n = \gamma$,

$$\Psi_{\gamma}^{-\gamma} = (\sin \alpha)^{\gamma}/(1.3.5 \cdot \cdot \cdot \cdot 2\gamma + 1). \tag{146}$$

Also

$$\Psi_{\gamma+1}^{-\gamma} = \cos \alpha \, \Psi_{\gamma}^{-\gamma}, \tag{147}$$

$$\Psi_{\gamma+2}^{-\gamma} = \left(1 - \frac{2\gamma + 4}{2\gamma + 3} \sin^2 \alpha\right) \Psi_{\gamma}^{-\gamma}. \tag{148}$$

Hence, the radial functions of the lowest modes are

$$\Psi_{1}^{0} = \cos \alpha,$$

$$\Psi_{1}^{-1} = \frac{1}{3} \sin \alpha,$$

$$\Psi_2^0 = 1 - \frac{4}{3}\sin^2 \alpha,$$

$$\Psi_2^{-1} = \frac{1}{3}\sin\alpha\cos\alpha,$$

$$\Psi_2^{-2} = \frac{1}{15}\sin^2 \alpha,$$

$$\Psi_3^0 = \cos\alpha\,(1 - 2\sin^2\alpha),$$

$$\Psi_3^{-1} = \frac{1}{3}\sin\alpha\left(1 - \frac{6}{5}\sin^2\alpha\right),$$

$$\Psi_3^{-2} = \frac{1}{15}\cos\alpha\sin^2\alpha,$$

$$\Psi_3^{-3} = \frac{1}{105}\sin^3\alpha. \tag{149}$$

In a space of uniform positive curvature the eigenvalues are

$$k^2 = \gamma(\gamma + 2), \ \gamma = 1, 2, 3, \cdots \tag{150}$$

and $m \le n \le \gamma$. The fundamental mode has an eigenvalue of $k^2 = 3$.

Negative curvature (κ = -1). For the eigenvalues in this case we use

$$k^2 = \gamma^2 + 1, \tag{151}$$

and from (138) $\lambda_{1,2} = -\frac{1}{2} \pm i\gamma$. $P_\nu^{-\mu}$ and Q_ν^μ are linearly dependent because $\mu = \frac{1}{2} + n$ is half integral, and for the linearly independent solutions P_ν^μ and $P_\nu^{-\mu}$ are again chosen. Since $P_{\lambda_1}^\mu = P_{\lambda_2}^\mu$, we chose $\nu = \lambda_1$, and therefore

$$\Psi_\gamma^n = (\pi/2 \ \sinh\alpha)^{1/2} P_{-1/2+i\gamma}^{1/2+n} (\cosh\alpha),$$

$$\Psi_\gamma^{-n} = (\pi/2 \ \sinh\alpha)^{1/2} P_{-1/2+i\gamma}^{-1/2-n} (\cosh\alpha). \tag{152}$$

The hypergeometric expression

$$P_{-1/2+i\gamma}^{\pm(1/2+n)} (\cosh\alpha) =$$

$$\frac{\left(\coth\frac{1}{2}\alpha\right)^{\pm(1/2+n}}{\Gamma\left(1 \mp \left(\frac{1}{2}+n\right)\right)} F\left(\frac{1}{2} - i\gamma, \frac{1}{2} + i\gamma; 1 \mp \left(\frac{1}{2}+n\right); -\sinh^2\frac{1}{2}\alpha\right),$$

is real, and as $\alpha \rightarrow 0$,

$$\Psi_\gamma^{\pm n} \rightarrow \sim \frac{\left(\coth\frac{1}{2}\alpha\right)^{\pm(1/2+n)}}{(\sinh\alpha)^{1/2}\,\Gamma\left(1 \mp \left(\frac{1}{2}+n\right)\right)},$$

and therefore only Ψ_γ^{-n} is regular at the origin. Thus,

$$\Psi_\gamma^{-n} = a_{n i \gamma} (\pi/2 \sinh \alpha)^{1/2} P_{-1/2 + i\gamma}^{-1/2 - n} (\cosh \alpha), \tag{153}$$

and because space is open, there are no periodic conditions to satisfy, and γ can have any real value.

For integral values of n,

$$\Psi_\gamma^{-n} = \frac{\sinh^n \alpha}{N_n} \frac{d^{n+1} (\cos \gamma \alpha)}{d (\cosh \alpha)^{n+1}}$$

$$N_n = -\gamma^2 (\gamma^2 + 1) \cdots\cdots (\gamma^2 + n^2), \tag{154}$$

and for $n = 0$,

$$\Psi_\gamma^{\circ} = \sin \gamma \alpha / \gamma \sinh \alpha, \tag{155}$$

$n = 1$,

$$\Psi_\gamma^{-1} = (\gamma^2 + 1)^{-1} (\gamma \cot \gamma \alpha - \coth \alpha) \Psi_\gamma^{\circ}, \tag{156}$$

and so forth.

We have found that the eigenvalue spectra for $\kappa = 0$, 1, and -1 are

$$\kappa = 0: \quad k^2 = \gamma^2, \qquad \gamma^2 \geq 0,$$

$$\kappa = 1: \quad k^2 = \gamma(\gamma + 2), \qquad \gamma = 1, 2, 3 \cdots,$$

$$\kappa = -1: \quad k^2 = \gamma^2 + 1, \qquad \gamma^2 \geq 0,$$

and the lowest eigenvalues are $k^2 = 0$, 3, and 1, respectively. The eigenvalues form continuous spectra for zero and negative curvatures, and a discrete spectrum for positive curvature.

5.8 Time Dependence of Modes in Einstein's Static Universe

From Section 5.1 we have

$$(8\pi G\rho + \Lambda) R^2 = 3(\dot{R}^2 + \kappa c^2),$$

$$(\Lambda - 8\pi Gp/c^2) R^2 = 2R\ddot{R} + \dot{R}^2 + \kappa c^2. \qquad (157)$$

In Einstein's static model $\dot{R} = \ddot{R} = 0$, $\kappa = 1$, and therefore

$$8\pi G\rho + \Lambda = 3c^2/R_0^2,$$

$$\Lambda - 8\pi Gp/c^2 = c^2/R_0^2. \qquad (158)$$

With the equation of state $p = (\nu - 1)\rho c^2$, it is found that

$$\Lambda = c^2(3\nu - 2)/\nu R_0^2. \tag{159}$$

Equations (125), (123), and (122), become

$$\ddot{\psi} + c^2 \left\{(\nu - 1)\ [k^2 + 2(3\nu - 2)\ \nu^{-1}] - (3\nu - 2)\right\} R_0^{-2}\ \psi = 0, \tag{160}$$

$$4\pi G R_0^2\ \delta\rho = -\ [k^2 - 3 + 2(3\nu - 2)\ \nu^{-1}]\ \psi, \tag{161}$$

$$\dot{\psi} = 4\pi G \nu \rho \varphi. \tag{162}$$

The case of $\nu \to 1$ has been considered in Section 4.2; we now consider a supernuclear model of $\nu = 4/3$. Equations (160)-(162) are therefore

$$\ddot{\psi} + \frac{2}{3}\Lambda \left[\frac{1}{3}\gamma(\gamma + 2) - 1\right] \psi = 0, \tag{163}$$

$$6\pi G \Lambda c^2\ \delta\rho = -\ \gamma(\gamma + 2)\ \psi, \tag{164}$$

$$\dot{\psi} = 16\pi G\rho\varphi/3, \tag{165}$$

in which $k^2 = \gamma(\gamma + 2)$ is used. The solutions are

$$\psi \propto \delta\rho \propto \exp \pm i \left\{\frac{2}{3}\Lambda \left[\frac{1}{3}\gamma(\gamma + 2) - 1\right]\right\}^{1/2} t. \tag{166}$$

Thus all modes of $\gamma > 1$ are oscillatory, and the fundamental mode $\gamma = 1$ has a solution which grows linearly with time:

$$\delta\rho = At + Bt^{-1}. \tag{167}$$

If the Einstein universe were static for an indefinite period of time the linear growth of the fundamental mode would count as a serious instability. It is well known,[60] however, that the Einstein universe is unstable against perturbations in R. Perturbing (157) and using the equilibrium equations (158) we find, to a first order, $\delta\ddot{R} = \nu\Lambda\,\delta R$. Therefore,

$$\delta R \propto \exp\,(3\nu - 2)^{1/2}\, ct/R_0, \tag{168}$$

and for $\nu > 2/3, \delta R$ grows exponentially. Hence the growth of the fundamental mode (167) must be re-examined in a nonstatic universe.

It was thought[53] that condensations in a static universe would increase the volume to $V_0 + \delta V$, thus launching the unstable universe on a career of expansion. Using the line element (119), we find

$$\delta V = (3R_0^2/c^2) \int \sin^2\alpha \sin\theta\, \psi(\alpha, \theta, \phi)\, d\alpha\, d\theta\, d\phi, \tag{169}$$

in the linear approximation, and therefore $\delta V = 0$. Thus, in the linear theory condensations do not guarantee that the Einstein universe evolves into an expanding Lemaitre model.

5.9 Time Dependence of Modes in Nonstatic Models

The time dependence of the modes for $p \ll \rho c^2$ has been considered in Section 4, and here we shall consider only the supernuclear model of $\nu = 4/3$ and $\Lambda = 0$. The equations for ψ and $\delta\rho$ are

$$\ddot{\psi} + 5\dot{R}R^{-1}\,\dot{\psi} + (c^2/R^2)\left(\frac{1}{3}k^2 - 4\kappa\right)\psi = 0, \tag{170}$$

$$4\pi G c^2\,\delta\rho = -\,3\dot{R}R^{-1}\,\dot{\psi} - R^{-2}\,(k^2c^2 - 3\kappa c^2 + 3\dot{R}^2)\,\psi, \tag{171}$$

from (128) and (123). With the transformation (93), $\kappa = 0$, these equations are

$$\frac{d^2\psi}{d\chi^2} + \frac{4}{\chi}\,\frac{d\psi}{d\chi} + \left(\frac{1}{3}k^2 - 4\kappa\right)\psi = 0, \tag{172}$$

$$\frac{1}{2}\,\frac{\delta\rho}{\rho} = -\,\chi\,\frac{d\psi}{d\chi} - \chi^2\left[\left(\frac{1}{3}k^2 - \chi\right) + 1\right]\psi. \tag{173}$$

Because χ is generally a small quantity ($\chi \leq 10^{-8}$), these equations are valid for $\kappa = 0, \pm 1$. In the supernuclear state there is no simple relation between ψ and $\delta\rho$. The solution of (172) is

$$\psi \propto \chi^{-3/2} J_{\pm 3/2}(s\chi),$$

$$s = \left(\frac{1}{3} k^2 - 4\kappa\right)^{1/2}. \tag{174}$$

For the low order modes $s\chi \ll 1$, and for $\kappa = 0, \pm 1$

$$\psi = A + BR^{-3}, \tag{175}$$

where $\chi \propto R \propto t^{1/2}$, and

$$\delta\rho/\rho = AR^2 + BR^{-1}, \tag{176}$$

and A and B are different in the two equations. Thus ψ is either constant or diminishes in an expanding universe, and $\delta\rho/\rho$ increases linearly with time, or

$$\delta\rho/\rho \propto R^2 \propto \rho^{-1/2}. \tag{177}$$

In general,

$$\psi = \frac{1}{\chi^2}\left[A\left(\frac{\sin s\chi}{s\chi} - \cos s\chi\right) + B\left(\sin s\chi + \frac{\cos s\chi}{s\chi}\right)\right] \tag{178}$$

We consider first flat space: $\kappa = 0$, $s = k/\sqrt{3}$, and assume that $s\chi \gg 1$. It follows

$$\psi \propto \chi^{-2} e^{ik\chi/\sqrt{3}},$$

$$\delta\rho/\rho \propto e^{ik\chi/\sqrt{3}}, \tag{179}$$

and $\delta\rho/\rho$ oscillates with constant amplitude during expansion and contraction. These equations, because of (94), hold for a wavelength $\lambdabar = Rk^{-1}$ of $\lambdabar << R^{3/2}/R_0^{1/2}$, where $R < R_n$. For a photon or neutrino universe the condition $s\chi >> 1$ is simply $\lambdabar << \chi R \sim c/(\rho G)^{1/2}$.

In curved space: $\kappa = \pm 1$, and for $s\chi >> 1$,

$$\psi \propto \chi^{-2} \exp i \left(\frac{1}{3} k^2 \mp 4\right)^{1/2} \chi, \tag{180}$$

and $\delta\rho/\rho \propto \psi\chi^2$. These results are the same as (179) in a universe in which the supernuclear state holds for $\chi << 1$. But in a photon or neutrino universe of positive curvature the modes

$$k^2 = \gamma(\gamma + 2) < 12,$$

or the first two modes $\gamma = 1$ and 2, grow exponentially. However, since $0 < \chi < \pi$, their growth is limited.

We can summarize by saying that the contrast density oscillates with constant amplitude for short wavelengths and increases linearly with time for long wavelengths.

6. DISCUSSION

The vibrations of the universe are moderately well behaved and show no signs of a catastrophic growth in amplitude. This conclusion has been reached by several authors in various ways. Within the limitations of simple, linearized theory the expanding universe is reasonably stable. This is illustrated in an approximate manner by Jeans' theory. The gravitational frequency is of the order $i(\rho G)^{1/2}$, and the age of the universe is of the order $(\rho G)^{-1/2}$, and hence perturbations cannot grow significantly in the time available.

In the supernuclear state the contrast density oscillates at constant amplitude for short wavelengths and increases for long wavelengths according to

$$\delta\rho/\rho \propto \rho^{-1/2}. \tag{181}$$

In the subnuclear state the contrast density depends on curvature and the ratio of specific heats. The maximum growth rate occurs in a cold universe and

$$\delta\rho/\rho \propto \rho^{-1/3}, \tag{182}$$

for expansion at zero curvature. The significance of these results depends on the magnitude of the initial disturbance that can be legitimately assumed. For arbitrarily small initial disturbances the results (181) and (182) tend to be meaningless. If the disturbances originate from thermal fluctuations, the mean square fluctuation in N particles is[40]

$$\left\langle (\Delta N)^2 \right\rangle = kT(\partial N/\partial \mu)_{T,V} \tag{183}$$

at constant temperature T and volume V, where μ is the chemical potential. The right-hand side of (183) is equal to N for a Maxwellian distribution, and is of the same order for a relativistic gas of bosons and fermions in which the Fermi energy is equal to kT. Therefore,

$$\delta\rho/\rho \sim N^{-1/2},$$

and this is exceedingly small for large masses containing ordinary gas particles.

A redeeming feature of the present theory is that the universe in the large tends to remain homogeneous and isotropic. If the growth rates (181) and (183) adequately accounted for irregularity, then the fundamental modes would also increase in the same manner, and in all probability the universe would develop pronounced macroscopic anisotropy. What is required is that the universe is unstable for an intermediate range of wavelengths which grow rapidly in an exponential fashion. At some stage in the expansion the density in various regions ceases to diminish with time, and thereafter condensations occur. But such a concept demands that the gravitational potential of the disturbance increases with time and attains a value of $\psi \sim GM/\lambda$ where M is an island mass and λ its radius. But nowhere in the linear analysis is ψ an increasing function of time in an expanding universe.

A linearized theory limited to irrotational motions and gravitational interactions is open to several criticisms. The neglect of all forms of rotation is a gross simplification, since angular momentum is a common and indispensible feature of galactic and stellar systems. It seems plausible that at subnuclear density, at least, a treatment based on rotational motions will lead to even slower rates of growth owing to the presence of inertial forces. Gamov [61] has proposed a primordial turbulent state of large amplitude fluctuations for the initial conditions. Similarly, Weizäcker[62] assumes an initial state of turbulent gas clouds. Bonnor [56] points out, however, that turbulence is more likely to be the result rather than the cause of condensations. Furthermore, initial conditions of this nature add to the mystery rather than clarifying it, and their postulation falls within the province of hypothesis A and renders invalid a linear treatment.

The assumption that initial disturbances are small, as in hypothesis B, demands that rapid growth is possible at some stage in the expansion of the universe. This is impossible unless we abandon the rudimentary fluid prescription. Following a speculative vein, we might suppose that because of complex fluid properties (such as energy transfer by radiation or neutrinos) the equation of state for perturbations is $\delta p = -(1 - \tilde{\nu})\,\delta\rho c^2$, and $\tilde{\nu} < 1$. In effect, the velocity of sound is imaginary. Then, for $\dot{\psi} >> \dot{R}/R$,

$$\psi \propto \exp \sim \left[(1 - \tilde{\nu})\, c^2 k^2 / G\rho R^2\right]^{1/2}.$$

If such an approach to the problem were possible and could yield rapid and large growth, then as $|\psi| \longrightarrow c^2$, the linear theory breaks down. It is interesting to notice that such growth not only could account for differentiation but might also lead to the development of phenomena such as the partial concealment of dense masses encapsulated in the metric.[63] Such bodies, consisting of matter in a primordial state, would then burst forth at subsequent epochs as quasi stellar objects,[64] and possibly as explosive nuclei in galaxies.[65] To an observer in such a body the universe is still quite young. These concepts have much in common with Ambartsumian's hypothesis,[66] and provide a physical mechanism as its basis.

Clearly, the origin of structure offers a challenging problem, and from this point of view a study of the modes of vibration of the universe has merely emphasized the dilemma that confronts us. If there is no way in which we can recover our faith in the efficacy of hypothesis B, then we must turn to hypothesis A in the hope that the primordial universe contains structure as the natural consequence of the properties of matter at very high density.

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