THE GROUP OF PERMUTATIONS ON 3 LETTERS AND THE PROPER ROTATIONS OF A 3-DIMENSIONAL ORTHOGONAL FRAME

by

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The ensuing mathematical discussion concerning the group of permutations on 3 letters and the proper rotations of a 3-dimensional orthogonal Frame is a study of relationships existing between the group of permutations on 3 letters, $\mathfrak{S}_3$, and the 6 different elements of the unimodular group SL(3) which are products of 3 unimodular elements of the 3! products formed by permuting the order of the multiplication of these 3 proper rotation matrices; where M_1, M_2, and M_3 describe rotation of the Frame ${}_3\mathfrak{F}$ about the x-, y-, and z-coordinate axis of ${}_3\mathfrak{F}$. Rotation shall not be constrained to the planes formed by the coordinate axes of ${}_3\mathfrak{F}$. This condition is equivalent to a choice of angles of rotation in such wise, that no one of the unimodular elements M_1, M_2, and M_3 is an identity. The angles θ_1, θ_2, and θ_3 of rotation about the x-, y-, and z-coordinate axis respectively are elements of the field $R^{\#}$ of real numbers. ${}_3\mathfrak{F}$ itself is embedded in a 3-dimensional continuum.

Consider the set of proper rotation matrices $\{M_1, M_2, M_3\}$ describing rotations of an orthogonal Frame ${}_3\mathfrak{F}$ embedded in a 3-dimensional continuum about the x-, y-, and z-coordinate axes of ${}_3\mathfrak{F}$, respectively. The explicit representation of these matrices are

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These unimodular matrices may of course, be multiplied together in 6 different order arrangements to form 6 different products $\sim$ to the set of permutations on 3 letters.

I shall let $M_1 \sim 1$, $M_2 \sim 2$, and $M_3 \sim 3$, and list the ordering of the 6 multiplications of these matrices as

[redacted]

1. $M_1M_2M_3 \sim 123 \sim \sigma_1$, 4. $M_2M_3M_1 \sim 231 \sim \sigma_4$

2. $M_1M_3M_2 \sim 132 \sim \sigma_2$, 5. $M_3M_1M_2 \sim 312 \sim \sigma_5$

3. $M_2M_1M_3 \sim 213 \sim \sigma_3$, 6. $M_3M_2M_1 \sim 321 \sim \sigma_6$

After computation of the corresponding traces of the matrix group elements, I find that

$\cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 + \cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 = \psi_- =$

$\mathrm{tr}(M_1M_2M_3) = \mathrm{tr}(M_2M_3M_1) = \mathrm{tr}(M_3M_1M_2)$, and

$\cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 + \cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3 = \psi_+ =$

$\mathrm{tr}(M_1M_3M_2) = \mathrm{tr}(M_2M_1M_3) = \mathrm{tr}(M_3M_2M_1)$

The permutational arrangement of unimodular group products $\sim$ to the trace $\psi_- \sim$ to a cyclic permutation of letters from left to right; and the cyclical (or permutational) arrangement of unimodular group products $\sim$ to $\psi_+ \sim$ to a cyclic permutation of letters from right to left. The array of cyclic permutations $\sim \psi_+$ may be achieved to by an initial inversion of 2 letters from the array of cyclic permutations $\sim \psi_-$.

Again, suppose I let $M_1 \sim 1$, $M_2 \sim 2$, and $M_3 \sim 3$, and consider the permutation group of order 3! with respect to multiplication, $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6; \cdot\}$: letting the group elements be expressed as

$$\sigma_1 = 1 \quad \sigma_2 = (23),\ \sigma_3 = (12),\ \sigma_4 = (123),\ \sigma_5 = (132),\ \sigma_6 = (13).$$

Then I can arrange the following correspondences,

$$\{\sigma_1, \sigma_4, \sigma_5\} \sim \mathrm{tr}(M_1M_2M_3) = \mathrm{tr}(M_2M_3M_1) = \mathrm{tr}(M_3M_1M_2)$$

$$\{\sigma_2, \sigma_3, \sigma_6\} \sim \mathrm{tr}(M_1M_3M_2) = \mathrm{tr}(M_2M_1M_3) = \mathrm{tr}(M_3M_2M_1)$$

where $\{1, (123), (132)\} \sim \psi_-$ and $\{(23), (12), (13)\} \sim \psi_+$ are the sets consisting of permutation elements corresponding to their respective traces. The inverses of the permutation elements may be expressed as

$\sigma_1^{-1} = 1,\ \sigma_2^{-1} = (32),\ \sigma_3^{-1} = (21),\ \sigma_4^{-1} = (213),\ \sigma_5^{-1} = (312),\ \sigma_6^{-1} = (31).$

I can now easily determine the permutation elements in terms of their inverses as a necessary step prior to formation of a classifying list of mutually conjugate group elements. Observe that $1 \equiv 1 \Rightarrow \sigma_1^{-1} = \sigma_1$, $(32) \equiv (23) \Rightarrow \sigma_2^{-1} = \sigma_2$, $(21) \equiv (12) \Rightarrow \sigma_3^{-1} = \sigma_3$, $(213) \equiv (132) \Rightarrow \sigma_4^{-1} = \sigma_5$, $(312) \equiv (123) \Rightarrow \sigma_5^{-1} = \sigma_4$, and $(31) \equiv (13) \Rightarrow \sigma_6^{-1} = \sigma_6$.

A listing of the calculations of the mutually conjugate group elements of the symmetric group on 3 letters.

$\sigma_1^{-1}\sigma_1\sigma_1 = \sigma_1$

$\sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2$

$\sigma_1^{-1}\sigma_3\sigma_1 = \sigma_3$

$\sigma_1^{-1}\sigma_4\sigma_1 = \sigma_4$

$\sigma_1^{-1}\sigma_5\sigma_1 = \sigma_5$

$\sigma_1^{-1}\sigma_6\sigma_1 = \sigma_6$

$\sigma_2^{-1}\sigma_1\sigma_2 = \sigma_1$

$\sigma_2^{-1}\sigma_2\sigma_2 = \sigma_2$

$\sigma_2^{-1}\sigma_3\sigma_2 = (32)(12)(23) = (13) = \sigma_6$

$\sigma_2^{-1}\sigma_4\sigma_2 = (32)(123)(23) = (132) = \sigma_5$

$\sigma_2^{-1}\sigma_5\sigma_2 = (32)(132)(23) = (123) = \sigma_4$

$\sigma_2^{-1}\sigma_6\sigma_2 = (32)(13)(23) = (12) = \sigma_3$

$\sigma_3^{-1}\sigma_1\sigma_3 = \sigma_1$

$\sigma_3^{-1}\sigma_2\sigma_3 = (21)(23)(12) = (13) = \sigma_6$

$\sigma_3^{-1}\sigma_3\sigma_3 = \sigma_3$

$\sigma_3^{-1}\sigma_4\sigma_3 = (21)(123)(12) = (132) = \sigma_5$

$\sigma_3^{-1}\sigma_5\sigma_3 = (21)(132)(12) = (123) = \sigma_4$

$\sigma_3^{-1}\sigma_6\sigma_3 = (21)(13)(12) = (23) = \sigma_2$

$\sigma_4^{-1}\sigma_1\sigma_4 = \sigma_1$

$\sigma_4^{-1}\sigma_2\sigma_4 = (213)(23)(123) = (13) = \sigma_6$

$\sigma_4^{-1}\sigma_3\sigma_4 = (213)(12)(123) = (23) = \sigma_2$

$\sigma_4^{-1}\sigma_4\sigma_4 = \sigma_4$

$\sigma_4^{-1}\sigma_5\sigma_4 = (213)(132)(123) = (132) = \sigma_5$

$\sigma_4^{-1}\sigma_6\sigma_4 = (213)(13)(123) = (12) = \sigma_3$

$\sigma_5^{-1}\sigma_1\sigma_5 = \sigma_1$

$\sigma_5^{-1}\sigma_2\sigma_5 = (312)(23)(132) = (12) = \sigma_3$

$\sigma_5^{-1}\sigma_3\sigma_5 = (312)(12)(132) = (13) = \sigma_6$

$\sigma_5^{-1}\sigma_4\sigma_5 = (312)(123)(132) = (123) = \sigma_4$

$\sigma_5^{-1}\sigma_5\sigma_5 = \sigma_5$

$\sigma_5^{-1}\sigma_6\sigma_5 = (312)(13)(132) = (23) = \sigma_2$

$\sigma_6^{-1}\sigma_1\sigma_6 = \sigma_1$

$\sigma_6^{-1}\sigma_2\sigma_6 = (31)(23)(13) = (12) = \sigma_3$

$\sigma_6^{-1}\sigma_3\sigma_6 = (31)(12)(13) = (23) = \sigma_2$

$\sigma_6^{-1}\sigma_4\sigma_6 = (31)(123)(13) = (132) = \sigma_5$

$\sigma_6^{-1}\sigma_5\sigma_6 = (31)(132)(13) = (123) = \sigma_4$

$\sigma_6^{-1}\sigma_6\sigma_6 = \sigma_6$

A listing which consists of mutually conjugate group elements of the symmetric group in 3 letters.

$$1 = \sigma_1 \sim C_1 = \{1\}$$

$$(23) = \sigma_2 \sim C_{(23)} = \{(23),\ (13),\ (12)\}$$

$$(12) = \sigma_3 \sim C_{(12)} = \{(12),\ (13),\ (23)\}$$

$$(123) = \sigma_4 \sim C_{(123)} = \{(123),\ (132)\}$$

$$(132) = \sigma_5 \sim C_{(132)} = \{(132),\ (123)\}$$

$$(13) = \sigma_6 \sim C_{(13)} = \{(13),\ (12),\ (23)\}$$

PROPOSITION. If M_1, M_2, and M_3 are proper rotations of a 3-dimensional orthogonal Frame ${}_3\mathfrak{F}$, where these unimodular elements of the group SL(3) are multiplied together in 6 different permutational arrangements, the elements of which the mutually conjugate classes $C_1, C_{(123)}$, and $C_{(132)}$, consist of are inner automorphisms of the elements of the alternating group $\mathfrak{A}_3$; and the automorphisms of C_1, $C_{(123)}$, and $C_{(132)}$, correspond to ψ_- where the permutations $\sigma_1 \sim M_1 M_2 M_3$, $\sigma_4 \sim M_2 M_3 M_1$, and $\sigma_5 \sim M_3 M_1 M_2$. The group $\{C_1, C_{(123)}, C_{(132)}, \cdot\cdot\} \cong \mathfrak{a}_3$.

DEMONSTRATION. The permutation symbols could be considered as products of transpositions. It is then easy for me to determine which of the permutations are elements of the alternating group $\mathfrak{a}_3$.

$$\sigma_1 = (1)(2)(3) = 1 \qquad \sigma_4 = (123) = (12)(13)$$

$$\sigma_2 = (23) \qquad \sigma_5 = (132) = (13)(12)$$

$$\sigma_3 = (12) \qquad \sigma_6 = (13)$$

$\therefore \{\sigma_1, \sigma_4, \sigma_5; \cdot\} \equiv \mathfrak{a}_3 \sim tr(M_1M_2M_3) = tr(M_2M_3M_1) = tr(M_3M_1M_2)$. In other words, the alternating group $\{1, (123), (132); \cdot\} \equiv \mathfrak{a}_3 \sim \psi_-$. With respect to the group $\mathfrak{a}_3$, the classes of the mutually conjugate elements are determined by calculation and correspondence to be:

$$C_1 = \{1\}$$

$$C_{(123)} = \{(123), (132)\}$$

$$C_{(132)} = \{(132), (123)\}$$

Thus the inner automorphisms $= \sigma_1$, $= \sigma_4$, and $= \sigma_5$ of the classes of mutually conjugate elements, C_1, $C_{(123)}$, and $C_{(132)}$ respectively, correspond to $\psi_- = tr(M_1M_2M_3) = tr(M_2M_3M_1) = tr(M_3M_1M_2)$, and the group $\{C_1, C_{(123)} C_{(132)}; \cdot\} \cong \mathfrak{a}_3$ as can be demonstrated by direct calculation.

Q.E.D.

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