Neutron Stars I. Properties at Absolute Zero Temperature

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ABSTRACT $\partial 7045$

The properties of a neutron star at absolute zero temperature are discussed. The problem of determining the ground state of a neutron star is formulated in a general way and the conditions are described under which one might reasonably hope that an individual-particle model (which we adopt) is valid. The effects of the strong interactions on the number densities and production thresholds of the various hadrons are illustrated with several examples. The modification of the energy spectrum of neutrons and protons in a neutron star is calculated using an effective mass approximation adapted from the theory of nuclear matter. Crude estimates are made of the contributions of hadrons other than nucleons to the equation of state and specific heat.

I. INTRODUCTION

In the present paper (I), we discuss the properties of a neutron star at absolute zero temperature. In the following paper (II),[1] we calculate, using the ideas discussed in I, the rates in a hot neutron star of some of the most important neutrino-cooling reactions. We also attempt to determine in II if the recently observed discrete x-ray sources can be identified, as many authors have suggested, with hot neutron stars.

Our approach in the present paper is to discuss a neutron star as if it were a huge nucleus, neglecting the thin outer shell from which the photons are emitted. Some of the most important properties of a typical neutron-star nucleus are:

$$B \approx - 2 I_z \qquad , \qquad (1a)$$

$$\approx N_n \qquad , \qquad (1b)$$

$$\approx 10^{+57} \qquad , \qquad (1c)$$

and

$$Q \approx 0 \qquad , \qquad (1d)$$

$$\rho \gtrsim \rho_{nucl} \ (= 3.7 \times 10^{+14} \ gm/cm^3) \qquad , \qquad (1e)$$

$$R \approx 10 \ km \qquad . \qquad (1f)$$

Here B, I_z, N_n, Q, and R are, respectively, the baryon number, z-component of the isotopic spin, neutron number, charge, mass density, and radius of the neutron star. The above numbers obtain for a star of approximately one solar mass, with $R/R_\odot \sim 10^{-5}$. In addition, a neutron star has a small admixture of leptons ($\approx 1\%$ by number of e^- and μ^-). All hadrons and leptons present in a neutron star are highly degenerate.

In Section II, we formulate in a general way the problem of determining the ground state of a neutron star and discuss the conditions under

which one might reasonably hope that an individual particle model (which we adopt) is valid. We also summarize the results obtained by other authors using a non-interacting gas model for the nucleons in a neutron star. In Section III, we show how the strong interactions can affect the equilibrium number densities and production thresholds of the various hadrons. In Section IV, we calculate the effect of the strong interactions on the energy spectrum (assuming no superconductivity) of the neutrons and protons in a neutron star. In Section V, we make crude estimates of the contribution of hadrons other than nucleons to the equation of state and specific heat.

II. THE GROUND STATE OF A NEUTRON STAR

A. General Statement and Remarks

The problem of determining the ground state of a neutron star can be stated in the following general form:[2] Find the state that minimizes the total energy for a given baryon number, mass density, and zero net charge. This general statement is obviously insufficient, by itself, to enable one to perform any practical calculations. All calculations[2-6] to determine the properties of the ground state that have been carried out so far lean heavily on the concept of individual particles supposed to exist inside the huge nucleus-like neutron star.

One is led to use a particle model of a neutron star because most of our laboratory knowledge of hadrons is expressed in terms of the properties of independent particles, much of the experimental information regarding strong interactions having been obtained by studying the inter-actions of free hadrons. To regard a neutron star as composed of indivi-dual particles is, of course, an oversimplification, but this

oversimplification possesses considerable self-consistency and some experimental justification. The self-consistency results from the action of the exclusion principle and the experimental justification can be found in the successes of the independent-particle model in describing nuclei.

The exclusion principle prohibits true scattering among the degenerate baryons in a neutron star at zero degrees K because all the energetically accessible states are occupied. A collision between two baryons in a neutron star can therefore be pictured as follows. Initially, when the separation is large compared to a Fermi, the two-particle wave function is a product of plane waves. During the collision, when the particles are close together, the product wave function is distorted because of the strong interactions. Since all energetically accessible states are occupied, the two-particle wave function must resume after the collision its original form as a product of plane waves. Thus baryons in a neutron star behave somewhat like conduction electrons in a metal, namely, they propagate like plane waves with some extra wiggles in the wave function when two particles are close together. The reason is the same in both cases (electrons in a metal or baryons in a neutron star); the effective strength of the forces (electromagnetic or strong) is greatly decreased in the medium of degenerate fermions by the exclusion principle.

The above picture is expected to be valid[7] if the wave number, $\tilde{k}$, which the average nuclear potential impresses upon a nucleon is smaller than $P_F/\hbar$, where P_F is the Fermi momentum of the neutrons or protons. The relevant criterion is therefore

$$\tilde{k} \approx (m_n V/\hbar^2)^{1/2}$$

$$< P_F/\hbar \qquad , \qquad (2)$$

where m_n is the mass of a neutron and V is the depth of the nuclear potential.

If one ignores for the moment a possible hard-core repulsion, then inequality (2) is approximately equivalent to the condition that $P_F > 170$ MeV/c; this condition is always satisfied for neutrons in a neutron star. The fact that inequality (2) is satisfied for neutrons is sufficient for the validity of the model since (for $\rho < 8 \, \rho_{nucl}$) most collisions in a neutron star are between pairs of neutrons or between a neutron and some other hadron.

We have suggested previously[8] that a necessary condition for the validity of any independent-particle model for hadrons is that the average separation, d, between hadrons satisfy the following inequality:

$$d > 0.5 \text{ F} \qquad . \qquad (3)$$

Inequality (3) is equivalent to the condition that $\rho < 8 \, \rho_{nucl}$. We now show in three different but related arguments why inequality (3) must be satisfied for valid calculations to be carried out, with our present knowledge of strong interactions, on the basis of an independent particle model. The arguments given in subsections (ii) and (iii) assume that the effects of strong interactions can be important in a neutron star; this is shown explicitly by means of examples in Section III.

(i) Hard Core

Our original argument[8] assumed the existence of a hard core in, for example, the nucleon-nucleon interaction. We again assume in this subsection a hard core. If inequality (3) is not satisfied, then pairs of hadrons spend most of their time within each other's hard cores. Because of the high-momentum components that are present in a hard-core interaction, any pair of neighboring hadrons will continually produce other kinds of virtual hadrons; thus the state vector of any particular particle will

contain large admixtures of various hadrons. A "neutron" at such high densities will spend a large fraction of its time as, e.g., a $\pi^- + n\pi^0\text{'s} + p$ or $K^+ + \pi^- + \Lambda^0$. Thus the concept of distinct strongly interacting particles is not meaningful for densities greater than or of the order of eight times nuclear densities.

This conclusion is easily understood in terms of the following simple example. Imagine a collection of alpha particles at a density for which $d > R_\alpha$, where R_α is the "radius" of an alpha particle. If the density of alpha particles is now increased so that $d \leq R_\alpha$, the alpha particles will come apart into their constituents, primarily neutrons and protons, as they do in actual nuclei. This simple example also suggests that the distinction between fermions and bosons probably disappears for densities in excess of eight times nuclear densities. Thus pions (bosons) will spend a large fraction of their time as fermion anti-fermion pairs (e.g., $N + \bar{N}$). In this situation, one must regard the star as one gooey mess and try to discuss the excitations of the star (or large nucleus) as a single entity.

(ii) Strange Forces

The forces due to the exchange of strange particles are expected to be important when d is of the order of $\hbar/m_K c$, i.e., 0.4 F. Since these forces are not well known at present, one cannot calculate reliably the strong interactions among hadrons at densities for which $d \leq 0.4$ F.

(iii) Strange Particles

The mass splittings between members of the baryon octet are of the order of a few hundred MeV. Thus strange particles such as Σ's, Λ^0's, etc., will be produced in profusion in a neutron star when the neutron Fermi energy is of the order of, say, 400 MeV. The condition that

$$\frac{P_F^2(n)}{2m_n} \approx 400 \text{ MeV} \quad , \quad (4)$$

implies an average separation between neutrons of the order of 0.4 F. Since the forces between various members of the baryon octet are not well known (except perhaps for the nucleon-nucleon forces), one can not carry out reliable calculations for densities such that $d \lesssim 0.4$ F.

Note that Eq. (4) also shows that relativistic effects, which can not be reliably included in dynamical calculations involving the strong interactions, are important for $d \lesssim 0.4$ F.[9]

B. The Non-Interacting Gas Model

The non-interacting gas model of a neutron star was proposed independently by Ambartsumyan and Saakyan[2] and Salpeter[4] in 1960 and has been investigated in great detail by Cameron[3] and Tsuruta.[5] This model represents the state of a neutron star in terms of the numbers of degenerate, but non-interacting hadrons and leptons that are present. Tsuruta has calculated detailed tables, on the basis of the non-interacting gas model, for the number densities of the various hadrons, Σ^-, Λ^0, Ξ^-, Δ^-, Σ^0, etc., as a function of stellar density for $\rho \lesssim 300 \, \rho_{nucl}$. A principal result of these calculations[2-5] is that only fermions are present at densities for which stable neutron-star models are expected to exist ($\rho < 300 \, \rho_{nucl}$); no pions are present on the basis of the non-interacting gas model until $\rho \gtrsim 300 \, \rho_{nucl}$.

The following approximate numerical results can easily be obtained, for $\rho \lesssim 4 \, \rho_{nucl}$, on the basis of the non-interacting gas model:

$$n(n) = 2 \times 10^{+38} \left(\rho / \rho_{nucl} \right) \text{ cm}^{-3} \qquad ; \qquad (5a)$$

$$n(e) = n(p) = 2 \times 10^{+36} \left(\rho / \rho_{nucl} \right)^2 \text{ cm}^{-3} \qquad ; \qquad (5b)$$

$$E_F(n) \approx E_F(e) = 7 \times 10^{+1} \left(\rho / \rho_{nucl} \right)^{2/3} \text{ MeV} \qquad ; \qquad (5c)$$

$$E_F(p) \approx 3 \left(\rho / \rho_{nucl} \right)^{4/3} \text{ MeV} \qquad ; \qquad (5d)$$

$$P_F(n) = 4 \times 10^{+2} \left(\rho / \rho_{nucl} \right)^{1/3} \text{ MeV/c} \qquad ; \qquad (5e)$$

$$P_F(e) = P_F(p) = 7 \times 10^{+1} \left(\rho / \rho_{nucl} \right)^{2/3} \text{ MeV/c} \quad . \qquad (5f)$$

Here $n(i)$, $E_F(i)$, and $P_F(i)$ are, respectively, the number density, Fermi kinetic energy, and Fermi momentum for particles of type i. Equations (5) will be used for order of magnitude estimates in this and the succeeding paper.

The number of electrons and protons is much less than the number of neutrons because of two facts: (1) The Fermi momentum of the electrons equals the Fermi momentum of the protons (the condition of zero charge); and (2) The mass of an electron is much less than the mass of a nucleon. The way in which these facts conspire to produce a relatively small number of electrons and protons can be seen easily from the equilibrium relation between neutrons, protons, and electrons, which is ($n + p + e^- \rightarrow n + n' + \nu_e$):

$$c \, P_F(e) + P_F^2(p)/2m_p \approx (m_n - m_p)c^2 + P_F^2(n)/2m_n \quad . \qquad (6)$$

Thus:

$$\left(n(e)/n(n) \right) \approx \left(P_F(e)/P_F(n) \right)^3 \qquad (7a)$$

$$\approx \left(P_F(n)/2m_n c \right)^3 \qquad (7b)$$

$$\ll 1 \qquad , \qquad (7c)$$

which is the origin of the name "neutron star".

The non-interacting gas model has been used to calculate[2-5] the equation of state, heat capacity, and other properties of dense matter for $\rho_{nucl} \leq \rho \leq 300 \ \rho_{nucl}$. These results have been applied to a number of problems including hydrodynamic models of supernova collapse.[10] Many of the papers in which these applications are made discuss situations for which $d \lesssim 0.5$ f; hence the crudest version of an independent-particle model has been used in a domain in which no independent-particle model is valid. The conclusions of these papers must be reexamined in order to determine how much the conclusions depend upon unjustified inferences from an independent-particle model.

III. PARTICLE MODELS WITH STRONG INTERACTIONS

A. General Formalism

The problem of determining the constituents of a neutron star can easily be formulated for any model that assumes the existence of individual particles inside the star. One defines a function

$$\Phi = \sum_i \int d^3 n_i \left[W_i \left\{ N_j \right\} + \alpha \ Q(i) + \beta \ B'(i) \right] \qquad , \qquad (8)$$

where the summation over the particle label i extends over all types of particles that are present, $d^3 n_i$ is the number of particles of type i in a given momentum or energy interval, W_i is the energy of a particle of type i and momentum p, N_j is the number density of particles of type j, $Q(i)$ and $B'(i)$ are the charge and baryon numbers of particles of type i, and α and β are Lagrange multipliers introduced in order to satisfy the constraints of conservation of charge and baryon number. The state of the neutron star is then determined by requiring that

$$\left.\frac{\partial \Phi}{\partial N_i}\right)_\Omega = 0 \qquad , \qquad (9)$$

where the minimization implied by Eq. (9) is carried out at constant volume Ω.

Note that Eqs. (8) and (9) can be used to determine the equilibrium state of matter even if the matter is not in the form of an electrically neutral neutron star.

B. Examples

The function $W_i(N_j)$ has a simple form for the leptons (e^- and μ^-) that are present because the average electrostatic energy is small ($\approx 0.2 \, (\rho/\rho_{nucl})^{2/3}$ MeV) compared to the Fermi energies (see Eqs. (5)). Therefore,

$$W_e(p) = c(m_e^2 c^2 + p^2)^{1/2} \qquad , \qquad (10a)$$

and

$$W_\mu(p) = c(m_\mu^2 c^2 + p^2)^{1/2} \qquad . \qquad (10b)$$

From Eqs. (8) - (10), one finds:

$$W_e(P_F(e)) = -\alpha \qquad , \qquad (11a)$$

$$= W_\mu(P_F(\mu)) \qquad , \qquad W_e(P_F(e)) \geq m_\mu c^2 \quad . \qquad (11b)$$

The functions W_i, where i is a hadron, depend on the number densities of all the hadrons present because of the strong interactions that obtain among all hadrons; the magnitudes of these interactions are comparable with the hadronic binding energies. A significant part of each hadronic function W_i will, nevertheless, be given by the simple expression:

$m_1 c^2 + p^2/2m_1$. Thus one obtains by differentiating Eq. (8) with respect to N_n:

$$m_n c^2 + E_F(n) + B(n) = -\beta \qquad . \qquad (12a)$$

Here:

$$B(n) \equiv \frac{\partial}{\partial N_n} \left[\int d^3\underset{\sim}{n}_n \ (W_n - m_n c^2 - p^2/2m_n) \right] \qquad (12b)$$

represents the average energy due to the strong interactions between the neutrons and all other hadrons present. The quantity $B(n)$ is negative and less than $-E_F(n)$ ($\equiv -P_F^2(n)/2\ m_n$) if the neutrons are bound independent of the gravitational forces. As a first approximation, one can neglect in computing $B(n)$ all interactions except those among the many neutrons present. In this simplified case, $B(n)$ is the average energy due to interactions of the neutrons in a neutron gas. Even in this case, the quantity $B(n)$ is uncertain by a factor of two or more depending upon which form is chosen for the nuclear forces in a nuclear-matter calculation.[11]

The equilibrium equation can be obtained by combining Eqs. (11) and (12) with a similar relation for protons. One finds:

$$W_F(e) + E_F(p) = E_F(n) + (m_n - m_p) c^2 + (B(n) - B(p)) \qquad , \qquad (13)$$

where $B(p)$ is defined by Eq. (12b) with n replaced by p. Note that Eq. (13) reduces, if $(B(n) - B(p))$ is set equal to zero, to the relation (Eq. (6)) valid in the non-interacting gas model. Preliminary estimates suggest that $(B(n) - B(p))$ is, however, rather large because of the great disparity between neutron and proton number densities.

If Σ^-'s are present,

$$m_{\Sigma^-}c^2 + E_F(\Sigma^-) = W_F(e) + m_n c^2 + E_F(n) + (B(n) - B(\Sigma^-)) \qquad , \qquad (14)$$

and if π^-'s are present,

$$m_\pi c^2 + B(\pi^-) = W_F(e) \qquad\qquad\qquad (15a)$$

$$\equiv \omega_\pi \qquad , \qquad\qquad (15b)$$

where we have defined ω_π to be the energy of the lowest pionic excitation. In writing Eq. (15), we have made use of the fact that pions are bosons and hence all the pions that are present (at zero temperature) will be in the lowest energy state.

Equations (13), (14), and (15) can be obtained by inspection from the equilibrium reactions, $n + e^- + p \to n + n + \nu_e$, $e^- + n + n \to \Sigma^- + \nu_e + n'$, and $n + n \to \pi^- + p + n'$. The reason why neutron stars can contain Σ^-'s, Λ^0's, and possibly many other hadrons in abundance, although these strange particles are not present to a good approximation in ordinary nuclei, is that the Fermi kinetic energy, $P_F^2/2m$, in neutron stars can be of the order of the mass differences (300 MeV) between the hadrons ($P_F^2/2m < 50$ MeV for ordinary nuclei).

C. Shifts in Threshold Densities

Strong interactions shift[8] the threshold densities at which various hadrons are produced from the values these threshold densities have in the non-interacting gas model. The crucial way in which these threshold shifts occur is most clearly understood by discussing a few examples. Pions are produced at densities such that ($n + n \to \pi^- + p + n'$):

$$W_e \left(P_F(e)\right) \geq m_{\pi^-} c^2 + B(\pi^-) \qquad . \qquad (16)$$

Sigmas are produced at densities such that $(e^- + n + n \rightarrow \Sigma^- + \nu_e + n')$:

$$W_e(P_F(e)) + E_F(n) \geq (m_{\Sigma^-} - m_n) \, c^2 + (B_0(\Sigma^-) - B(n)) \qquad . \qquad (17)$$

Pions are produced before sigmas if:

$$B(\pi^-) \leq 0.5 \left[(m_\Sigma - m_p - 2m_\pi) \, c^2 + (B_0(\Sigma^-) - B(p) - E_F(p)) \right] \qquad . \qquad (18)$$

Inequality (18) follows from Eqs. (13), (16), and (17). It is useful to rewrite Eq. (18), expressing all energies in MeV and estimating the proton Fermi energy from the non-interacting gas model. One finds in this way that the criterion for pions being produced before sigmas is (energies in MeV):

$$B(\pi^-) \leq -10 - 1.4 \, (\rho/\rho_{nucl})^{4/3} + 0.5 \left[B_0(\Sigma^-) - B(p) \right] \qquad . \qquad (18')$$

The question of whether or not this inequality is satisfied has great practical significance since the presence of a large number of pions changes the predicted cooling rates of a hot neutron star by a large factor $(\sim 10^{+7})$.[1,8] Note that inequality (18) or (18') can never be satisfied if one neglects, as one does with the non-interacting gas model, the

effects of the strong interactions (i.e., sets $B(\pi^-) = B_0(\Sigma^-) = B(p) \equiv 0$). The reason that the threshold density for the production of pions is so high ($\approx 300\ \rho_{nucl}$) on the basis of the non-interacting gas model is that the excess negative particles, electrons, are drained off into Σ^-'s before the Fermi energy of the electrons becomes high enough to make pions.

D. General Remarks About Models that Include Strong Interactions

The equations given in Section III A-C are valid for any model that assumes the existence of individual particles in a neutron star. Of course, these particles will have, as a result of their continuous strong interactions, properties that are different from their free-particle analogues which are studied in most laboratory experiments. Unfortunately, one must invoke a detailed theory of strong interactions in order to calculate quantities such as $B(\pi^-)$ and $B_0(\Sigma^-)$. We hope that some high-energy theorists will apply their methods to the calculation of these interaction energies which are vital to an understanding of neutron stars.[12]

IV. THE ENERGY SPECTRUM OF A NEUTRON STAR

A. General Discussion

The specific heat and neutrino luminosity of a neutron star depend critically on the spectrum of energy states available to the star. In the present work (papers I and II), we describe the states of the star in terms of its constituent particles, adopting the model that Gomes, Walecka, and Weisskopf used to describe nuclear matter.[7]

We assume that the nucleons in a neutron star do not form a superfluid: that is, we assume that there is no energy gap between the ground state and the first excited state of the nucleon gas.[13] An energy gap of

more than 0.1 MeV in the neutron energy spectrum would greatly reduce both the neutrino luminosity and the specific heat of the star.

We are now trying to determine theoretically whether a dense nucleon gas forms a superfluid and to estimate the effects of superfluidity on the cooling rates of hot neutron stars; we expect to report on this work at a later date.

B. The Nucleon Effective Masses

(i) Definitions

According to the individual-particle model, the expression for the density of states available to a single nucleon is given by

$$\rho(E) = 2^{-1} \pi^{-2} \hbar^{-3} p^2 \, dp/dE \qquad (19)$$

where $\rho(E)$ is the number of states per unit energy interval per unit volume, and p and E are the momentum and energy of the nucleon. For a non-relativistic nucleon, the free-particle model implies that

$$\rho(E) = 2^{-1} \pi^{-2} \hbar^{-3} pm \qquad , \qquad (20)$$

where m is the mass of the nucleon. The effect of interparticle interactions on the energy spectrum of a star can be represented approximately by writing the energy of each individual nucleon in the form

$$E(p) = c \sqrt{m^2 c^2 + p^2} - mc^2 + U(p) \qquad , \qquad (21)$$

where $U(p)$ is the change in the single-particle energy produced by interactions with neighboring nucleons. We define the effective mass $m^*(p)$ by the relation

$$\frac{1}{m^*(p)} = \frac{1}{(m^2 + p^2 c^{-2})^{1/2}} + \frac{1}{p} \frac{dU(p)}{dp} \quad , \quad (22)$$

which leads to the expression

$$\rho = 2^{-1} \pi^{-2} \hbar^{-3} p \, m^*(p) \quad (23)$$

for the density of single-particle states. Note that Eq. (22) reduces to the usual[7] non-relativistic definition of an effective mass if p is neglected relative to m in the first term on the right-hand side of Eq. (22). The additional relativistic correction $(-\frac{1}{2} p^2 m^{-2} c^{-2})$ is small ($\sim 5\%$) for nuclear matter. We are interested primarily in the density of states near the Fermi momentum P_F, because this is the quantity that enters into neutrino cooling rates. Thus we need calculate only $m_n^* \left[P_F(n) \right]$ and $m_p^* \left[P_F(p) \right]$, which we can now write more compactly as m_n^* and m_p^*, respectively.

(ii) Calculation of the Effective Masses

We need the effective masses of both the neutron and the proton for our calculations of cooling rates. There are, however, two important simplifications that result from the fact that the number density of protons is much smaller than the number density of neutrons; one can, with sufficient accuracy, neglect the effect of neutron-proton interactions on the neutron energy as well as the effect of proton-proton interactions on the proton energy.

The nucleons are only slightly relativistic for the densities at which an individual-particle treatment is valid, and the term $p^{-1} dU/dp$ in Eq. (22) is not large compared to m^{-1}. We thus treat both the relativistic correction $(-\frac{1}{2} p^2 m^{-2} c^{-2})$ and the interaction-correction as small perturbations and do not consider relativistic corrections to the interaction term in Eq. (22). Following the non-relativistic treatment of Gomes et al.,[7] we make several simplifying assumptions:

(1) The potential acting in an odd-parity nucleon-nucleon state is negligibly small;

(2) The potential acting in even-parity states is spin-independent and consists of a short-range hard-core potential, $V^{core}(r)$, and a long-range attractive potential, $V^{att}(r)$;

(3) The repulsive core makes a negligible contribution to dU/dp;

(4) The Born approximation provides an accurate estimate of the expectation value of the attractive potential (because of the effect of the exclusion principle on the nucleon wave functions).

Gomes et al.[7] have shown that the above approximations result in small errors at densities near nuclear density.

The four assumptions listed above imply a simple correspondence between nuclear matter and a neutron star with the same number density of neutrons. In computing $U(p)$ for a neutron in a neutron star, we include interactions with only half the neutrons in the star, because assumption (1) and the exclusion principle imply that there is no interaction between neutrons with parallel spin. The corresponding $U(p)$ for nuclear matter (which contains equal numbers of neutrons and protons) includes contributions from half the neutrons and all the protons present. Thus we conclude that

$$U_n^{n.s.}(p;\ \rho_n) \approx \frac{1}{3}\ U_n^{n.m.}(p;\ \rho_n) \tag{24}$$

where superscripts "n.s." and "n.m." denote, respectively, "neutron star" and "nuclear matter", and the subscript "n" represents "neutron". One can use a similar argument to show that

$$U_p^{n.s.}(p;\ \rho_n) \approx \frac{2}{3}\ U_p^{n.m.}(p;\ \rho_n) \ , \tag{25}$$

The assumptions (1) - (4) can be used to show that the neutron and proton energies have the form:

$$U_n^{n.s.} (p; \rho_n) = \frac{1}{2} U_p^{n.s.} (p; \rho_n) \tag{26a}$$

$$= (2\pi\hbar)^{-3} \int d^3q \int d^3r \cos^2(\underline{k} \cdot \underline{r}) V^{att}(r) , \tag{26b}$$

$$|\underline{q}| < P_F(n)$$

where
$$\underline{k} = (2\hbar)^{-1} (\underline{p} - \underline{q}) \tag{26c}$$

and $P_F(n)$ is the neutron Fermi momentum.

The effective masses of the neutron and proton have been calculated using Eqs. (22) and (26). The computations have been carried out for the following potentials: (1) an attractive square well with a repulsive core (the potential used by Gomes et al.); and (2) several combinations of attractive Yukawa potentials and repulsive cores (the potentials suggested by Preston[14]). There is a significant variation in the values of the effective masses calculated using these potentials, in spite of the fact that all the potentials were chosen to fit the low-energy nucleon-nucleon scattering data. In the next two paragraphs, we describe the general behavior of the effective masses as functions of density, indicating the extent to which the numerical results depend on the particular potential chosen. The errors introduced in our calculations of the specific heat and cooling rates by the uncertainties in the effective masses are small compared to the other uncertainties that exist.

(iii) Neutron Effective Mass

The neutron effective mass takes on its minimum value at a density of the order of ρ_{nucl}. For $0.5 < \rho/\rho_{nucl} < 5$, the neutron effective mass $m_n^{*n.s.}$ is in the range

$$0.90 \, m_n < m_n^{*n.s.} < 1.15 \, m_n \qquad . \qquad (27)$$

Our present estimates for $m_n^{*n.s.}$ are somewhat higher than in our previous work[8] since we did not include the relativistic correction in our earlier estimate. For $\rho \ll \rho_{nucl}$, the effective mass can be expressed in the form

$$m_n^{*n.s.} \approx m_n \left[1 - \alpha \, (\rho/\rho_{nucl}) + 0.08 \, (\rho/\rho_{nucl})^{2/3} \right] \quad , \quad (28)$$

where $\alpha = 2.5 \pm 0.5$.

(iv) Proton Effective Mass

The proton effective mass reaches its minimum value m_{min}^* at a density ρ_{min}, where

$$0.5 \, m_n < m_{min}^* < 0.75 \, m_n \qquad (29)$$

and

$$0.9 \, \rho_{nucl} < \rho_{min} < 2 \, \rho_{nucl} \qquad . \qquad (30)$$

For $\rho \ll \rho_{nucl}$, the effective mass can be expressed in the form

$$m_p^{*n.s.} \approx m_n \left[1 - \gamma \, (\rho/\rho_{nucl}) \right] \qquad , \qquad (31)$$

where

$$\gamma = 5.0 \pm 1.0 \qquad . \qquad (32)$$

At high densities, $m_p^{*n.s.}$ is given approximately by

$$m_p^{*n.s.} \approx m_n \left[1 - \delta \, (\rho_{nucl}/\rho)^{1/3}\right] \quad , \tag{33}$$

where $\qquad\qquad 0.6 < \delta < 2.0$.

The effective masses can thus be calculated with reasonable accuracy, despite the fact that our present ignorance of the strong interactions makes the accurate calculation of the energy of a neutron gas difficult (or impossible). The energy U_{tot} of a neutron gas is the sum of a negative part U_n, which results from the attractive well, and a positive part U_p, which results from the repulsive core and the ordinary kinetic energy. The positive and negative contributions to U_{tot} tend to cancel, and $|U_{tot}|$ is generally small compared to either $|U_n|$ or $|U_p|$. Thus, small errors in U_n or U_p can cause large fractional errors in U_{tot}. On the other hand, the strong interactions cause only a relatively small change in the effective mass. Thus it is possible to calculate the effective masses to within about 10% despite the uncertainty in the treatment of the strong interactions.

C. Electrons and Muons

The energy spectra of the electrons and muons in a neutron star are essentially the same as their corresponding free-particle spectra, because the energies of the electromagnetic interactions are small (< 1 MeV) compared to the relevant Fermi energies.

V. THE EQUATION OF STATE AND SPECIFIC HEAT

The strong interactions among the hadrons present in a neutron star make it difficult to find an accurate equation of state for neutron-star matter. The equations of state based on various theoretical estimates[15] of the energy of a neutron gas differ by as much as a factor of five at

typical neutron-star densities. The presence of hadrons other than the nucleons can be estimated, on the basis of the non-interacting gas model, to change the pressure by less than a factor of two.

The specific heat also depends on the threshold densities for the production of various species of strange particles. The specific heat can be shown to increase by a factor of the order of 1.5 near the threshold density for the production of each new baryon. The densities at which these increases occur are somewhat uncertain (cf. Section III) because the strong interactions can cause large shifts in the threshold densities of the strange baryons.

The presence of pions will not affect the specific heat directly. A pion gas becomes degenerate at a temperature T_c, where

$$T_c \approx (4 \times 10^{12} \, {}^{\circ}K) \, (n_\pi/n_n)^{2/3} \, (\rho/\rho_{nucl})^{2/3} \quad , \qquad (34)$$

and n_π/n_n is the ratio of the number density of pions to the number density of neutrons. Pions are therefore highly degenerate if $n_\pi > 0.1 \, n_n$ and $T < 10^{10} \, {}^{\circ}K$. The ratio of the pion specific heat C_π to the nucleon specific heat C_n is given by

$$\frac{C_\pi}{C_n} \approx 0.1 \left(\frac{n_\pi}{n_n}\right)^{1/3} \left(\frac{T}{T_c}\right)^{1/2} \quad . \qquad (35)$$

Thus, C_π is negligible compared to C_n if $T \ll T_c$.

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