THE RESPONSE OF AN UNBOUNDED ATMOSPHERE TO POINT

DISTURBANCES II. IMPULSIVE DISTURBANCES

[redacted]

Shoji Kato*

Institute for Space Studies
Goddard Space Flight Center
National Aeronautics and Space Administration
New York, New York

Received ~~Rec. August 16, 1965~~

*NAS-NASA Resident Research Associate on leave of absence from the University of Tokyo.

ABSTRACT

The response of an unbounded and isothermal atmosphere to a point impulse of force is studied. Fronts of disturbances propagate with two different speeds corresponding to acoustic waves and to gravity waves. The distributions of kinetic energy density in the two wakes behind the two fronts are calculated. A striking result is that behind the acoustic front, the radiation density is very intense above and below the source, if the atmosphere is sufficiently stable to convection. This result may be of interest in connection with the formation of spicule structures in the solar upper chromosphere.

[redacted]

1. INTRODUCTION

In a previous paper (Kato 1965; hereafter this paper will be referred to as I) we studied the response of an atmosphere to time-harmonic point disturbances. This atmosphere was assumed to be isothermal and unbounded and subjected to an uniform field of gravity. In the present paper we study the response of the same kind of atmosphere to impulsive point disturbances, and emphasize some interesting effects of the degree of stability of the atmosphere upon the response.

The impulsive disturbance treated here will be assumed isotropic for two reasons. First, in paper I, we have already learned something of the directional response of the isothermal atmosphere by treating linear, oscillating forces aligned in different directions. In the case of impulsive disturbances we may expect that the differences in the direction will produce similar differences in the response of the atmosphere. Second, for an isotropic point impulse, the response of the atmosphere is spherically symmetric if the atmosphere is convectively neutral ($\gamma = 1$), as we shall see. Thus, we can concentrate our attention upon effects of the degree of stability of the atmosphere by studying deviation from spherical symmetry in the response. In the Appendix, however, we indicate how directed impulses may be treated and show how they may be combined to yield the results obtained with an isotropic source.

Now, Whitham (1961) showed that in the initial value problems of propagation of waves, the amplitude of distances can be calculated easily and directly with the help of some properties of group velocity. We apply

Whitham's method to study of the response of an atmosphere to an isotropic impulse, without solving an inhomogeneous wave equation. An advantage of such a direct argument is that it may be taken over to a slightly non-isothermal atmosphere.

II. GROUP VELOCITY

In order to apply Whitham's approach, it is necessary to obtain an expression for the group velocity. If the dispersion relation is given by $G(\underset{\sim}{k}, \omega) = 0$, the group velocity, $\underset{\sim}{U}$, can be written as

$$\underset{\sim}{U} \equiv \frac{\partial \omega}{\partial \underset{\sim}{k}} = - \nabla G \Big/ \frac{\partial G}{\partial \omega} \quad , \tag{1}$$

because $\partial G/\partial \underset{\sim}{k} + (\partial G/\partial \omega)\, \partial \omega/\partial \underset{\sim}{k} = 0$. In an isothermal atmosphere stratified under a constant gravity, $\underset{\sim}{g} = (o, o, -g)$, the dispersion relation, $G(\underset{\sim}{k}, \omega) = 0$, is given by (e.g., see equation (5) in I)

$$G(k, \omega) \equiv - \omega^4 + \omega^2 (c^2 k^2 + \omega_1^2) - c^2 \omega_2^2 k_1^2 = 0 \; , \tag{2}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$, $k_1^2 = k_x^2 + k_y^2$, and

$$\omega_1 = \gamma\, g/2\, c, \text{ and } \omega_2 = (\gamma - 1)^{1/2}\, g/c \; , \tag{3}$$

where γ, g and c are the ratio of the specific heats, the magnitude of gravity, and the adiabatic sound velocity, respectively.

If the angle between the direction of the group velocity and the positive z-axis is Θ and the longitude measured from x-axis is ϕ, the relation, $\underset{\sim}{U} /\!/ -\nabla G$, shown in equation (1) can be written as

$$\frac{2c^2(\omega^2 - \omega_2^2)\, k_x}{\sin\Theta \cos\phi} = \frac{2c^2(\omega^2-\omega_2^2)\, k_y}{\sin\Theta \sin\phi} = \frac{2c^2\, \omega^2\, k_z}{\cos\Theta} \; . \tag{4}$$

Expressing, for example, k_y and k_z by k_x with the use of equation (4), and inserting them into equation (2), we have an expression for k_x written by ω, θ and ϕ. Quite similarly we can have expressions for k_y and k_z. These expressions are[1)]

$$\begin{Bmatrix} k_x \\ k_y \end{Bmatrix} = \frac{1}{c}\left[\frac{\omega^4(\omega^2-\omega_1^2)}{(\omega^2-\omega_2^2)(\omega^2-\omega_2^2\cos^2\theta)}\right]^{1/2}\sin\theta\begin{Bmatrix}\cos\phi \\ \sin\phi\end{Bmatrix} \tag{5}$$

and

$$k_z = \frac{1}{c}\left[\frac{(\omega^2-\omega_2^2)(\omega^2-\omega_1^2)}{(\omega^2-\omega_2^2\cos^2\theta)}\right]^{1/2}\cos\theta \quad . \tag{6}$$

Inserting the values for k_x, k_y and k_z given in equations (5) and (6) into a component of equation (1), the magnitude of the group velocity can be expressed as (see also Moore and Spiegel 1964)

$$U^2(\omega,\theta) = c^2\,\frac{(\omega^2-\omega_2^2)^3(\omega^2-\omega_1^2)(\omega^2-\omega_2^2\cos^2\theta)}{\omega^2[(\omega^2-\omega_2^2)^2+\omega_2^2(\omega_1^2-\omega_2^2)\sin^2\theta]^2} \quad . \tag{7}$$

Of course, the group velocity has meaning only in the frequency ranges of acoustic waves $(\omega^2 > \omega_1^2)$ and of gravity waves $(\omega^2 < \omega_2^2\cos^2\theta)$. Here we should mention that in waves radiating energy in the critical direction, θ_c $(\cos^2\theta_c = \omega_2^2/\omega^2)$, of gravity waves the group velocity is zero (see equation (7)) and the wavelength is infinite (see equations (5) and (6)).

III. FREQUENCY DISTRIBUTION IN WAKES

We assume a point impulse of force at the origin, $\underset{\sim}{r} = 0$, and at the time, $t = 0$. As the medium is dispersive, there remain wakes after the

---

1) We consider the plane wave $\exp(i\omega t - i\underset{\sim}{k}.\underset{\sim}{r})$ for $\omega > 0$. If we consider the plane wave $\exp(i\omega t + i\underset{\sim}{k}.\underset{\sim}{r})$ for $\omega > 0$ the signs of k_x, k_y and k_z in equations (5) and (6) should be changed.

passage of fronts of disturbances. At points in the wakes far from the fronts the disturbances are broken down into wave trains, the different wave number components or frequency components distribute over the physical $\underset{\sim}{r}$-space, and each $\underset{\sim}{k}$ (or ω) propagates with the group velocity (Whitham 1961, see also Appendix B in Lighthill's paper 1960). That is, far from the fronts of disturbances, $\underset{\sim}{k}$ (or ω) and $\underset{\sim}{r}$ are related through the group velocity $\underset{\sim}{U}$ by

$$\underset{\sim}{r} = \underset{\sim}{U}(\omega, \theta)\ t\ . \tag{8}$$

This relation enables us to calculate the frequency distribution in wakes except for the region near the fronts. We introduce a sphere of radius $r_c = ct$ (this sphere represents the front of acoustic waves). Then, from equation (8), we have $r/r_c = U(\omega,\theta)/c$ or by solving this for ω we obtain $\omega = \omega(r/r_c, \theta; \gamma)$, which shows the frequency distribution in wakes. In general, ω is not a single valued function because there are two wave modes (the acoustic and the gravity wave modes). The surfaces of constant values of ω in the $\underset{\sim}{r}$-space are drawn in Figures 1 and 2 (see also Figure 4 in Moore and Spiegel's paper (1964)) for the acoustic wave mode ($\omega \geq \omega_1$) and for the gravity wave mode ($\omega \leq \omega_2$), respectively. The three curves for each value of ω in Figure 1 are for $\gamma = 1$, 4/3 and 5/3 respectively. The left hand side in Figure 2 is for $\gamma = 4/3$, and the right hand side is for $\gamma = 5/3$.

Figure 1 shows that a front of acoustic waves propagates spherically from the source with the sound velocity $(\gamma R T_o)^{1/2}$. After the passage of the front there remains a wake. In the wake frequencies of disturbances tend rapidly to the upper characteristic frequency of the atmosphere, ω_1, as shown in Figure 1. The result that there is a wake behind the acoustic front, and the frequencies of disturbances in the wake tend to the upper characteristic frequency is qualitatively the same as in the one-dimensional case studied by Lamb (1908). (Lamb studied the vertical propagation of disturbances from an impulsive force acting uniformly on an infinite horizontal plane.) The other front (front of gravity waves) is the one formed by the envelope of disturbances propagating from the source as gravity waves, and after the passage of the front, wave motions at a point in the wake consist of two different frequencies as shown in Figure 2. The gravity waves, of course, do not appear in Lamb's one-dimensional case.

Here we note that Figures 1 and 2 can also be interpreted as polar diagrams of magnitude of the group velocity for acoustic waves and for gravity waves, respectively. That is, the radial distance from the origin of a point on a curve in Figures 1 and 2 denotes the magnitude of the group velocity in that direction and for that frequency.

IV. DISTRIBUTION OF KINETIC ENERGY DENSITY IN WAKES

a) Whitham's Method

Whitham (1961) has shown that in the initial value problem the amplitude variation of disturbances can be calculated easily and directly with the help of some properties of group velocity. We shall apply here Whitham's method to calculations of the distribution of kinetic energy density in wakes radiated from an isotropic impulse. Following Whitham, we consider plane wave solutions, $\exp(i\omega t - i\underset{\sim}{k}.\underset{\sim}{r})$, where ω and $\underset{\sim}{k}$ are related by the dispersion relation, $G(\underset{\sim}{k}, \omega) = 0$. In our present problem, the dispersion relation is given by equation (2). This dispersion relation shows that for a given $\underset{\sim}{k}$ there are always two positive values of ω^2; one corresponds to the acoustic wave mode and the other to the gravity wave mode. For disturbances associated with the each wave mode, the magnitude of velocity, u, times one-half of the undisturbed density, $\rho_o^{\frac{1}{2}}$, may be analyzed into plane waves as

$$\rho_o^{\frac{1}{2}} u(\underset{\sim}{r},t) = \int_{-\infty}^{\infty} F(\underset{\sim}{k}) \exp(i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r})\, dk, \quad (\omega \gtrless 0) \tag{9}$$

where ω is related to $\underset{\sim}{k}$ by the dispersion relation, where F can be expressed in terms of the initial conditions by

$$F(\underset{\sim}{k}) = \frac{1}{(2\pi)^3} \int \rho_o^{\frac{1}{2}} u(\underset{\sim}{r},o) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r})\, d\underset{\sim}{r} \;. \tag{10}$$

The right hand side of equation (9) is complex and its real part gives the magnitude of the disturbance.

If an impulse is isotropic the form of the initial condition is obvious, and we can apply Whitham's method easily. That is, the initial condition can be written as

$$\rho_o^{\frac{1}{2}} u(\underset{\sim}{r},o) = A\, \delta(\underset{\sim}{r}), \quad A = \text{complex constant.} \tag{11}$$

Thus, inserting equation (11) into equation (10), we have

$$F(\underset{\sim}{k}) = \frac{A}{(2\pi)^3} \tag{12}$$

That is, $F(\underset{\sim}{k})$ depends neither on the direction of $\underset{\sim}{k}$ nor on its magnitude.

The total kinetic energy of disturbances at the time, t, is given by

$$\int \frac{1}{2} \rho_o^{\frac{1}{2}} u(\underset{\sim}{r},t) \rho_o^{\frac{1}{2}} u^*(\underset{\sim}{r},t)\, d\underset{\sim}{r} , \tag{13}$$

and, on substitution of equation (9), this can also be written as

$$(2\pi)^3 \int_{-\infty}^{\infty} \frac{1}{2} F(\underset{\sim}{k}) F^*(\underset{\sim}{k})\, d\underset{\sim}{k} . \tag{14}$$

As mentioned in the previous section, after the initial disturbances have been broken down into wave trains the different values of $\underset{\sim}{k}$ are distributed over the physical $\underset{\sim}{r}$-space and each value of $\underset{\sim}{k}$ propagates in the physical $\underset{\sim}{r}$-space with the group velocity. Therefore, we have a direct correspondence between equations (13) and (14). That is, if the wave number at position $\underset{\sim}{r}$ is $\underset{\sim}{k}$ and the volume element $\Delta\underset{\sim}{r}$ contains waves with wave numbers in an element $\Delta\underset{\sim}{k}$ of wave number space, then the contribution of that element to energy is

$$\frac{1}{2} \rho_o^{\frac{1}{2}} u(r, t) \rho_o^{\frac{1}{2}} u^* (r, t) \Delta \underset{\sim}{r} , \tag{15}$$

or in wave number space,

$$\frac{1}{2} (2\pi)^3 F(k) F^* (k) \Delta \underset{\sim}{k} \tag{16}$$

and the two must be equal. Therefore the amplitude of $\rho_o^{\frac{1}{2}} u (r, t)$ is

$$\left|\rho_o^{\frac{1}{2}} u(\underset{\sim}{r},t)\right| = (2\pi)^3 \left| F(\underset{\sim}{k})\right| \left\{\frac{\Delta\underset{\sim}{k}}{\Delta\underset{\sim}{r}}\right\}^{\frac{1}{2}} . \tag{17}$$

If we follow the wave number $\underset{\sim}{k}$ and retain a constant $\Delta\underset{\sim}{k}$, the volume element $\Delta\underset{\sim}{r}$ will vary due to the small differences in the group velocity for the different wave numbers in the element. Using this fact, we can calculate $\Delta\underset{\sim}{k}/\Delta\underset{\sim}{r}$. Inserting the expression for $\Delta\underset{\sim}{k}/\Delta\underset{\sim}{r}$ calculated in this way into equation (17), Whitham (1961) obtained

$$\left|\rho_o^{\frac{1}{2}}\, u(\underset{\sim}{r},t)\right| = \left(\frac{2\pi}{t}\right)^{3/2} \left|F(\underset{\sim}{k})\right| \left\{ \det\left|\frac{\partial^2\omega}{\partial k_i \partial k_j}\right| \right\}^{-1/2} . \quad (18)$$

The above is a direct and simple derivation of the intensity of disturbances (compared with the method of stationary phase approximation (Lighthill, 1960)), although the derived information is restricted. (Equation (18) shows only the magnitude of velocity, and components or the phase of the velocity are not obtained.)

b) Application to an Isothermal Atmosphere

To apply Whitham's method to an atmosphere we must calculate $\det\, \partial^2\omega/\partial k_i \partial k_j$ explicitly as a function of ω. Considering that in an isothermal atmosphere the dispersion relation, $G(\underset{\sim}{k}, \omega) = o$, is given by equation (2) and also using equations (5) to (7), after rather lengthy calculations we have the following relation

$$\det\left|\frac{\partial^2\omega}{\partial k_i \partial k_j}\right| = \frac{C^2}{8} \; \frac{(\omega^2 - \omega_2^2 \cos^2\theta)^2}{\omega^3(\omega^2 - \omega_1^2)} \; U^2 \frac{\partial U^2}{\partial \omega^2} , \; (\omega \gtrless 0). \quad (19)$$

With the use of equation (19), the amplitude of $\rho_o^{\frac{1}{2}} u(\underset{\sim}{r}, t)$ given by equation (18) can be written as

$$\left|\rho_o^{\frac{1}{2}}\; u(\underset{\sim}{r},t)\right| = \left(\frac{2\pi}{t}\right)^{3/2} \left(\frac{8}{C^2}\right)^{\frac{1}{2}} \left|F(\underset{\sim}{k})\right| \left[\frac{\omega^3\;(\omega^2 - \omega_1^2)}{(\omega^2 - \omega_2^2\cos^2\theta)^2} \; \frac{1}{U^2 \partial U^2/\partial\omega^2}\right]^{\frac{1}{2}} . \quad (20)$$

For an isotropic impulse $F(\underset{\sim}{k})$ is constant and given by equation (12). Consequently, the kinetic energy density $E_K\,(\underset{\sim}{r}, t)$, associated with a wave mode is

$$E_K'(\underset{\sim}{r},t) = \frac{1}{4}\left|\rho_o^{\frac{1}{2}}\, U(\underset{\sim}{r},t)\right|^2$$

$$= \frac{4}{(2\pi)^3\, C^2\, t^3}\, |A|^2\, \frac{\omega^3(\omega^2-\omega_1^2)}{(\omega^2-\omega_2^2\cos^2\theta)^2}\, \frac{1}{U^2\partial U^2/\partial\omega^2}, \quad (\omega > 0) \tag{21}$$

(cf. equation (36) in the Appendix) where ω is the frequency of the waves at the position $\underset{\sim}{r}$ and the time t and is given by equation (8) as discussed in Section III.

Here it should be remembered that there are three cases where equation (8) has zero, one and two real positive solutions of ω for a given $\underset{\sim}{r}$ and t (see Figures 1 and 2). These three cases respectively correspond to the three cases; i) where the front of acoustic waves has not reached the observation point ($\underset{\sim}{r}$, t) yet; ii) where the front of acoustic waves has passed the observation point, but the front of gravity waves has not reached there yet; and iii) where both fronts have passed. Equation (21) is valid for each wave mode. Thus, in case iii), the total kinetic energy density, at $\underset{\sim}{r}$ and t is given by summing equation (21) for the acoustic wave and equation (21) for the gravity wave (cf. equation (36) in the Appendix).

Finally, it should be mentioned that the attenuation of intensity like t^{-3} or r^{-3} (r^{-n} in the n-dimensional space) along any radius vector is characteristic of impulses propagated three dimensionally outward in a dispersive medium. An impulse contains a range of frequencies, so disturbances propagate as a volume and their volume increases as r^3.

V. NUMERICAL RESULTS FOR INTENSITY DISTRIBUTION IN WAKES

Equation (21) can be rewritten as

$$E_K(\underset{\sim}{r}, t) = \frac{4}{(2\pi)^3 t^3}\, |A|^2 \cdot \frac{(\gamma^{-\frac{1}{2}}\omega_1)^3}{C_i^{\,6}}\, F(\omega,\theta;\gamma), \tag{22}$$

where

$$F(\omega,\theta;\gamma) = \frac{\omega^3(\omega^2-\omega_1^2)}{\omega_1^3(\omega^2-\omega_2^2\cos^2\theta)^2} \; \frac{\gamma^{-\frac{1}{2}} C_i^{\;4}}{U^2\partial U^2/\partial\omega^2} \quad , \tag{23}$$

and C_i denotes the isothermal sound velocity. The coefficient of F in equation (22) is independent of both the ratio of specific heats, γ, and position in the $\underset{\sim}{r}$-space. The space and γ dependencies of $E_K(\underset{\sim}{r}, t)$ are represented by the non-dimensional quantity, $F(\omega,\theta;\gamma)$, defined by equation (23). We will calculate $F(\omega,\theta;\gamma)$ due to acoustic waves and gravity waves, separately.

a) Behind the Front of Acoustic Waves

As discussed in Section III, the distribution of ω in the $\underset{\sim}{r}$-space is a function of r/r_c, θ and γ. So, F is also a function of r/r_c, θ and γ. The calculated distribution of $F(\omega, \theta; \gamma)$ in the $\underset{\sim}{r}$-space is shown in Figures 3(a), 3(b) and 3(c) for cases of γ = 1, 4/3 and 5/3, respectively. Here we mention that we do not calculate the intensity of disturbances near the front because near the front disturbances do not break down into wave trains yet and the approximation used to obtain equation (21) is not applicable. That is, a point in the $\underset{\sim}{k}$-space does not correspond to a point in the $\underset{\sim}{r}$-space exactly and energy within a given volume in the $\underset{\sim}{k}$-space is not conserved.

Figure 3(a) shows that intensity of disturbances in the wake decreases monotonically and spherically after the passage of the front in a convectively neutral atmosphere (γ = 1). This is natural because an isotropic impulse acts in an atmosphere having no particular direction (convective neutrality). However, figures 3(b) and 3(c) show that after the passage of the front radiation density is very intense above and below the source if the degree of stability of the atmosphere increases (i.e. if the value of γ increases). This result is suggested by the following two facts. First, an impulse contains all frequency components. Second, the results of paper I show that the atmosphere can respond strongly to a time-harmonic disturbance whose frequency is near the upper characteristic frequency, ω_1, in a sharply restricted region above and below the source, if the atmosphere is sufficiently stable to convection (see Figure 1(a) and Figure 3 in paper I).

This striking result shown in Figure 3(b) and 3(c) (radiation density is very intense sharply above the source) may be of interest here, because this may suggest a reason of formation of spicule structures in the solar upper chromosphere. That is, vertical focussing is a possible result of stratification alone.

b) Behind the Front of Gravity Waves

The distribution of $F(\omega, \theta; \gamma)$ behind the front of gravity waves are calculated for the case of $\gamma = 5/3$, and added in Figure 2. On the front, $\partial U^2/\partial\omega^2$ is zero because the front is defined as the envelope of disturbances of all frequency components. So, if equation (23) is applied literally, intensity becomes infinite on the front. Equation (23) is, however, invalid near the front for the same reason mentioned in the previous sub-section for the case of the front of acoustic waves.

Here we should mention that intensity shown in Figure 2 comes mainly from waves radiated near the critical direction θ_c ($\cos^2\theta_c = \omega^2/\omega_2^2$). As mentioned before (Section II or paper I), the wave number of waves radiating energy in the critical direction is infinite. Thus, a careful discussion whether intensities of gravity waves shown in Figure 2 can be radiated to a great distance from the source in actual physical situations may be necessary.

In conclusion, the author would like to express his sincere thanks to Dr. Edward A. Spiegel for kind advice and discussions, and for reading the manuscript. A part of this work was made when the author attended the Summer School of Geophysical Fluid Dynamics held at Woods Hole Oceanographic Institution, 1964. The author would like to thank Prof. Willem V. R. Malkus and other participants for discussions. He would like to thank the National Academy of Sciences for its support and Dr. R. Jastrow for extending to him the hospitality of the Goddard Institute for Space Studies.

APPENDIX: RESPONSE TO A POINT IMPULSE ACTING IN A PARTICULAR DIRECTION

a) The Asymptotic Solution of the Inhomogeneous Wave Equation

As done in this paper, the response of the atmosphere to an impulse can be studied as an initial value problem. However, the same problem can be also studied by solving the inhomogeneous wave equation. The later method will be simpler in the case where the impulse is a directed point source. Let us assume an external force, f $(\underset{\sim}{x}, t)$, in an unbounded and isothermal atmosphere stratified under a constant gravity. Then, the inhomogeneous wave equation represented by the pressure variation, p_1, is (see equations (5) and (6) in 1)

$$\left[- \frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2} (c^2\nabla^2 - \omega_1^2) + c^2\, \omega_2^2 \nabla_1^2 \right] (p_1/\rho_o^{\frac{1}{2}}) = S \, , \qquad (24)$$

where

$$S(\underset{\sim}{x}, t) = - \rho_o^{\frac{1}{2}} c^2 \frac{\partial^2}{\partial t^2} \left[\frac{\partial}{\partial z} + \omega_2^2/g \right] f_z(\underset{\sim}{x}, t)$$

$$- \rho_o^{\frac{1}{2}} c^2 \left[\frac{\partial^2}{\partial t^2} + \omega_2^2 \right] \left(\frac{\partial}{\partial x} f_x (\underset{\sim}{x}, t) + \frac{\partial}{\partial y} f_y(\underset{\sim}{x}, t) \right) . \qquad (25)$$

We solve equation (24) under the condition that dependent variable, $p_1/\rho_o^{\frac{1}{2}}$, is everywhere zero until an initial instant, t = o, and at the time t = o the force, $\underset{\sim}{f}$, acts instantly.

We will introduce a four dimensional space $(\underset{\sim}{r}, ct)$, and a four dimensional wave vector space $(\underset{\sim}{k}, \omega/c)$. Then, using the approximation of stationary phase as $t \to \infty$ and $r' \to \infty$ (where r' is the distance between the observation point, $\underset{\sim}{r}$, and the source region $\underset{\sim}{x}$; hereafter we take the origin in the source region and do not distinguish r' and r because $r' \to \infty$), the asymptotic solution of equation (24) is (Lighthill, 1960)

$$p_1/\rho_o^{\frac{1}{2}}(\underset{\sim}{r}, t) \cong \Sigma \frac{(2\pi)^{5/2}}{\hat{r}^{3/2}} \frac{C_o \Phi(\underset{\sim}{k}, \omega/c) \exp[i(-\underset{\sim}{k}\cdot\underset{\sim}{r} + \omega t)]}{|\Box G| \cdot |K|^{1/2}} , \quad (26)$$

where $\hat{r}$ is the four dimensional distance, $(r^2 + c^2 t^2)^{1/2}$, and where the sum Σ is over all points $(\omega \gtrless o)$ on the hyper-surface, $G(\underset{\sim}{k}, \omega/c) = 0$ (see equation (2)), whose normal is anti-parallel to the four dimensional observation point $(\underset{\sim}{r}, ct)$; that is, over all points where

$$\underset{\sim}{r} : ct = \nabla G(\underset{\sim}{k}, \omega/c) : -\frac{\partial}{\partial(\omega/c)} G(\underset{\sim}{k}, \omega/c) . \quad (27)$$

(cf. equations (1) and (8)). Moreover, in equation (26), $\Phi(\underset{\sim}{k}, \omega/c)$ is a Fourier component of $S(\underset{\sim}{x}, t)$;

$$\Phi(\underset{\sim}{k}, \omega/c) = \frac{1}{(2\pi)^4} \int S(\underset{\sim}{x}, t) \exp[-i(-\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega t)]\, d\underset{\sim}{x}\, d(ct), \quad (28)$$

C_o is a phase factor of modulus 1; $\Box G$ is the gradient of G in Cartesian four space; and K is the Gaussian curvature of the hypersurface - each element of the matrix of products of first derivatives of G being multiplied into the corresponding co-factor of the matrix of second derivatives, and divided by G^5.

b) Response in Kinetic Energy

As a measure of the response we adopt kinetic energy dentisy, $E_K(\underset{\sim}{r}, t)$. Kinetic energy density is related to pressure fluctuation by an equation (see equation (13) in paper I). Inserting equation (26) into the relation, after a lengthy calculation we have

$$E_K(\underset{\sim}{r}, t) = \frac{(2\pi)^5}{4t^3} \Sigma' \; \frac{1}{c^7} \; \frac{(\omega^2-\omega_1^2)^{1/2}}{(\omega^2-\omega_2^2)^{3/2} \; (\omega^2-\omega_2^2\cos^2\theta)^{1/2}}$$

$$\cdot \; |\phi|^2 \; \frac{1}{U\partial U^2/\partial\omega^2} \tag{29}$$

where the prime added to Σ means that the summation is performed only for points where $\omega > o$ of the points satisfying equation (28).

In deriving equation (29) we used

$$|\Box G|^2 = 4\, c^2\, \omega^4 \; \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{(\omega^2-\omega_2^2\cos^2\theta)} \left(1 + \frac{c^2}{U^2}\right) \tag{30}$$

and

$$K = \frac{32}{|\Box G|^6}\, c^9\, \omega^6 \left[\frac{\omega^2(\omega^2-\omega_1^2)^3(\omega^2-\omega_2^2)^5}{(\omega^2-\omega_2^2\cos^2\theta)}\right]^{1/2} \; \frac{1}{U^3}\,\frac{\partial U^2}{\partial\omega^2} \tag{31}$$

which are obtained from the definition of G and K and the relation between $(\underset{\sim}{k}, \omega/c)$ and $(\underset{\sim}{r}, ct)$ shown in equation (27).

c) Response to Three Kinds of Impulses

To calculate $|\phi|^2$ in equation (29) explicitly, we must specify the external force, $\underset{\sim}{f}$. We study about the following three kinds of sources, separately; that is, point impulses acting to x, y and z directions at the origin, $\underset{\sim}{r} = 0$, and at the time, $t = 0$:

$$f_i = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \delta(\underset{\sim}{x})\delta(t) \begin{Bmatrix} \delta_{ix} \\ \delta_{iy} \\ \delta_{iz} \end{Bmatrix} \quad \begin{matrix} \text{case (a)} \\ \text{case (b)} \\ \text{case (c)} \end{matrix} \quad . \tag{32}$$

In case (a), for example, inserting equation (32) into equation (25) and performing integrations with respect to $\underset{\sim}{x}$ and t to obtain Φ defined by equation (28), then we have Φ expressed by $\underset{\sim}{k}$ and ω. We eliminate $\underset{\sim}{k}$ from this expression for Φ with the help of equation (27) and square the expression. Then, we have

$$|\Phi_1|^2 = \frac{A_1^2}{(2\pi)^8} \rho_{oo}^{-1} c^4 \omega^4 \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{(\omega^2-\omega_2^2\cos^2\theta)} \sin^2\theta \cos^2\phi, \quad \text{case (a)} \tag{33}$$

where the suffix 1 of Φ denotes case (a), and where ρ_{oo} denotes the undisturbed density at the position of the impulse. Quite similarly, for cases (b) and (c), we have

$$|\Phi_2|^2 = \frac{A_2^2}{(2\pi)^8} \rho_{oo}^{-1} c^4 \omega^4 \frac{(\omega^2 - \omega_1^2)(\omega^2-\omega_2^2)}{(\omega^2-\omega_2^2\cos^2\theta} \sin^2\theta \sin^2\phi, \quad \text{case (b)} \tag{34}$$

and

$$|\Phi_3|^2 = \frac{A_3^2}{(2\pi)^8} \rho_{oo}^{-1} c^4 \omega^4 [(\omega_1^2-\omega_2^2) + \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{(\omega^2-\omega_2^2\cos^2\theta)} \cos^2\theta], \quad \text{case (c)} \tag{35}$$

If equations (33), (34) or (35) are inserted into equation (29), we have expressions for distribution of kinetic energy in the $\underset{\sim}{r}$-space resulting from impulses directed in the x, y, or z directions, respectively.

d) A Relation to the Responce to an Isotropic Impulse

To show a relation between the response of the atmosphere to an isotropic impulse and to the directed impulses, we assume three atmospheres of the same kind, and assume that in each atmosphere an impulse acts in the x, y and z-direction with a same amplitude ($A_o \equiv A_1 = A_2 = A_3$) at the same point and time, respectively. We sum the kinetic energy density at the corresponding points in these three systems at the same time. That is, we sum equations (33) to (35) and insert the sum into equation (29) to obtain

$$E_K(\underset{\sim}{r},t) = \frac{1}{4(2\pi)^3 t^3} \frac{(\rho_{oo}^{-1} A_o)^2}{C^2} \Sigma' \frac{\omega^3(\omega^2-\omega_1^2)}{(\omega^2-\omega_2^2 \cos^2\theta)^2} \frac{1}{U^2\partial U^2/\partial\omega^2} . \quad (36)$$

As anticipated, the intensity distribution of the kinetic energy density in the $\underset{\sim}{r}$-space shown in equation (36) is quite same with that for an isotropic point impulse shown in equation (21).

REFERENCES

Lamb, H. 1908, Proc. London Math. Soc., 7, 122.

Lighthill, M. J. 1960, Phil. Trans. R. Soc. London, A, 252, 397.

Moore, D. W., and Spiegel, E. A. 1964, Ap. J., 139, 48.

Kato, S. 1966, to be published (paper I).

Whitham, G. B. 1961, Comm. Pure Applied Math., 14, 675.

Figure Captions

Fig. 1. Frequency distribution in wave motions in the wake of acoustic waves. r_c denotes the position of the front of disturbances. Three cases (γ = 1, 4/3 and 5/3) are drawn on the same map.

Fig. 2. Frequency distribution in wave motions in the wake of gravity waves. The envelopes denote the front of disturbances. The left hand side is for $\gamma = 4/3$; the right hand side is for $\gamma = 5/3$. The broken lines denote intensity of disturbances radiated by an isotropic point impulse, and the numerical values represent values of $F(\omega, \theta; \gamma)$. The circular arcs at both sides of the figure denote the position of the front of acoustic waves.

Fig. 3. Distribution of kinetic energy density in the wake of acoustic waves radiated by an isotropic point impulse. The numerical values added to iso-intensity curves represent values of $F(\omega, \theta; \gamma)$ defined by equation (23). r_c denotes the position of the front of disturbances. Figures 3(a), 3(b) and 3(c) correspond to the cases γ = 1, 4/3 and 5/3, respectively.