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TRANSIENT RESPONSE FROM THE LYAPUNOV STABILITY EQUATION

by William G. Vogt

George C. Marshall Space Flight Center
Huntsville, Ala.

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TRANSIENT RESPONSE FROM THE LYAPUNOV STABILITY EQUATION

By

William G. Vogt

SUMMARY

For a certain class of linear and nonlinear asymptotically stable systems and a certain class of Lyapunov functions $v(\underline{x})$, estimates of transient response,

$$\hat{\gamma} = \max_{\underline{x}} \left[\frac{-\dot{v}(\underline{x})}{v(\underline{x})} \right], \qquad \check{\gamma} = \min_{\underline{x}} \left[\frac{-\dot{v}(\underline{x})}{v(\underline{x})} \right]$$

are related to the real parts of the characteristic values of the system matrix, $\underline{A}$, of the linear approximation system. It is shown that

$$\hat{\gamma} \geqq -2 \min_{\underline{i}} \left[\operatorname{Re} \lambda_i(\underline{A}) \right]$$

$$\check{\gamma} \leqq -2 \max_{\underline{i}} \left[\operatorname{Re} \lambda_i(\underline{A}) \right]$$

and a formula is given to choose $v(\underline{x})$ such that $\hat{\gamma}$, $\check{\gamma}$ approach these limiting values as closely as desired.

INTRODUCTION

In stability studies of control systems describable by a vector differential equation of the type

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad \underline{f}(\underline{0}) = \underline{0} \tag{1}$$

where $\underline{x}$ and $\underline{f}(\underline{x})$ are n-vectors and $\underline{f}(\underline{x})$ satisfies certain conditions [1] assuring the existence, uniqueness, and continuity of solutions to equation (1), a common procedure is to obtain a Lyapunov function $v(\underline{x})$ which, along with its derivative $\dot{v}(\underline{x})$, satisfies conditions sufficient to assure stability or instability of the null solution $\underline{x} = \underline{0}$ of equation (1). More specifically, for most control systems, asymptotic stability is desired. A sufficient condition for asymptotic stability of the null solution (NS) of equation (1) is the existence of a Lyapunov function $v(\underline{x})$ which, along with its derivative $\dot{v}(x)$, satisfies the following conditions in some neighborhood of the origin [2-4]:

$$\begin{aligned} &(a)\quad v(\underline{0}) = 0 \\ &(b)\quad v(\underline{x}) > 0, \quad \underline{x} \neq \underline{0} \\ &(c)\quad \underline{\text{grad}}\, v(\underline{x}) \text{ is continuous} \\ &(d)\quad \dot{v}(\underline{x}) = [\underline{\text{grad}}\, v(\underline{x})]' \; \underline{f}(\underline{x}) < 0, \quad \underline{x} \neq \underline{0}. \end{aligned} \tag{2}$$

If such a $v(\underline{x})$ exists, the asymptotic stability of the NS of equation (1) is assured. For such $v(\underline{x})$, it has been shown that an estimate of the transient response of equation (1) can be obtained [2]. This estimate consists of determining two constants $\check{\gamma}$, $\hat{\gamma}$ by the following formulas

$$\check{\gamma} = \min_{\underline{x}} \left[\frac{-\dot{v}(\underline{x})}{v(\underline{x})} \right]; \qquad \hat{\gamma} = \max_{\underline{x}} \left[\frac{-\dot{v}(\underline{x})}{v(\underline{x})} \right] \tag{3}$$

in some region $0 < |\underline{x}| < r$, $r > 0$ where the conditions of equation (2) are satisfied. Denoting by $\underline{\theta}(t; \underline{x}_0)$ a solution of equation (1) starting from x_0 at time t=0, the time behavior of $v(\underline{\theta}(t; \underline{x}_0))$ must satisfy [2]

$$v(\underline{x}_0) \exp[-\hat{\gamma}t] \leqq v(\underline{\theta}(t; \underline{x}_0)) \leqq v(\underline{x}_0) \exp[-\check{\gamma}t]. \tag{4}$$

Thus $\hat{\gamma}$ and $\check{\gamma}$ correspond in some sense to the largest and smallest time constants of the system described by equation (1), but the exact relationship is not clearly understood.

The significance of $\hat{\gamma}$ and $\check{\gamma}$ is obscured largely because the Lyapunov functions are not unique. Different Lyapunov functions yield different values for $\hat{\gamma}$ and $\check{\gamma}$. These estimates depend on the system of equations and the Lyapunov function chosen.

The object of this paper is to relate the estimates in question to a certain class of equation (1) and a certain class of Lyapunov functions satisfying equation (2).

FORMULATION

System

The following investigation is limited to that class of systems of equation (1) for which the right side $\underline{f}(\underline{x})$ satisfies the following conditions

$$\underline{f}(\underline{x}) = \underline{A}\,\underline{x} + \underline{g}\,(\underline{x}) \tag{5}$$

$$\text{Re}\,[\lambda_i(\underline{A})] < 0 \qquad i=1,2,\ldots n. \tag{6}$$

In words, $\underline{f}(\underline{x})$ is limited to that class of vector functions, each component of which is expressible in a convergent power series expansion in some neighborhood of the origin and the linear approximation of which has only negative real part characteristic values. If $\underline{A}$ is a matrix that satisfies equation (6) it is called a stability matrix.

The ordinary linear constant coefficient differential equation which approximates equation (5) in some sufficiently small neighborhood of the NS of equation (5) is given by

$$\dot{\underline{x}} = \underline{A}\,\underline{x} \tag{7}$$

and the solutions are designated by $\underline{\phi}\,(t; \underline{x}_0)$.

Lyapunov Functions

Attention is restricted to Lyapunov functions of the form

$$v(\underline{x}) = \underline{x}'\,\underline{V}\,\underline{x} + v_1(\underline{x}) \tag{8}$$

where $\underline{V}$ is a real, symmetric, positive-definite (RSPD) matrix and $v_1(\underline{x})$ contains only terms of degree greater than two in the variables $x_1, x_2, \ldots, x_n$.

Thus, the first approximation to $v(\underline{x})$ is given by

$$V(\underline{x}) = \underline{x}'\,\underline{V}\,\underline{x}\,. \tag{9}$$

The derivative of $v(\underline{x})$ along solutions to equation (1) is given by

$$\dot{v}(\underline{x}) = \underline{x}'\,(\underline{A}'\,\underline{V} + \underline{V}\,\underline{A})\,\underline{x} + \dot{v}_1(\underline{x}) \tag{10}$$

where $\dot{v}_1(\underline{x})$ consists only of terms of degree greater than two in the variables $x_1, x_2, \ldots, x_n$. The first approximation of $\dot{v}(\underline{x})$ is therefore given by

$$\dot{V}(\underline{x}) = \underline{x}' (\underline{A}' \underline{V} + \underline{V} \underline{A}) \underline{x} \quad . \tag{11}$$

For $v(\underline{x})$, $\dot{v}(\underline{x})$ to satisfy the requirements of equation (2) in some neighborhood of the origin, $\dot{V}(\underline{x})$ must be negative definite, thus implying that

$$\underline{A}' \underline{V} + \underline{V} \underline{A} = -2\underline{R} \tag{12}$$

where $\underline{R}$ is an RSPD matrix.

Equation (12) is referred to as the Lyapunov stability equation. To illustrate its fundamental importance, a theorem is stated.

Theorem 1 [2]: $\underline{A}$ is a stability matrix if, and only if, $\underline{V}$, as a solution to equation (12), is unique and RSPD wherever $\underline{R}$ is RSPD.

Thus, instead of choosing an RSPD $\underline{V}$ in equation (9) such that the resulting $\underline{R}$ in equation (12) is RSPD, choosing any RSPD $\underline{R}$ and solving for $\underline{V}$ in equation (12) settles the stability question of the NS of equation (7). If there is asymptotic stability of the NS of equation (7), $\underline{V}$ provides an appropriate Lyapunov function $V(\underline{x})$ in equation (9) which meets all the conditions of equation (2). Thus, in the system formulation, equation (6) can be replaced by the condition that $\underline{V}$, as a solution to equation (12), is unique and RSPD wherever $\underline{R}$ is RSPD.

Note that the conditions of equation (5) and (6) are sufficient to assure the existence of Lyapunov functions of the type required by equation (8) [3]. With this formulation, the problem originally posed can be reduced to a simpler problem.

REDUCTION OF THE PROBLEM

The original problem was to determine how the constants $\hat{\gamma}$ and $\check{\gamma}$ defined in equation (3) are related to the systems and the Lyapunov functions chosen. From the definition

$$\gamma(\underline{x}) = \frac{-\dot{v}(\underline{x})}{v(\underline{x})} \tag{13}$$

it is clear that for $0 < |\underline{x}| < r$

$$\hat{\gamma} = \max_{\underline{x}} [\gamma(\underline{x})] \qquad \check{\gamma} = \min_{\underline{x}} [\gamma(\underline{x})] \quad . \tag{14}$$

Substitution of equations (8) and (10) into equation (13) yields

$$\gamma(\underline{x}) = - \frac{\dot{V}(\underline{x}) + \dot{v}_1(\underline{x})}{V(\underline{x}) + v_1(\underline{x})} = 2\Gamma(\underline{x}) \frac{1 + \dfrac{\dot{v}_1(\underline{x})}{\dot{V}(\underline{x})}}{1 + \dfrac{v_1(\underline{x})}{V(\underline{x})}} \tag{15}$$

where

$$\Gamma(\underline{x}) = -\frac{1}{2} \frac{\dot{V}(\underline{x})}{V(\underline{x})} = \frac{\underline{x}' \underline{R} \underline{x}}{\underline{x}' \underline{V} \underline{x}} \quad . \tag{16}$$

From equation (15) and the nature of the Lyapunov function and its derivative in equations (8) and (10), it is clear that for all $0 < |x| < r_1 < r$ there exists an $\epsilon(r_1)$ such that

$$2\Gamma(x) \ (1-2\epsilon) \leqq \gamma(\underline{x}) \leqq 2\Gamma(\underline{x}) \ (1+2\epsilon) \tag{17}$$

and further that

$$\lim \epsilon(r_1) = 0 \ \text{ as } r_1 \to 0. \tag{18}$$

Thus, for $0 < |\underline{x}| < r$

$$\hat{\gamma} \geqq 2\hat{\Gamma} \ ; \ \check{\gamma} \leqq 2\check{\Gamma} \tag{19}$$

where

$$\hat{\Gamma} = \max_{\underline{x}} \left[\frac{\underline{x}' \underline{R} \underline{x}}{\underline{x}' \underline{V} \underline{x}} \right] ; \ \check{\Gamma} = \min_{\underline{x}} \left[\frac{\underline{x}' \underline{R} \underline{x}}{\underline{x}' \underline{V} \underline{x}} \right] . \tag{20}$$

Clearly, for the class of systems and the class of Lyapunov functions considered, $\hat{\Gamma}$ and $\check{\Gamma}$ give the limiting values of $\hat{\gamma}$ and $\check{\gamma}$. The original problem is thereby reduced to the consideration of the following:

Problem: Given a stability matrix, $\underline{A}$, what are the interrelationships between $\underline{A}$, $\underline{R}$, and $\underline{V}$ satisfying equation (12) in terms of $\hat{\Gamma}$ and $\check{\Gamma}$ defined by equation (20)?

The answer to this reduced problem, while still incomplete, is scattered throughout many publications [5-7], often in forms which are not suitable to the problem described here. The treatment below is an attempt to organize this material to give an answer to this problem and at the same time to introduce some new results.

Note that equation (4) is not satisfied with $2\hat{\Gamma}$ and $2\check{\Gamma}$ substituted for $\hat{\gamma}$ and $\check{\gamma}$. The relations of equation (19) indicate that the fastest (slowest) system "transient" in terms of equation (4) is as fast (slow) or faster (slower) than that characterized by exp $[-2\hat{\Gamma}\, t]$ (exp $[-2\check{\Gamma}\, t]$) and further that near the origin the fastest and slowest response are characterized by "time constants" $1/2\hat{\Gamma}$ and $1/2\check{\Gamma}$ respectively.

PENCILS OF QUADRATIC FORMS [5]

Definitions and Theorems

The real form

$$\underline{x}'\,\underline{R}\,\underline{x} - \underline{x}'\,\underline{V}\,\underline{x} \quad (\underline{R} = \underline{R}',\ \underline{V} = \underline{V}') \tag{21}$$

is called a pencil of quadratic forms. If $\underline{x}'\,\underline{V}\,\underline{x}$ is positive definite, the pencil is called regular. The equation

$$|\underline{R} - \lambda \underline{V}| = 0 \tag{22}$$

is called the characteristic equation of the pencil. If λ_i satisfies equation (22), it is called a characteristic value of the pencil. If there exists a real vector, $\underline{z}_i$, such that

$$\underline{R}\ \underline{z}_i = \lambda_i\ \underline{V}\,\underline{z}_i\,, \tag{23}$$

$\underline{z}_i$ is called a characteristic vector belonging to the pencil. The following theorem holds.

Theorem 2: The characteristic equation (22) of the regular pencil of forms, equation (21), always has n real roots, λ_i, with corresponding real characteristic vectors satisfying equation (23). The $\underline{z}_i$ can be chosen such that

$$\underline{z}_i' \, \underline{V} \, \underline{z}_k = \delta_{ik} \quad (i,k = 1, 2, \ldots, n). \tag{24}$$

Since $|\underline{V}| \neq 0$, the matrix $\underline{R}\,\underline{V}^{-1}$ has the same characteristic values as the pencil, equation (21), does. Hence, these characteristic values will be designated by $\lambda_i(\underline{R}\,\underline{V}^{-1})$, $i = 1, 2, \ldots, n$. The matrix $\underline{R}\,\underline{V}^{-1}$ has n real roots and is similar to a diagonal matrix [5].

Let $\underline{Z}$ designate the matrix composed of the columns $\underline{z}_i$ which satisfy equation (24). Then

$$\underline{Z}' \, \underline{V} \, \underline{Z} = \underline{E} \quad \text{(Identity Matrix)} \tag{25}$$

$$\underline{Z}' \, \underline{R} \, \underline{Z} = \operatorname{diag}\{\lambda_i(\underline{R}\,\underline{V}^{-1})\} \tag{26}$$

Extremal Properties

Order the characteristic values of $\underline{R}\,\underline{V}^{-1}$ as

$$\min_i [\lambda_i(\underline{R}\,\underline{V}^{-1})] = \lambda_1(\underline{R}\,\underline{V}^{-1}) \leqq \ldots \leqq \lambda_n(\underline{R}\,\underline{V}^{-1}) = \max_i [\lambda_i(\underline{R}\,\underline{V}^{-1})]$$

and let the corresponding characteristic vectors satisfying equation (24) be $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_n$. With this ordering, the following holds [5].

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = \min_{\underline{x}} \left[\frac{\underline{x}' \, \underline{R} \, \underline{x}}{\underline{x}' \, \underline{V} \, \underline{x}} \right] = \frac{\underline{z}_1' \, \underline{R} \, \underline{z}_1}{\underline{z}_1' \, \underline{V} \, \underline{z}_1} \tag{27}$$

$$\lambda_n(\underline{R}\,\underline{V}^{-1}) = \max_{\underline{x}} \left[\frac{\underline{x}' \, \underline{R} \, \underline{x}}{\underline{x}' \, \underline{V} \, \underline{x}} \right] = \frac{\underline{z}_n' \, \underline{R} \, \underline{z}_n}{\underline{z}_n' \, \underline{V} \, \underline{z}_n} \tag{28}$$

Relations to $\hat{\Gamma}$ and $\check{\Gamma}$

From the definitions of $\hat{\Gamma}$ and $\check{\Gamma}$ given by equation (20) it is clear that

$$\hat{\Gamma} = \lambda_n (\underline{R}\, \underline{V}^{-1}) \tag{29}$$

$$\check{\Gamma} = \lambda_1 (\underline{R}\, \underline{V}^{-1}). \tag{30}$$

Thus the problem has been reduced to finding the minimum and maximum characteristic values of the matrix $\underline{R}\, \underline{V}^{-1}$ where $\underline{R}$ and $\underline{V}$ are related to $\underline{A}$ by equation (12).

<u>Remark</u>: In this development, it was not assumed that $\underline{R}$ was an RSPD matrix, but merely that $\underline{V}$ was RSPD. Thus if $\underline{R}$ is chosen RSPSD (real, symmetric, positive-semidefinite) such that $\underline{V}$ in equation (12) is RSPD, the same relations hold.

RELATIONS BETWEEN $\lambda_i (\underline{R}\underline{V})^{-1}$ AND $\lambda_j(\underline{A})$

Notation

Adopt the notation

$$\lambda_i(\underline{A}) = \alpha_i(\underline{A}) + \underline{i}\ \beta_i(\underline{A}) \quad (\underline{i} = \sqrt{-1}). \tag{31}$$

Note that $\alpha_i < 0$ since $\underline{A}$ is a stability matrix. Order the eigenvalues of $\underline{A}$ as follows:

$$\max_i [\alpha_i(\underline{A})] = \alpha_1(\underline{A}) \geqq \ldots \geqq \alpha_n(\underline{A}) = \min_i [\alpha_i(\underline{A})]. \tag{32}$$

For all $\lambda_j(\underline{A})$ having α_i as a real part, some of these may have simple elementary divisors and some may not. In the case that there is at least one non-simple elementary divisor associated with the roots having $\alpha_1(\underline{A})$ $(\alpha_n(\underline{A}))$ as a real part, designate one of these as $\lambda_1(\underline{A})$ $(\lambda_n(\underline{A}))$. Then $\alpha_1(\underline{A})$ $(\alpha_n(\underline{A}))$ either corresponds to characteristic values having simple or non-simple elementary divisors depending on whether $\underline{A}$ has or has not at least one non-simple elementary divisor corresponding to roots with real part $\alpha_1(\alpha_n)$.

Complex-Valued Vectors

For convenience in the later development, the following theorem is stated.

<u>Theorem 3:</u> If $\underline{H}$ is any real symmetric matrix and $\underline{w}$ is any complex-valued vector, a real vector $\underline{x}$ can always be found such that

$$\underline{x}'\,\underline{H}\,\underline{x} = \underline{w}^*\,\underline{H}\,\underline{w} \quad (\underline{w}^* = \overline{\underline{w}}') \; . \tag{33}$$

Proof: Let

$$D(\underline{x},\underline{w}) = \underline{x}'\,\underline{H}\,\underline{x} - \underline{w}^*\,\underline{H}\,\underline{w} \tag{34}$$

and $\underline{w} = \underline{u} + \underline{i}\,\underline{v}$, $\underline{x} = \underline{u} + b\,\underline{v}$ where b is a real constant which is to be selected such that $D(\underline{x},\underline{w}) = 0$. Note that $\underline{w}^*\,\underline{H}\,\underline{w}$ is a hermitian form and thereby takes on only real values.

$$\begin{aligned} D(\underline{x},\underline{w}) &= (\underline{u} + b\,\underline{v})'\,\underline{H}(\underline{u} + \underline{b}\,\underline{v}) - (\underline{u} - \underline{i}\,\underline{v})'\,\underline{H}(\underline{u} + \underline{i}\,\underline{v}) \\ &= \underline{u}'\,\underline{H}\,\underline{u} + 2b\,\underline{v}'\,\underline{H}\,\underline{u} + b^2\,\underline{v}'\,H\,\underline{v} - \underline{u}'\,\underline{H}\,\underline{u} - \underline{v}'\,\underline{H}\,\underline{v} \\ &= b^2\,\underline{v}'\,\underline{H}\,\underline{v} + 2\,\underline{v}'\,\underline{H}\,\underline{u}\,b - \underline{v}'\,\underline{H}\,\underline{v} \end{aligned}$$

If b is selected as follows, $D(\underline{x},\underline{w})$ will be zero.

$$b = \begin{cases} 0 \text{ if } \underline{v}'\,\underline{H}\,\underline{v} = 0 \\ \dfrac{-\underline{v}'\,\underline{H}\,\underline{u} + \left[(\underline{v}'\,\underline{H}\,\underline{u})^2 + (\underline{v}'\,\underline{H}\,\underline{v})^2\right]^{\frac{1}{2}}}{\underline{v}'\,\underline{H}\,\underline{v}} \\ \qquad\qquad \text{if } \underline{v}'\,\underline{H}\,\underline{v} \neq 0 \end{cases} \tag{35}$$

Choosing b according to equation (35) makes $D(\underline{x},\underline{w}) = 0$. Thus $\underline{x} = \underline{u} + b\,\underline{v}$ is the real vector such that equation (33) holds.

Relations Between $\lambda_i(\underline{R}\underline{V})^{-1}$ And $\lambda_j(\underline{A})$

<u>Theorem 4:</u> Given a real stability matrix $\underline{A}$, and any RSPD or RSPSD $\underline{R}$ and RSPD $\underline{V}$ satisfying equation (12), the following inequalities hold:

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) \leqq -\alpha_1(\underline{A}) = -\operatorname{Re}\left[\lambda_1(\underline{A})\right] \tag{36}$$

$$\lambda_n(\underline{R}\,\underline{V}^{-1}) \geqq -\alpha_n(\underline{A}) = -\operatorname{Re}\,[\lambda_n(\underline{A})] \quad . \tag{37}$$

Proof: By equations (27) and (28), and Theorem 3, it is necessary only to show the existence of complex-valued vectors $\underline{w}_i$ such that

$$-\alpha_i = \frac{\underline{w}_i^* \,\underline{R}\,\underline{w}_i}{\underline{w}_i^* \,\underline{V}\,\underline{w}_i} \quad (i = 1, \ldots, n). \tag{38}$$

From equation (12)

$$\underline{w}_i^* \,(\underline{A}'\,\underline{V} + \underline{V}\,\underline{A})\,\underline{w}_i = -2\,\underline{w}_i^* \,\underline{R}\,\underline{w}_i \,. \tag{39}$$

Let $\underline{w}_i$ be a characteristic vector of $\underline{A}$; i.e. $\underline{A}\,\underline{w}_i = \lambda_i(\underline{A})\,\underline{w}_i$. Then equation (39) yields, since $\underline{A}' = \underline{A}^*$,

$$(\underline{A}\,\underline{w}_i)^* \,\underline{V}\,\underline{w}_i + \underline{w}_i^* \,\underline{V}\,(\underline{A}\,\underline{w}_i) = -2\,\underline{w}_i^* \,\underline{R}\,\underline{w}_i$$

$$(\bar{\lambda}_i + \lambda_i)\,\underline{w}_i^* \,\underline{V}\,\underline{w}_i = -2\,\underline{w}_i^* \,\underline{R}\,\underline{w}_i \,. \tag{40}$$

Since $\bar{\lambda}_i + \lambda_i = 2\,\alpha_i$, Theorem 3 shows that there exists a real vector $\underline{x}_i$ with

$$-\alpha_i = \frac{\underline{x}_i' \,\underline{R}\,\underline{x}_i}{\underline{x}_i' \,\underline{V}\,\underline{x}_i} \qquad i = 1, 2, \ldots, n. \tag{41}$$

Since equation (41) holds in particular for i = 1, n and since the $\underline{x}_i$ may not be the principal vectors $\underline{z}_i$, equations (36) and (37) must be true. Thus the theorem is true.

Can the equalities of equations (36) and (37) be made strict for some choices of $\underline{R}$ and $\underline{V}$? The answer to this question must await the results of the next section.

A TRANSFORMATION OF THE LYAPUNOV STABILITY EQUATION

If a conjunctive transformation by the complex matrix $\underline{T}$, $|\underline{T}| \neq 0$ is performed on equation (12) yielding

$$\underline{T}^* \; (\underline{A}' \, \underline{V} + \underline{V} \, \underline{A}) \; \underline{T} = -2 \, \underline{T}^* \; \underline{R} \, \underline{T}, \tag{42}$$

equation (42) may be rewritten as

$$(\underline{T}^{-1} \, \underline{A} \, \underline{T})^* \; \underline{T}^* \; \underline{V} \, \underline{T} + \underline{T}^* \; \underline{V} \, \underline{T} \; (\underline{T}^{-1} \, \underline{A} \, \mathrm{T}) = -2 \, \underline{T}^* \; \underline{R} \; \underline{T}$$

or

$$\underline{B}^* \; \underline{U} + \underline{U} \, \underline{B} = -2 \, \underline{Q}. \tag{43}$$

Where

$$\underline{B} = \underline{T}^{-1} \, \underline{A} \, \underline{T}, \quad \underline{U} = \underline{T}^* \; \underline{V} \, \underline{T}, \quad \underline{Q} = \underline{T}^* \; \underline{R} \, \underline{T},$$

it is clear that the definiteness properties of $\underline{U}$ and $\underline{Q}$ are the same as those of $\underline{V}$ and $\underline{R}$, respectively, and the elementary divisors of $\underline{B}$ are the same as those of $\underline{A}$.

In particular, let $\underline{T} = \underline{T}(\epsilon)$ transform $\underline{A}$ to its Jordan normal form

$$\underline{T}^{-1} \, \underline{A} \, \underline{T} = \mathrm{diag} \, \{\underline{J}_1, \ldots, \underline{J}_s\} = \underline{J}\,(\epsilon) \tag{44}$$

$$\underline{J}_i = \begin{bmatrix} \lambda_i(\underline{A}) & \epsilon & & \underline{0} \\ & \ddots & \ddots & \\ & & \ddots & \epsilon \\ \underline{0} & & & \lambda_i(\underline{A}) \end{bmatrix} = \lambda_i(\underline{A}) \; \underline{E}_i + \epsilon \, \underline{H}_i \tag{45}$$

where $\underline{E}_i$ and $\underline{H}_i$ are of dimension $p_i \times p_i$ corresponding to the degree of the elementary divisor of $\underline{A}$, $(\lambda_i - \lambda_i(\underline{A}))^{p_i}$.

In equation (43), choose $\underline{Q}$ to be $\underline{Q} = \mathrm{diag} \, \{\underline{Q}_1, \ldots, \underline{Q}_s\}$

where

$$\underline{Q}_i(\epsilon) = - \, \alpha_i(\underline{A}) \; \underline{E}_i + \left(\frac{\epsilon}{2}\right) (\underline{H}_i' + \underline{H}_i).$$

Thus, $\underline{U} = \underline{E}$ (the identity matrix).

For $\epsilon > 0$, but sufficiently small, it is clear that $\underline{Q}$ so defined is positive-definite. The characteristic values of the pencil $(\underline{Q} - \lambda\underline{E})$ are the roots of the equation

$$|\underline{Q} - \lambda\underline{E}| = 0 \tag{46}$$

which are the roots of the equivalent expression

$$\prod_{i=1}^{s} |\lambda\underline{E}_i - \underline{Q}_i| = 0. \tag{47}$$

By a theorem of Gershgorin [8], the roots of each of these factors of equation (47) must be within an ϵ distance of the corresponding $-\lambda_i(\underline{A})$; i.e.

$$-\alpha_i(\underline{A}) - \epsilon \leqq \lambda_{j_i}(\underline{Q}_i) \leqq -\alpha_i(\underline{A}) + \epsilon \quad (j_i = 1, 2, \ldots p_i). \tag{48}$$

In particular, for a $\underline{J}_i$ corresponding to a simple elementary divisor,

$$\lambda_{j_1}(\underline{Q}_i) = -\alpha_i(\underline{A}). \tag{49}$$

Since $\underline{R}\,\underline{V}^{-1}$ is similar to $\underline{Q}$, i.e.

$$\underline{R}\,\underline{V}^{-1} = \underline{T}^{*-1}\,\underline{Q}\,\underline{T}^{-1}\,\underline{T}\,\underline{T}^{*} = \underline{T}^{*-1}\,\underline{Q}\,\underline{T}^{*} \tag{50}$$

the $\lambda_j(\underline{R}\,\underline{V}^{-1})$ are the same as the $\lambda_j(\underline{Q})$. The foregoing, along with Theorem 4, has demonstrated the following theorem.

<u>Theorem 5</u>: Given a stability matrix $\underline{A}$, and an $\epsilon > 0$, it is always possible to find an $\underline{R}$ and $\underline{V}$ satisfying equation (12) such that

$$-\alpha_1 - \epsilon \leqq \lambda_1(\underline{R}\,\underline{V}^{-1}) \leqq -\alpha_1 \tag{51}$$

$$-\alpha_n \leqq \lambda_n(\underline{R}\,\underline{V}^{-1}) \leqq -\alpha_n + \epsilon. \tag{52}$$

In particular, choosing

$$\underline{V} = \frac{1}{2}\left(\underline{T}^{*-1}\,\underline{T}^{-1} + \underline{T}'^{-1}\,\overline{\underline{T}}^{-1}\right) \tag{53}$$

where $\underline{T}$ transforms $\underline{A}$ to $\underline{J}(\epsilon)$ as in equation (44), yields the required $\underline{R}$ and $\underline{V}$. Note that $\underline{R}$ and $\underline{V}$ depend on ϵ. Consideration of equation (49) yields the following corollary.

Corollary 1: Given a stability matrix $\underline{A}$, an RSPD $\underline{R}$ and $\underline{V}$ satisfying equation (12) can be found such that

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = -\alpha_1(\underline{A}) \tag{54}$$

$$[\lambda_n(\underline{R}\,\underline{V}^{-1}) = -\alpha_n(\underline{A})] \tag{55}$$

(a) if and (b) only if the elementary divisors of $\underline{A}$ corresponding to the $\lambda_i(\underline{A})$ having real part $\alpha_1(\underline{A})$ $[\alpha_n(\underline{A})]$ are all simple.

Proof: If they are all simple, the implication (equation (54)) follows directly from equation (49), establishing (a) of the corollary. To establish (b), it is only necessary to show that if one $\lambda_i(\underline{A})$ with Re $[\lambda_i(\underline{A})] = \alpha_1(\underline{A})$ has a non-simple elementary divisor, there is at least one vector, say $\underline{x}$, such that

$$\frac{\underline{x}'\,\underline{R}\,\underline{x}}{\underline{x}'\,\underline{V}\,\underline{x}} < -\alpha_1(\underline{A}) \;.$$

Let $\underline{u}$ and $\underline{v}$ be complex such that

$$\underline{A}\,\underline{u} = [\alpha_1(\underline{A}) + \underline{i}\,\beta_i(\underline{A})]\,\underline{u} = \lambda_i(\underline{A})\,\underline{u} \tag{56}$$

$$\underline{A}\,\underline{v} = [\alpha_1(\underline{A}) + \underline{i}\,\beta_i(\underline{A})]\,\underline{v} + \underline{u} = \lambda_i(\underline{A})\,\underline{v} + \underline{u}\,. \tag{57}$$

Thus $\lambda_i(\underline{A})$ has a non-simple elementary divisor and real part $\alpha_1(\underline{A})$. Let $\underline{w} = \underline{u} + \epsilon\,\underline{v}$ where ϵ is real. From equation (12)

$$\underline{w}^*\,(\underline{A}'\,\underline{V} + \underline{V}\,\underline{A})\,\underline{w} = -2\,\underline{w}^*\,\underline{R}\,\underline{w}\,. \tag{58}$$

From equations (56), (57), and (58)

$$[\bar{\lambda}_i + \lambda_i]\,\underline{w}^*\,\underline{V}\,\underline{w} + 2\,\epsilon\underline{w}^*\,\underline{V}\,\underline{w} - 2\,\epsilon^2\,(\mathrm{Re}[\underline{v}^*\,\underline{V}\,\underline{w}]) = -2\,\underline{w}^*\,\underline{R}\,\underline{w}$$

or

$$-\alpha_1 - \epsilon\left(1 - \epsilon\,\frac{\mathrm{Re}\,[\underline{v}^*\,\underline{V}\,\underline{w}]}{\underline{w}^*\,\underline{V}\,\underline{w}}\right) = \frac{\underline{w}^*\,\underline{R}\,\underline{w}}{\underline{w}^*\,\underline{V}\,\underline{w}}\;. \tag{59}$$

For sufficiently small ϵ, say $|\epsilon_0| > |\epsilon| > 0$, the parenthesized term on the left of equation (59) will be positive. Thus, for $|\epsilon_0| > \epsilon > 0$,

$$\frac{\underline{w}^* \ \underline{R} \ \underline{w}}{\underline{w}^* \ \underline{V} \ \underline{w}} < - \ \alpha_1 \quad .$$

By Theorem 3, a corresponding inequality will hold for some real $\underline{x}$. This implies

$$\lambda_1 (\underline{R} \ \underline{V}^{-1}) < - \ \alpha_1 \tag{60}$$

which completes the proof of Corollary 1.

OTHER BOUNDS ON THE $\lambda_i(\underline{R}\underline{V}^{-1})$

Bounds on $\lambda_1(\underline{R}\underline{V}^{-1})$

The preceding sections have shown that $\lambda_1(\underline{R} \ \underline{V}^{-1})$ is bounded above by $- \alpha_1(\underline{A})$. If $\underline{A}$ is a stability matrix and $\underline{R}$ and $\underline{V}$ are to be RSPD matrices related by equation (12), $\lambda_1(\underline{R} \ \underline{V}^{-1})$ must be bounded below by zero. Thus

$$0 < \lambda_1(\underline{R} \ \underline{V}^{-1}) \leqq - \alpha_1(\underline{A}) \quad . \tag{61}$$

If the conditions are relaxed to include RSPSD $\underline{R}$, while maintaining the requirement that $\underline{V}$ be unique and RSPD as a solution to equation (12), equation (61) takes the form

$$0 \leqq \lambda_1 (\underline{R} \ \underline{V}^{-1}) \leqq - \alpha_1 (\underline{A}) \ . \tag{62}$$

According to a theorem by LaSalle [9], asymptotic stability of the null solution of equation (1) is assured if the conditions of equation (2) hold with (d) relaxed to

$$(d_1) \ \dot{v}(x) \leqq 0 \tag{63}$$

provided the set of $\underline{x}$ such that $\dot{v}(\underline{x}) = 0$ contains as a largest invariant set of equation (1), the null solution, $\underline{x} = \underline{0}$. In terms of $V(\underline{x})$ and $\dot{V}(\underline{x})$, this requires that the null space of $\underline{R}$ contain no invariant space of $\underline{A}$ except $\underline{x} = \underline{0}$. Provided these conditions are met, $\underline{R}$ can be chosen RSPSD, justifying the equality on the left of equation (62).

Bounds on $\lambda_n(\underline{R}\,\underline{V}^{-1})$

$\lambda_n(\underline{R}\ \underline{V}^{-1})$ is bounded below by $-\alpha_n(\underline{A})$. It is also bounded above as long as $\underline{R}$ is at least RSPSD. In particular,

$$-\alpha_n(\underline{A}) \leqq \lambda_n(\underline{R}\ \underline{V}^{-1}) \leqq -\operatorname{tr}(\underline{A}) = -\sum_{i=1}^{n} \alpha_i(\underline{A}) \tag{64}$$

Equation (64) follows from equation (12) by considering

$$\underline{A}' + \underline{V}\ \underline{A}\ \underline{V}^{-1} = -2(\underline{R}\ \underline{V}^{-1})$$

$$\operatorname{tr}(\underline{A}' + \underline{V}\ \underline{A}\ \underline{V}^{-1}) = 2\operatorname{tr}(\underline{A}) = -2\operatorname{tr}(\underline{R}\ \underline{V}^{-1})$$

or

$$-\operatorname{tr}(\underline{A}) = \operatorname{tr}(\underline{R}\ \underline{V}^{-1}) .$$

Since $\lambda_i(\underline{R}\ \underline{V}^{-1}) \geqq 0$ and since $\operatorname{tr}(\underline{R}\ \underline{V}^{-1}) = \sum_{i=1}^{n} \lambda_i(\underline{R}\ \underline{V}^{-1})$, equation (64) follows directly.

RELATIONS TO $\hat{\Gamma}$ AND $\check{\Gamma}$

The following estimates have been obtained.

$$-\alpha_n(\underline{A}) \leqq \hat{\Gamma} \leqq -\operatorname{tr}(\underline{A}) \tag{65}$$

$$0 \leqq \check{\Gamma} \leqq -\alpha_1(\underline{A}). \tag{66}$$

These estimates are valid for any RSPD $\underline{R}$ and $\underline{V}$ which satisfy equation (12).

If a judicious choice of $\underline{R}$ and $\underline{V}$ are made, $\hat{\Gamma}$ and $\check{\Gamma}$ can be made as close to $-\alpha_n(\underline{A})$ and $-\alpha_1(\underline{A})$ as desired, i.e. for $\epsilon > 0$.

$$-\alpha_n(\underline{A}) \leqq \hat{\Gamma} \leqq -\alpha_n(\underline{A}) + \epsilon \tag{67}$$

$$-\alpha_1(\underline{A}) - \epsilon \leqq \check{\Gamma} \leqq -\alpha_1(\underline{A}) \tag{68}$$

Such a $\underline{V}$ is given by equation (53).

In equation (67), ϵ can be set to zero if all $\lambda_i(\underline{A})$ with $\mathrm{Re}[\lambda_i(\underline{A})] = \alpha_n(\underline{A})$ have simple elementary divisors; similarly, for ϵ in equation (68).

RELATIONS TO $\hat{\Gamma}$ AND $\check{\Gamma}$

From equation (19) it is clear that for the class of systems and the class of Lyapunov functions,

$$\hat{\gamma} \geqq 2\hat{\Gamma} \geqq -2\alpha_n(\underline{A}) \tag{69}$$

$$0 \leqq \check{\gamma} \leqq 2\check{\Gamma} \leqq -2\alpha_1(\underline{A}). \tag{70}$$

Equations (69) and (70) can be thought of in two ways. If $\hat{\gamma}$ and $\check{\gamma}$ are known, they are estimates of $-2\alpha_n(\underline{A})$ and $-2\alpha_1(\underline{A})$. Conversely, if $-\alpha_n(\underline{A})$ and $-\alpha_1(\underline{A})$ are known, they serve as limiting values for $\hat{\gamma}/2$ and for $\gamma/2$ for all Lyapunov functions of the class considered.

EXAMPLES

To illustrate the theory just stated, several examples are given. For ease in calculation and clarity in presentation, only the most simple matrices are selected for $\underline{A}$.

The Lyapunov stability equation is

$$\underline{A}'\,\underline{V} + \underline{V}\,\underline{A} = -\,2\,R. \tag{12}$$

In each example below, for the $\underline{A}$ and $\underline{R}$ given, $\underline{V}$ is solved for and the $\lambda_i(\underline{R}\,\underline{V}^{-1})$ are calculated.

Example 1:

$$\underline{A} = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \quad ; \qquad \underline{R} = \begin{bmatrix} 1 & k \\ k & r^2 \end{bmatrix} \quad ;$$

$$\underline{V} = \begin{bmatrix} 1/a_1 & 2k/(a_1 + a_2) \\ 2k/(a_1 + a_2) & r^2/a_2 \end{bmatrix}$$

where $a_2 \geqq a_1 > 0$, $r^2 - k^2 > 0$. After some manipulations

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = a_1 - K(a_2 - a_1) \leqq a_1$$

$$\lambda_2(\underline{R}\,\underline{V}^{-1}) = a_2 + K(a_2 - a_1) \geqq a_2$$

where

$$K = \tfrac{1}{2}\left[\left(\frac{r^2}{r^2 - k^2\,\dfrac{4a_1a_2}{(a_1+a_2)^2}}\right)^{\frac{1}{2}} - 1\right].$$

Observe that $K = 0$, if $k = 0$; $K > 0$ if $r^2 - k^2 > 0$, and $K(a_2 - a_1) = a_1$ if $r^2 - k^2 = 0$.

Note that even if $r^2 - k^2 = 0$, that is, $\underline{R}$ is RSPSD, $\underline{V}$ is still RSPD and according to the theorem by LaSalle, this is sufficient for the asymptotic stability of $\underline{x} = \underline{0}$ of equation (7), because the null space of $\underline{R}$ contains no invariant subspace of $\underline{A}$. Also note that if $r^2 - k^2 = 0$,

$$\lambda_1(\underline{R}\ \underline{V}^{-1}) = 0$$

$$\lambda_2(\underline{R}\ \underline{V}^{-1}) = a_1 + a_2$$

or, in other words, the limiting values on the $\lambda_i(\underline{R}\ \underline{V}^{-1})$ predicted by equations (62) and (64) are actually attained for this choice of $\underline{R}$.

Example 2:

$$\underline{A} = \begin{bmatrix} -a & 1 \\ 0 & -a \end{bmatrix} ; \qquad \underline{R} = \begin{bmatrix} 1 & k \\ k & r^2 \end{bmatrix} ;$$

$$\underline{V} = \begin{bmatrix} 1/a & (2ak+1)/2a^2 \\ (2ak+1)/2a^2 & (2a^2k^2+2ak+1)/2a^3 \end{bmatrix}$$

where $a > 0$, $r^2-k^2 > 0$. The $\lambda_i(\underline{R}\ \underline{V}^{-1})$ are

$$\lambda_1(\underline{R}\ \underline{V}^{-1}) = a\ [1 - (4a^2(r^2-k^2) + 1)^{-\frac{1}{2}}] < a$$

$$\lambda_2(\underline{R}\ \underline{V}^{-1}) = a\ [1 + (4a^2(r^2-k^2) + 1)^{-\frac{1}{2}}] > a$$

It is clear that no value of k will make $\lambda_1(\underline{R}\ \underline{V}^{-1}) = a = \lambda_2(\underline{R}\ \underline{V}^{-1})$. However, given any $\epsilon > 0$, the relation

$$a > \lambda_1(\underline{R}\ \underline{V}^{-1}) > a - \epsilon$$

$$a < \lambda_2(\underline{R}\ \underline{V}^{-1}) < a + \epsilon$$

can be made to hold by choosing

$$r^2 - k^2 > \max [\, 0, \frac{a^2 - \epsilon^2}{4a^2 \epsilon^2} \,]$$

which illustrates Theorem 5 and its Corollary 1.

Also note that if $r^2 - k^2 = 0$, that is, $\underline{R}$ is RSPD, $\underline{V}$ is still RSPD, and since the null space of $\underline{R}$ contains no invariant subspace of $\underline{A}$, this is still sufficient to show that $\underline{x} = \underline{0}$ of equation (7) is asymptotically stable. In this case

$$\begin{aligned} \lambda_1(\underline{R}\,\underline{V}^{-1}) &= 0 \\ & \qquad\qquad\qquad (r^2 - k^2 = 0) \\ \lambda_2(\underline{R}\,\underline{V}^{-1}) &= 2a \end{aligned}$$

which also illustrates equations (62) and (64).

Example 3:

$$\underline{A} = \begin{bmatrix} -a_1 & a_2 \\ -a_2 & -a_1 \end{bmatrix} \qquad\qquad \underline{R} = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}$$

$$\underline{V} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} \dfrac{a_1^2 - a_1 a_2 k + a_2^2}{a_1} & a_1 k \\ a_1 k & \dfrac{a_1^2 + a_1 a_2 k + a_2^2}{a_1} \end{bmatrix}$$

where $a_1 > 0$, $1 - k^2 > 0$. The $\lambda_i(\underline{R}\,\underline{V}^{-1})$ are

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = a_1 \,[\, 1 - \left(\frac{a_2^2\, k^2}{a_1^2 + a_2^2 - a_1^2 k^2}\right)^{\frac{1}{2}} \,] \leqq a_1$$

$$\lambda_2(\underline{R}\,\underline{V}^{-1}) = a_1 \left[1 + \left(\frac{a_2^2\, k^2}{a_1^2 + a_2^2 - a_1^2 k^2} \right)^{\frac{1}{2}} \right] \geqq a_1 .$$

Note that

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = a_1 = \lambda_2(\underline{R}\,\underline{V}^{-1}) \text{ if } k = 0$$

and

$$\lambda_1(\underline{R}\,\underline{V}^{-1}) = 0, \quad \lambda_2(\underline{R}\,\underline{V}^{-1}) = 2a_1 \text{ if } k = 1.$$

CONCLUSIONS

It is evident that the estimates of transient response $\hat{\gamma}$ and $\check{\gamma}$ will always give upper and lower bounds for the smallest and largest "time constants" of the system considered, but that such estimates may not be very good. Moreover, a judicious choice of the Lyapunov functions will give as good an estimate as desired. However, this choice of the required Lyapunov functions is intrinsically dependent on a knowledge of the characteristic vectors of the matrix $\underline{A}$. This is clearly undesirable. Work is now underway to find some method to utilize digital computation to obtain this judicious choice of Lyapunov function without prior knowledge of the characteristic vectors of $\underline{A}$.

George C. Marshall Space Flight Center
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