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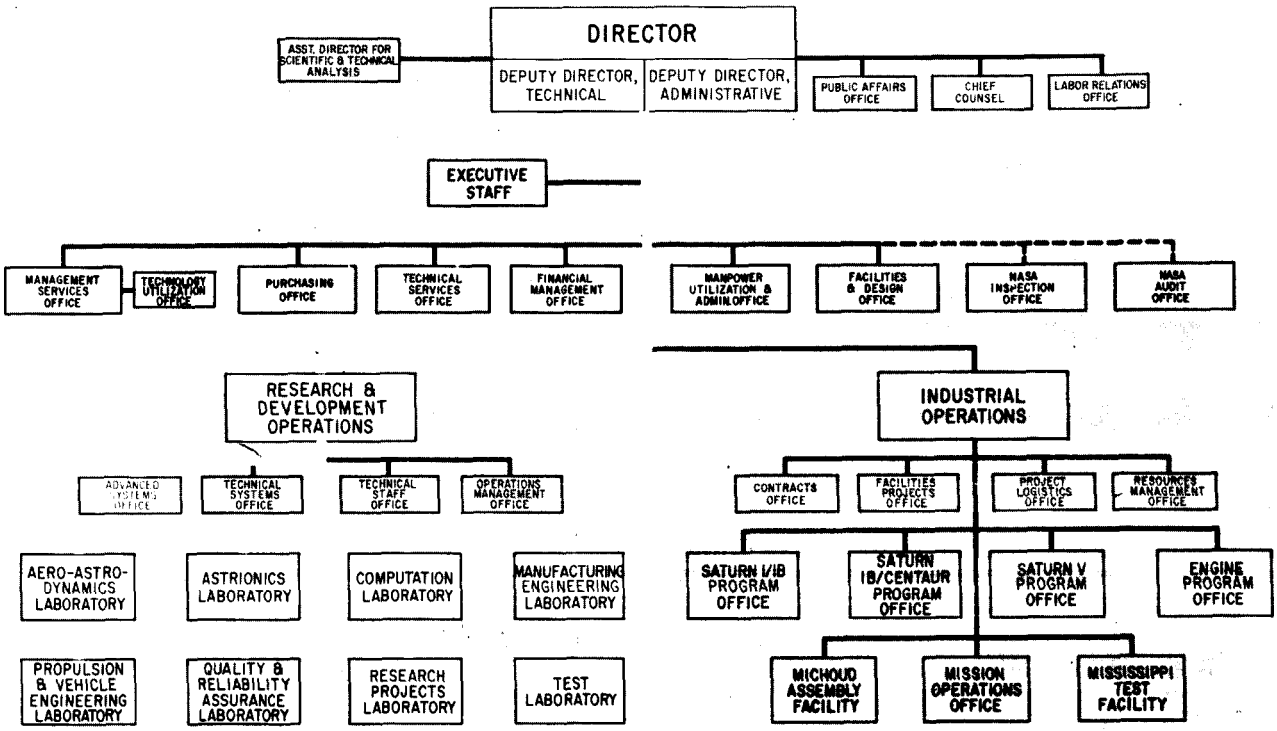

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UNITS OF MEASURE

In a prepared statement presented on August 5, 1965, to the U. S. House of Representatives Science and Astronautics Committee (chaired by George P. Miller of California), the position of the National Aeronautics and Space Administration on Units of Measure was stated by Dr. Alfred J. Eggers, Deputy Associate Administrator, Office of Advanced Research and Technology:

"In January of this year NASA directed that the international system of units should be considered the preferred system of units, and should be employed by the research centers as the primary system in all reports and publications of a technical nature, except where such use would reduce the usefulness of the report to the primary recipients. During the conversion period the use of customary units in parentheses following the SI units is permissible, but the parenthetical usage of conventional units will be discontinued as soon as it is judged that the normal users of the reports would not be particularly inconvenienced by the exclusive use of SI units."

The International System of Units (SI Units) has been adopted by the U. S. National Bureau of Standards (see NBS Technical News Bulletin, Vol. 48, No. 4, April 1964).

The International System of Units is defined in NASA SP-7012, "The International System of Units, Physical Constants, and Conversion Factors," which is available from the U. S. Government Printing Office, Washington, D. C. 20402.

SI Units are used preferentially in this series of research reports in accordance with NASA policy and following the practice of the National Bureau of Standards.

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WASHINGTON, D. C.

MATHEMATICS AND COMPUTATION RESEARCH

RESEARCH ACHIEVEMENTS REVIEW SERIES NO. 20

RESEARCH AND DEVELOPMENT OPERATIONS
GEORGE C. MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA

1966

PREFACE

In 1955, the team which has become the Marshall Space Flight Center (MSFC) began to organize a research program within its various laboratories and offices. The purpose of the program was two-fold: first, to support existing development projects by research studies and second, to prepare future development projects by advancing the state of the art of rockets and space flight. Funding for this program came from the Army, Air Force, and Advanced Research Projects Agency. The effort during the first year was modest and involved relatively few tasks. The communication of results was, therefore, comparatively easy.

Today, more than ten years later, the two-fold purpose of MSFC's research program remains unchanged, although funding now comes from NASA Program Offices. The present yearly effort represents major amounts of money and hundreds of tasks. The greater portion of the money goes to industry and universities for research contracts. However, a substantial research effort is conducted in house at the Marshall Center by all of the laboratories. The communication of the results from this impressive research program has become a serious problem by virtue of its very voluminous technical and scientific content.

The Research Projects Laboratory, which is the group responsible for management of the consolidated research program for the Center, initiated a plan to give better visibility to the achievements of research at Marshall in a form that would be more readily usable by specialists, by systems engineers, and by NASA Program Offices for management purposes.

This plan has taken the form of frequent Research Achievements Reviews, with each review covering one or two fields of research. These verbal reviews are documented in the Research Achievements Review Series.

Ernst Stuhlinger
Director, Research Projects Laboratory

These papers presented January 6, 1966

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RESEARCH IN COMPUTATIONAL MATHEMATICS AND LANGUAGES

By

C. L. Bradshaw*

SUMMARY

The problems associated with effectively relating machine languages and problem oriented language for efficient use of computers is discussed. The need for improvements in the man-machine relationship and economic improvements of trade-offs that can be achieved between presently developed machine languages to broaden the participation and usefulness of the computer in space programs is emphasized.

The impact of standardization on computer utilization, the consequent reduction in redundant effort and relief from the continual need for the reformulation of the problem is shown as an important objective of the Computation Laboratory and some achievements in this area are discussed.

I. INTRODUCTION

Computers have become an essential tool in the research and development programs of our nation. They have also become a very expensive and sensitive item in our nation's budget. Research into computational mathematics and languages can lead to a more effective use of this most important tool. This research effort can attack the overall problems on five main fronts:

1. Improving the mathematics involved in obtaining a computer solution to an engineering or scientific problem,

2. Improving the computer programming languages used in problem solution,

3. Obtaining a more effective use of computing hardware and software as relates to specific classes of problems,

4. Development of more efficient computer hardware, and

5. Improving the man-machine relationship as relates to automatic computing devices.

MSFC has done or sponsored considerable research in areas which have direct application to the areas mentioned above. This survey will show the MSFC efforts in these areas. This presentation will be followed by two papers which will be more explicit in two of these areas.

II. COMPUTATIONAL LANGUAGES

As mentioned earlier, computers are now being used to solve many diverse problems in the fields of engineering, science, and business. The fast computing speed and large internal memory of general purpose computers are valuable assets to those who prepare the problem solution. However, the language of computers is a sequence of numbers which usually is reduced by the machine to a sequence of ones and zeros, and this machine language is generally foreign to the problem solution prepared by an expert in a specialized area. Thus, there is a gap between the language of a problem and the language of a machine for solving it. The seriousness of this gap is intensified by the fact that there are almost as many machine languages as there are kinds of computers. Therefore, work performed in machine language at one site is often of little value elsewhere if the machines are different.

To reduce this gap, many programs have been prepared so that on the one hand they can be understood by computers and on the other hand they will accept as input a higher level language which is closer to the problem and is called the source language. The techniques to implement programs which accept source languages fall into two overlapping categories. The simplest is to have the computer interpret statements of the source language and process the intent when it is recognized. The more popular technique is to translate the source program into an object program which is either in machine language or closer

* Deputy Director, Computation Laboratory

to machine language. In the latter technique, the source program may have several equivalent representations since the object language from one translation may be the source language for another.

Pragmatically, a language is defined in terms of what a processor (whether it be called compiler, translator, or assembler) will recognize correctly for the user. In other words, a computer language is directly or indirectly the language for input to a computer. Consequently, standardization with a group of users is necessary to determine what is mutually agreed to as being the language.

When the users of a language are allowed to participate in the formulating process, a more mature version of the language can be produced more quickly. The definition of ATOLL II (Automatic Test Oriented Launch Language), which we will now describe, illustrates how the changing needs of the user can be incorporated into the language design during definition to avoid producing numerous languages. In such an environment, the trade-offs between features desired and the implementation cost can be evaluated realistically.

ATOLL II is a problem-oriented language for real-time launch vehicle testing. The language is structured like FORTRAN (Formula Translation) and includes, in addition, real-time test-oriented statements, a more elaborate data description capability, a limited ability to manipulate symbolic or other non-numeric values, and an ability to include inline symbolic coding.

The language provides the capability to manipulate both the ground support equipment and the launch vehicle. It provides for real-time delays, for control based on maintaining a sequence of events where event execution is time related with respect to previous functions.

The language is open ended in that the user may define what appears to the user to be additional source statement types. Thus, the compiler can be adapted to the problem area as new specifications are prepared in the language.

The language is block structured to permit dynamic allocation of variable and temporary storage. This feature, in combination with a provision to automatically segment a test into independent programs, assures that the object space required for test programs can be kept very small.

The language has been designed to satisfy many of the requirements and desires of management, the launch system engineer, and the computer programmer. According to dynamic definition techniques which have been developed, this language has evolved rapidly over the past year. ATOLL II is fully documented and available for study.

We now would like to mention the problem of language translation. Since much programming effort has already been expended in languages which are now obsolete, or for which processors are not readily available, there is interest in the capability to translate a program into another language which is available. Also, it is desirable to minimize the number of different languages a user must learn.

Many super-processors have been proposed which are "machine independent." The purpose of such a processor is to allow preparation of compilers for classes of languages rather than for only a specific one. Until recently, insufficient information was available to determine what classes are sufficiently defined for implementing in such general terms. However, some efforts have been quite successful in the areas of assemblers and context-free languages. How to proceed with context-sensitive languages is not yet clear, although preliminary efforts in this direction are being made.

Some of the most dramatic developments in software have been seen in operating systems, or so-called control programs. Control programs are being given the tasks of handling computer interrupts, doing bookkeeping on jobs, servicing remote terminals on a priority basis, scheduling memory and computer time, editing and merging of programs, and total data management. As techniques of mechanizing such tasks are developed, the user is freed from meticulous operations and his turn-around time is shortened so that his time can be used more productively.

One problem area which has experienced many attempts but little success is that of preparing language processors. Early attempts such as Jovial and Neliac have proved to be educational but economically unfavorable. Recent developments in assembly level languages have improved expressibility, stratified the control of symbol expressions, incorporated list structures, and refined recursive macro capability with conditional parameter substitution. Yet, more work is needed to fill the gap between the kinds of languages which are easily implemented and the kinds of statements which users in specialized areas find most appropriate for the problem at hand.

An example of a specialized area which has justified the development of a new language is trajectory programming, which is discussed in the next section.

III. THE MSFC TRAJECTORY LANGUAGE

In the past, MSFC scientific programmers have worked individually with the engineer in the development of programs which were designed, programmed, checked out, and documented specifically for that engineer's need. This relationship of programmer to engineer has proved effective because of the nature of past problems and the limited language and systems capabilities. Specifically, in the past, many programs were large and involved with long production lifetimes. More recently, however, desired programs cover a wider range of applications with more limited use. This fact makes it imperative that our scientific programmers produce and maintain many more programs. Also, software advances have been made which open the door to a more general and sophisticated approach to the trajectory applications area. There seems to be no choice but to re-evaluate our overall procedures and optimize where possible. As the result of careful investigation, it has been determined that the function of setting up and maintaining trajectory programs can and must be optimized to a maximum reasonable level. At this point, the maximum level of optimization or automation cannot be ascertained; however, some optimization can be realized. This is possible when one realizes that the entire area of trajectory computation, when taken as a whole, is a set of associated problems with many elements in common.

The form that a problem may acquire in the process of being prepared for computation will vary widely within the range of imagination, experience, and other resources possessed by individuals who perform this task. For this reason, it is frequently difficult and time consuming for one person to use or become familiar with a program that was written by someone else. It is also a time consuming task for a programmer to modify his own program. Detailed documentation relieves this problem to some extent, but the more documentation there is to be studied, the more complicated the task becomes. It is necessary, or at least desirable, to restrict the general overall structure of programs to conform as nearly as possible to a general well defined standard model. As a result of the language and systems improvements and increased workloads, it was imperative that research be done to establish a faster, less expensive, and more useful service by developing a trajectory oriented programming system.

With the increasing demands for faster results from man and the computer, it has become obvious that these demands cannot be met with present resources. With the cost of manhours increasing and the cost of computing machine time decreasing, more burden must be placed on the computer by the use of more problem oriented systems. The trajectory programmer must be enabled to do a better job in a shorter time and at less cost to meet these demands.

We answer the question, "How can a trajectory oriented programming system help eliminate some of the effort required by the man in the man-to-machine cycle?" as follows:

1. Construct the programs in modular form. Programs could consist of elements called modules and can be thought of as building blocks for many programs. This will eliminate repetitive efforts since modules may be interchanged to create completely different programs without major reprogramming effort.

2. Standardize nomenclature. To implement the modular concept, standardization is necessary. It enables the programmer and the user to communicate in well defined terms, eliminating confusion in definition of coordinate systems, mathematical models and units. Documentation of work done will be more effective and meaningful. Programs written by other programmers will be easier to interpret and understand.

3. Standardize organization of programs. The user and programmer can communicate at a common level. The programmer would be free to do more useful and creative work in other areas. New users and new programmers can become familiar with programs and trajectory concepts earlier. Interchanging of programs will be easier. Programs will be easier to modify and maintain simply by changing and modifying only the necessary modules. Programs will be easier to evaluate since the programming effort will be isolated from system functions. Organization of the problem will be simplified since much of the logic will be handled by the system preprocessor.

We next look at the impact on utilization.

A problem oriented language and system will enable the programmer to drastically cut the time required in setting up and maintaining a trajectory related program. This savings results from the programmer being able to use precoded subroutines and sub-programs as the need arises. These precoded elements will be fully documented and completely checked out beforehand, thus freeing the programmer from these routine tasks.

A library will be established for the programmer and the user. Therefore, the engineer will be relieved

of the never ending problem of reformulation and rechecking of requests and proposals. As routines are developed and pooled by users, much redundant effort will be eliminated.

Better documentation of work will be provided by a standardization of program logic.

We now look at the difficulties in developing a trajectory system of this type. To be widely useful and accepted, a trajectory system must satisfy the requirements of all the users. Each user usually has a special interest which emphasizes certain areas more than others; for instance, the engineer whose primary concern is trajectory optimization puts a different emphasis on the guidance package than will the engineer whose primary concern is simulating the on-board guidance computer. In actual practice, it is difficult to satisfy the needs of the many users with one trajectory system. Even though difficult, sufficient research has been done to demonstrate the feasibility of further research of such effort.

We must recognize the research that must be carried on in the development of a problem oriented system.

1. Problem areas must be analyzed to determine the feasibility of developing and implementing a trajectory language, in our case, to see if the trajectory area is sufficiently large to justify such an effort. We have determined that it is large enough.

2. Work done by other laboratories and installations must be investigated to determine if they are doing work in this area. If so, we must establish why the effort, amount of effort, state of development, and evaluation of their effort relative to our needs.

3. Our own needs, present and future, must be investigated considering customer contacts, past requirements, survey of existing programs, what future needs are expected to be.

4. Methods must be developed for providing the following:

a. Program logic - to allow the programmer to direct the flow of the program in extremely involved logical paths.

b. Events - all complex trajectory programs involve some type of events or interrupts; much planning and analysis are required prior to writing a program. Examples of events are engine cutoffs, weight drops, high-q and tilt arrests.

c. Integration - much research needs to be done in this area to allow the user to select integration schemes to give the desired accuracy for his problems.

d. Input and Output - the present laborious effort of writing long lists and involved format statements must be simplified.

Our future plans are as follows:

1. Develop and implement an upward compatible trajectory oriented programming system. It must also be emphasized that all future programs written in the MSFC trajectory system will be FORTRAN IV compatible with other installations.

2. Prepare abstracts for all modules both mathematical and program.

3. Prepare a users manual for the system including complete description of statements and their source output.

4. Provide training for the users.

5. Study feasibility of adapting preprocessor to another machine. The Vectran Engineering Simulation System (VESS) preprocessor has already been translated to ALGOL and will process FORTRAN IV statements on the Burroughs B-5500 computer.

The trajectory oriented programming system planned by MSFC, Marshall Vectran Engineering Simulation System (MARVESS), contains the only preprocessor which actually provides a system function that will recognize a set of statements and create a trajectory program. No other system which we have studied can provide these necessary features.

IV. AMTRAN

I would like to mention at this time one other research effort which is underway at our Center and has as its goal the improvement of the man-machine relationship. This effort, which has been given the name Automatic Mathematical Translator (AMTRAN), is directed by Dr. Robert Seitz of the Research Projects Laboratory. AMTRAN is designed to be an automatic programming, on-line, multi-terminal computer system which should afford marked improvements in programming, debugging and turn-around times when it is fully developed. The system permits

a scientist or engineer to enter mathematical equations in their natural mathematical format as they appear in a textbook and, barring complications, to obtain an immediate graphical display of the solution on an output display device. The system is intended to be used for straightforward problem solution by the engineer or scientist with little computer experience while at the same time providing the flexibility required by the experienced programmer to solve non-routine problems. A "sampler" version of the system is now available using a modified IBM 1620 computer.

V. RANDOM PROCESS THEORY

The purpose of research in this area is to examine the existing Computation Laboratory techniques used to reduce and analyze random process data toward the objective of devising new or improved applications of statistics and random process theory. The specific goals of this research are to reduce the data editing and computer usage time, to increase the "accuracy" of the statistical estimates of the processed data, and to recommend future applications of existing data reduction equipment. These improvements are to be a result of the investigation of the techniques used by the Computation Laboratory and the appropriate application of:

1. Digital filtering techniques
2. Correlation function analysis
3. Spectral smoothing techniques
4. Special functions or processing
5. Spectrum analysis of nonstationary functions.

Research contracts were undertaken to study numerical smoothing and differentiation methods. With these studies, digital filtering techniques were developed and investigated. The main effort was devoted to linear digital (numerical) filters for performing smoothing, differentiation, and integration of discrete data and to do error analysis for these filters.

The mathematical foundations were rigorously justified by beginning with classical Fourier theory and following through with the development of generalized functions which led to specific functions used for filtering. This work is well documented in NASA Contractor Report CR-136. These desired digital filtering techniques were derived and are now being successfully applied to test data.

The Computation Laboratory also initiated a research study with the Cornell Aeronautical Laboratory to do research in areas which would satisfy the Laboratory requirements of:

1. Studying and applying the available random data processing techniques to the existing MSFC problems, and

2. Developing new and improved techniques of data processing.

The following discussion indicates that the above requirements are being satisfied.*

A. DIGITAL FILTERING

Selection of an appropriate sampling interval which produces negligible frequency folding is paramount to accurate digital data processing. The vast amount of literature available which describes digital simulation of transfer functions from the time response point of view can be used to produce prewhitening filters having specific frequency characteristics.

Taking the Tustin Transform of an analog notch filter will produce a digital filter which can be used for pre-whitening, with the possibility of total rejection of one frequency. These notch filters contain relatively few weights.

In situations where the power spectral density function of only a band of frequencies is of interest, digital heterodyning may provide a computational time savings in data processing.

B. CORRELATION FUNCTIONS

After reading the analysis of different methods of estimating correlation functions, one should conclude that modifications should be made to any existing computational technique that does not consider both the accuracy of estimates and the computer time required. Many types of correlation function estimators are given (autocorrelation being a special case of cross-correlation). Extensive study of the "half-polarity" correlator is presented. Computer programs are outlined, which will calculate, in minimum time, the "half-polarity" and "full-precision" correlation functions. It is also suggested that correlation computational techniques given in the reference are applicable.

* Research Studies of Random Process Theory and Physical Application, NASA CR-61081.

C. OPTIMAL SMOOTHING OF POWER SPECTRAL DENSITIES (PSD)

The appropriate application of proper techniques will produce spectral estimates with greater accuracy and also eliminate the need for pre-whitening of the signal prior to processing.

In June, 1965, a 12-month extension to the project was initiated. The objectives are to extend and expand the techniques under the initial effort. Primary investigations will be the application of the non-stationary correlation function theory and digital correlation function computation techniques. The following list summarizes the technical effort and indicates the order of priority:

1. Non-stationary data processing

2. Stationary data processing

3. Block diagrams covering application of data processing techniques developed.

The following are the major accomplishments to date:

1. The discrete data non-stationary correlation function theory has been developed.

2. A solution for the form of the optimum filters to be used in the discrete data correlation function detector has been obtained.

VI. DISCRETE OPTIMIZATION TECHNIQUES

This laboratory has a contract with the University of Tennessee to do research in Discrete Optimization Techniques. The principal investigator is Dr. Gordon Sherman of the University's Computing Center and Mathematics Department.

The problem is to maximize (minimize) a function defined on a given finite set. Typical examples are: the shortest tour problem, the job shop scheduling problem, and the transportation problem. Satisfactory solutions are available for some problems of this class, while complete enumeration of all alternatives, if it were possible to do so, is the only known way of producing solutions for other cases.

Dr. Sherman has taken a stochastic approach to the problem with the basic idea of combining intelligent search with random search. He has produced a family of algorithms that are quite efficient in the shortest tour type problem. These problems were used as test cases since the most research had already been done on them. Detailed explanation of the method, algorithms, and results may be found in an article called "Discrete Optimizing" by Reiter and Sherman in the September 1965 issue of the Journal of Industrial and Applied Mathematics.

VII. ANALOG COMPUTATION AND SIMULATION

The traditional tool for simulation of dynamic systems has been the analog computer. The similarity between the real system and the program on the analog computer, and the possibility of identifying a block of the real system as a group of computer components, gives the simulation technique the advantage of a model-like representation. This allows for an easy introduction of modifications and immediate observation of the effects of these changes. There are certain shortcomings in the use of analog computers. These are in the lack of random access memory, limited arithmetic precision, awkwardness in performing complex arithmetic, and others. These shortcomings led to a combination of the analog with the general purpose digital computers, thus preserving the advantages of the analog while overcoming most of the shortcomings. This type of system is called a hybrid system.

Hybrid computation, however, introduces problems itself. Even though at many different places detailed investigations have been conducted, it was felt necessary to secure the support of an academic institution for basic studies in the area of error analysis of hybrid computation.

Since this is a difficult and complex field, these studies are expected to become a long range effort. Some investigations have already been conducted by the Georgia Institute of Technology. The time limit for this review allows us to report only on the problem area, the approach, and the more important results. Dr. Finn (of Georgia Tech) has investigated the errors introduced by sampling, by hold operation (zero and first order) for periodic, pulse shaped, and stationary band limited random functions of time. The sampling rate must be at least high enough to avoid fold over. When the highest frequency present in the continuous signal is f maximum, then the sampling frequency must be more than 2 f maximum to avoid fold over. This is well known and follows directly from a frequency presentation of the sampled signal.

When we concentrate on wide sense stationary random processes as time functions, we can make statistical predictions about the expected error and, what is more meaningful, the expected error square since the probability of positive and negative errors is equal. We can determine sampling rates, limiting the probability of our error to exceed a preset limit. The theory allows us to determine upper and lower bounds for the ratio mean square error for a band limited random process as a function of the sampling rate (zero hold). These investigations are intended to be extended to higher order hold sampling techniques where similarly interesting results can be expected.

For sampling periodic functions, upper bounds for the ratio mean square error can also be given under the assumption that the initial phase is uniformly distributed over all possible values in a random fashion.

Dr. Finn has concentrated on investigating the error introduced by replacing the continuous function of time X(t) with a sampled representation using zero order and first order hold. Dr. Hammond, also of Georgia Tech, has worked on a system of n first order differential equations. With little loss in generality, he starts with a class in which the first derivative is represented explicitly as a function of position and time. The hybrid computer uses its analog part for integration and its digital part for function generation. This allows one to derive for the error a system of linear differential equations. For short intervals the coefficients in these equations can be considered as constant and the forcing function can be approximated by a staircase function. This not only allows one to use Laplace techniques for their analytical solution, but also provides a computer program of moderate complexity which can be incorporated with the hybrid program. A test program is presently set up in the Simulation Branch of the Computation Laboratory to study the usefulness of this approach.

VIII. NUMERICAL INTEGRATION

Because of the tremendous cost of modern computing equipment and the considerable amount of time used to perform certain types of studies, for example, orbit calculations, very substantial savings in computer time and dollars can be realized by even modest improvements in numerical integration techniques. The Computation Laboratory, in a continual search for better integration methods, has a dynamic program in mathematical and numerical analysis. This program is carried out by in-house staff members, specialists on a consulting basis, and through contracts with universities and some industrial firms.

The numerical integration of differential equations demands quite a large amount of computing time. Therefore, a great deal of attention has been given to devising more efficient methods of integration. The laboratory has a research contract with Vanderbilt University, Nashville, Tennessee, for the investigation of improved techniques for numerical integration of differential equations. The principal investigator on the contract has been Professor E. B. Shanks of the University's Mathematics Department. Professor Shanks has devoted his efforts primarily to a study of Runge-Kutta type processes. At the time the contract began, there existed the well-known Runge-Kutta formulas of fourth order requiring four evaluations of the function; the Kutta-Nystrom formulas of fifth order requiring six evaluations; and the less well-known Huta formulas of sixth order requiring eight evaluations.

A paramount problem in trying to increase the order of the formulas in the Runge-Kutta sense is that the number of conditions to be satisfied increase exponentially (essentially) and by the fact that the degree of the resulting algebraic conditions increases by two at each stage; for example, a seventh-order formula with nine evaluations involves 58 algebraic conditions with about half of them of twelfth degree. In such a complex system the notation becomes cumbersome and a problem in itself. However, the problem became tractable through use of the tensor calculus notation.

Dr. Shanks has been able to develop formulas of the sixth order with seven evaluations; seventh order with nine evaluations; eighth order with twelve evaluations; and ninth order with seventeen evaluations.

By adopting a new view point in which not all of the algebraic conditions were exactly satisfied, Dr. Shanks has been able to develop formulas of fifth order accuracy with five evaluations; sixth order with six evaluations; seventh order with seven evaluations; and eighth order with ten evaluations. All experience to date indicates that these formulas are more efficient than any of this type known previously. Additional details may be found in NASA Technical Note D-2920 and "Solution of Differential Equations by Evaluation of Functions," Mathematics of Computation, January 1966.

NEW ONE-STEP INTEGRATION METHODS OF HIGH ACCURACY

By

Erwin Fehlberg*

SUMMARY 27308

New numerical methods for the solution of periodic trajectories for the restricted three body problem are presented and factors affecting both the accuracy of results and the reduction of electronic computer time through the use of the new methods are discussed. Extension of the Runge-Kutta method to higher order of accuracy and the establishment of a pure series expansion method with transformation of the original differential equations to a second-degree algebraic system and application of recurrence formulas has provided a method to more effectively use computer capability and point the way for use of the new methods in many space problems.

I. INTRODUCTION

The development of the electronic computer has created a need, and that need is becoming increasingly urgent, for more accurate, more powerful numerical methods of computation. The computer is, of course, nothing more than a piece of hardware, however complex. It obeys whatever numerical methods are programmed into it. Most numerical methods were developed long ago when the only tools available to mathematicians were pencil and paper, and perhaps a few tables of pre-calculated values (sines, cosines, logarithms, etc.). The advent of the simple desk calculator helped. But even with the desk calculator, computational procedures had to be kept simple. Complicated operations, no matter how refined or necessary for the solution of certain complex problems, were impractical or impossible.

The rapid development of the modern electronic computer caught most mathematicians unprepared to use other than their old methods on the new hardware. Even now, the numerical methods used at many computer facilities are still the old desk calculator methods. For example, the standard method of Runge and Kutta, still widely used for the numerical integration of differential equations, was developed around 1900. The method was quite suitable for use with hand-operated desk calculators.

But many of the scientific and engineering problems at Marshall Space Flight Center have become so involved that use of the standard, turn-of-the-century methods is completely out of the question. Not only are these methods often extremely slow, consuming excessive amounts of expensive computer running time; they are inaccurate, producing unreliable results. Thus there is a pressing need at the Marshall Space Flight Center for new, more advanced computational methods designed especially for use with the modern electronic computer.

The Computation Laboratory has been actively seeking modern numerical methods suitable, in particular, for the solution of problems in astronautics and celestial mechanics. In recent years, several new approaches to the solution of ordinary differential equations have been developed in which such problems are expressed. One new approach is based on a power series expansion combined with a sophisticated, high-order Runge-Kutta procedure. Unlike the old methods still in widespread use, these new methods can conveniently be extended to any high-order accuracy desired.

These powerful, high-order methods drastically reduce the errors involved in the numerical integration of differential equations. Such errors originate both in the physical limitations of the computer, i.e., round-off errors, and in the limitations of the numerical method programmed into the computer, i.e., truncation errors. Moreover, in problems like the three-body problem, the new methods proceed in large integration steps without impairing accuracy. Thus they are also much faster than conventional methods, which must proceed in extremely small steps to preserve some accuracy.

The Marshall Space Flight Center is conducting extensive theoretical and numerical studies of periodic orbits of vehicles in the earth-moon system.

* Technical Staff, Computation Laboratory

Data precisely defining a large number of such periodic orbits have been obtained using the new methods. Considering the effect of the truncation error, and to give some idea of the accuracy and speed possible with these methods, a sample periodic orbit in the restricted three-body problem was computed and found to retain its periodicity to within 0.01 millimeter (the distance from the earth to the moon is 384,000 kilometers). The computation took only about 5 percent as long as with the conventional Runge-Kutta-Nyström method.

These new methods can, of course, be used to solve many other problems in addition to problems in celestial mechanics. They are fully reported in the literature.

II. AVAILABLE INTEGRATION METHODS

A. MULTISTEP METHODS

Methods for the numerical integration of differential equations are, broadly speaking, either multistep or one-step. Multistep methods were developed as early as the nineteenth century, mainly for problems in astronomy. As their name indicates, these methods use the information from several backward computation steps in calculating the solution for the current step. Multistep methods (e.g., the methods of Adams, Cowell, Gauss, etc.) are very efficient for problems that can be integrated in steps of constant size. Since many such problems are encountered in astronomy, it is not at all surprising that a number of multistep methods have been developed by astronomers. Multistep methods also have the very great advantage that they generally require only one or two evaluations of the differential equations per step, and they can be extended to any order of accuracy simply by adding higher-order difference terms to the formulas. Hence, unlike the Runge-Kutta method, which is the classical one-step method, they are quite fast on an electronic computer; they are economical, and they can be made very accurate.

But multistep methods do have a number of major disadvantages. They are not self-starting, but require a special starting procedure. A history of known values is needed before computation can begin. Thus a number of backward values must be created by means, for example, of an iterative procedure. And if the integration step size has to be changed during the computation, if, for instance, the step size must be reduced to preserve accuracy, additional time-consuming iterations are needed to build a new difference scheme—time-consuming because the iterated values for the changed step size must be of high accuracy. For these reasons, among others, multistep methods are largely restricted to problems that can be integrated entirely in steps of the same size. This is, of course, the case in the determination of astronomical orbits, where the distances between the attracting bodies are changing, but not radically.

This is not at all the case in, say, the interesting orbits of the restricted three-body problem. Figure 1 shows a typical periodic orbit of the restricted three-body problem. The earth and the moon are the two attracting masses. These and the space vehicle are shown in the rotating coordinate system, in which the x-axis always extends from the earth to the moon. Every fifth integration step is indicated, except in the vicinity of the earth, where the steps were too numerous to show. Obviously, the integration steps in the vicinity of both the earth and the moon are much smaller than those where the vehicle is far from either attracting body.

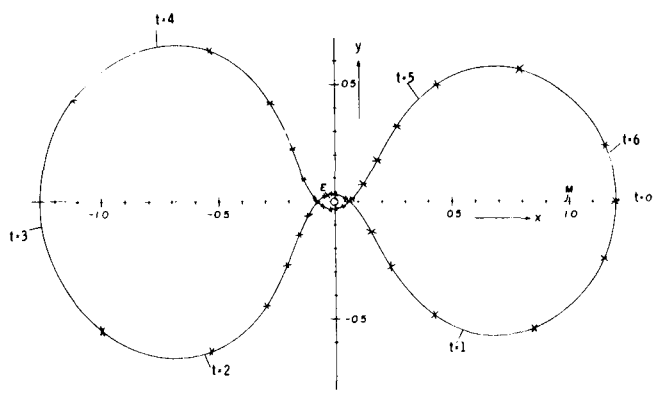

FIGURE 1. PERIODIC ORBIT OF THE RESTRICTED THREE-BODY PROBLEM

This kind of flexibility—to be able to increase the step size as much as possible or to be able to decrease it as much as necessary—is essential for efficient integration of such problems as the restricted three-body problem. It speeds the integration during the large part of the orbit where the space vehicle is near one or the other of the attracting masses.

This need for flexibility in the size of the integration step is caused by two kinds of error that accumulate during a computation. Too large a step size results in an unacceptably large truncation error because truncation error is proportional to a certain power of the step size. Too small a step size not only slows the computation, thus wasting

second degree. For special differential equations the procedure has been outlined in earlier papers by Steffensen [4], Rabe [5], and the author [3]. The procedure is based on the fact that the high-order derivatives of a second-degree system of differential equations can be conveniently obtained on a computer by recurrence formulas.

This can best be illustrated by the transformation procedure in a simple example. Consider the differential equation

$$\frac{dx}{dt} = e^{-x} . \tag{1}$$

Introduce the auxiliary function

$$e^{-x} = u \tag{2}$$

and obtain from equations (1) and (2) a system of second-degree algebraic differential eaquations

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = -u^2 . \tag{3}$$

Substituting the power series expansions

$$x = \sum_{\nu=0} X_\nu \cdot (t - t_0)^\nu , \quad u = \sum_{\nu=0} U_\nu \cdot (t - t_0)^\nu \tag{4}$$

into equation (3) and comparing coefficients for the terms with $(t - t_0)^n$ results in the following recurrence formulas for the coefficients in equation (4):

$$\left.\begin{aligned} (n+1)\, X_{n+1} &= U_n \\ (n+1)\, U_{n+1} &= -\sum_{\nu=0}^{n} U_\nu \cdot U_{n-\nu} \end{aligned}\right\} \quad (n=1,2,3,\dots) \tag{5}$$

Since the first coefficient X_0 is known from the initial value $x(t_0)$ for the step and the first coefficient U_0 can be obtained from equation (2), all following coefficients X_ν, U_ν ($\nu = 1, 2, 3, \dots$) can easily be computed from the recurrence formulas of equation (5), a very convenient procedure for electronic computers.

It is quite obvious, too, that the power series expansion method allows for an extremely simple automatic step-size control. Assuming truncation of the expansion in equation (4) for x after the term $X_n(t-t_0)^n$, the leading term of the truncation error of x can easily be found by extending the computation to the next coefficient X_{n+1}. If the truncation error turns out to be too large or too small, the step size at Δt can be adjusted immediately in such a way that $|X_{n+1}\,(\Delta t)^{n+1}|$ remains within prescribed limits. (For safety it might sometimes be advisable to consider more than just one term of the truncation error.) Unlike Runge-Kutta or multistep methods, no repetition of any computation is necessary if the step size fails to meet the requirements for the magnitude of the truncation error. The Computation Laboratory knows of no other method that offers such easy step-size control.

In this simple example, there is no real need to introduce auxiliary functions, since a repeated differentiation of the differential equation (1) can be performed without difficulty. A more representative example follows to illustrate how convenient the method can be.

IV. POWER SERIES EXPANSION METHOD APPLIED TO THE RESTRICTED THREE-BODY PROBLEM

Clearly, the following equations, for the restricted three-body problem in the rotating coordinate system, are not nearly so simple as equation (1):

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= x + 2\frac{dy}{dt} - (1-\mu)\frac{x+\mu}{[(x+\mu)^2 + y^2]^{3/2}} \\ &\quad - \mu\frac{x-(1-\mu)}{[(x-1+\mu)^2 + y^2]^{3/2}} \\ \frac{d^2y}{dt^2} &= y - 2\frac{dx}{dt} - (1-\mu)\frac{y}{[(x+\mu)^2 + y^2]^{3/2}} \\ &\quad - \mu\frac{y}{[(x-1+\mu)^2 + y^2]^{3/2}} \end{aligned}\right\} \tag{6}$$

where μ = the relative mass of the moon in the earth-moon system.

There exists a first integral of these equations of motion, the so-called Jacobi integral

$$J = \frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - x^2 - y^2\right] - \frac{1-\mu}{[(x-1+\mu)^2 + y^2]^{1/2}}$$

$$- \frac{\mu}{[(x-1+\mu)^2 + y^2]^{1/2}} = \text{Const.} \tag{7}$$

Auxiliary functions are again introduced

$$\left.\begin{aligned} r^2 &= (x+\mu)^2 + y^2, \quad s^2 = (x-1+\mu)^2 + y^2 \\ u &= \frac{(1-\mu)}{r^3} , \qquad v = \frac{\mu}{s^3} \end{aligned}\right\} \tag{8}$$

expensive machine time, even more seriously, it results in an unacceptably large round-off error because round-off error is a direct function of the number of steps that must be taken to compute the entire problem.

For use with computers, the ability to selectively vary or to automatically control the integration step size is fundamental to the efficient, fast-and-accurate integration of problems like the restricted three-body problem. This cannot be done with multistep methods except at considerable expense in increased complexity and increased computer running time.

B. ONE-STEP METHODS

One-step methods lend themselves more readily to step-size variation. In fact, the step size can be changed at any time and can immediately be accommodated to local conditions at any point in an integration. One-step methods are also self-starting. But they, too, have a number of disadvantages. The classical Runge-Kutta method is of only fourth-order accuracy. Several extensions have been made in the last decade, notably by Shanks [1, 2], but Shanks' method, the most accurate Runge-Kutta procedure developed to date, is still of relatively low-order accuracy (up to eighth-order). Runge-Kutta methods also tend to be slow because of their great complexity. The differential equation must be evaluated many times for each integration step. Shanks' eighth-order method, for example, requires 12 evaluations per step. If, in addition, the differential equations are complicated, if they contain transcendental functions (sine, cosine, exponential functions, etc.), the method becomes excessively slow and the cost in computer running time will certainly be high.

Another weakness of Runge-Kutta methods is that they, too, like the multi-step methods, lack an economical procedure for automatically adjusting the step size to the local conditions of the problem. As with all one-step methods, the step size can be changed at any time, but no economical control procedure seems to exist to do it automatically. In fact, using Runge-Kutta methods, one never knows whether the proper step size (for a combination of maximum accuracy and speed) is being used. There is no easy way to determine this. Apart from somewhat doubtful rule-of-thumb control procedures, there exists only Richardson's well-known method of the deferred approach to the limit. A step is computed, recomputed with half the step size (or double the step size), and then, by an extrapolation procedure, the results of the two computations are compared. This, however, doubles the computational effort or, more exactly, doubles computer running time merely for the benefit of step-size control.

This, briefly, was the state-of-the-art in methods for integrating differential equations when the Computation Laboratory began research in the field. The available methods were rather inaccurate and slow, and costly in machine time even for the solution of a relatively simple problem like the restricted three-body problem. For more complex problems in orbital mechanics, like the n-body problem, they could well turn out to be prohibitively slow and, worse, intolerably inaccurate. An original, starting 16-digit accuracy could easily dwindle to two or one or no accurate digits at the end of a long, complex computation. Also, it seemed absurd to pay heavily for auxiliary features like step-size control.

III. POWER SERIES EXPANSION METHOD

As a first quick improvement, the range of problems normally solved by pure power series expansions was expanded [3]. The use of pure power series expansions to solve differential equations is not exactly new, of course.* But unless the differential equation under consideration was extremely simple, pure power series expansion methods had generally been discarded as leading to cumbersome and lengthy computations for the derivatives they require. And, in fact, the method suggested requires a repeated total differentiation of the differential equation(s) with respect to the independent variable to obtain the necessary coefficients of the power series expansion.

Only a few years ago, the repeated total differentiation of a differential equation was not considered feasible, since, with increasing order, the derivatives become rather unwieldy expressions. But with fast electronic computers such a procedure is generally quite feasible. It is well-known that in the last few years considerable progress has been made in the automatic differentiation of formulas by computers.

Moreover, as an even more effective approach, apart from a straight-forward differentiation of the differential equation(s), a great number of these can be differentiated in a rather simple way by first transforming them, through introduction of auxiliary functions, into algebraic differential equations of the

* All commonly used integration procedures are, in fact, based in part on a power series expansion. The coefficients of Runge-Kutta formulas or the coefficients of multistep methods are obtained by expanding in power series (Taylor series, etc.) and then finding systems of equations of condition for these coefficients.

Introducing equation (8) into equation (6) transforms the original system into the following second-degree algebraic system, which can be integrated directly by power series expansions:

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= x + 2\frac{dy}{dt} - u(x+\mu) - v(x-1+\mu), \\ \frac{d^2y}{dt^2} &= y - 2\frac{dx}{dt} - uy - vy \\ r\frac{du}{dt} + 3u\frac{dr}{dt} &= 0, \quad s\frac{dv}{dt} + 3v\frac{ds}{dt} = 0, \\ r^2 &= (x+\mu)^2 + y^2, \quad s^2 = (x-1+\mu)^2 + y^2 \end{aligned}\right\} \quad (9)$$

Again, of course, new differential equations for the auxiliary functions have been added, but the new system is completely algebraic, containing only products of two functions throughout.

Let the power series expansions be

$$\left.\begin{aligned} x &= \sum_{\nu=0} X_\nu (t-t_o)^\nu, \quad y = \sum_{\nu=0} Y_\nu (t-t_o)^\nu \\ u &= \sum_{\nu=0} U_\nu (t-t_o)^\nu, \quad v = \sum_{\nu=0} V_\nu (t-t_o)^\nu \\ r &= \sum_{\nu=0} R_\nu (t-t_o)^\nu, \quad s = \sum_{\nu=0} S_\nu (t-t_o)^\nu \end{aligned}\right\} \quad (10)$$

The first coefficients X_o, X_1 and Y_o, Y_1, are known at the beginning of the integration step. The forst coefficients R_o, S_o, U_o, and V_o are then determined from equation (8).

Inserting power series expansions from equation (10) into the original system of equation (9), the following recurrence formulas for the succeeding coefficients are obtained:

$$\left.\begin{aligned} 2R_oR_n &= \sum_{\nu=0}^{n} X_\nu X_{n-\nu} + 2\mu X_n + \sum_{\nu=0}^{n} Y_\nu Y_{n-\nu} - \sum_{\nu=1}^{n-1} R_\nu R_{n-\nu} \\ 2S_oS_n &= \sum_{\nu=0}^{n} X_\nu X_{n-\nu} - 2(1-\mu)X_n + \sum_{\nu=0}^{n} Y_\nu Y_{n-\nu} - \sum_{\nu=1}^{n-1} S_\nu S_{n-\nu} \\ nR_oU_n &= -3\sum_{\nu=1}^{n} \nu R_\nu U_{n-\nu} - \sum_{\nu=1}^{n-1} \nu U_\nu R_{n-\nu} \\ nS_oV_n &= -3\sum_{\nu=1}^{n} \nu S_\nu V_{n-\nu} - \sum_{\nu=1}^{n-1} \nu V_\nu S_{n-\nu} \\ (n+1)nX_{n+1} &= X_{n-1} + 2nY_n - \mu U_{n-1} + (1-\mu)V_{n-1} - \sum_{\nu=0}^{n-1} (U_\nu + V_\nu)X_{n-1-\nu} \\ (n+1)nY_{n+1} &= Y_{n-1} - 2nX_n - \sum_{\nu=0}^{n-1} (U_\nu + V_\nu)Y_{n-1-\nu} \end{aligned}\right\} \quad (n = 1, 2, 3, \ldots) \quad (11)$$

Thus all the Taylor coefficients of X and Y can be obtained. These expressions can be extended to as many terms as desired. There is no restriction whatsoever on the order of the formulas. This is in distinct contrast to Runge-Kutta formulas, in which each advance of only one order in accuracy has taken many years to establish and, as mentioned earlier, the highest known order is only the eighth. There is no such problem with recurrence formulas. It is such recurrence formulas, evaluated automatically on the computer and extended to any order desired, that form the basis of both the pure power series expansion method and, as will be shown in the next section, the improved Runge-Kutta method developed by the Computation Laboratory. By a suitable transformation the original differential equation(s) is reduced to a second-degree algebraic system and then the recurrence formulas are applied whose coefficients are determined automatically on the computer.

The Computation Laboratory has solved many problems—restricted three-body, motion of an electron in the field of a magnetic dipole, and others—by a pure power series expansion method. However, while the method proved superior to other existing methods, there still seemed to be some room for improvement. For example, the number of terms in all the sums in equation (11) increases with increasing n. Hence computer running time gets longer for higher-order coefficients.

It is well-known, too, that power series expansions have certain limitations with respect to the truncation error. When one truncates the expansion, there is no way of covering the remainder of the error, which is roughly equal to the leading term. This is unavoidable in a power series expansion, although it is even more of a problem in multistep methods. This need not be such a problem with Runge-Kutta methods, and it is for this reason that the Computation Laboratory has developed the Runge-Kutta transformation method, which combines the high-order accuracy of power series expansions with a good coverage of the truncation error. In fact, since, to a certain extent at least, the leading term of the truncation error can be covered, the Runge-Kutta transformation method radically reduces the truncation error.

Thus a combination of the two methods should not only be more accurate than the pure power series method, it should also provide an advantage in speed because larger step sizes can be used. The combined method is described in the next section.

V. RUNGE-KUTTA TRANSFORMATION METHOD

A. FORMULAS OF ANY DESIRED HIGH ORDER

Consider only second-order differential equations since it is these that are most frequently encountered in physics and mechanics. (The method works for first-order systems as well [6].) For brevity, formulas for only a single equation will be written, although the method holds in exactly the same way for systems. Letting x be the original dependent variable, then:

$$\left.\begin{aligned} \ddot{x} &= f(t,\ x,\ \dot{x}) \\ x(t_0) &= x_0,\ \dot{x}(t_0) = \dot{x}_0 \end{aligned}\right\} \quad . \tag{12}$$

Next a transformed variable x_T is introduced, which equals the original variable minus the first m+2 terms of the power series expansion for x

$$x_T = x - \sum_{\nu=1}^{m+2} X_\nu\ (t - t_0)^\nu \tag{13}$$

$$\dot{x}_T = \dot{x} - \sum_{\nu=1}^{m+2} \nu X_\nu\ (t - t_0)^{\nu-1} \quad . \tag{14}$$

Performing this subtraction results in a function with zero derivatives for $t=t_0$ up to the m+2nd order. The following differential equation is obtained for the transformed function x_T:

$$\left.\begin{aligned} \ddot{x}_T &= f_T = f - \sum_{\nu=2}^{m+2} \nu(\nu-1)\ X_\nu\ (t - t_0)^{\nu-2} \\ x_T(t_0) &= x(t_0) = x_0,\ \dot{x}_T(t_0) = 0 \end{aligned}\right\} \quad . \tag{15}$$

That the first m+2 derivatives equal zero considerably faciliates establishment of Runge-Kutta equations of condition. Furthermore, accuracy can be of any high order desired simply by subtracting enough terms from the original function. In other words, a very simple function is always created that has zero derivatives up to the m+2nd order. Runge-Kutta formulas of any high order desired can then be obtained for this transformed function merely by choosing m sufficiently large. For example, the Runge-Kutta formulas for the transformed differential equation (15) just given would read

$$\left.\begin{aligned} k_1 &= f_T(t_0 + \alpha_1 h,\ x_0,\ 0)\ h \\ k_2 &= f_T(t_0 + \alpha_2 h,\ x_0 + \beta_0 k_1 h,\ 0 + \beta_1 k_1) h \\ k_3 &= f_T(t_0 + \alpha_3 h,\ x_0 + \gamma_0 k_1 h + \delta_0 k_2 h,\ 0 + \gamma_1 k_1 \\ &\quad + \delta_1 k_2) h \\ k_4 &= f_T(t_0 + \alpha_4 h,\ x_0 + \epsilon_0 k_1 h + \zeta_0 k_2 h + \eta_0 k_3 h,\ 0 + \epsilon_1 k_1 \\ &\quad + \zeta_1 k_2 + \eta_1 k_3) h \end{aligned}\right\} \tag{16}$$

and

$$\left.\begin{aligned} x_T &= x_0 + (C_1 k_1 + C_2 k_2 + C_3 k_3)\ h + 0(h^{m+5}) \\ \dot{x}_T &= 0 + C_1' k_1 + C_2' k_2 + C_3' k_3 + 0(h^{m+5}) \\ \hat{x}_T &= x_0 + (\hat{C}_1 k_1 + \hat{C}_2 k_2 + \hat{C}_3 k_3 + \hat{C}_4 k_4) h + 0(h^{m+6}) \end{aligned}\right\} \quad . \tag{17}$$

Three substitutions yield an accuracy in x, $\dot{x}$ to the $h^{m+4\text{th}}$ term, where m is the number of differentiations performed in equation (13) before x can be replaced by x_T. The fourth substitution in equation (16) yields the truncation error term required for automatic step-size control.

Thus an additional advantage of this approach, which also distinguishes it from any other Runge-Kutta formulas, is the very simple, economical (in computer running time) procedure for control of the truncation error. Only the first three evaluations are needed for the actual computation; the fourth evaluation gives an improved value for $\hat{x}$ which is accurate to one further power of h. By subtracting these terms, the leading term of the truncation error is represented with sufficient accuracy

$$T_x \approx x_T - \hat{x}_T = [\ (C_1 - \hat{C}_1) k_1 + (C_2 - \hat{C}_2) k_2 + (C_3 - \hat{C}_3) k_3 - \hat{C}_4 k_4]\ h\ . \tag{18}$$

Full details on these new high-order Runge-Kutta formulas are given in reference 6.

B. FORMULAS WITH AN ARBITRARILY SMALL TRUNCATION ERROR

It may be noted, without going into detail, that in more recent work the Computation Laboratory has established Runge-Kutta formulas in which a parameter σ and the absolute value of all members of the

leading term of the truncation error, for x_T as well as for $\dot{x}_T$, can be made as small as desired, but not zero since some coefficients would then become infinite. Full details on these high-order formulas with an arbitrarily small truncation error are given in reference 7.

These new formulas required again a suitable transformation of the original differential equations. This transformation is based on a power series expansion. Any desired degree of accuracy can be obtained by doing only three or four evaluations of the differential equations. This is not possible with other Runge-Kutta type formulas now found in the literature. There is a small penalty to be paid in computer time since the recurrence formulas must be evaluated. The additional computation is not great and, because the method is of high order, the integration can proceed in larger steps without impairing accuracy. The computer running time is much faster than for other known integration methods. This is shown in Section VII.

VI. SOME OTHER MODERN RUNGE-KUTTA FORMULAS

Briefly, for comparison, consider two other modern Runge-Kutta methods: the Shanks explicit method and the Butcher implicit method. To simplify comparison, both methods are presented in eighth-order form. As mentioned earlier, Shanks' formulas are available only to the eighth order. Butcher's implicit formulas are available to any order.

First, consider Shanks' explicit formulas [1, 2] for $\dot{x} = f(x)$

$$
\left.
\begin{aligned}
k_1 &= f(x_0)\,h \\
k_2 &= f(x_0 + \alpha_{21}k_1)\,h \\
k_3 &= f(x_0 + \alpha_{31}k_1 + \alpha_{32}k_2)\,h \\
k_4 &= f(x_0 + \alpha_{41}k_1 + \alpha_{42}k_2 + \alpha_{43}k_3)\,h \\
&\vdots \\
k_{12} &= f(x_0 + \alpha_{12,1}k_1 + \ldots + \alpha_{12,11}k_{11})\,h \\
\hline
x &= x_0 + \sum_{\nu=1}^{12} C_\nu k_\nu + 0(h^9)
\end{aligned}
\right\} \quad . \quad (19)
$$

Each integration step here requires 12 substitutions, k_1 through k_{12}, which are multiplied by certain weight factors and summed to obtain the new value for x. But because these formulas include no procedure for controlling the truncation error, each integration step really requires 23 substitutions if Richardson's extrapolation procedure is used for step-size control, i. e., 2·12 substitutions with one substitution omitted since the first substitution occurs twice in the computation.

In 1964, Butcher [8, 9] published two noteworthy papers on implicit Runge-Kutta methods. Following are his eighth-order formulas for $\dot{x} = f(x)$:

$$
\left.
\begin{aligned}
k_1 &= f(x_0)\,h \\
\hline
k_2 &= f(x_0 + \beta_{21}k_1 + \beta_{22}k_2 + \beta_{23}k_3 + \beta_{24}k_4)\,h \\
k_3 &= f(x_0 + \beta_{31}k_1 + \beta_{32}k_2 + \beta_{33}k_3 + \beta_{34}k_4)\,h \\
k_4 &= f(x_0 + \beta_{41}k_1 + \beta_{42}k_2 + \beta_{43}k_3 + \beta_{44}k_4)\,h \\
\hline
k_5 &= f(x_0 + \beta_{51}k_1 + \beta_{52}k_2 + \beta_{53}k_3 + \beta_{54}k_4)\,h \\
\hline
x &= x_0 + \sum_{\nu=1}^{5} C^1_\nu k_\nu + 0(h^9)
\end{aligned}
\right\} \quad . \quad (20)
$$

Unlike explicit formulas, where the increments for x: k_1, k_2, k_3, etc., are successively computed, each value depending only a previous values, implicit formulas require an iterative computation. Any increment k_ν depends not only on the preceding increments k_1, k_2, ... $k_{\nu-1}$ but also on k_ν itself and on the succeeding increments $k_{\nu+1}$, $k_{\nu+2}$, ... Naturally, this iterative computation is more involved than the straightforward procedure for explicit Runge-Kutta formulas. But implicit formulas do require considerably fewer substitutions than explicit formulas. Formulas (20) require only five substitutions, only three of which are iterative, per step versus 12 for the comparable Shanks' formulas. However, the iteration tends to be slow. This is demonstrated in Table I.

TABLE I

COMPARISON OF TWELFTH-ORDER METHODS, RESTRICTED THREE-BODY PROBLEM. (1)

Method	Final x	Final $\dot{y}$	Number of Steps	Computer Running Time (min)
RKB (2)	1.20000 00000 00013	-1.04935 75098 30328	216	0.88
PSE (3)	1.19999 99999 99981	-1.04935 75098 30303	493	0.21
RKT 1 (4)	1.20000 00000 00001	-1.04935 75098 30321	389	0.15
RKT 2 (5)	1.20000 00000 00013	-1.04935 75098 30332	290	0.13

(1) Corresponding results for Shanks' eighth-order formulas are: 1.20000 00000 00002; - 1.04935 75098 30310; 814 steps; and 0.46 minute. Note that eighth-order formulas are, of course, not competitive in speed or accuracy with twelfth-order formulas. This is far more obvious in more complex differential equations, as can already be seen in the more complex, but still relatively simple, case of the restricted four-body problem.

(2) RKB = Runge-Kutta-Butcher method [8, 9]
(3) PSE = Power series expansion method [3]
(4) RKT 1 = Runge-Kutta-transformation method [6]
(5) RKT 2 = Runge-Kutta-transformation method [7]

VII. CONCLUSIONS

As an example of the computing speeds possible with the modern methods described, the periodic orbit shown in Figure 1 was computed. The orbit has the following initial values:

$$x_0 = 1.2,\ y_0 = 0,\ \dot{x}_0 = 0,\ \dot{y}_0 = -1.04935\ 75098\ 30320 \quad (\mu = 1/82.45)\ . \tag{21}$$

To preserve sufficient accuracy for a true comparison of the methods, the initial value $\dot{y}_0$ was computed in 20-digit arithmetic. The computations were executed on an IBM 7094, Model II computer (16 digits). Table I shows the results for one complete orbit.

For all methods compared in Table I, we lose about two digits on a 16-digit computer. The Butcher method, which uses the fewest integration steps, is extremely accurate but it is also extremely slow. In all cases, however, the deviations are negligible, being of the order of 0.01 millimeter for this particular orbit. But the method is nearly seven times as fast as the Butcher method, i.e., it would cost about seven time as much in computer rentals to use the Butcher method. Thus, even in a relatively simple problem like this example, it pays to use the most efficient integration method available.

A group, headed by Mr. Mert C. Davidson, is being set up in the Computation Laboratory to explore, in detail, applications of these methods to practical problems that will exploit their full possibilities. For example, a program has already been written to solve the complete n-body problem (including oblateness terms), as a whole, with no reliance on data from relatively inaccurate external sources like ephemeris tables, etc. The computation of this problem has been rapid and highly successful. But many new worthwhile applications still need to be developed and exploited.

Finally, it should be noted that the subject matter of this paper is given much more thorough coverage in the paper New One-Step Integration Methods of High-Order Accuracy Applied to Some Problems in Celestial Mechanics, which will be published shortly by NASA.

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RECENT DEVELOPMENTS IN ANALYTICAL CELESTIAL MECHANICS

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SUMMARY

For the restricted problem of three bodies the derivation of periodic solutions different from the classically known ones is discussed and the ideas used for their existence proofs are briefly outlined and related to classical methods. In particular, the derivation of closed perturbed precessing elliptic orbits of arbitrary eccentricity and small major axis about the smaller one of the two attracting bodies with arbitrary mass ratio is indicated. Two numerical examples of recently discovered closed trajectories are included.

I. INTRODUCTION

Among the problems of celestial mechanics, which are important for space flight applications, the restricted three body problem plays a central role. This problem is concerned with the description of the possible trajectories of a particle (manned or unmanned satellite, meteorite, planetoid) of negligible mass under the gravitational attraction from two heavy celestial bodies, which are assumed revolving according to Kepler's laws on circles about each other. Limiting our attention to the two-dimensional case the equations of motion of the particle can be written in the form

$$\ddot{x} + 2i\dot{x} - x = -\nu(x+\mu)\,|x+\mu|^{-3} - \mu(x-\nu)\,|x-\nu|^{-3} \quad , \qquad (\dot{} = d/dt \ , \ i^2 = -1) \qquad (1)$$

where $x = x_1 + ix_2$ is the complex position vector of the particle, referred to a rectangular coordinate system, which rotates with unit angular velocity about the center of mass of the two heavy bodies with masses μ and $\nu = 1-\mu$ as origin.

The attempt, among others, to exhibit periodic solutions of equation (1) has received great emphasis and led to some success through the work of Hill, Poincaré, Birkhoff and others, and is still being pursued vigorously. It will be our sole concern in this presentation. The study of periodic solutions of equation (1) is of interest for several reasons. First, since the restricted three body problem presents a non-integrable dynamical system, every contribution toward an understanding and a description of its general solution afforded by particular solutions is highly welcome. Second, recent advances by Kolmogorov, Moser and Arnold in the areas of stability and almost periodic motions have led to an understanding of the behavior of dynamical systems in the vicinity of its periodic motions. And third, some periodic solutions of equation (1) are of great practical interest in dynamical astronomy or in space flight mechanics.

Our present knowledge of periodic solutions of equation (1) is still modest. Without discussing the classically known solutions, which are either near the libration points, or are close to circular solutions (for small $\mu > 0$, or, for arbitrary μ, when near one of the masses or far away from both masses), or which are inside a closed zero-velocity oval about the heavier mass closing only after many revolutions, etc., we will give a description of some recently discovered periodic solutions and of the ideas used for their existence proofs. These new solutions are characterized by their relationship to Keplerian elliptic motions of positive and possibly large eccentricities, presenting relative to equation (1), a situation which classical researchers attempted in vain to illuminate although they had essentially created the methods with which to attack such problems.

II. PROBLEM

Let us describe our problem. Equation (1) approximates the equation of motion for the Kepler problem (two body problem with one mass transformed to rest)

$$\ddot{z} = -mz\,|z|^{-3} \ , \quad m > 0 \qquad (2)$$

in an inertial coordinate system after a rotation $z = e^{it}x$; for example, if $\mu > 0$ is very small, $m = \nu$

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and the particle does not approach the smaller body too close (planetary case), or when the particle moves in the vicinity of the smaller one of the attracting bodies (satellite case), etc. Now equation (2) has elliptic solutions, say with major half axis $a > 0$ and eccentricity ϵ, $0 < \epsilon < 1$. Such a solution is closed in the rotating x_1, x_2 coordinate system, if its period $T_0 = 2\pi a^{3/2}$ is a rational multiple of the period 2π of e^{it}, i.e., if $a = (m/k)^{2/3}$ with integers m, $k \neq 0$. Under this assumption the resulting rotating elliptic orbit to be described by $x = x^*(t)$ is closed after $k - m$ revolutions about the origin, having the period $T^* = 2\pi m$. The problem is to find periodic solutions $x = x(t)$ of equation (1) which are near $x^*(t)$.

This problem can, in the simpler planetary case, be solved with the classical methods devised by Poincaré. The only additional idea needed consists of the application of an appropriate periodicity condition instead of the classical condition of return of the motion to its initial state after time $T > 0$. This classical condition leads to a singular case, even after reduction with the help of the Jacobian integral of equation (1), when applied to the generation of periodic solutions of equation (1) from the above $x^*(t)$. The difficulty can be overcome by using the condition

$$x(t) = \text{real}, \ \dot{x}(t) = \text{pure imaginary at } t = 0 \text{ and } t = \tfrac{1}{2}T > 0 \qquad (3)$$

for a solution $x = x(t)$ of equation (1). Equation (3) implies that the curve $x = x(t)$ $(0 \leq t \leq T)$ becomes symmetric about the real axis of the x-plane and closed. Thus, $x(t)$ becomes periodic with period T. To satisfy equation (3), the solutions x of equation (1) represent analytic functions not only of t, but also of the parameter μ in equation (1) and of the initial position and velocity coordinates

$$x_j(0) = \xi_j, \ \dot{x}_j(0) = \eta_j, \ (j = 1,2). \qquad (4)$$

Then equation (3) can be rewritten as

$$x_2(\tfrac{1}{2}T, \xi_1, \xi_2, \eta_1, \eta_2, \mu) = \dot{x}_1(\tfrac{1}{2}T, \xi_1, \xi_2, \eta_1, \eta_2, \mu) = 0,$$

$$\xi_2 = \eta_1 = 0 \qquad (5)$$

giving two scalar real equations for the unknowns T, ξ_1, η_2. When $\mu = 0$, these equations have a known solution (say T^*, ξ_1^*, η_2^*) belonging to the generating solution $x^*(t)$ of equation (1) with $\mu = 0$ after proper choice of its initial values. Since the respective Jacobian determinant with respect to T and η_2 does not vanish (to establish this fact constitutes the decisive part of the existence proof), the implicit function theorem leads to the existence of solutions for T, ξ_1, η_2 of equation (5) near T^*, ξ_1^*, η_2^* for sufficiently small $\mu > 0$ and thus to periodic solutions of equation (1) near $x^*(t)$ [1].

By proper choice of m, k, and ϵ above the precessing elliptic orbit $x^*(t)$, $(0 \leq t \leq T^*)$ can be made to pass the attractive bodies at prescribed small distances and this property will still hold for the resulting periodic trajectories $x(t)$, $(0 \leq t \leq T)$ of equation (1) with $\mu > 0$, since μ is small. Such trajectories are of great astronautical interest for space flight in the Earth-Moon system.

III. SATELLITE CASE

We now come to the more difficult satellite case of our problem. This time we need new ideas to show that periodic solutions $x(t)$ of equation (1) near $x^*(t)$ exist (since $\nu > 0$ is small, for instance) and the periodic motion is to take place in the near vicinity of the body of mass ν (called planet), where the disturbance exerted by the other more massive body, being nearly at rest in the inertial coordinate system, causes large deviations from Keplerian motion for the third body near the revolving planet. If $x^*(t)$ is simplified to a circular solution by putting $\epsilon = 0$ and dispensing with the condition that $T_0/2\pi$ be rational, our problem has been solved already in different ways by Hill, Brown, Moulton, Wintner and Siegel. But these results, which are based on power series expansions of the coordinates x_1, x_2 in powers of the small period, give no indication of the existence of periodic solutions near $x^*(t)$ with $\epsilon > 0$ and small $a = (m/k)^{2/3}$. The first result in the direction of the present problem, although only in the planetary case, was obtained by Birkhoff using the Poincaré - Birkhoff fixed point theorem for annulus mappings, and more recently by Moser, who used the Birkhoff - Siegel fixed point theorem for local area-preserving mappings to get a more accurate description of the location of the obtained solutions of equation (1). A similar result in the satellite case, giving for each sufficiently small value of the Jacobian integral the

existence of countably many periodic solutions which close only after many revolutions about the planet, was recently established by Conley using the Poincaré-Birkhoff fixed point theorem and a new construction of the classically known nearly circular solutions mentioned above. These periodic solutions of equation (1) still lack a more accurate geometrical description. Adequate references to the literature are contained in references 1 and 2.

We shall now give a brief description of the ideas and techniques which lead to the existence of countably many families of 1-parametric solutions $x(t)$ near $x^*(t)$ in the satellite case of equation (1), belonging to sufficiently small values of $m/k = a^{3/2}$ and having the family parameter ϵ which ranges over suitable closed intervals contained on $0 < \epsilon < 1$. These solutions exist for all $0 < \nu < 1$.

We transform equation (1) by a translation $w = x + \mu$ into

$$\ddot{w} + 2i\dot{w} - w + \nu w|w|^{-3}$$

$$= -\mu(1+(w-1)|w-1|^{-3}) = P_o(w). \qquad (6)$$

Here the right hand term $P_o(w)$ is the disturbing function, which vanishes at $w = 0$, i.e., at the location of the small planet of mass ν. At $w = 0$, the left side is singular, however. Replacing $P_o(w)$ by 0 in equation (6) leads, after the rotation $z = e^{it}w$, to equation (2) with $m = \nu$. Thus equation (6) is close to the integrable Kepler problem for small $|w|$ even though $\mu = 1 - \nu$ is not small. Again, use of the periodicity conditions of equation (3), with x replaced by w, is decisive and assures a non-vanishing Jacobian relative to the unperturbed elliptic motion $x^*(t)$. Rewriting equation (3) in the form of equation (5), however, is of no use since now μ is not considered as a small available parameter, but is fixed and nearly 1. Despite this we shall solve equation (3) with equation (4) for T and η_2, as in the case of equation (5), with the help of an implicit function theorem by application of the following idea.

We replace in the right side of equation (6) the given function $P_o(w)$ by an arbitrary function $P(w)$ from a suitable set F of functions, which contains $P_o(w)$ especially. The resulting solutions $w(t)$ (or $x(t) = w(t) - \mu$ just as well) then depend upon their initial values and upon $P(w)$. Therefore equation (3) can be rewritten in the form of equation (5), but with μ replaced by P if P is the name of the chosen function $P(w)$ from F. Now P can be considered as a generalized parameter varying over F, instead of the real parameter μ. When $P = 0$ (the zero element in F), equation (5) again has a known solution (say T^*, ξ^*_1, η^*_2) determined by $x^*(t)$. Thus, using an appropriate implicit function theorem we arrive at the existence of solutions T, ξ_1, η_2 of equation (5) near T^*, ξ^*_1, η^*_2 for sufficiently small P of F (say for $\|P\| \leq r^*$) after having introduced a suitable norm $\|..\|$ on F. But then it is decisive that the given P_o in equation (6) satisfies $\|P_o\| \leq r^*$ to obtain periodic solutions of equation (6). Since P_o is not available to choice, a lower estimate is needed for r^* and not merely the existence of an $r^* > 0$ with the above property.

The derivation of this estimate requires not only a precise application of an implicit function theorem, but also sufficiently accurate knowledge of the general solution of equation (6) for initial values near ξ^*_j, η^*_j, $(j = 1,2)$ and a time range at least as large as the anticipated period T near T^*, so that sharp estimates of the perturbation of $w(t)$ from Keplerian motion $w^*(t)$ can be obtained. For this purpose the solutions of equation (6) have to be constructed by a suitable method of perturbation theory. Because of the singularity of equation (6) at $w = 0$ the motion of the satellite is considered in an annulus about the planet which contains the precessing Keplerian elliptic orbit $w^*(t)$. The value of $\|P_o\|$ depends on the size of this annulus and thus on m, k, ϵ, and μ. The main difficulty arises from the fact that $\|P_o\|$ becomes small along $w^*(t)$ only, when $a = (m/k)^{4/9}$ becomes small, leading to an increase of the required range (from 0 to at least $2\pi k$) of the independent variable, for which the eccentric or the true anomaly can be taken. But thereby the above r^* decreases with decreasing m/k, almost defeating our goal $\|P_o\| \leq r^*$. This difficulty does not appear in the classical case of circular $w^*(t)$, or $\epsilon = 0$, mentioned earlier.

IV. CONCLUSION

Summarizing, we can say that generalization of Poincaré's small parameter method to the non-parametric case by considering the disturbing function itself as a generalized small parameter belonging to a normed function space leads to applicability of classical methods again, and, together with suitable periodicity conditions, for example equation (3), and with sufficiently accurate convergent methods of perturbation theory, to an existence proof for periodic solutions $x(t)$ of equation (1) near $x^*(t)$ in the elliptic satellite case with arbitrary μ in $0 < \mu < 1$ [2].

Finally, by numerical extension of the latter families of periodic orbits to greater distances from the small planet, interesting new trajectories of the restricted three body problem have been found which do not belong to the satellite case or to the planetary case. Some of these pass repeatedly near both attractive bodies [3]; others have been found by my collaborator, M. C. Davidson [4]. Among the latter ones are trajectories which demonstrate the phenomenon of temporary capture with satellite motion about each one of the attractive bodies and periodically alternating transitions from the vicinity of one body to the vicinity of the other. Two examples (Figs. 1 and 2) will be included here. They are drawn in the rotating x_1, x_2 coordinate system under the assumption that the attractive bodies represent earth E and moon M with $\mu = 0.0123 \approx 1/82$. They constitute numerical solutions of equation (1).

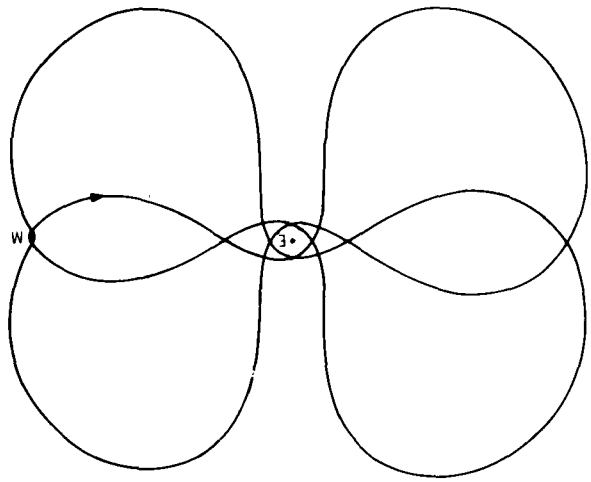

FIGURE 1

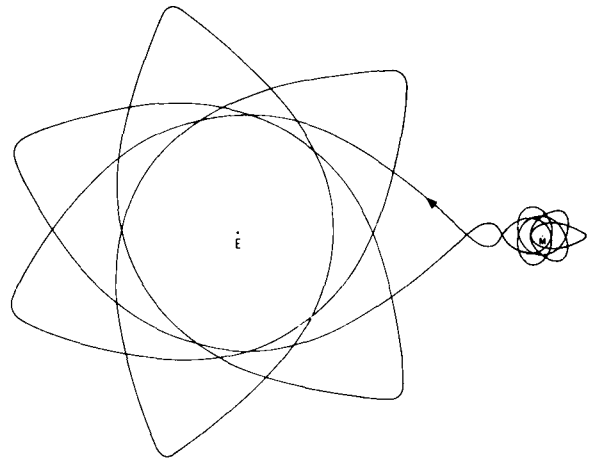

FIGURE 2

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By C. L. Bradshaw, Erwin Fehlberg and Richard F. Arenstorf

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