STABILITY OF INTERSTELLAR GAS AND FIELD INCLUDING ROTATION

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Preprint Number
EFINS-66-48

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ABSTRACT

The stability of the interstellar gas to perturbations in the gas pressure, magnetic field and cosmic ray pressure is discussed from a hydromagnetic point of view taking rotation into account. For a particular model, it is shown that the system is unstable to transverse perturbations in the sense described by Parker (1966). This result holds true for a uniformly rotating system bound in a stellar parabolic gravitational well provided only that the Coriolis acceleration is everywhere less than the stellar gravitational acceleration. The tendency for the interstellar gas to drain along the lines of force into the lowest regions of magnetic field is reduced, but not removed, by rotation. The e-folding time is of the order of the free-fall time under an effective gravity $(g - r\Omega^2)$.

The results are derived for a particular model, but their physical content should be of general validity for all systems.

1. INTRODUCTION

In a recent paper (Parker, 1966), the possible galactic magnetic field configurations were discussed from a theoretical point of view. Both the magnetic field and cosmic ray gas exert pressures of the order of 10^{-12} dynes cm^{-2}. It was assumed that the intergalactic medium (if any) exerted a negligible pressure compared to either the interstellar magnetic field or the cosmic ray gas.

It was shown that the interstellar gas, cosmic ray, magnetic field system, which is held in a stellar gravitational field, was unstable to a Rayleigh-Taylor instability of such a nature that the interstellar gas tended to concentrate in "pockets" suspended in the magnetic field. Several possible configurations were discussed by Parker; e.g., plane and circular geometries, twisted and uniform fields. No matter which particular situation was chosen, nor how the gravitational field varied--within wide limits-- it was found that the system was always unstable with a growth time of the order of the free-fall time.

It occurred to the author that one possible stabilizing influence which had been neglected was the Coriolis force due to galactic rotation. The reason that we expect rotation to slow down or stop the instability is that the Coriolis force will try to eject material from the pockets.

In order to be specific, we will examine the stability of a cylindrical two-dimensional atmosphere which possesses an ambient magnetic field, $B_o(r)$, whose lines of force are concentric circles about the origin. Here r is the radial distance from the origin. While this case is not general, it is sufficient to show the tendency of rotation to stabilize the system.

The results obtained below show that the Coriolis force will reduce the rate of instability, but will not stop it. It is shown that the e-folding time is of the order of the free-fall time appropriate to an effective gravity $(g - r\Omega^2)$.

2. GENERAL ANALYSIS

We choose our particular geometry to be an infinitely long cylinder which we treat as a two-dimensional atmosphere in the sense that all quantities are independent of Z and depend only upon r and θ where (r, θ, Z) are the usual cylindrical coordinates. The ambient magnetic field can then be written

$$\underset{\sim}{B}_o = \left(0, B_o(r), 0\right) . \tag{1}$$

The composition of matter in the cylinder is taken to be three-fold.

1. We assume some cylindrically symmetric spatial distribution of stars exists throughout the cylinder. These stars provide a gravitational acceleration, $g(r)$, which is purely in the radial direction. Apart from providing the gravity, the stars are assumed to take no part in any motion.

2. We assume that there is an isothermal distribution of gas embedded in the magnetic and gravitational fields. This gas has r.m.s. velocity u in any given direction and an equilibrium mass density $\rho_0(r)$. Thus, the equilibrium gas pressure is $p_0(r) = \rho_0(r)\, u^2$. This thermal gas is assumed to have a sufficiently low mass density that its contribution to the gravitational field of the system is negligible. In fact, including self-gravitation would just enhance the rate of instability.

3. We assume that the system also contains a cosmic ray gas whose equilibrium pressure is $P_o(r)$. It is further assumed that the cosmic ray gas is so hot as to be essentially unaffected by the stellar gravitational field. Further, we shall be concerned only with perturbations whose phase velocities are considerably smaller than the sound speed in the pure cosmic ray gas ($\sim c/\sqrt{3}$). The volume of a tube of force does not vary to first order in a perturbation. Hence.

$$\frac{d\,\delta P}{dt} = 0 \quad , \tag{2}$$

where $\delta P(r, \theta, t)$ is the first order perturbation in the cosmic ray pressure.

In order to include rotation, we assume that the cylinder of material is rotating with an equilibrium velocity of rotation, $V_o(r)$, which is taken to be parallel to the unperturbed magnetic field lines. The equilibrium pressure condition is

$$\frac{d}{dr}\left(p_0 + P_0 + B_0^2/8\pi\right) + \frac{B_0^2}{4\pi r} = -\rho_0\left(g - V_0^2 r^{-1}\right). \tag{3}$$

In the cases which we shall discuss, the cylinder will be on infinite, rather than finite, extent in the (r, θ) plane. Hence, the perturbation quantities tend to zero as $r \to \infty$ independently of θ, in order to preserve physical sense.

The equations of motion for the thermal gas can be written

$$\rho\left[\frac{\partial \underset{\sim}{v}}{\partial t} + (\nabla \times \underset{\sim}{v}) \times \underset{\sim}{v} + \frac{1}{2}\nabla(\underset{\sim}{v}\cdot\underset{\sim}{v})\right] = -\nabla\left(p + P + \underset{\sim}{B}\cdot\underset{\sim}{B}/8\pi\right) + \frac{1}{4\pi}(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B} - \underset{\sim}{g}\,\rho \;. \tag{4}$$

We introduce a transverse velocity perturbation $\delta\underset{\sim}{v}\,(r, \theta, t) \equiv (\delta v_r, \delta v_\theta, 0)$ and a perturbation magnetic vector potential, $\delta\underset{\sim}{A}\,(r, \theta, t)$, which will take to be functions of r, θ, and t only. We also take $\delta\underset{\sim}{A} = \hat{z}\delta A$ where $\hat{z}$ is a unit vector in the Z-direction. The linearized equations of motion for the thermal gas then become

$$\rho_0\left(\frac{\partial \delta v_r}{\partial t} + \Omega\frac{\partial \delta v_r}{\partial \theta} - 2\Omega\, \delta v_\theta\right)$$
$$= -\frac{\partial}{\partial r}\left(\delta p + \delta P\right) - g\,\delta\rho + \frac{1}{4\pi}\left[B_0 \nabla^2 \delta A + \frac{1}{r}\frac{\partial \delta A}{\partial r}\frac{\partial}{\partial r}(rB_0)\right], \tag{5}$$

$$\rho_0\left(\frac{\partial \delta v_\theta}{\partial t} + \Omega\frac{\partial \delta v_\theta}{\partial \theta} + 2\Omega\,\delta v_r + r\frac{d\Omega}{dr}\,\delta v_r\right)$$
$$= -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\delta p + \delta P\right) + \frac{1}{4\pi r^2}\frac{\partial \delta A}{\partial \theta}\frac{\partial}{\partial r}(rB_0) . \tag{6}$$

Here δp is the perturbation to the thermal gas pressure and the angular frequency of rotation, Ω, is defined by $V_0 = r\Omega$.

The conservation of mass is described by

$$\frac{\partial \delta\rho}{\partial t} + \Omega\frac{\partial \delta\rho}{\partial \theta} + \frac{d\rho_0}{dr}\,\delta v_r + r^{-1}\rho_0\left[\frac{\partial}{\partial r}(r\delta v_r) + \frac{\partial \delta v_\theta}{\partial \theta}\right] = 0 . \tag{7}$$

From Eq. (2), we have that

$$\frac{\partial \delta P}{\partial t} + \Omega\frac{\partial \delta P}{\partial \theta} + \delta v_r\frac{dP_0}{dr} = 0 . \tag{8}$$

We make the assumption that the pressure variations in each element of thermal gas are adiabatic so that

$$\delta p/p_0 = \gamma\,\delta\rho/\rho_0 , \tag{9}$$

where γ is a constant. Thus

$$\frac{\partial \delta p}{\partial t} + \Omega \frac{\partial \delta p}{\partial \theta} + \delta v_r \frac{dp_0}{dr} + \gamma r^{-1} p_0 \left[\frac{\partial}{\partial r}(r \delta v_r) + \frac{\partial \delta v_\theta}{\partial \theta} \right] = 0 \quad . \tag{10}$$

Since the system under discussion is a two-dimensional atmosphere in the (r, θ) plane, we must have the net force in the Z-direction vanishing. Hence

$$\frac{\partial \delta A}{\partial t} = \delta v_r B_0 - \Omega \frac{\partial \delta A}{\partial \theta} \quad . \tag{11}$$

In order to exhibit the basic instabilities due to the presence of rotation, magnetic field and cosmic ray gas, we shall consider three simple situations.

a) We assume that only the rotating thermal gas is present and that it possesses a finite pressure.

b) We assume that the thermal gas is completely cold and that no cosmic ray gas is present. Thus the only force comes from the magnetic field. This system is allowed to rotate since it is known (Parker, 1966) that, in the absence of rotation, this system is unstable to transverse waves.

c) We assume that the thermal gas is completely cold and that the magnetic field pressure is an infinitesimal fraction of the cosmic ray pressure. Thus, the field exerts no force and serves only to couple the cosmic ray gas to the interstellar medium. In this case, the only force comes from the cosmic ray pressure. This system is known to be unstable, in the absence of rotation, to transverse waves, so we allow rotation. We shall see later that this system is ill-determined in some sense to be specified. This will result in an inhomogeneous Riemann-Hilbert problem which is extremely difficult to solve. As a consequence, we will make several restrictive approximations. Nevertheless, it will still be possible to see how rotation alters this otherwise unstable system.

In all three cases, we assume that the cylinder has constant angular frequency, i.e., $d\Omega/dr = 0$. While this is a restrictive condition, it simplifies the analysis. It suffices to show how rotation affects the three systems which are known to be unstable if $\Omega = 0$.

We shall consider each of the above situations in turn.

3. THERMAL GAS INCLUDING CONSTANT ROTATION

Here we set $B_0 = \delta P = P_0 = \delta A = d\Omega/dr = 0$. In such a case, the appropriate linearized equations of § 2 become

$$\frac{dp_0}{dr} = -\rho_0\left(g - r\Omega^2\right) , \tag{12}$$

$$\rho_0\left(\frac{\partial \delta v_r}{\partial t} + \Omega \frac{\partial \delta v_r}{\partial \theta} - 2\Omega\, \delta v_\theta\right) = -\frac{\partial \delta p}{\partial r} - g\,\delta\rho , \tag{13}$$

$$\rho_0\left(\frac{\partial \delta v_\theta}{\partial t} + \Omega \frac{\partial \delta v_\theta}{\partial \theta} + 2\Omega\delta v_r\right) = -\frac{1}{r}\frac{\partial \delta p}{\partial \theta} \quad , \tag{14}$$

$$\frac{\partial \delta\rho}{\partial t} + \Omega\frac{\partial \delta\rho}{\partial \theta} + \delta v_r \frac{d\rho_0}{dr} + r^{-1}\rho_0\left[\frac{\partial}{\partial r}(r\delta v_r) + \frac{\partial \delta v_\theta}{\partial \theta}\right] = 0 \, , \tag{15}$$

$$\frac{\partial \delta p}{\partial t} + \Omega\frac{\partial \delta p}{\partial \theta} + \delta v_r \frac{dp_0}{dr} + \gamma r^{-1} p_0\left[\frac{\partial}{\partial r}(r\delta v_r) + \frac{\partial \delta v_\theta}{\partial \theta}\right] = 0 \, . \tag{16}$$

We now assume that all perturbation quantities vary as

$$f(r)\exp\left[i(m\theta - \omega t)\right] \quad , \tag{17}$$

where m is integer and, for instability, we require $\mathrm{Im}(\omega) > 0$.

Making use of Eqs. (15), (16) and (17), we can eliminate $\delta p(r)$ and $\delta\rho(r)$ from Eqs. (13) and (14) in favor of $\delta v_r(r)$ and $\delta v_\theta(r)$. From the two resulting equations, we eliminate $\delta v_\theta(r)$ in favor of $\delta v_r(r)$. Upon so doing, it can be shown, after some algebra, that δv_r satisfies the equation

$$\begin{aligned}
&\gamma p_0 \Psi^{-1}(1 + imBr^{-1})\,\delta v_r'' \\
&+\delta v_r'\begin{bmatrix} 2i\Omega B\rho_0 + g\rho_0\Psi^{-1}(1+imBr^{-1}) \\ +\Psi^{-1}\left(p_0' + \gamma p_0 r^{-1} + \gamma p_0' + im\gamma p_0' r^{-1}B + im\gamma p_0 r^{-1}(A+B') - im\gamma p_0 B r^{-2}\right)\end{bmatrix} \\
&+\delta v_r\begin{bmatrix} 2i\Omega A\rho_0 + \rho_0\Psi + g\Psi^{-1}\left(\rho_0' + \rho_0 r^{-1} + im\rho_0 A r^{-1}\right) \\ -\Psi^{-1}\left(p_0'' + \gamma p_0' r^{-1} - \gamma p_0 r^{-2} + im\gamma p_0' A r^{-1} - im\gamma p_0 A r^{-2} + im\gamma p_0 A' r^{-1}\right)\end{bmatrix} = 0 \, ,
\end{aligned} \tag{18}$$

where

$$i\left(\rho_0\Psi^2 - m^2\gamma p_0 r^{-2}\right)A = m\left(p_0' r^{-1} + \gamma p_0 r^{-2}\right) - 2\Omega\rho_0\Psi \quad , \tag{19}$$

$$ir(\rho_0\psi^2 - m^2\gamma p_0 r^{-2})B = m\gamma p_0 , \tag{20}$$

$$\psi = m\Omega - \omega , \tag{21}$$

and primed quantities denote differentiation w.r.t. r., i.e.,

$$p_0' \equiv dp_0/dr \quad , \text{ etc.}$$

If we set $\Omega = 0$ in Eq. (18) and search for modes with $\omega = i\tau^{-1}$ and τ large it is a simple matter to show that Eq. (18) reduces to

$$g\rho_0\delta v_r\,(\gamma\omega)^{-1}(\gamma\rho_0'/\rho_0 - p_0'/p_0) = 0 \ . \tag{22}$$

Since the interstellar gas is taken to be isothermal

$$\rho_0'/\rho_0 = p_0'/p_0 \ . \tag{23}$$

Thus, a marginally stable solution exists if, and only if, $\gamma = 1$. In fact, if $\gamma \leq 1$, it is well known that the system is unstable.

In the case where Ω is finite, we search for marginally stable solutions in the form

$$\psi \equiv m\Omega - \omega = -i\tau^{-1}$$

and τ large. Then it can be seen that Eq. (18) reduces to

$$g\rho_0'\psi^{-1}\delta v_r\,(\gamma - 1) = 0 , \tag{24}$$

where use has been made of Eq. (23). Thus, once again, a marginally stable solution exists only if $\gamma = 1$. For $\gamma > 1$, it can be shown that the system is stable provided $g > r\Omega^2$. This conditional stability arises because

$$\rho_0'/\rho_0 = -u^{-2}(g - r\Omega^2) \, . \tag{25}$$

It is clear that in order to define the cylinder, we require some form of central condensation in density. Hence, we require $g > r\Omega^2$ at all points if the material is to be bounded. If $g < r\Omega^2$, it is clear that we have infinite density as $r \rightarrow \infty$ and this is a non-physical situation.

Having demonstrated that the rotating interstellar gas is stable by itself, unless $\gamma \leq 1$, the obvious question to ask is whether the magnetic field and cosmic ray gas destabilize this system in the case of constant, but finite, angular frequency.

To exhibit the basic response of the interstellar gas due to the magnetic field and cosmic ray gas, it suffices to treat the interstellar gas as completely cold, i.e. $u^2 = 0$.

4. COLD GAS INCLUDING ROTATION AND MAGNETIC FIELD

Here, we ignore the cosmic ray gas pressure and the thermal gas pressure so that we have $p_0 = \delta p = \delta P = P_0 = d\Omega/dr = 0$.

In this case, the appropriate linearized equations of § 2 may be written

$$\frac{1}{2}\frac{d}{dr}(B_0^2) + B_0^2 r^{-1} = -4\pi\rho_0\left(g - r\Omega^2\right) , \tag{26}$$

$$\rho_0\left(\frac{\partial\delta v_r}{\partial t} + \Omega\frac{\partial\delta v_r}{\partial\theta} - 2\Omega\,\delta v_\theta\right) = -g\,\delta\rho + (4\pi)^{-1}\left[B_0\nabla^2\delta A + \frac{1}{r}\frac{\partial(\delta A)}{\partial r}\frac{\partial}{\partial r}(rB_0)\right] , \tag{27}$$

$$\rho_0\left(\frac{\partial\delta v_\theta}{\partial t} + \Omega\frac{\partial\delta v_\theta}{\partial\theta} + 2\Omega\,\delta v_r\right) = (4\pi r^2)^{-1}\frac{\partial\delta A}{\partial\theta}\frac{\partial}{\partial r}(rB_0) , \tag{28}$$

$$\frac{\partial\delta\rho}{\partial t} + \Omega\frac{\partial\delta\rho}{\partial\theta} + \delta v_r\,\rho_0' + r^{-1}\rho_0\left[\frac{\partial}{\partial r}(r\,\delta v_r) + \frac{\partial\delta v_\theta}{\partial\theta}\right] = 0 , \tag{29}$$

$$\frac{\partial\delta A}{\partial t} + \Omega\frac{\partial\delta A}{\partial\theta} = B_0\,\delta v_r . \tag{30}$$

If we again assume that all first order perturbation quantities vary as

$$f(r)\exp\left[i(m\theta - \omega t)\right] , \tag{31}$$

we can eliminate all the perturbation quantities in Eqs. (27) through (30) in favor of $\delta A(r)$. Upon doing so, it can be shown, after some algebra, that $\delta A(r)$ is determined from the equation

$$\begin{aligned}
&\delta A'' + \delta A'\left(2r^{-1} + B_0' B_0^{-1} + 4\pi g\rho_0 B_0^{-2}\right) \\
&+ \delta A\Big\{-m^2 r^{-2} + 4\pi\rho_0\psi^2 B_0^{-2} + 2\Omega\psi^{-1}\left[m r^{-1}\left(r^{-1} + B_0' B_0^{-1}\right) - 8\pi\Omega\rho_0\psi B_0^{-2}\right] \\
&+ g\Big[4\pi B_0^{-2}\left(\rho_0' + \rho_0 r^{-1}\right) - 4\pi\rho_0 B_0' B_0^{-2} - m^2 r^{-2}\psi^{-2}\left(r^{-1} + B_0' B_0^{-1}\right) \\
&+ 8\pi\Omega m\rho_0\left(rB_0^2\psi\right)^{-1}\Big]\Big\} = 0 ,
\end{aligned} \tag{32}$$

where, as before, $\psi = m\Omega - \omega$.

In order to make a direct comparison with the results obtained by Parker for this case, in the absence of rotation, we investigate Eq. (32) under the assumption of constant Alfven velocity. Then

$$2 B_0' B_0^{-1} = \rho_0' \rho_0^{-1} \tag{33}$$

Use of Eq. (33) in (26) enables us to write

$$\rho_0' + 2\rho_0 r^{-1} = -2\rho_0 V^{-2}(g - r\Omega^2) \tag{34}$$

where V^2 = constant = $B_0^2/(4\pi\rho_0)$.

We further restrict the system by demanding that the distribution of stars be chosen so that the interstellar gas is bounded in a parabolic gravitational potential well. Then

$$g = g_0 (r/R) \quad . \tag{35}$$

We also require $g_0 > R\Omega^2$ in order that the cylinder have some physical meaning. In such a case, we see that Eq. (34) yields

$$\rho_0(r) = \rho_c (R/r)^2 \exp\left[-r^2 V^{-2}(g_0 R^{-1} - \Omega^2)\right] \quad . \tag{36}$$

Thus the cylinder is of infinite radius. The solution of Eq. (32) required is that which gives $\delta A(r)$ regular at $r = 0$ and $\delta A(r) \to 0$ as $r \to \infty$.

Use of Eqs. (33), (34) and (35) enables Eq. (32) to be written

$$\delta A'' + \delta A'(r^{-1} + ar) + \delta A\,(b - cr^2 - m^2 r^{-2}) = 0 \quad , \tag{37}$$

where

$$a = \Omega^2 V^{-2}, \quad c = g_0 (RV^4)^{-1} (g_0 R^{-1} - \Omega^2)$$

and

$$b = \psi^2 V^{-2} - 2\Omega^2 V^{-2} \psi^{-1}(2\psi - m\Omega) + g_0 R^{-1} m^2 (g_0 R^{-1} - \Omega^2)(\psi V)^{-2}.$$

The solution of Eq. (37) which is regular at r = 0, is shown in Appendix I to be

$$\delta A(r) = \text{constant} \times r^m e^{\mu r^2} \times {}_1F_1\left(\tfrac{1}{2}m + \frac{(\mu + b)}{2(4\mu + a)}, m+1, -(2\mu + a/2)r^2\right). \tag{38}$$

Here μ is one of the two values

$$\mu = (4V^2)^{-1}\left\{-\Omega^2 \pm \sqrt{[\Omega^4 + 4g_0 R^{-1}(g_0 R^{-1} - \Omega^2)]}\right\} , \tag{39}$$

and ${}_1F_1$ (α, β, γ) is the degenerate hypergeometric function. In Appendix I we have given some of the asymptotic properties of ${}_1F_1$ (α, β, γ) for various limiting values of α, β or γ. These properties are made use of below.

If we choose the lower sign in Eq. (39) it can be seen that, for b finite, as $r \to \infty$ we have

$$\delta A(r) \sim \exp\left(-\frac{r^2\Omega^2}{4V^2}\right) \times r^{-\left[1+\frac{(\mu+b)}{(4\mu+a)}\right]} \tag{40}$$

Thus $\delta A(r) \to 0$ as $r \to \infty$ for this μ.

If we now look for the behavior of Eq. (38) with $\omega = m\Omega + i\tau^{-1}$ and τ real, positive and sufficiently large, we see that

$$\frac{m}{2} + \frac{(\mu+b)}{2(4\mu+a)} \simeq \frac{-g_0 m^2\tau^2(g_0R^{-1}-\Omega^2)}{2R(4V^2\mu+\Omega^2)} > 0 \tag{41}$$

since $(4\mu V^2 + \Omega^2) < 0$ because μ is the solution of Eq. (39) with the lower sign.

For any small enough r, we see that $\delta A \sim r^m e^{-\lambda r^2}$ where $\lambda > 0$.

Also for any finite, but reasonably large, r, we can always choose a τ such that

$$ {}_1F_1\left(\frac{g_0m^2\tau^2(g_0R^{-1}-\Omega^2)}{2R\sqrt{}}, m+1, \frac{r^2\sqrt{}}{2V^2}\right) \sim \frac{e^{r^2\sqrt{}/4V^2}}{r^{m+1/2}} \cos\left[\frac{mr\tau g_0\sqrt{(1-R\Omega^2g_0^{-1})}}{VR} - m\pi/2 - \pi/4\right], \tag{42}$$

where

$$\sqrt{} \equiv \sqrt{[\Omega^4 + 4g_0R^{-1}(g_0R^{-1}-\Omega^2)]}\,. \tag{43}$$

Hence, for reasonably large, but finite, r, we can always find a τ such that

$$\delta A(r) \sim \frac{e^{-\Omega^2 r^2/4V^2}}{r^{1/2}} \cos\left[\frac{m r \tau g_0 R^{-1}}{V}\sqrt{(1 - R\Omega^2 g_0^{-1})} - \pi m/2 - \pi/4\right] . \tag{44}$$

It is evident from Eqs. (38) and (44) that δA(r) is well behaved satisfying δA(0) = 0 and oscillating with decreasing amplitude with increasing radius. Within the limitation of the boundary conditions, the system is unstable for sufficiently large, real, positive τ which are given by

$$\omega = m\Omega + i\tau^{-1} \tag{45}$$

The wavelength of the oscillation is increased over its value in the absence of rotation by the factor $(1 - R\Omega^2 g_0^{-1})^{-1/2}$. Thus the e-folding time for a fixed wavelength is increased over that which obtains in the absence of rotation by the same factor. This result is subject to the condition, $g_0 > R\Omega^2$.

5. COLD GAS INCLUDING ROTATION AND COSMIC RAYS

In this case we neglect the thermal gas pressure and magnetic field in favor of the cosmic ray pressure. Thus we set $p_0 = \delta p = B_0 = \delta A = d\Omega/dr = 0$. The appropriate linearized equations of § 2 now become

$$P_0' = -\rho_0 (g - r\Omega^2) \quad , \tag{46}$$

$$\rho_0 \left(\frac{\partial \delta v_r}{\partial t} + \Omega \frac{\partial \delta v_r}{\partial \theta} - 2\Omega \delta v_\theta \right) = -g\delta\rho - \frac{\partial \delta P}{\partial r} , \tag{47}$$

$$\rho_0 \left(\frac{\partial \delta v_\theta}{\partial t} + \Omega \frac{\partial \delta v_\theta}{\partial \theta} + 2\Omega \delta v_r \right) = -\frac{1}{r} \frac{\partial \delta P}{\partial \theta} , \tag{48}$$

$$\frac{\partial \delta\rho}{\partial t} + \Omega \frac{\partial \delta\rho}{\partial \theta} + \delta v_r \rho_0' + r^{-1} \rho_0 \left[\frac{\partial}{\partial r} (r \delta v_r) + \frac{\partial \delta v_\theta}{\partial \theta} \right] = 0 , \tag{49}$$

$$\frac{\partial \delta P}{\partial t} + \Omega \frac{\partial \delta P}{\partial \theta} + \delta v_r P_0' = 0 . \tag{50}$$

In this section of the paper, we will again assume a particular form for the gravitational acceleration. In fact, we let g(r) be given by Eq. (35). It is a relatively simple matter to show that, with all perturbation quantities varying as

$$f(r,t)\, e^{im\theta} \tag{51}$$

we can write the equation for $\delta P(r, t)$ in the form

$$D^4 \delta P + 4\Omega^2 D^2 \delta P + 2im\Omega g_0 R^{-1} (2 - R\Omega^2 g_0^{-1}) D \delta P - m^2 g_0^2 R^{-2} (1 - R\Omega^2 g_0^{-1}) \delta P = r\Omega^2 D^2 \frac{\partial \delta P}{\partial r} , \tag{52}$$

where

$$D \equiv \frac{\partial}{\partial t} + im\Omega .$$

We can simplify Eq. (52) by setting

$$\delta P(r,t) = \varphi(r,t)\, e^{-im\Omega t} , \tag{53}$$

when it follows that

$$\frac{\partial^4\varphi}{\partial t^4} + 4\Omega^2\frac{\partial^2\varphi}{\partial t^2} + 2im\Omega g_0 R^{-1}(2 - R\Omega^2 g_0^{-1})\frac{\partial\varphi}{\partial t}$$

$$- m^2 g_0^2 R^{-2}(1 - R\Omega^2 g_0^{-1})\varphi = r\Omega^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial\varphi}{\partial r}\right) \quad . \tag{54}$$

It can be seen by inspection that, for $\Omega = 0$, the solution to Eq. (54) is indeterminate as far as the radial dependence is concerned. If we are not to have $\Omega = 0$ as a singular case, it is clear that the appropriate solution to Eq. (54) must incorporate this indeterminateness for $\Omega = 0$ as a "boundary condition."

We must also demand that the appropriate solution of Eq. (54) be regular at $r = 0$ and vanish as $r \to \infty$ since the cylinder is of infinite extent.

The solution of Eq. (54) which satisfies these conditions is difficult to obtain. In Appendix II, the general method of finding such a solution is given. In view of the formidable amount of mathematics involved in obtaining the required solution, we feel that sufficient physical insight will be gained if we iterate the general solution of Appendix II by a Neumann series and terminate the solution after one iteration. This is a valid technique for small Ω. The method of generating this solution is presented in Appendix III. Such a solution is only approximate, but it demonstrates how a small amount of rotation alters the instability rate derived by Parker (1966).

The approximate solution of Appendix III is

$$\varphi(r,t) \simeq a_1(r)\exp\left[\sigma_1 t\left(1+\frac{R\Omega^2}{4mg_0}\frac{\partial \ln a_1(r)}{\partial \ln r}\right)\right] \qquad (55)$$

where

$$\sigma_1 \simeq \sqrt{(mg_0R^{-1})} - \Omega^2\sqrt{(\tfrac{1}{2}mRg_0^{-1})} - 2i\Omega \quad . \qquad (56)$$

This solution is valid for small Ω, large m and reasonably short times.

We see that the system is unstable with an e-folding time given by

$$\tau(r) \simeq \sqrt{(R/mg_0)} \times \left[1+\frac{R\Omega^2}{2g_0}\left(1-\frac{1}{2m}\frac{\partial \ln a_1(r)}{\partial \ln r}\right).\right] \quad . \qquad (57)$$

For large m and provided $\partial \ln a_1(r)/\partial \ln r$ is reasonably well behaved we see that

$$\tau(r)^2 \simeq R(mg_0)^{-1}\left(1+R\Omega^2 g_0^{-1}\right) \quad . \qquad (58)$$

In the absence of rotation, the e-folding time, say τ_0, is given by

$$\tau_0^2 = R\,(mg_0)^{-1}$$

Thus the time scale for instability is increased over the value which obtains in the absence of rotation by a factor $\left(1+\frac{1}{2}R\Omega^2 g_0^{-1}\right)$ for weak rotation.

For strong rotation, but $g_o > R\,\Omega^2$ we expect (but have been unable to prove) that the e-folding time will be increased even more than the value given by Eq. (58).

Within the limitation of the approximations made, we have shown that this system is unstable.

6. DISCUSSION

The basic point established by the above calculations is that if the lines of force of the large scale galactic, or spiral arm, magnetic field are confined by the weight of the interstellar gas, then the gas always tends to drain downward along the lines of force towards the lowest regions. This result is true even when rotation is present.

The speed with which the gas drains can be slowed, but not stopped, by rotation provided only that $g > r\Omega^2$. The e-folding time for instability is of the order of the free-fall time under an effective gravity $(g - r\Omega^2)$.

Since the galactic free-fall time is of the order of 3×10^7 years we expect that the magnetic field and cosmic ray gas will drive the instability with a time scale of order $10^7 - 10^8$ years in regions where $R\Omega^2 \ll g_o$ and slightly longer times in regions where $R\Omega^2 < g_o$.

ACKNOWLEDGEMENTS

The author is particularly grateful to Professor E. N. Parker for many stimulating discussions concerning numerous aspects of this paper. He is also grateful to Professor W. H. Reid who suggested that Equation (54) was best handled by singular perturbation techniques.

This work was supported by the National Aeronautics and Space Administration under Grant NASA NsG 96-60.

APPENDIX I

We wish to solve

$$\frac{d^2\delta A}{dr^2} + \frac{d\delta A}{dr}(ar + r^{-1}) + \delta A\,(b - cr^2 - m^2 r^{-2}) = 0 \qquad \text{(A1)}$$

subject to the conditions

i) $\delta A(r) \to 0$ as $r \to \infty$,

ii) $\delta A(r)$ is regular at $r = 0$.

We change variables through $r = \xi^2$ when (A1) becomes

$$4\xi \frac{d^2\delta A}{d\xi^2} + \frac{d\delta A}{d\xi}(4 + 2a\xi) + \delta A(b - c\xi - m^2\xi^{-1}) = 0\,. \qquad \text{(A2)}$$

Now let

$$\delta A = f(\xi)\, e^{\mu\xi}\, \xi^{m/2} \quad , \qquad \text{(A3)}$$

where

$$4\mu^2 + 2\mu a - c = 0. \qquad \text{(A4)}$$

Then it can easily be shown that $f(\xi)$ satisfies

$$\xi \frac{d^2 f}{d\xi^2} + \frac{df}{d\xi}\left[\left(2\mu + \frac{1}{2}a\right)\xi + m + 1\right]$$

$$+ f\left[\mu(m + 1) + \frac{1}{4}(am + b)\right] = 0. \qquad \text{(A5)}$$

This equation is simply the equation for the degenerate hypergeometric function. Since we require that $\delta A(r)$ be regular at $r = 0$ we see that the appropriate solution to Eq. (A5) is

$$f(\xi) = \text{constant} \times {}_1F_1\left(\frac{1}{2}m + \frac{(\mu+b)}{2(4\mu+a)},\ m+1,\ -(2\mu + a/2)\xi\right). \tag{A6}$$

Thus the solution to Equation (A1) which is regular at $r = 0$ is

$$\delta A(r) = \delta A_o r^m e^{\mu r^2} {}_1F_1\left(\frac{1}{2}m + \frac{(\mu+b)}{2(4\mu+a)},\ m+1,\ -(2\mu + a/2)r^2\right). \tag{A7}$$

Here δA_o is an arbitrary constant and μ is the solution of Eq. (A4) for which

$$\delta A(r) \to 0 \qquad \text{as } r \to \infty .$$

This value of μ is chosen in the text.

Some of the asymptotic properties of the degenerate hypergoemagnetic function which are made use of in the text are

i) for $|\lambda| >> 1$, $|\lambda| >> |x|$, $|\lambda| >> |\zeta|$, $|x| \neq 0$, $|\arg(\sqrt{x})| < 3\pi/4$ and $|\arg(\lambda)| < \pi/2$ we can write (Grashteyn and Ryzhik, 1965).

$${}_1F_1(a,c,x) \sim \frac{e^{x/2}\,\Gamma(2\zeta+1)\,x^{-c/2}\,|x|^{1/4}}{\sqrt{\pi}\ \ |\lambda|^{\zeta+1/4}} \cos\left(2\sqrt{|\lambda x|} - \zeta\pi - \pi/4\right) \tag{A8}$$

where

$$\lambda = \frac{1}{2}(c-2a),\quad 2\zeta = c-1.$$

ii) for $|x| >> 1$ and a, c finite, we can write (Erdeyli et. al., 1953)

$$ {}_1F_1(a,c,x) \sim \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \quad , \quad Re(x) \to +\infty \quad ; $$

$$ {}_1F_1(a,c,x) \sim \frac{\Gamma(c)}{\Gamma(c-a)} (-x)^{-a} , \quad Re(x) \to -\infty \quad ; $$

APPENDIX II

We wish to solve

$$\left(\frac{\partial^4}{\partial t^4} + 4\Omega^2\frac{\partial^2}{\partial t^2} + ib\Omega\frac{\partial}{\partial t} - c\right)\varphi(r,t) = r\Omega^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial\varphi}{\partial r}\right) , \tag{B1}$$

where

$$b = 2mg_0R^{-1}(2 - R\Omega^2g_0^{-1}) , \quad c = m^2g_0^2R^{-2}(1 - R\Omega^2g_0^{-1}) ,$$

subject to the conditions:

i) $\varphi(r,t)$ is regular at $r = 0$,

ii) $\varphi(r,t) \to 0$ for all finite t as $r \to \infty$,

iii) in the limit $\Omega \to 0$, $\varphi(r,t)$ must be an undetermined function of r but a determined function of t. This condition is enforced so that the solution of Eq. (B1) reduces to Parker's (1966) solution in the limit $\Omega \to 0$.

In order to solve Eq. (B1) subject to these conditions, it proves convenient to define a Green's function. We note that the operator in parentheses on the left hand side of Eq. (B1) is a linear operator in time. For convenience we set

$$T \equiv \frac{\partial^4}{\partial t^4} + 4\Omega^2\frac{\partial^2}{\partial t^2} + ib\Omega\frac{\partial}{\partial t} - c . \tag{B2}$$

We define the Green's function, G(t), and the homogeneous functions $\Psi_i(r,t)$, by

$$TG(t - t_1) = \delta(t - t_1) , \tag{B3}$$

and

$$T\Psi_i(r,t) = 0 \ , \qquad i = 1, 2, 3, 4. \tag{B4}$$

We note that the $\Psi_i(r,t)$ are completely undetermined as far as the spatial dependence is concerned since T is purely a time operator. Thus, we expect intuitively that the general solution of Eq. (B1) will reduce to the $\Psi_i(r,t)$ in the limit $\Omega \rightarrow 0$. This expectation is substantiated by the analysis.

Making use of Eqs. (B3) and (B4), we see that the solution to Eq. (B1) can be written

$$\varphi(r,t) - \sum_{j=1}^{4} \lambda_j \Psi_j(r,t) = r\Omega^2 \frac{\partial}{\partial r} \int_0^\infty G(t-t_1) \frac{\partial^2 \varphi(r,t_1)}{\partial t_1^2} dt_1 \ . \tag{B5}$$

That this is indeed the solution to Eq. (B1) can be seen by operating from the left in Eq. (B5) with T.

Here, the λ_j are arbitrary constants. We can simplify the problem of solving this implicit equation for $\varphi(r,t)$ if we write

$$\varphi(r,t) - \sum_{j=1}^{4} \lambda_j \Psi_j(r,t) = r\Omega^2 \frac{\partial}{\partial r} \int_0^\infty G(t-t_1) \chi(r,t_1) dt_1 , \tag{B6}$$

where $\chi(r,t)$ satisfies the equation

$$\chi(r,t) - \sum_{j=1}^{4} \lambda_j \frac{\partial^2 \Psi_j(r,t)}{\partial t^2} = r\Omega^2 \frac{\partial}{\partial r} \int_0^\infty g(t-t_1) \chi(r,t_1) dt_1 , \tag{B7}$$

and

$$g(t-t_1) = \frac{\partial^2 G(t-t_1)}{\partial t^2} \ . \tag{B8}$$

It is clear that if we can solve Eq. (B7) for $\chi(r, t)$ it is a relatively simple matter to find $\varphi(r,t)$ since we only have to perform one weighted integral of $\chi(r,t)$ with respect to time.

Even without solving Eq. (B7) it is obvious that if $\Omega = 0$ then

$$\varphi(r,t) = \sum_{j=1}^{4} \lambda_j \Psi_j(r,t) \quad , \tag{B9}$$

and, since the $\Psi_j(r,t)$ are completely unspecified as far as their radial dependence is concerned, so also is $\varphi(r,t)$.

Thus, the solution of Eq. (B1) given by Eq. (B6) does indeed reduce to Parker's solution in the absence of rotation (provided, of course, that $\Psi_j(r,t) \neq 0$ or $\lambda_j \neq 0$) .

If all the $\Psi_j(r,t) = 0$ and/or all the $\lambda_j = 0$, then the only solution in the limit $\Omega = 0$ is $\varphi(r,t) = 0$. This is a rather trivial case. Consequently, we shall assume that $\Psi_j(r,t) \neq 0$ for <u>any</u> Ω value including $\Omega = 0$. (We make no statement concerning the λ_j as yet.)

We do not propose to find explicitly the solution for Eq. (B1) in this Appendix since we do not know the $\Psi_j(r,t)$. We will set up a sequence of steps such that, once the $\Psi_j(r,t)$ are specified, a solution to Eq. (B7) can be found. Thus $\varphi(r,t)$ can be determined.

In order to solve Eq. (B7) we first Mellin transform $\varphi(r, t)$ and $\Psi_j(r,t)$ w.r.t.r. The Mellin transform is defined by

$$\lambda(\xi) = \int_0^\infty \Lambda(r)\, r^{\xi-1}\, dr \quad , \tag{B10}$$

and its inverse transform

$$2\pi i\, \Lambda(r) = \int_{\xi_0 - i\infty}^{\xi_0 + i\infty} \lambda(\xi)\, r^{-\xi} d\xi \quad . \tag{B11}$$

Here ξ_0 is to be chosen so that $\Lambda(r)$ satisfies any appropriate conditions. In our case these are that $\mathcal{X}(r,t)$ be regular at r = 0 and vanish as r → ∞ for all finite t.

For convenience, we write

$$\lambda(\xi) = \mathcal{M}[\Lambda(r)] \quad \text{and} \quad \Lambda(r) = \mathcal{M}^{-1}[\lambda(\xi)] \, .$$

Making use of Eq. (B10), it can be seen that Eq. (B7) becomes

$$\mathfrak{X}(\xi,t) - \sum_{j=1}^{4} \lambda_j \frac{\partial^2 \Psi_j(\xi,t)}{\partial t^2} = -\Omega^2 \xi \int_0^\infty g(t-t_1)\, \mathfrak{X}(\xi,t_1)\, dt_1$$

$$+\Omega^2 \int_0^\infty g(t-t_1)\, r^\xi\, \mathcal{X}(r,t_1) \Big|_{r=0}^{\infty} dt_1 \, , \tag{B12}$$

where

$$\mathfrak{X}(\xi,t) = \mathcal{M}[\mathcal{X}(r,t)] \quad \text{and} \quad \Psi_j(\xi,t) = \mathcal{M}[\Psi_j(r,t)] \, .$$

For simplicity, we shall assume that

$$r^\xi\, \mathcal{X}(r,t) \Big|_{r=0}^{r=\infty} \equiv 0 \, . \tag{B13}$$

Although Eq. (B13) is not a necessary requirement, it simplifies the analysis. In principle, it is possible to solve Eq. (B12) without the use of Eq. (B13), but this is a difficult task.

If Eq. (B13) is satisfied, we see that Eq. (B12) becomes

$$\chi(\xi,t) - \sum_{j=1}^{4} \lambda_j \frac{\partial^2 \Psi_j(\xi,t)}{\partial t^2} = -\Omega^2 \xi \int_0^\infty g(t-t_1)\chi(\xi,t_1)\,dt_1 . \tag{B14}$$

Before proceeding further with Eq. (B14), let us examine $G(t - t_1)$ and thus $g(t - t_1)$.

It can be seen from Eq. (B3) that

$$G(t-t_1) = (2\pi)^{-1} \int_{i\omega_0-\infty}^{i\omega_0+\infty} \frac{e^{-i\omega(t-t_1)}\,d\omega}{(\omega^4 - 4\Omega^2\omega^2 + b\Omega\omega - c)} , \tag{B15}$$

where ω_o has yet to be specified. Since the physical situation demands that $G(t - t_1)$ propagate the solution at t_1 forward to its value at t, it is clear that for $t_1 > t$ we require $G(t - t_1) = 0$. Thus we choose ω_o so that

$$\begin{aligned} G(t-t_1) &= \sum_{\ell=1}^{4} \frac{e^{\sigma_\ell(t-t_1)}}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} \qquad , t>t_1 \\ &= 0 \qquad , \qquad t<t_1 \end{aligned} \tag{B16}$$

where the σ_ℓ are the roots of

$$\sigma_\ell^4 + 4\Omega^2\sigma_\ell^2 + ib\Omega\sigma_\ell - c = 0. \tag{B17}$$

Thus we can write Eq. (B14) as

$$\chi(\xi,t) - \sum_{j=1}^{4} \lambda_j \frac{\partial^2 \Psi_j(\xi,t)}{\partial t^2} = -\Omega^2 \xi \int_0^t g(t-t_1)\chi(\xi,t_1)\,dt_1 . \tag{B18}$$

We also know that

$$\Psi_j(r,t) = \begin{cases} a_j(r)\, e^{\sigma_j t} , & t>0 \\ 0 , & t<0 \end{cases} \qquad j=1,2,3,4. \tag{B19}$$

where the σ_j are again the roots of Eq. (B17) and the $a_j(r)$ are completely undetermined.

Setting $A_j(\xi) = M[a_j(r)]$ we see that Eq. (B18) can be written

$$\chi(\xi,t) - \sum_{j=1}^{4} \lambda_j \sigma_j^2 A_j(\xi) e^{\sigma_j t} = -\Omega^2 \xi \int_0^t g(t-t_1)\chi(\xi,t_1)\,dt_1 . \tag{B20}$$

We note that

$$g(t-t_1) \equiv \frac{\partial^2 G(t-t_1)}{\partial t^2}$$

$$= \sum_{\ell=1}^{4} \frac{e^{\sigma_\ell (t-t_1)}}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} \left[\sigma_\ell^2 S(t-t_1) + 2\sigma_\ell \delta(t-t_1) + \frac{d\delta(t-t_1)}{dt}\right], \tag{B21}$$

where $S(x) = \begin{matrix} 1, & >0 \\ 0, & <0 \end{matrix}$; and $\delta(t)$ is the Dirac δ-function. Thus Eq. (B20) can be written

$$\chi(\xi,t) - \sum_{\ell=1}^{4} \lambda_\ell \sigma_\ell^2 A_\ell(\xi) e^{\sigma_\ell t}$$
$$= -\Omega^2 \xi\, \chi(\xi,t) \sum_{\ell=1}^{4} \frac{\sigma_\ell}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} - \Omega^2 \xi \frac{\partial \chi(\xi,t)}{\partial t} \sum_{\ell=1}^{4} (4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)^{-1}$$
$$- \Omega^2 \xi \sum_{\ell=1}^{4} \frac{\sigma_\ell^2}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} \int_0^t e^{\sigma_\ell (t-t_1)} \chi(\xi,t_1)\,dt_1 . \tag{B22}$$

Now it can easily be seen that

$$\frac{1}{2\pi i}\int_C \frac{dz}{(z^4+4\Omega^2 z^2+ib\Omega z-c)} = \sum_{\ell=1}^{4}(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)^{-1} \qquad \text{(B23)}$$

where the contour C in the complex z-plane is a circle centered on the origin of infinite radius. But this integral is zero. Then the right hand side of Eq. (B23) is also zero. Hence Eq. (B22) can be written

$$\chi(\xi,t)\left[1+\Omega^2\xi\sum_{\ell=1}^{4}\frac{\sigma_\ell}{(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)}\right]-\sum_{\ell=1}^{4}\lambda_\ell\sigma_\ell A_\ell(\xi)e^{\sigma_\ell t}$$

$$=-\Omega^2\xi\sum_{\ell=1}^{4}\frac{\sigma_\ell^2}{(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)}\int_0^t e^{\sigma_\ell(t-t_1)}\chi(\xi,t_1)\,dt_1 \qquad \text{(B24)}$$

Now let us consider

$$\frac{z^2}{(z^4+4\Omega^2z^2+ib\Omega z-c)}-\sum_{\ell=1}^{4}\frac{\sigma_\ell^2}{(z-\sigma_\ell)(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)}=\Phi(z)\ \text{say.} \qquad \text{(B25)}$$

It is clear that $\Phi(z)$ is an analytic function of z and is therefore constant. By considering $z \rightarrow \infty$, we see that $\Phi(z) = 0$. Thus

$$\frac{z^2}{(z^4+4\Omega^2z^2+ib\Omega z-c)}=\sum_{\ell=1}^{4}\frac{\sigma_\ell^2}{(z-\sigma_\ell)(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)}, \qquad \text{(B26)}$$

for all z. In particular, if z = 0, we see that

$$\sum_{\ell=1}^{4}\frac{\sigma_\ell}{(4\sigma_\ell^3+8\Omega^2\sigma_\ell+ib\Omega)}=0\ . \qquad \text{(B27)}$$

Thus, Eq. (B24) can be written

$$\chi(\xi,t) - \sum_{\ell=1}^{4} \lambda_\ell \sigma_\ell^2 A_\ell(\xi) e^{\sigma_\ell t}$$

$$= -\Omega^2 \xi \sum_{\ell=1}^{4} \frac{\sigma_\ell^2}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} \int_0^t e^{\sigma_\ell(t-t_1)} \chi(\xi,t_1)\, dt_1 \quad . \tag{B28}$$

We now Laplace transform Eq. (B28) where

$$L(\xi,p) = \int_0^\infty \ell(\xi,t) e^{-pt} dt \quad , \tag{B29}$$

and

$$2\pi i\, \ell(\xi,t) = \int_{p_0-i\infty}^{p_0+i\infty} L(\xi,p) e^{pt} dp \quad , \tag{B30}$$

where p_o is to be determined so that $\chi(\xi, t) = 0$ for $t < 0$.

For convenience we write

$$L(\xi,p) = \mathcal{L}[\ell(\xi,t)] \text{ and } \ell(\xi,t) = \mathcal{L}^{-1}[L(\xi,p)]$$

Then we see that Eq. (B28) becomes

$$X(\xi,p)\left[1 + \Omega^2 \xi \sum_{\ell=1}^{4} \frac{\sigma_\ell^2}{(p-\sigma_\ell)(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)}\right] = \sum_{\ell=1}^{4} \frac{\lambda_\ell \sigma_\ell^2 A_\ell(\xi)}{(p-\sigma_\ell)} \tag{B31}$$

From Eq. (B26) we see that

$$\sum_{\ell=1}^{4} \frac{\sigma_\ell^2}{(p-\sigma_\ell)(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} = \frac{p^2}{(p^4 + 4\Omega^2 p^2 + ib\Omega p - c)} \quad . \tag{B32}$$

Hence Eq. (B31) becomes

$$\chi(\xi, p)\left[1 + \frac{p^2\Omega^2\xi}{(p^4 + 4\Omega^2p^2 + ib\Omega p - c)}\right] = \sum_{\ell=1}^{4} \frac{\lambda_\ell \sigma_\ell^2 A_\ell(\xi)}{(p - \sigma_\ell)} \tag{B33}$$

Thus

$$\chi(\xi, p) = \frac{\displaystyle\sum_{\ell=1}^{4} \frac{\lambda_\ell \sigma_\ell^2 A_\ell(\xi)}{(p-\sigma_\ell)}}{\left[1 + \dfrac{p^2\Omega^2\xi^2}{(p^4 + 4\Omega^2 p^2 + ib\Omega p - c)}\right]} . \tag{B34}$$

Hence the general solution to Eq. (B1) under the assumption that Eq. (B13) is obeyed can be written

$$\varphi(r,t) - \sum_{\ell=1}^{4} \lambda_\ell \Psi_\ell(r,t)$$

$$= -\frac{r\Omega^2}{4\pi^2}\frac{\partial}{\partial r}\int_0^t dt_1 \int_{\xi_0 - i\infty}^{\xi_0 + i\infty} d\xi \int_{p_0 - i\infty}^{p_0 + i\infty} dp\, G(t-t_1)\, r^{-\xi} e^{pt_1} \chi(\xi, p) . \tag{B35}$$

Thus once we are given the $a_j(r)$ it is possible, in principle, to evaluate the triple integral in Eq. (B35) and hence to find the $\varphi(r,t)$ which has the required properties as $r \to 0$, $r \to \infty$, and vanishes for $t < 0$ for all r. Such a $\varphi(r,t)$ is found by choosing ξ_o and p_o appropriately.

If Eq. (B13) does not hold, the simplest method of solving Eq. (B1) is to treat Eq. (B12) as an inhomogeneous Carleman integral equation and then solve the corresponding inhomogeneous Riemann-Hilbert problem. Should such an approach be required, the reader is referred to Muskhelishvili (1953).

APPENDIX III

For small Ω and $\Psi_j(r, t) \neq 0$ we can approximate to the solution of Eq. (B1) by

$$\varphi(r,t) - \sum_{j=1}^{4} \lambda_j \Psi_j(r,t)$$

$$\simeq \sum_{j=1}^{4} \lambda_j r\Omega^2 \frac{\partial}{\partial r}\int_0^t G(t-t_1)\frac{\partial^2 \Psi_j(r,t_1)}{\partial t_1^2}dt_1 + O(\Omega^4), \quad \text{(C1)}$$

where

$$G(t-t_1) = \begin{cases} \displaystyle\sum_{\ell=1}^{4} \frac{e^{\sigma_\ell (t-t_1)}}{(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} & , t > t_1 \\ 0 & , t < t_1 \end{cases} \quad \text{(C2)}$$

Also, we know that

$$\Psi_j(r, t) = a_j(r)e^{\sigma_j t} \quad \text{(C3)}$$

Thus

$$\varphi(r,t) - \sum_{j=1}^{4} \lambda_j a_j(r) e^{\sigma_j t}$$

$$\simeq \sum_{j=1}^{4} \sigma_j^2 \lambda_j \Omega^2 r \frac{\partial a_j(r)}{\partial r}\left\{\frac{t e^{\sigma_j t}}{(4\sigma_j^3 + 8\Omega^2\sigma_j + ib\Omega)} + {\sum_{\ell=1}^{4}}' \frac{(e^{\sigma_j t} - e^{\sigma_\ell t})}{(\sigma_j - \sigma_\ell)(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)}\right\}, \quad \text{(C4)}$$

where the prime on the sum over ℓ means that the term $\ell = j$ is omitted.

Now it is known that, in the absence of rotation, the roots of Eq. (B17) are just

$$\sigma_{1,2} = \pm\sqrt{(mg_0 R^{-1})}\ ; \quad \sigma_{3,4} = \pm i\sqrt{(mg_0 R^{-1})}.$$

Of these four roots only σ_1 leads to an unstable situation. To see how this particular root is altered when a small amount of rotation is included, we set

$$\lambda_1 = 1, \ \lambda_j = 0 \ (j \neq 1) \quad .$$

Then Eq. (C4) becomes

$$\varphi(r,t) \simeq q_1(r)\, e^{\sigma_1 t} \left[\!\left[1 + \Omega^2 \frac{\partial \ln q_1(r)}{\partial \ln r} \times \right.\right.$$
$$\left.\left.\left\{ \frac{\sigma_1^2 t}{(4\sigma_1^3 + 8\Omega^2\sigma_1 + ib\Omega)} + \sum_{\ell=2}^{4} \frac{\sigma_1^2 \left[1 - e^{t(\sigma_\ell - \sigma_1)}\right]}{(\sigma_1 - \sigma_\ell)(4\sigma_\ell^3 + 8\Omega^2\sigma_\ell + ib\Omega)} \right\} \right]\!\right] . \tag{C5}$$

For reasonably short times, but not so short that the linear term in t in parentheses is cancelled by the sum of terms from $\ell = 2$ to 4, we may write

$$\varphi(r,t) \simeq q_1(r)\, e^{\sigma_1 t} \left[1 + \frac{\Omega^2 \sigma_1^2 t}{(4\sigma_1^3 + 8\Omega^2\sigma_1 + ib\Omega)} \frac{\partial \ln q_1(r)}{\partial \ln r} \right] . \tag{C6}$$

To order Ω^2, and m large, this may be written

$$\varphi(r,t) \simeq q_1(r)\, \exp\left[\sigma_1 t \left(1 + \frac{R\Omega^2}{4 m g_0} \frac{\partial \ln q_1(r)}{\partial \ln r} \right) \right] , \tag{C7}$$

where, to order Ω^2, σ_1 is given by

$$\sigma_1 \simeq \sqrt{(m g_0 R^{-1})} - \Omega^2 \sqrt{(\tfrac{1}{2} m g_0^{-1} R)} - 2i\Omega + O(\Omega^3) . \tag{C8}$$

The solution given by Eq. (C7) is the one made use of in § 5.

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