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by Norbert Stankiewicz

Lewis Research Center
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SUMMARY

The recombinative decay of collision-dominated plasmas is investigated theoretically, and the limit of validity of such radiationless models is determined. The density range considered conforms to those plasma densities usually encountered in the analysis of a magnetohydrodynamic generator. The results indicate that neglecting radiative processes may not always be justified at these densities. It is concluded that the radiationless model is valid for argon when the electron density is greater than 6.8×10^{15} per cubic centimeter and for cesium when it is greater than 1.1×10^{15} per cubic centimeter.

INTRODUCTION

One of the problems in plasma physics, and in particular, magnetohydrodynamic (MHD) power generation is the determination of the range of validity of the Saha equation. This equation gives the fractional ionization of a gas in equilibrium as a function of neutral density, temperature, and ionization potential. It is common practice in MHD generator analysis (e.g., ref. 1) to apply this relation to the nonequilibrium situation in which the plasma electrons are at a higher temperature than the heavier species. This temperature difference is caused by the electron heating phenomenon produced in a plasma flowing through a magnetic field (ref. 2). The electron density is normally calculated from the Saha equation at this higher electron temperature.

Previous work on this problem includes the hydrogen calculations of Ul'yanov (ref. 3) using Born approximation cross sections; the model atom (based on cesium) calculations of BenDaniel and Tamor (ref. 4) using cross sections based on the Townsend α coefficient; and cesium and potassium (in helium and argon atmospheres) data of Hiramoto (ref. 5) based on the J. J. Thompson theory of recombination (ref. 6); and the work of Griem (ref. 7) using semiclassically derived cross sections.

This study investigates the Saha assumption for a decaying (i.e., recombining) plasma, whose electrons are initially at a higher temperature than the ions and neutrals.

The initial conditions are assumed to be such that the dominant mode of plasma relaxation is by three-body recombination in which the third body is another electron. In this regime, the bound states are in equilibrium with the free electrons and the Saha equation therefore holds. The conclusion that optically dense plasmas (collisionally dominated) obey the Saha relation at the electron temperature has been previously reached in the theoretical discussions of reference 8. Experimentally, this behavior has been observed in reference 9 for a highly ionized hydrogen plasma Likewise, the argon data presented in reference 10 indicate that for the higher electron densities the decay follows the Saha relation.

As the decay continues and the electron density decreases, collisions with the neutral atoms are not frequent enough to maintain this equilibrium state. Radiative processes then become competitive with collisional effects in determining the bound level populations. An estimate of this threshold point is made based on the assumption that the ground state of the atom is the first level to fall out of equilibrium with the free electrons. The Gryzinski expression for the inelastic cross sections (ref. 11), is used and the threshold point is calculated for argon and cesium.

ANALYSIS

Basic Premises

As in references 3, 4, and 7, electron-electron interactions are assumed to occur so rapidly that a Maxwellian velocity distribution of electrons is assured. Accordingly, the number of electrons in a six-dimensional phase cell $dq_e^3\, dv_e^3$ is given by

$$f_e\, dq_e^3\, dv_e^3 = n_e\left(\frac{m}{2\pi kT_e}\right)^{3/2} \exp(-mv_e^2/2kT_e) \tag{1}$$

where f_e is the electron distribution function; T_e, the electron temperature; dq_e^3, the infinitesimal volume in coordinate space; and dv_e^3, the infinitesimal volume in velocity space. (All symbols are defined in appendix A.) The distribution function in equation (1) is normalized to the electron number density n_e by integrating over all velocity space.

Hence,

$$n_e = \int f_e\, dv_e^3 \tag{2}$$

The local electron thermal energy u_e is given by a similar integration:

$$u_e = \int \frac{1}{2} m v_e^2 f_e \, d\vec{v}_e^3 = \frac{3}{2} n_e k T_e \tag{3}$$

The number density of neutrals in the j^{th} excited state is denoted as n_j, and each is regarded as an independent species of the plasma. Since the total number density of nuclei N is a constant, the number of independent equations describing the plasma is reduced by one because of the normalizing equation:

$$\sum_j n_j + n_i = N = \sum_j n_j + n_e \tag{4}$$

The last step is a consequence of the assumed charge neutrality of the system. (Only singly charged ions are considered.)

If the neutral levels happen to be in equilibrium with the free electrons, each excited state will obey a Saha relation of the form

$$n_j^s = n_e^2 \frac{h^3}{2\omega_i} \left(\frac{1}{2\pi m k T_e}\right)^{3/2} \omega_j \exp(E_j/kT_e) \tag{5}$$

The superscript s is used to denote Saha equilibrium; ω_i is the ion degeneracy and ω_j and E_j are the degeneracy and the ionization energy of the j^{th} state.

General Formulation of Problem

The following processes are considered to be most likely for the plasma densities of interest in this study:

(1) Three-body (electron-electron-ion) recombination to the j^{th} state and its inverse two-body (electron-neutral) ionization from the j^{th} state:

$$e + e + i \rightleftarrows e + n_j$$

(2) Two-body (electron-ion) recombination with photoemission and its inverse, photoionization:

$$e + i \rightleftarrows n_j + h\nu$$

(3) Superelastic and inelastic collisions:

$$e + n_j \rightleftarrows e + n_k \qquad j \neq k$$

(4) Radiative deexcitation and photoexcitation:

$$n_k \rightleftarrows n_j + h\nu \qquad k > j$$

(5) Recoil energy loss by the electrons in elastic collisions with the ions and neutrals:

$$e_1 + \binom{i}{n} \rightarrow e_2 \qquad E_1 > E_2$$

For these processes, the coupled rate equations describing the decaying plasma can be written as

$$\frac{dn_e}{dt} = \left(\frac{\partial n_e}{\partial t}\right)_{IR}^{c} + \left(\frac{\partial n_e}{\partial t}\right)_{IR}^{r} \tag{6a}$$

$$\frac{dn_j}{dt} = \left(\frac{\partial n_j}{\partial t}\right)_{IR}^{c} + \left(\frac{\partial n_j}{\partial t}\right)_{IR}^{r} + \sum_k \left[\left(\frac{\partial n_j}{\partial t}\right)_{j \rightleftarrows k}^{c} + \left(\frac{\partial n_j}{\partial t}\right)_{j \rightleftarrows k}^{r}\right] \tag{6b}$$

$$\frac{du_e}{dt} = \left(\frac{\partial u_e}{\partial t}\right)_{IR}^{c} + \left(\frac{\partial u_e}{\partial t}\right)_{IR}^{r} + \left(\frac{\partial u_e}{\partial t}\right)_{N}^{c} + \sum_{j,k} \left(\frac{\partial u_e}{\partial t}\right)_{j \rightleftarrows k}^{c} \tag{6c}$$

The superscripts c and r refer to collisional and radiative processes, respectively, while the subscripts IR, $j \rightleftarrows k$, and N refer to ionization recombination processes, transitions between levels, and elastic recoil energy losses that the electrons suffer during collisions with the ions and neutrals. The differential equation describing the temperature decay is obtained by differentiating equation (3) and substituting equations (6a) and (6c):

$$\frac{dT_e}{dt} = \frac{2}{3kn_e}\left[\left(\frac{\partial u_e}{\partial t}\right)^c_{IR} + \left(\frac{\partial u_e}{\partial t}\right)^r_{IR} + \left(\frac{\partial u_e}{\partial t}\right)^c_N + \sum_{j,k}\left(\frac{\partial u_e}{\partial t}\right)^c_{j \rightleftarrows k}\right] - \frac{T_e}{n_e}\left[\left(\frac{\partial n_e}{\partial t}\right)^c_{IR} + \left(\frac{\partial n_e}{\partial t}\right)^r_{IR}\right] \tag{7}$$

The following forms of the collisional terms in the rate equations (6) are derived in appendix B:

$$\left(\frac{\partial n_e}{\partial t}\right)^c_{IR} = n_e \sum_j \left(n_j - n_j^s\right)\langle v\sigma_I(E_j \rightarrow E^*)\rangle \tag{8}$$

$$\left(\frac{\partial n_j}{\partial t}\right)^c_{IR} = n_e\left(n_j^s - n_j\right)\langle v\sigma_I(E_j \rightarrow E^*)\rangle \tag{9}$$

$$\sum_k \left(\frac{\partial n_j}{\partial t}\right)^c_{j \rightleftarrows k} = n_e \sum_{k<j}\left\{n_k - \frac{\omega_k}{\omega_j} n_j \exp[(E_k - E_j)/kT_e]\right\}\langle v\sigma_{ex}(k \rightarrow j)\rangle$$

$$+ n_e \sum_{k>j}\left\{\frac{\omega_j}{\omega_k} n_k \exp[(E_j - E_k)/kT_e] - n_j\right\}\langle v\sigma_{ex}(j \rightarrow k)\rangle \tag{10}$$

$$\left(\frac{\partial u_e}{\partial t}\right)^c_{IR} = n_e \sum_j \left(n_j^s - n_j\right)\left\langle\frac{1}{2} mv^2 v\sigma_I(E_j \rightarrow E^*)\right\rangle \tag{11}$$

$$\sum_{j,k}\left(\frac{\partial u_e}{\partial t}\right)^c_{j \rightleftarrows k} = n_e \sum_j \sum_{k>j}\left\{\frac{\omega_j}{\omega_k} n_k \exp[(E_j - E_k)/kT_e] - n_j\right\}\left\langle\frac{1}{2} mv^2 v\sigma_{ex}(j \rightarrow k)\right\rangle \tag{12}$$

From reference 1, the recoil energy losses can be written as

$$\left(\frac{\partial u_e}{\partial t}\right)^c_N = -\frac{3m}{M} n_e k(T_e - T_0)(\nu_{ei} + \nu_{en}) \tag{13}$$

where T_0 is the temperature of the ions and neutrals (taken to be equal), ν_{ei} and ν_{en} are the electron-ion and electron-neutral collision frequency for momentum transfer, m is the electron mass, and M is the mass of the neutral (or ion).

Finally, from equation (4), it is clear that

$$\frac{dn_e}{dt} = -\sum_j \frac{dn_j}{dt} \tag{14}$$

Radiationless Case

For the radiationless case, consideration of equations (6), (8), (10), and (15) shows that

$$\sum_j \sum_k \left(\frac{\partial n_j}{\partial t}\right)^c_{j \rightleftarrows k} = 0 \tag{15}$$

This equation implies that the inelastic-superelastic collision rates vanish by detailed balancing. Equation (10) then yields

$$\frac{n_k}{n_j} = \frac{\omega_k}{\omega_j} \exp\left[(E_j - E_k)/kT_e\right] \qquad E_j > E_k \tag{16}$$

which indicates that under these conditions, equilibrium exists among the bound states at the free-electron temperature. Consequently, the Saha relation in equation (5) holds for each bound state.

If equation (5) is summed and inserted into the normalizing condition, equation (4), the result gives the usual form of the Saha equation:

$$n_e^2 \frac{h^3}{2\omega_i}\left(\frac{1}{2\pi mkT_e}\right)^{3/2} \exp(E_o/kT_e) \sum_j \omega_j \exp\left[(E_o - E_j)/kT_e\right] = N - n_e \tag{17}$$

For moderate temperatures, the summed term (the partition function) is equal to the ground-state degeneracy ω_o and therefore equation (17) may be written as

$$\frac{n_e^2}{N - n_e} = \frac{2\omega_i}{\omega_o h^3}(2\pi mkT_e)^{3/2} \exp(-E_o/kT_e) \tag{18}$$

For the radiationless case then, the electron density is only a function of the electron temperature, and a complete description of the decaying plasma is formulated by solving equation (7). Because of the equilibrium conditions given by equations (16) and (5), equation (7) reduces to

$$\frac{dT_e}{dt} = \frac{2}{3kn_e}\left(\frac{\partial u_e}{\partial t}\right)_N^c \tag{19}$$

and substituting equation (13) gives

$$\frac{dT_e}{dt} = -\frac{2m}{M}(T_e - T_0)(\nu_{ei} + \nu_{en}) \tag{20}$$

Although the results given in equations (17) and (20) are well known, the proof presented here is useful for determining the extent that a competing process, such as the loss of energy through radiation, causes a departure from detailed balancing, as will be shown in the next section.

Conditions of Validity of Radiationless Case

As the temperature decreases, the number of electrons capable of maintaining the superelastic and inelastic processes in detailed balancing also decreases. The ground state will be the first level to depart from Saha equilibrium primarily because of its larger energy separation from the continuum. Therefore, in order to estimate the equilibrium threshold, all states above the ground state are assumed to remain in equilibrium; that is,

$$n_j = n_j^S \qquad j \neq 0 \tag{21}$$

All emitted radiation arising from transitions to the ground state is also assumed lost, that is, the plasma is optically thin.

Since the total gain in neutral population in all levels is equal to the rate that free electrons leave the continuum, the following is true:

$$\frac{dn_e}{dt} = \left(\frac{\partial n_e}{\partial t}\right)^c_{IR} + \left(\frac{\partial n_e}{\partial t}\right)^r_{IR} = - \sum_j \left[\left(\frac{\partial n_j}{\partial t}\right)^c_{IR} + \left(\frac{\partial n_j}{\partial t}\right)^r_{IR}\right] \tag{22}$$

Then, from considering equations (7) and (15), it is concluded that

$$\sum_j \sum_k \left[\left(\frac{\partial n_j}{\partial t}\right)^c_{j\rightleftarrows k} + \left(\frac{\partial n_j}{\partial t}\right)^r_{j\rightleftarrows k}\right] = 0 \tag{23}$$

It is no longer valid to argue that equation (23) holds because of detailed balancing, since by hypothesis some inverse processes are missing. In particular, for a transparent plasma the term $\left(\frac{\partial n_j}{\partial t}\right)^r_{j\rightleftarrows k}$ contains no reabsorption term from the ground state. For the states above ground, since they are in Saha equilibrium and therefore collisionally dominated, dismissing radiative processes between them as being improbable is consistent. The collisional terms vanish between states that are in equilibrium, as seen from equation (10), and equation (23) then becomes

$$\sum_k \left[\left(\frac{\partial n_o}{\partial t}\right)^c_{o\rightleftarrows k} + \left(\frac{\partial n_o}{\partial t}\right)^r_{o\rightleftarrows k}\right] = 0 \tag{24}$$

The first term in equation (24) is given by equation (10) and is equal to

$$\sum_k \left(\frac{\partial n_o}{\partial t}\right)^c_{o\rightleftarrows k} = n_e \sum_k \left\{\frac{\omega_o}{\omega_k} n_k \exp[(E_o - E_k)/kT] - n_o\right\} \langle v\sigma_{ex}(o \rightarrow k)\rangle \tag{25}$$

Since the states above the ground level are assumed to be in equilibrium, use of equation (6) yields

$$\frac{\omega_o}{\omega_k} n_k^s \exp[(E_o - E_k)/kT_e = n_e^2 \frac{h^3}{2\omega_i}\left(\frac{1}{2\pi mkT_e}\right)^{3/2} \omega_o \exp(E_o/kT_e) = n_o^s \tag{26}$$

Hence, equation (25) can be written as

$$\sum_k \left(\frac{\partial n_o}{\partial t}\right)^c_{o\rightleftarrows k} = n_e\left(n_o^s - n_o\right) \sum_k \langle v\sigma_{ex}(o \rightarrow k)\rangle \tag{27}$$

where $\sum_k \langle v\sigma_{ex}(o \rightarrow k)\rangle$ is the collision coefficient for inelastic collisions from the ground state to all excited states.

The radiative deexcitation term in equation (24) is proportional to the population of the k^{th} state; the constant of proportionality is the spontaneous emission probability to the ground state $A(k \rightarrow o)$. Hence,

$$\sum_{k>o} \left(\frac{\partial n_o}{\partial t}\right)^r_{o\leftarrow k} = \sum_{k>o} n_k^s A(k \rightarrow o) \tag{28}$$

Using equation (26) (with $\omega_o = 1$) will change equation (28) to

$$\sum_{k>o} \left(\frac{\partial n_o}{\partial t}\right)^r_{o\leftarrow k} = n_o^s \sum_{k>o} \omega_k \exp[-(E_o - E_k)/kT_e]A(k \rightarrow o) \tag{29}$$

Substituting equations (29) and (27) into equation (24) yields the ratio of the nonequilibrium ground-state density to the equivalent Saha density of the ground state:

$$\frac{n_o}{n_o^s} = 1 + \frac{\sum_{k>o} \omega_k \exp[-(E_o - E_k)/kT_e]A(k \rightarrow o)}{n_e \sum_{k>o} \langle v_e\sigma_{ex}(o \rightarrow k)\rangle} \tag{30}$$

When equation (B23) is substituted into equation (30), equation (30) becomes

$$\frac{n_o}{n_o^s} = 1 + \frac{\sum_{k>o} \omega_k \exp[-(E_o - E_k)/kT_e]A(k \rightarrow o)}{n_e \sum_{k>o} \omega_k \exp[-(E_o - E_k)/kT_e]\langle v_e \sigma_{dex}(k \rightarrow o)\rangle} \tag{31}$$

The second term of equation (31) is the ratio of the total radiative deexcitation rate to the total collisional deexcitation rate into the ground state. If the collisional rates greatly predominate, equation (31) shows that the ground state is in equilibrium, as expected.

For the gases and densities of interest, the contribution to the series in equation (31) from terms higher than the first is smaller by at least an order of magnitude. This change in contribution is due to the reduced population of higher levels, the smaller cross sections, and the smaller radiative transition coefficients.

Equation (31) then reduces to

$$\frac{n_o}{n_o^s} = 1 + \frac{A(1 \rightarrow o)}{n_e \langle v\sigma_{dex}(1 \rightarrow o)\rangle} \tag{32}$$

However, in order to use the summed rate coefficients as calculated in reference 9, equation (32) is finally written as

$$\frac{n_o}{n_o^s} = 1 + \frac{A(1 \rightarrow o)}{n_e \sum_{k>o} \langle v\sigma_{dex}(k \rightarrow o)\rangle} \tag{33}$$

The error in passing from equation (32) to equation (33) is about 10 percent. The effect of increasing optical thickness, however, is in the direction of reducing this error.

The conductivity of a plasma is of primary interest in MHD generators. Since the conductivity σ is directly proportional to the electron number density, the fractional decrease in conductivity from its Saha value can be written as

$$\frac{\sigma^s - \sigma}{\sigma^s} = \frac{n_e^s - n_e}{n_e^s} \tag{34}$$

In the region where deviations from the Saha values become important, it is expected that most of the neutrals are in their ground state. Equation (34) then becomes

$$\frac{\sigma^s - \sigma}{\sigma^s} = \left(\frac{n_o - n_o^s}{n_o^s}\right)\frac{n_o^s}{n_e^s} \tag{35}$$

Substituting equation (33) and defining F_i^s as the fraction of ionization calculated under Saha conditions allows equation (35) to be written as

$$\frac{\sigma^s - \sigma}{\sigma^s} = \frac{A(1 \rightarrow o)}{n_e \sum_{k>o} \langle v\sigma_{dex}(k \rightarrow o)\rangle} \frac{(1 - F_i)}{F_i} \tag{36}$$

Equations (33) and (36) form the basis of the next section in establishing a criterion for equilibrium.

RESULTS AND DISCUSSION

Equation (33) may now be applied to determine the threshold of validity of the collisionally dominated plasma for those conditions (pressure and temperature) that are required in MHD generators. Reference 1 indicates that argon seeded with cesium is a useful working fluid for such a generator. The remainder of this report will therefore be concerned with determining the equilibrium threshold for these two gases.

From reference 12 the spontaneous transition coefficient $A(1 \rightarrow 0)$ for cesium is 3×10^7 per second. Dugan (ref. 13) calculated the superelastic collision frequency for cesium for an electron density of 10^{15} per cubic centimeter using Gryziński cross sections (ref. 11). Tables II and III of reference 13 give these data, from which the term $\sum_{k>o} \langle v\sigma_{dex}(k \rightarrow o)\rangle$ may be obtained by dividing by 10^{15}. The resulting values are

$$\sum_{k>0} \langle v\sigma_{dex}(k \rightarrow o)\rangle = \begin{cases} 9.54\times10^{-8} \text{ cu cm/sec; } T = 1000^\circ \text{ K} \\ \\ 7.64\times10^{-8} \text{ cu cm/sec; } T = 5000^\circ \text{ K} \end{cases} \tag{37}$$

Since $A(1 \rightarrow o)$ is not a function of temperature, the ratio

$$\frac{A(1 \rightarrow o)}{\sum\limits_{k>o} \langle v\sigma_{dex}(k \rightarrow o)\rangle}$$

does not change drastically with temperature. The threshold being sought is expected to be nearer the 5000° K data point. Equation (33) then gives

$$\frac{n_o - n_o^s}{n_o^s} = \frac{3.93\times10^{14}}{n_e} \tag{38}$$

The criterion for equilibrium will be taken arbitrarily as a deviation $\left(n_o - n_o^s\right)/n_o^s$ from the ground state density of 1/e or less. Equation (38) therefore shows that for electron densities above 1.1×10^{15} per cubic centimeter, the cesium ground state is adequately given by the Saha expression.

For argon, Dugan (personal communication) estimated the resonance transition coefficient $A(1 \rightarrow o)$ by extrapolation from helium and neon data to be 5×10^{8} per second. The computer program for superelastic collisions used by Dugan in the cesium calculations of reference 13 was adapted for argon and the following information was obtained:

$$\sum_{k>o} \langle v\sigma_{dex}(k \rightarrow o)\rangle = \begin{cases} 2\times10^{-7} \text{ cu cm/sec; } T = 10\,000^{o} \text{ K} \\ 5.6\times10^{-7} \text{ cu cm/sec; } T = 5000^{o} \text{ K} \\ 1.6\times10^{-6} \text{ cu cm/sec; } T = 2000^{o} \text{ K} \end{cases} \tag{39}$$

Using the 10^4 °K data changes equation (33) for argon to

$$\frac{n_o - n_o^s}{n_o^s} = \frac{2.5\times10^{15}}{n_e} \tag{40}$$

Applying the arbitrary deviation of 1/e shows that the ground state is essentially in equilibrium for a free-electron density greater than 6.8×10^{15}.

Comparison of these results with the work of other authors is summarized in table I. The entries refer to electron num-

TABLE I. - EQUILIBRIUM LIMITS

Gas	Electron number density, electrons/cm^3			
	Present report	Griem (ref. 7)	BenDaniel and Tamor (ref. 4)	Chen experimental, ref. (10)
Argon	6.8×10^{15}	8.0×10^{16}	---------	$\sim3\times10^{15}$
Cesium	1.1×10^{15}	1.1×10^{16}	$\sim1.2\times10^{17}$	-------

ber densities below which equilibrium should not be expected. These values are comparable to the electron number densities typically used in MHD generators. However, in order to apply these results to actual generator conditions, a criterion based on the plasma conductivity rather than on electron number density will be established. According to reference 1, at an ambient temperature of 700° K and a pressure of 10^{-1} atmosphere, the fraction of seed required was less than 10^{-2}. Typical neutral number densities are 10^{18} argon and 10^{16} cesium atoms per cubic centimeter. The electron temperatures corresponding to the electron densities shown in table I are therefore 9250° K for argon and 3000° K for cesium (ref. 14). Since electron temperatures higher than 3200° K are unnecessary (ref. 15) in an actual generator, it can be concluded that the argon carrier gas will not contribute to the conductivity.

If the criterion for equilibrium is arbitarily taken to be a 10 percent deviation of conductivity, equation (36) for cesium can be written

$$\frac{\sigma^{s} - \sigma}{\sigma^{s}} = \frac{3.93\times10^{14}}{n_e} \frac{(1 - F_i)}{F_i} \leq 0.1 \tag{41}$$

When numerator and denominator are multiplied by N, the cesium seed density, equation (41) becomes

$$\frac{3.93\times10^{14}}{N} \frac{(1 - F_i)}{F_i^2} \leq 0.1 \tag{42}$$

or

$$F_i \geq \frac{3.93\times10^{15}}{2N} \left(\sqrt{1 + \frac{4N}{3.93+10^{15}}} - 1\right) \tag{43}$$

Substituting the value of 10^{16} per cubic centimeter for N gives finally

$$F_i \geq 0.46 \tag{44}$$

or

$$n_e \geq 4.6\times10^{15} \tag{45}$$

Hence, this result differs from the previous one by about a factor of four. Equation (44) refers to the fractional ionization of the cesium seed only. For the same conditions, the overall fractional ionization of the cesium-argon mixture is

$$F_{i,\,tot} \equiv \frac{n_e}{N_{Ar} + N_{Cs}} \geq \frac{4.6\times10^{15}}{1.01\times10^{18}} = 4.55\times10^{-3} \tag{46}$$

This is a fraction of ionization that is easily met in most MHD designs. However, equation (46) is the criterion for a particular choice of gas temperature and operating pressure. For other conditions, a different fractional ionization requirement will hold.

The analysis presented in this report still leaves unanswered the question of how the required conductivity will be achieved in a generator that relies on the electron heating phenomenon, and in which the initial electron densities are below the minimum values quoted here.

CONCLUSIONS

This study of the limits of radiationless decay of the high-density plasmas argon and cesium indicates that the neglect of radiative processes may not always be justified for those plasmas usually encountered in magnetohydrodynamic (MHD) generator analysis. Comparison of the population of the ground state with the equivalent equilibrium population (Saha) shows that serious departure from equilibrium occurs (1) for argon when the free-electron density falls below 6.8×10^{15} per cubic centimeter, and (2) for cesium when the free-electron density falls below 1.1×10^{15} per cubic centimeter. The result of ignoring radiative effects below these limits would be to overestimate the conductivity of an MHD generator to an extent that depends on its fractional ionization. This overestimation is especially true of those generators which are designed to employ the electron heating phenomenon to achieve the required conductivity, and which necessarily operate initially with low electron densities.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 4, 1966

APPENDIX A

SYMBOLS

$A(k \rightarrow j)$	spontaneous radiative transition coefficient from k^{th} to j^{th} state, sec^{-1}
b	impact parameter
dq^3	infinitesimal volume in coordinate space
dv^3	infinitesimal volume in velocity space
E	energy of free electron
E_j, E_k	ionization energy from j^{th} and k^{th} states
E_o	ionization energy from ground state
e	electronic charge; Napierian base 2.718
F_i	fractional ionization
$F_{i, tot}$	total fractional ionization
f_e	electron distribution function
h	Planck's constant
k	Boltzmann's constant
M	mass of neutral atom or ion
m	electron mass
N	equilibrium number density of atoms
n	number density
n_j, n_k	neutral number density in j^{th} and k^{th} states
n_o	neutral number density in ground state
P	reaction probability
T	temperature
T_o	final equilibrium temperature; ion and neutral temperature
t	time
u_e	average electron energy defined in eq. (3)
v	velocity
δ	unit mechanism (see p. 20)
λ	$1/kT_e$ (appendix B)
ν	collision frequency for momentum transfer
σ	conductivity
$\sigma_{dex}(k \rightarrow j)$	collision induced deexcitation cross section from k^{th} to j^{th} state $(k > j)$
$\sigma_{ex}(j \rightarrow k)$	collision induced excitation cross section from j^{th} to k^{th} state $(k > j)$
$\sigma_I(E_j \rightarrow E^*)$	ionization cross section from j^{th} state with emission of electron having energy E^*

$\sigma_R(E^* \rightarrow E_j)$	recombination cross section for capture of electron having energy E^* into j^{th} state
ω	degeneracy
$\langle\ \rangle$	average value
$\partial/\partial t$	change due to interaction processes

Subscripts:

dex	deexcitation
e	electron
ex	excitation
I	ionization
IR	ionization-recombination
i	ion
j	j^{th} state
k	k^{th} state
$j \rightleftarrows k$	process between j^{th} and k^{th} state
N	elastic collision
n	neutral
o	ground state
R	recombination
0	equilibrium

Superscripts:

c	collisional
r	radiative
s	Saha equilibrium
$*\dagger$	characterizing free electrons

APPENDIX B

DERIVATION OF COLLISION INTEGRALS

Throughout appendix B the parameter $\lambda \equiv 1/kT_e$ will be used for ease of writing. The superscripts *, † will be used to characterize the electrons involved in the collisions.

Two-Body Ionization - Three-Body Recombination

The energy diagram (fig. 1) depicts the energy levels of a typical ionization-recombination process.

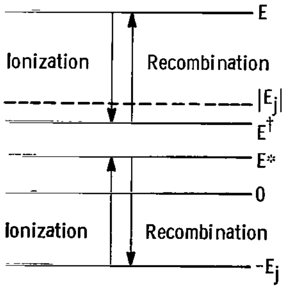

Figure 1. - Schematic diagram depicting energy levels of a typical ionization-recombination process.

The change in electron density in a time dt due to ionization from the j^{th} state is

$$\left(\frac{\partial n_e}{\partial t}\right)_{jI} dt = \int f_e \, d\overset{3}{v}_e \, n_j \, d\overset{3}{q}_n \, P(E_j \rightarrow E^*)dE^* \tag{B1}$$

in which the argument of f_e is the energy E of the ionizing electron; E^* is the final energy of the emitted electron; and $P(E_j \rightarrow E^*)dE^*$ is the probability of emitting into the E^* free state (see eq. (1919), ref. 16). The neutral particle must find itself in a volume element $d\overset{3}{q}_n$ equal to $d\overset{3}{q}_e$ if such a collision is to be effected; hence,

$$d\overset{3}{q}_n \, d\overset{3}{v}_e = \frac{16\pi^2}{m^2} b \, db \, E \, dE \, dt \tag{B2}$$

The ionization cross section for emission into an energy E^* is then

$$\sigma_I(E_j \rightarrow E^*) = 2\pi \int P(E_j \rightarrow E^*) b \, db \tag{B3}$$

With these substitutions and with the explicit form of f_e from equation (1), equation (B1) becomes

$$\left(\frac{\partial n_e}{\partial t}\right)_{jI} = 8\pi m n_e\left(\frac{\lambda}{2\pi m}\right)^{3/2}\int n_j e^{-\lambda E}\,\sigma_I(E_j \rightarrow E^*)dE^*\,E\,dE \tag{B4}$$

where the integral sign represents a double integration over E^* from zero to $E - E_j$ and over E from E_j to infinity.

The change in electron density due to recombination to the j^{th} state (in which another electron is the third body that carries away the excess energy) is given by

$$\left(\frac{\partial n_e}{\partial t}\right)_{jR} dt = -\int f_e^{\dagger}\,d\overset{3\dagger}{v_e}\,f_e^*\,d\overset{3*}{q_e}\,d\overset{3*}{v_e}\,n_i\,d\overset{3}{q_i}\,P(E^* \rightarrow E_j) \tag{B5}$$

where the argument of f_e^* is the energy E^* of the electron to be captured into the j^{th} state; $f_e^{\dagger}$ has the energy argument $E^{\dagger}$ and represents the electron that will carry away the excess energy of the reaction; and $P(E^* \rightarrow E_j)$ is the probability of capture into the j^{th} state.

Microscopically, the ionization and recombination processes of equations (B1) and (B5) are related by the energy equation

$$E = E^{\dagger} + E^* + E_j \tag{B6}$$

From equation (B5) it is noted that the volume element of the ion $d\overset{3}{q_i}$ must be equal to $d\overset{3\dagger}{q_e}$ if a collision is to be realized. Hence, transforming variables as in equation (B2) gives equation (B5) as

$$\left(\frac{\partial n_e}{\partial t}\right)_{jR} = -\int f_e^{\dagger} f_e^* n_i P(E^* \rightarrow E_j)(16\pi^2 m)^2 b^*\,db^*\,b^{\dagger}\,db^{\dagger}\,dt\,E^*\,dE^*\,E^{\dagger}\,dE^{\dagger} \tag{B7}$$

Defining, as in equation (1921) of reference 16, the cross section for recombination to be

$$\sigma_R(E^* \rightarrow E_j) = 4\pi^2\int P(E^* \rightarrow E_j)b^*\,db^*\,b^{\dagger}\,db^{\dagger}\,dt \tag{B8}$$

yields equation (B7) as

$$\left(\frac{\partial n_e}{\partial t}\right)_{jR} = -\,64\pi^2 m^2 n_e^3 \left(\frac{\lambda}{2\pi m}\right)^3 \int \exp[-\lambda(E^{\dagger} + E^{*})]\sigma_R(E^{*} \rightarrow E_j)E^{\dagger}\, dE^{\dagger}\, E^{*}\, dE^{*} \qquad \text{(B9)}$$

Equation (B6) can be used to change variables from $E^{\dagger}$ to E; and when equations (B9) and (B4) are added and summed over j, the net change in electron density due to ionization-recombination processes is found:

$$\left(\frac{\partial n_e}{\partial t}\right)_{IR} = 8\pi m n_e \left(\frac{\lambda}{2\pi m}\right)^{3/2} \sum_j \left[n_j \int \exp\,(-\lambda E)\sigma_I(E_j \rightarrow E^{*})dE^{*}\, E\, dE\right.$$

$$\left. -8\pi m n_e^2 \left(\frac{\lambda}{2\pi m}\right)^{3/2} e^{\lambda E_j} \int \exp(-\lambda E)\sigma_R(E^{*} \rightarrow E_j)E^{*}\, dE^{*}(E - E^{*} - E_j)dE\right] \qquad \text{(B10)}$$

Here the integral sign represents, as mentioned before, a double integration over E and E^{*} with their appropriate limits.

The relation between $\sigma_R(E^{*} \rightarrow E_j)$ and $\sigma_I(E_j \rightarrow E^{*})$ is given in equation (1924) of reference 16, and in the notation of this report is,

$$E\sigma_I(E_j \rightarrow E^{*}) = \frac{16\pi m}{h^3} \frac{\omega_i}{\omega_j} E^{*}(E - E^{*} - E_j)\sigma_R(E^{*} \rightarrow E_j) \qquad \text{(B11)}$$

Hence, equation (B10) becomes

$$\left(\frac{\partial n_e}{\partial t}\right)_{IR} = n_e \sum_j \left(n_j - n_j^s\right)\langle v\sigma_I(E_j \rightarrow E^{*})\rangle \qquad \text{(B12)}$$

where the definition of the Saha density in equation (5) was used and

$$\langle v\sigma_I(E_j \rightarrow E^{*})\rangle = 8\pi m \left(\frac{\lambda}{2\pi m}\right)^{3/2} \int \exp(-\lambda E)(E_j \rightarrow E^{*})dE^{*}\, E\, dE \qquad \text{(B13)}$$

The change in the population of the j^{th} state is derived in exactly the same way with the exception that equation (B1) now represents the number of electrons leaving, and equation (B5) represents the number entering the bound state. Hence,

$$\left(\frac{\partial n_j}{\partial t}\right)_{IR} = n_e\left(n_j^S - n_j\right)\langle v\sigma_I(E_j \rightarrow E^*)\rangle \qquad \text{(B14)}$$

The energy transferred to and from the electron gas due to ionization-recombination results in an expression similar to equation (B13). The unit mechanism implicit in the derivation of (B13) is denoted as $\delta(\partial f_e/\partial t)_{IR}$ where the δ means that none of the indicated integrations in equations (B1) and (B5) are performed; that is, the averaging is not yet carried out. Multiplying $\delta(\partial f_e/\partial t)_I$ by E and averaging gives the energy loss in ionization; and multiplying $\delta(\partial f_e/\partial t)_R$ by E and again averaging gives the energy gained by recombination. Hence,

$$\left(\frac{\partial u_e}{\partial t}\right)_{IR} = \int E\left(\frac{\partial f_e}{\partial t}\right)_{IR} d^3v_e = n_e\sum_j \left(n_j^S - n_j\right)\left\langle\frac{1}{2} mv^2 v\sigma_I(E_j \rightarrow E^*)\right\rangle \qquad \text{(B15)}$$

where

$$\left\langle\frac{1}{2} mv^2 v\sigma_I(E_j \rightarrow E^*)\right\rangle = 8\pi m\left(\frac{\lambda}{2\pi m}\right)^{3/2}\int \exp(-\lambda E)\sigma_I(E_j \rightarrow E^*)E^2 \, dE \, dE^* \qquad \text{(B16)}$$

Inelastic-Superelastic Collisions

It is assumed first that $E_j > E_k$ (or $k > j$). The change in population of the j^{th} state due to inelastic and superelastic collisions proceeds as follows: The number of particles that leave the j^{th} state for the k^{th} state is equal to the inelastic collision rate (since $E_j > E_k$), that is,

$$\left(\frac{\partial n_j}{\partial t}\right)_{j \rightarrow k} = - 8\pi m\left(\frac{\lambda}{2\pi m}\right)^{3/2} n_e n_j \int \sigma_{ex} \exp(-\lambda E)E \, dE \qquad \text{(B16)}$$

where the explicit form of f_e was used, and the differentials were transformed as in equation (B2). The inverse to this process is the superelastic collision rate, equal to

$$\left(\frac{\partial n_j}{\partial t}\right)_{k \rightarrow j} = 8\pi m\left(\frac{\lambda}{2\pi m}\right)^{3/2} n_e n_k \int \sigma_{dex} \exp(-\lambda E^*)E^* \, dE^* \qquad \text{(B17)}$$

where

$$E^* = E - (E_j - E_k) \tag{B18}$$

According to equation (1902) of reference 16, the relation between σ_{ex} and σ_{dex} is given by

$$\omega_k E^* \sigma_{dex} = \omega_j E \sigma_{ex} \tag{B19}$$

Hence, summing over all $k > j$ gives the partial result

$$\sum_{k>j} \left(\frac{\partial n_j}{\partial t}\right)_{j \rightleftarrows k} = 8\pi m n_e \left(\frac{\lambda}{2\pi m}\right)^{3/2} \sum_{k>j} \left\{n_k \frac{\omega_j}{\omega_k} \exp[(E_j - E_k)] - n_j\right\} \int_{Ej-Ek}^{\infty} \sigma_{ex}(j \rightarrow k) E \exp(-\lambda E)\, dE \tag{B20}$$

Observing the convention that E^* is the energy of the free electron before superelastic collision (or after inelastic collision) and expressing the rates in terms of the excitation cross section shows easily that for $E_k > E_j$ (or $k < j$) the following rate equation holds:

$$\sum_{k<j} \left(\frac{\partial n_j}{\partial t}\right)_{j \rightleftarrows k} = 8\pi m n_e \left(\frac{\lambda}{2\pi m}\right)^{3/2} \sum_{k<j} \left\{n_k - \frac{\omega_k}{\omega_j} n_j \exp[\lambda(E_k - E_j)]\right\} \int_{E_k - E_j}^{\infty} \sigma_{ex}(k \rightarrow j) E \exp(-\lambda E)\, dE \tag{B21}$$

The following relations are noted:

$$\langle v^* \sigma_{dex} \rangle = n_e \int f_e^* v_e^* \sigma_{dex}\, dv_e^{3*} = 8\pi m \left(\frac{\lambda}{2\pi m}\right)^{3/2} \int_0^{\infty} e^{-\lambda E} \sigma_{dex} E^*\, dE^* \tag{B22}$$

Using equation (B19) and the energy relation equation (B18) converts equation (B22) into

$$\langle v^*\sigma_{dex}\rangle = 8\pi m\left(\frac{\lambda}{2\pi m}\right)^{3/2} \frac{\omega_j}{\omega_k} \exp[\lambda(E_j - E_k)] \int_{E_j - E_k}^{\infty} \exp(-\lambda E)\sigma_{ex} E \, dE$$

$$= \frac{\omega_j}{\omega_k} \exp[\lambda(E_j - E_k)]\langle v\sigma_{ex}\rangle \tag{B23}$$

Retention of the asterisk in equation (B23) is spurious since both v and v* are dummy variables that lose their distinction after averaging.

Adding equations (B20) and (B21) then gives the result quoted in equation (11):

$$\sum_k \left(\frac{\partial n_j}{\partial t}\right)^c_{j\rightleftarrows k} = n_e \sum_{k<j} \left\{n_k - \frac{\omega_k}{\omega_j} n_j \exp[\lambda(E_k - E_j)]\right\}\langle v\sigma_{ex}(k \rightarrow j)\rangle$$

$$+ n_e \sum_{k>j} \left\{\frac{\omega_j}{\omega_k} n_k \exp[\lambda(E_j - E_k)] - n_j\right\}\langle v\sigma_{ex}(j \rightarrow k)\rangle \tag{B24}$$

The energy exchange due to inelastic-superelastic collisions is simply computed by considering that for every electron excited from the state j to the state k a free electron originally having energy E leaves the phase cell characterized by E. Thus

$$\sum_{j,k} \left(\frac{\partial u_e}{\partial t}\right)^c_{j\rightleftarrows k} = \sum_{j,k} \int E\left(\frac{\partial f_e}{\partial t}\right)^c_{j\rightleftarrows k} dv_e^3$$

$$= n_e \sum_j \sum_{k>j} \left\{\frac{\omega_j}{\omega_k} n_k \exp[\lambda(E_j - E_k)] - n_j\right\}\left\langle\frac{1}{2} mv^2 v\sigma_{ex}(j \rightarrow k)\right\rangle \tag{B25}$$

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