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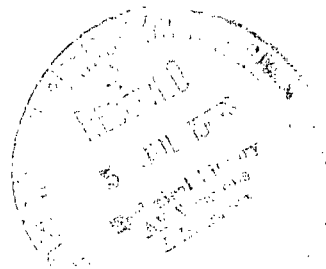
A REVIEW OF SOME EXISTING LITERATURE CONCERNING DISCONTINUOUS STATE VARIABLES IN THE CALCULUS OF VARIATIONS

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DEFINITION OF SYMBOLS

Latin Symbols	Definition
A	algebraic substitution used to simplify an expression
C	integration constant used in the DuBois-Reymond equation when subscripted
C	fixed discontinuity in a state variable when unsubscripted
D	specified relationship between variables at a point of discontinuity
E	Weierstrass excess function
F	fundamental function appearing in the formulation of the multiplier rule
g	a function of the end points to be extremized
J	the numerical value of g
ℓ	Lagrange multipliers associated with the end point constraints
R_{1j}	a region of 2n+1 dimensional space in which the constraints are assumed to have three orders of continuous derivatives
R_1	the union of all R_{1j} regions
R_2	a region such that the m(n+1) dimensional point $[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)]$ are interior to it
t	the independent variable
x_i	any of the set of original state variables
x	the entire set of x_i's
X	used only as $\dot{X}$: a set not identical to $\dot{x}$ in the Weierstrass test

DEFINITION OF SYMBOLS (Cont'd)

Latin Symbols	Definition

z_i — variables employed after the transformation which include both the initial state variables and the value of the independent variable at which discontinuities occur

z — the entire set of z_i's

Z — used only as $\dot{Z}$: a set not identical to $\dot{z}$ in the Weierstrass test

Greek Symbols

Δ — indicates the magnitude of discontinuity in the ith state variable at the jth time point as in Δx_{ij}

ϵ — a small number

λ_i — a Lagrange multiplier

λ — the entire set of λ_i's

π_i — a set of numbers used in the Clebsch condition

τ — independent variable in the transformed problem

ϕ — a dynamical constraint

ψ — an end point constraint

Subscripts

f — indicates a final point

h — dummy summation variable: range $1, \ldots, n$

i — index used on a member of the set $\{x_1, \ldots, x_n\}$

j — index used on a member of the set $\{t_1, \ldots, t_m\}$

k — index used on a member of the set $\{z_1, \ldots, z_N\}$ or dummy summation index

Subscripts	Definition
m	the number of values of the independent variable at which discontinuities occur in the state variables
n	the number of state variable in the original problem
N	the number of state variables in the transformed problem
p	the number of end point constraints
P	the number of end point constraints in the transformed problem
q	the number of constraints in the original problem
Q	the number of constraints in the transformed problem
r	dummy summation index: $r = 1, \ldots, m$
s	dummy summation index: $s = 1, \ldots, N$
α	index on constraints in transformed problem: $\alpha = 1, \ldots, Q$
β	index on constraints in original problem: $\beta = 1, \ldots, q$
ν	index on end point constraints: $\nu = 1, \ldots, p$ (original problem) or $\nu = 1, \ldots, P$ (transformed problem)
γ	index on constraints in original problem: $\gamma = 1, \ldots, q$
0	indicates an initial point in the original problem
1	indicates an initial point in the transformed problem
2	indicates a final point in the transformed problem

Superscript	
j	indicates the particular arc under consideration

DEFINITION OF SYMBOLS (Concluded)

Other Notation	Definition
*	indicates a quantity expressed in terms of the transformed variables
•	indicates a derivative with respect to the independent variable of the initial problem
'	indicates a derivative with respect to the independent variable of the transformed problem

A REVIEW OF SOME EXISTING LITERATURE CONCERNING DISCONTINUOUS STATE VARIABLES IN THE CALCULUS OF VARIATIONS

SUMMARY

The important problem of determining optimal trajectories for problems with discontinuous state variables has been treated by a few authors in rather esoteric papers. This paper summarizes the contents of two of these in a fairly detailed discussion. An attempt has been made to develop explicitly those relationships which are not contained in material dealing with the case of continuous state variables.

The essential feature of the discussion is a transformation from a set of state variables with points of discontinuity to a set of new variables which, though greater in number, are continuous. The necessary conditions of Euler, Weierstrass, and Clebsch, along with the transversality conditions, are discussed in the transformed problem; the results are rewritten in the original variables.

The bulk of material covered assumes that the magnitudes of the discontinuities are known a priori, but the case of unknown discontinuities is treated in the latter portion of the paper.

INTRODUCTION

Most of the modern texts dealing with the subject of classical mechanics are oriented to those areas which form a basis for quantum mechanics. The recent interest in the application of classical mechanics to astronautics cannot be adequately treated by the texts which deal primarily with groundwork for quantum mechanics, and for this reason much research has been done into the older literature of mechanics.

One such area which arises in the aerospace field is the necessity of considering discontinuous integrands in the problems formulated via calculus of variations techniques. This problem has been treated only very recently, and the formulation rests on the doctoral dissertation of C. H. Denbow which was published in 1937 (Ref. 1).

Denbow did not assume discontinuous variables but rather a problem situation in which unknown corner points existed. The basic idea of this paper was extended by Hunt (Ref. 2) to a case in which the variables were assumed to be discontinuous. The points at which the discontinuities appear are unknown, but the magnitudes of the discontinuities are assumed known. Mason, et. al. (Ref. 3), have treated the problem in which both the locations and magnitudes of the discontinuities are unknown. The work of Boyce (Ref. 4) extends Hunt's treatment to explicit recognition of the control variables and to the inclusion of inequality constraints in the formulation.

In order to obtain some insight into the necessity of considering such problems, we note that it is often desirable to determine a so-called optimum rocket trajectory, that is the trajectory which maximizes - or minimizes - some quantity depending upon the end points of that trajectory. Within this framework we can treat the problem of finding the maximum payload that a vehicle can deliver to a specified orbit.

The problem developed by Hunt covers the case in which there is a known vehicle, i.e., the dry weights of the individual stages are known. By this assumption we know, at launch, what the discontinuity in the mass must be at each staging point. (Note that the thrust actually "tails off" then "builds up" so that thrust may be regarded as a continuous variable. Then the accelerations are discontinuous only at points where the mass is discontinuous.)

The problem of Mason et. al. is applicable to the case in which an existing rocket is not specified, but rather one in which individual stages are designed in such a way that a maximum payload with given liftoff thrust is obtained. It is well known that optimum weight ratios exist between successive stages; since these are coupled with the trajectory to be flown, we have a full calculus of variations problem.

This paper does not add anything new to the above theories. The purpose is, rather, to expand the above work to such a level that most workers familiar with variational methods (at the level of Bolza's problem) can obtain a knowledge of the discontinuous problem with a minimum of time expenditure. All of the above papers are rather formidable and leave much of the developmental work to the reader.

The notation used in Hunt's work will be retained, with the exception that certain conventions (such as summation on repeated indices and a subscript notation for partial derivatives) will be replaced by more explicit notation for clarity. The subscript notation is probably the most difficult part of the study, and any simplification proves most beneficial.

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Finally, it will prove worthwhile to carry references to a specific problem throughout the development in order to both illustrate and motivate the theory. The example that will be chosen is, of course, the multistage rocket problem.

FORMULATION OF THE PROBLEM

The problem of Mayer considers the techniques of finding a maximum or minimum of some function of the end points

$$J = g[\, t_0, t_f, x(t_0), x(t_f)\,]$$

(where t_0, t_f are the initial and final times; $x(t_0)$, $x(t_f)$ are some functions of these time points). It is assumed that the x's must satisfy certain differential equations and end point conditions. In our present problem we shall be concerned with a finite number of time points at each of which certain conditions shall be specified. (These can be thought of as the points at which staging occurs, for example.)

To illustrate the problem, we assume that we have n variables, $x_i(t)$ $(i = 1, \ldots, n)$ and m time points t_j, $j = 1, \ldots, m$. Assuming that we already know the answer, as is usual in calculus of variations, the following diagram illustrates the $x_i(t)$ for 3 variables and 4 junctions, i.e., $n = 3$, $j = 4$.
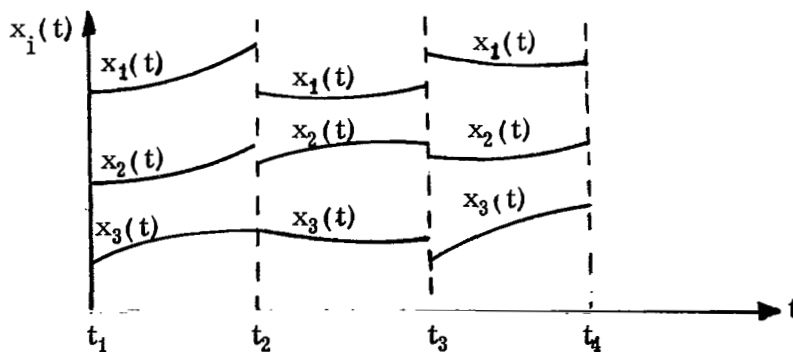
In this case the $x_i(t)$ have been chosen to maximize a function

$$J = g[\, t_1, t_2, t_3, t_4, x_1(t_1), x_1(t_2), x_1(t_4), \ldots, x_3(t_1), \ldots, x_3(t_4)\,]$$

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In general, we write

$$J = g[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)]$$

where $x(t_1) = [x_1(t_1), x_2(t_1), \ldots, x_n(t_1)]$ etc.

The x's are, again as usual, subject to certain differential equation constraints along the trajectory. If we have q such constraints we can write these as

$$\phi_\beta(t, x, \dot{x}) = 0 \qquad\qquad (\beta = 1, \ldots, q)$$

But if, for example, we wish to change the nature of a differential equation at some time point t_j then we must append an additional label to obtain something like

$$\phi_\beta^{(j)}(t, x, \dot{x}) = 0 \qquad\qquad (\text{valid for } t \geq t_j)$$

(Specifically, we might wish to let $\phi_\beta^{(j)}$ be a flight path equation which includes aerodynamic effects and $\phi_\beta^{(j+1)}$ be the corresponding flight path equation with the atmospheric terms deleted.) It should be noted that the number of constraints must be <u>less</u> than the number of variables or there will be no freedom to choose an optimum path. Thus, we require

$$q < n$$

with the strict inequality assumed to hold.

To obtain the range on j we relate this index to our number of end points. For arc number one, with $t_1 \leq t < t_2$, we have j = 1; that is, all our equations on the first arc are labeled with a 1, and the differential equations are

$$\phi_\beta^{1}(t, x, \dot{x}) = 0.$$

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The second arc is labeled with a 2, etc. We must include the last end point, t_m, for some arc, and since we have excluded the right-hand end point on each arc we write

For $j = 1, 2, \ldots, m-2$, we have $t_j \leq t < t_{j+1}$

For $j = m-1$, we have $t_j \leq t \leq t_{j+1}$

(Note that m points give m-1 arcs, so in the $\phi_\beta^{(j)} = 0$ equations we have j never equal to m.)

Other constraints on our problem are the conditions which must be met at the points t_j. (This condition allows us to specify whatever seems appropriate at each staging point such as initial latitude, final orbital inclination, altitude of second stage cutoff, etc.) If we label our constraints at the t_j points by ψ_ν, we have

$$\psi_\nu [t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)] = 0$$

where, again, the x's are assumed to represent the entire set $(x_1, \ldots, x_n)$. Now the range on ν must be determined. This procedure essentially counts how many things we can fix. We may, if necessary or desirable, fix all time points t_j, which gives m conditions. We may also fix all the x's at t_1 and t_m, giving 2n more points. Finally, we can specify all the x's at either (not both) side of an intermediate time point, t_j. Since we have n x's fixed at either side of m-2 points, n(m-2) more conditions are added. (We cannot fix the x's at both sides of t_j since we shall later specify the magnitude of the discontinuity; this procedure could yield internal inconsistencies.) We obtain a total of

$$m + 2n + n(m-2) = m(n+1).$$

This is the maximum number of ψ_ν constraints we may have. Thus,

$$\nu = 1, \ldots, p; \quad p \leq m(n+1).$$

These remarks preface the following statement of our problem.

Find in a class of admissible* arcs

$$x_i(t) \quad (i = 1, \ldots, n; \ t_1 \le t \le t_m) \tag{1}$$

satisfying differential equations of the form

$$\phi_\beta^{(j)} (t, x, \dot{x}) = 0 \qquad (\beta = 1, \ldots, q < n) \tag{2}$$

$$(\text{for } j = 1, \ldots, m-2 \text{ we have } t_j \le t < t_{j+1}$$

$$\text{for } j = m - 1 \qquad \text{we have } t_j \le t \le t_{j+1})$$

and end conditions of the form

$$\psi_\nu[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)] = 0 \quad (\nu = 1, \ldots, p \le m(n+1)) \tag{3}$$

one which minimizes

$$J = g[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)] . \tag{4}$$

We have now isolated all discontinuities at the points t_j. Since we shall apply the multiplier rule, we require that the $\phi_\beta^{(j)}$ functions are of class C^3 (i. e., $\phi_\beta^{(j)} \epsilon C^3$). This requirement is assumed to hold in some region of a $2n+1$ dimensional space (n x's, n $\dot{x}$'s and one t) which we label as R_{1j} (the j is discussed below). Furthermore, the multiplier rule requires that the $\phi_\beta^{(j)}$ equations are independent. Since this condition must hold on each arc, we may state independence by requiring that the matrix

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* The term "admissible" requires some explanation and will be defined on page 7, paragraph 3.

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$$\left(\frac{\partial \phi^{(j)}_\beta}{\partial \dot{x}_i} \right)$$

be of rank q. (Remember we have q equations.) Thus, we are assured that we can uniquely solve for the q $\dot{x}_i$'s.

Another requirement which must be assumed for the following theory to be valid is that our 2n+1 dimensional points are all interior to the region R_{1j}.

We require an independence relationship among the ψ_ν's for similar reasons. That is, we require that the matrix

$$\left(\frac{\partial \psi_\nu}{\partial t_j} \quad , \quad \frac{\partial \psi_\nu}{\partial x_i(t_j)} \right) \tag{5}$$

be of rank p (we had p end conditions). Also, let R_2 be a region such that the m(n+1) dimensional points $[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)]$ are interior to R_2. In this region we require that all ψ_ν's and g are of class C^3.

To return to the subscript which was placed on R_1, the raison d'être for j is that we considered each individual arc on which the differential equations held as separate segments. If we now take the union of all R_{1j}'s, we obtain the total region R_1. In set notation

$$R_1 = \bigcup_{j=1}^{m-1} R_{1j}$$

(m t_j's give m-1 arcs). An arc in R_1 can be defined as an arc $x_i(t)$ over the range $t_1 \leq t \leq t_m$ which is continuous except possibly at a finite number of points, t_j (j = 1, \ldots, m); $x_i(t)$ consists of a finite number of arcs, and $x_i(t)$ is assumed to have continuous first order derivatives ($x_i(t) \epsilon C^1$). The term "admissible" arc can now be defined.

Definition: An admissible arc for our problem is a set $[t_1, \ldots, t_m]$ and an arc interior to R_1, all of whose ends and corners $[t_j, x(t_j)]$ are interior to R_2.

By this definition an admissible arc gives well defined values to $\phi_\beta^{(j)}$, g, ψ_ν, and J.

This information is preliminary. We turn now to the transformation of our stated problem into a form tractable to classical methods.

TRANSFORMATION OF THE PROBLEM

As is well known, the theory of variational techniques cannot handle problems involving discontinuous variables. These could be applied if a technique existed to make all variables continuous. The observation of the diagram on page 3 and a little reflection might give rise to the following concept. The variable x_1 is obviously discontinuous at the point t_2. But if we were to regard x_1 on the first arc and x_1 on the second arc as two variables instead of one discontinuous variable, then we have a problem involving only continuous variables. We bought this condition by adding another variable, but that is not a strong penalty since we already know how to treat the multiple variable problem.

The discontinuity of x_1 at t_2 still exists, of course, since we have not altered the physics of the situation. We must eventually find a method of connecting our path across this point, but, as will be later demonstrated, the natural corner conditions will do just this.

This, in essence, is the philosophy behind the manipulations that follow: transfer from a problem involving discontinuous variables to one involving an expanded set of continuous variables.

We initiate the procedure by transforming the time. To do this, introduce a ficticious time τ and define $t(\tau)$ as

$$t = t_j + (t_{j+1} - t_j)\tau \quad (j = 1,\ldots,m-1)$$

with τ having a range $0 \leq \tau \leq 1$. For our first arc we have

$$t_1 \leq t \leq t_2,$$

so $\quad t = t_1 + (t_2 - t_1)\tau$ which gives $j = 1$. For our last arc,

$$t_{m-1} \leq t \leq t_m,$$

so $t = t_{m-1} + (t_m - t_{m-1})\tau,$

so $j = m - 1.$

The variables x_i are functions of t, and we introduce new variables and call them z's, which are functions of τ. They are defined in terms of the x's by

$$z_{(j-1)n+i}(\tau) = x_i(t). \tag{6}$$

That subscript on z looks a bit odd, but the original approach shows that it can be explained. On the first arc ($t_1 \leq t \leq t_2$) we have variables $x_1, \ldots, x_n$. For this arc, as noted above, $j = 1$. So on the first arc

$$z_i(\tau) = x_i(t) \quad (i = 1, \ldots, n)$$

Now we must introduce new z's for the second arc, and since we have used up $z_1, \ldots, z_n$ we start the second arc variables as

$$z_{n+1}(\tau) = x_1(t).$$

Therefore,

$$z_{n+i}(\tau) = x_i(t) \quad (\text{second arc}),$$

which can also be written as

$$z_{(2-1)n+i}(\tau) = x_i(t).$$

Since $j = 2$ on the second arc, we have

$$z_{(j-1)n+i}(\tau) = x_i(t) \quad (\text{second arc}).$$

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Continuing in this way we find that

$$z_{(j-1)n+i}(\tau) = x_i(t) \quad (j^{\text{th}} \text{ arc}).$$

As noted above, for the last arc, $j = m - 1$. So we obtain our j range as

$$j = 1, \ldots, m-1.$$

To see how many variables we really have introduced, we set j and i to their maximum values and find

$$z_{(m-2)n+n} = z_{(m-1)n},$$

which tallies with the figure where we had $n = 3$, $m = 4$ giving z_9. A simple count on this diagram shows nine variables. (Here the first cloud appears on the horizon; for it is not unusual to use 12 x's and 5 staging points in a trajectory – which means 48 variables to consider.)

Now we consider our discontinuity in terms of the z's. Our variables are defined at the left end of an arc, which is the right side of t_j denoted by t_j^+. The value – which is unspecified – of $x_i(t)$ at the left side of t_j (i.e., the right end of the previous arc) we shall call t_j^-. Here we consider that

$$\left. \begin{array}{l} t_j^+ = t_j + \epsilon \\[2mm] t_j^- = t_j - \epsilon \end{array} \right\} \epsilon \text{ small}$$

Directly from the definition of the z's (eq. 6) we have

$$x_i(t_j^+) = z_{(j-1)n+i}(0).$$

At the other end of this interval we have the same x_i evaluated at t_{j+1}^-. We still have $z_{(j-1)n+i}$ since we don't start our new set of z's until we move to the right of t_{j+1}. So

$$x_i(t_{j+1}^-) = z_{(j-1)n+i}(1).$$

For the value of x_i at the right of t_{j+1}, i.e., $x_i(t_{j+1}^+)$, we note that we are into the next set of z's; thus we add n to our previous value giving

$$x_i(t_{j+1}^+) = z_{(j-1)n+n+i}(0) = z_{jn+i}(0).$$

Adding the last two equations we have

$$z_{(j-1)n+i}(1) + x_i(t_{j+1}^+) = x_i(t_{j+1}^-) + z_{jn+i}(0)$$

or

$$z_{(j-1)n+i}(1) + [x_i(t_{j+1}^+) - x_i(t_{j+1}^-)] = z_{jn+i}(0)$$

$$= z_{(j-1)n+n+i}(0).$$

(7)

The term in parentheses is our known discontinuity of the x_i at the point t_{j+1}.

Now this is good for any x, so $i = 1, \ldots, n$. Our last discontinuity in the x's occurs at t_{m-1}, which in our equations shows up as $x_i(t_{j+1})$. So we must have

$$j + 1 = m - 1 \rightarrow j = m - 2.$$

Our first discontinuity occurs at t_2, so $x_i(t_2) \leftrightarrow x_i(t_{j+1}) \rightarrow j = 1$. Thus our range of j in equation (7) is

$$j = 1, \ldots, m-2.$$

This defines our first $(m-1)n$ variables. We now add to this the time points t_j and make them variables of our problem. Since we have numbered our z's from $z_1, \ldots, z_{(m-1)n}$ we start numbering the new z variables (which are really just the t_j's) with $z_{(m-1)n+1}, \ldots, z_{(m-1)n+m}$ (we have m t_j's). Then

$$t_j = z_{(m-1)n+j}.$$

Since our t_j's are <u>constant</u> we have

$$z'_{(m-1)n+j} = 0 \quad (j = 1, \ldots, m) \tag{8}$$

where $z' = \dfrac{dz}{d\tau}$ for any z. These are all the variables that we shall need. We have collected a total of

$$(m-1)n + m = N$$

variables.

The function g, which we wish to extremize is

$$J = g[t_1, \ldots, t_m, x(t_1), \ldots, x(t_m)].$$

The values of all the t_i's in this equation and the values of x_i's at the corner points (at one side or the other of the corner points – and we chose the right side for every point except the last one) have now been expressed in terms of the $z_i(0)$, $z_i(1)$, which we shall denote by $z(0)$, $z(1)$. Thus J becomes

$$J* = g*[z(0), z(1)]$$

where the $*$ indicates a functional dependence of z.

Next we transform the $\phi_\beta^{(j)}$'s. As in the case of the z's there will be many more $\phi_\beta^{(j)}$'s in the representation in terms of τ, z, z' than there were in terms of $t, x, \dot{x}$. The new functions are defined as

$$\phi^*_{(j-1)q+\beta}(\tau, z, z') = \phi_\beta^{(j)}[t(\tau), x(z), \dot{x}(z', \tau)].$$

(9)

The justification for the $(j-1)q+\beta$ subscript is much the same as before. On the first arc, $j = 1$; we range β from 1 to q (q constraints). On the next arc we start by numbering $\phi^*_{q+1} = \phi^*_{(2-1)q+1} = \phi^*_{(j-1)q+1}$ ($j=2$ on the second arc), etc. Again, for m time points t_j we obtain m-1 arcs. So in equation (9) we have

$$j = 1, \ldots, m-1$$

$$\beta = 1, \ldots, q$$

and we still have

$$t = t_j + (t_{j+1} - t_j)\tau.$$

This condition accounts for our original differential equation constraints. But equation (8) also produces additional constraints of certain z''s, so we can write

$$\phi^*_{(m-1)q+j}(\tau, z, z') = z'_{(m-1)n+j}.$$

(10)

The subscript of the left side was chosen to pick up directly after the last subscript of equation (9). That is, we insert our maximum value of j(= m-1) and β(= q) in this equation giving

$$\phi^*_{(m-1-1)q+q} = \phi^*_{(m-1)q},$$

and start equation (10) at $\phi^*_{(m-1)q+1}$. The right side subscript comes directly from equation (8). Note that the two subscripts do not agree numerically. The only variable subscript in this equation, j, to account for all the time points t_j, is

$$j = 1, \ldots, m.$$

We had previously defined $(m-1)n+m = N$ variables and now we have defined a total of $(m-1)q+m$ constraints. If we subscript all of our $\phi*$'s with an α, we see that the first $(m-1)n$ of these

$$\phi^*_\alpha \qquad (\alpha = 1, \ldots, (m-1)q)$$

are defined for $0 \leq \tau < 1$ (excluding the right end of our interval again). The second set

$$\phi^*_\alpha \qquad (\alpha = (m-1)q+1, \ldots, (m-1)q+m)$$

are defined for $0 \leq \tau \leq 1$. (It does not matter which value of τ is plugged in to equation (8).) The total number of constraints we label as

$$(m-1)q+m = Q.$$

For the end conditions we had $\psi_\nu = 0$, where $\nu = 1, \ldots, p \leq m(n+1)$. We can now get ψ^*_ν 's - i.e., transform ψ_ν into dependence upon z, τ variables - just as we got g* from g. This gives us our first p conditions. But there are others.

In equation (7) we define

$$x_i(t^+_{j+1}) - x_i(t^-_{j+1}) = \Delta x_{ij}.$$

(Note that our discontinuity here is at the right end of the jth arc. Thus it occurs at the j+1'st point t_{j+1}. This accounts for the difference of the j subscript across the equation.) Equation (7) may now be written

$$\psi^*_{(j-1)n+i+p} = z_{(j-1)n+i}(1) + \Delta x_{ij} - z_{(j-1)n+n+i}(0) = 0. \qquad (11)$$

The left subscript starts with $j = 1$, $i = 1$, so we pick up at ψ^*_{1+p} as we should since above we used up $p\,\psi$'s. For the last point we have $t_j = t_{m-1}$ so our last arc is the $j-2$'nd one giving a range to j of

$$j = 1, \ldots, m-2.$$

i, as usual, refers to our variables $x_1, \ldots, x_n$ so we have

$$i = 1, \ldots, n.$$

Putting j and i to their maximum values we find we have accumulated $(m-2-1)n+n+p = (m-2)n+p$.

Our final two end point constraints are on the end values of τ. Since our last constraint was $\psi^*_{(m-2)n+p}$ we have

$$\psi^*_{(m-2)n+p+1} = \tau_1 = 0$$

$$\psi^*_{(m-2)n+p+2} = \tau_2 - 1 = 0.$$

Thus our new (transformed) end conditions are

$$\psi^*_\nu [\tau_1, z(\tau_1), \tau_2, z(\tau_2)] = 0$$

where

$$\nu = 1, \ldots, (m-2)n+p+2 = P.$$

The original problem has now been transformed into the problem of finding, in a class of arcs defined by $z_j(\tau)$ satisfying $\phi^*_\alpha = 0$ and $\psi^*_\nu = 0$,

an arc which minimizes $J^* = g^*[\,z(\tau_1)\,,z(\tau_2)\,]$. This now satisfies the hypothesis of Bolza's problem, and we can apply the standard formulation as given by Bliss (Ref. 5). The proof that the solution to our original problem has a solution which is equivalent to that of the transformed problem when written in the original coordinates is not obvious from a mathematical viewpoint. The argument which justifies this statement is given in Reference 2, and it will not be reproduced at this time since no real gain would come of it. We shall now proceed to examine in detail the derivation which produces the Euler equations, transversality conditions, Weierstrass condition, and Clebsch condition for the discontinuous state variable problem.

THE EULER EQUATIONS AND TRANSVERSALITY CONDITIONS

Since the problem of Bolza is now applicable to our task, we can apply the multiplier rule as given by Bliss (Ref. 4). An admissible solution, E^*, of the equations $\phi_\alpha^* = 0$ ($\alpha = 1,\ldots,Q$) defined on $[\tau_1,\tau_2]$, is said to satisfy the multiplier rule if there exist constants λ_0^*, ℓ_ν ($\nu = 1,\ldots,P$) not all zero and a function $F^*\,(\tau,z,z',\lambda^*) = \sum_{\alpha=1}^{Q} \lambda_\alpha^* \phi_\alpha^*$ with multipliers $\lambda_\alpha^*(\tau)$ continuous on $[\tau_1,\tau_2]$ except possibly at values of τ defining corners of E^* where they have well defined right and left limits, such that the equations

$$\frac{\partial F^*}{\partial z_k'} = \int_{\tau_1}^{\tau} \frac{\partial F^*}{\partial z_k}\, d\tau + C_k \quad (k = 1,\ldots,N) \tag{12}$$

are satisfied along E^*, and such that the equation

$$\left[\left(F^* - \sum_{k=1}^{N} z_k' \frac{\partial F^*}{\partial z_k'}\right) d\tau + \sum_{k=1}^{N} \frac{\partial F^*}{\partial z_k'}\, dz_k\right]_1^2 ,\ + \lambda_0^*\, dg^* \tag{13}$$

$$+ \sum_{\nu=1}^{P} \ell_\nu d\psi_\nu^* = 0$$

holds at the ends of E^* for every choice of differentials $d\tau_1, dz_j(\tau_1), d\tau_2, dz_j(\tau_2)$. Every minimizing arc, E^*, for the problem of Bolza must satisfy the multiplier rule.

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The function F* defined in the above statement can be written by use of equations (9) and (10) as

$$F^*_j = \sum_{\beta=1}^{q} \lambda^*_{(j-1)\,q+\beta}(\tau)\, \phi^*_{(j-1)\,q+\beta}(\tau, z, z')$$

$$(14)$$

$$+ \sum_{r=1}^{m} \lambda^*_{(m-1)\,q+r}\, z'_{(m-1)\,n+r}$$

where we have split our constraints into the two groups defined by equations (9) and (10). The j subscript affixed to F* is (mechanically) necessary because of the free j subscript on the right hand side of the equation; its physical significance corresponds to the possibility of F^*_j differing from F^*_{j+1} by variations in the ϕ^*'s. Note that j is not summed. (Obtaining a non-subscripted j would require an additional summation.) The equations produced from application of equation (12) to the function F* have different characteristics depending upon whether we consider the first (m-1)n of these equations or the last m equations.

Application of the Euler Equations to the First (m-1)n Variables

For the first (m-1)n equations, we begin by consideration of the term

$$\frac{\partial F^*_j}{\partial z'_k} = \frac{\partial F^*_j}{\partial z'_{(j-1)\,n+i}} = \sum_{s=1}^{n} \frac{\partial F}{\partial \dot{x}_s}\, \frac{d\dot{x}_s}{dz'_{(j-1)\,n+i}} \, .$$

But

$$x_i(t) = z_{(j-1)\,n+i}(\tau)$$

so that

$$\dot{x}_i(t) = z'_{(j-1)\,n+i}\, \frac{d\tau}{dt} \, .$$

Since

$$t = t_j + (t_{j+1} - t_j)\tau,$$

we have

$$\frac{dt}{d\tau} = t_{j+1} - t_j .$$

Then

$$z'_{(j-1)n+i} = (t_{j+1} - t_j)\dot{x}_i$$

giving

$$\frac{d\dot{x}_i}{dz'_{(j-1)n+i}} = \frac{1}{t_{j+1} - t_j} .$$

In the above sum we obtain a non-zero term for only one value of s. Then

$$\frac{\partial F^*_j}{\partial z'_k} = \frac{\partial F}{\partial \dot{x}_i} \frac{1}{t_{j+1} - t_j} .$$

From equation (14), by direct differentiation,

$$\frac{\partial F^*_j}{\partial \dot{x}_i} = \sum_{\beta=1}^{q} \lambda^*_{(j-1)q+\beta} \frac{\partial \phi^*_{(j-1)q+\beta}}{\partial \dot{x}_i} ,$$

and since

$$F^*_j[\tau(t), z(x), z'(x, \dot{x})] = F[t, x, \dot{x}],$$

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we have

$$\frac{\partial F_j^*}{\partial z_k'} = \sum_{\beta=1}^{q} \frac{\lambda^*_{(j-1)q+\beta}}{t_{j+1} - t_j} \frac{\partial \phi^*_{(j-1)q+\beta}}{\partial \dot{x}_i}.$$

The next term of equation (12) which we develop is

$$\frac{\partial F_j^*}{\partial z_k} = \sum_{k=1}^{n} \frac{\partial F_j^*}{\partial x_s} \frac{dx_s}{dz_k} = \frac{\partial F_j^*}{\partial x_i} \quad (k = 1, \ldots, (m-1)n)$$

(again, x_s depends only upon z and $\dfrac{dx_i}{d z_{(j-1)n+i}} = 1$)

$$\frac{\partial F_j^*}{\partial x_i} = \sum_{\beta=1}^{q} \lambda^*_{(j-1)q+\beta} \frac{\partial \phi^*_{(j-1)q+\beta}}{\partial x_i}.$$

Equation (12) thus reads

$$\sum_{\beta=1}^{q} \frac{\lambda^*_{(j-1)q+\beta}}{t_{j+1} - t_j} \frac{\partial \phi^*_{(j-1)q+\beta}}{\partial \dot{x}_i} - \int_{\tau_1}^{\tau} \sum_{\beta=1}^{q} \lambda^*_{(j-1)q+\beta} \frac{\partial \phi^*_{(j-1)q+\beta}}{\partial x_i} d\tau$$

(15)

$$- C_{(j-1)n+i} = 0.$$

We now transform multipliers to convert equation (15) into a more useful form. Since λ_0^* is a constant, we have the definition

$$\lambda_0 = \lambda_0^*$$

(16a)

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and the transforms of the multipliers $\lambda_\alpha^*(\tau)$ are <u>defined</u> by

$$\lambda_\beta^{(j)}(t) = \frac{\lambda_{(j-1)q+\beta}^*(\tau)}{t_{j+1}-t_j} = \frac{\lambda_{(j-1)q+\beta}^*\left(\dfrac{t-t_j}{t_{j+1}-t_j}\right)}{t_{j+1}-t_j} \qquad (16b)$$

Note that in the right hand term we have simply replaced τ by $\dfrac{t-t_j}{t_{j+1}-t_j}$ — that is, $\lambda_{(j-1)q+\beta}^*\left(\dfrac{t-t_j}{t_{j+1}-t_j}\right)$ indicates a <u>functional dependence</u>. Note that $\dfrac{1}{t_{j+1}-t_j}$ is a constant multiplying all multipliers by the same value for a given j.

Equation (15) now reads

$$\sum_{\beta=1}^{q} \lambda_\beta^{(j)} \frac{\partial \phi_\beta^{(j)}}{\partial \dot{x}_i} - \int_{t_j}^{t} \sum_{\beta=1}^{q} \lambda_\beta^{(j)} \frac{\partial \phi_\beta^{(j)}}{\partial x_i} \, dt - C_{(j-1)n+i} = 0 \qquad (17)$$

where we have substituted

$$\phi_\beta^{(j)} = \phi_{(j-1)q+\beta}^* .$$

To explain the change of the integration limits, we note first that since

$$t = t_j + (t_{j-1} - t_j)\tau$$

we have

$$t|_{\tau_1} = t|_{\tau=0} = t_j$$

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which explains the lower limit. The upper limit, τ, is

$$\tau = \frac{t - t_j}{t_{j+1} - t_j}$$

which is actually our upper limit. But since the integral is taken to any value of the independent variable, now t, we simply set

$$\frac{t - t_j}{t_{j+1} - t_j} \rightarrow t .$$

We can now write our function F as $F(t, x, \dot{x}, \lambda)$ with

$$F = \sum_{\beta=1}^{q} \lambda_\beta \phi_\beta$$

where it is now understood that we restrict ourselves to the arc $t_j \le t \le t_{j+1}$ so that

$$\phi_\beta = \phi_\beta^{(j)}, \; \lambda_\beta = \lambda_\beta^{(j)} \quad \text{for} \quad t_j \le t < t_{j+1} \; (j = 1, \ldots, m-1) .$$

The range on j, as usual, is due to the m-1 arcs generated by m points $t_1, \ldots, t_m$. Equation (17) now assumes the fairly simple form

$$\frac{\partial F}{\partial \dot{x}_i} - \int_{t_j}^{t} \frac{\partial F}{\partial x_i} \, dt - C_{(j-1)n+i} = 0. \quad (t_j \le t < t_{j+1}) . \tag{18}$$

Application of the Euler Equations to the
Last m Variables

Thus far we have applied equation (12) (a portion of the multiplier rule) to the set of variables corresponding to our original x's. But we have a number of other variables to consider; these are the points $t_1, \ldots, t_m$, and we now investigate the application of the Euler equations to these. This will result in determination of the multipliers $\lambda^*_{(m-1)q+r}$.

To reiterate the applicable equations, we have, first,

$$\frac{\partial F^*}{\partial z'_k} - \int_{\tau_1}^{T} \frac{\partial F^*}{\partial z_k} - C_k = 0 \qquad (k = 1, \ldots, N)$$

where N (= $(m-1)n+m$) is the total number of constraints. In deriving equation (18) we used up only the first $(m-1)n$ variables (i.e., the original set of x's). To proceed to the last m equations, we use the DuBois-Reymond equation (12) (which is really a generalized form of the Euler equations). As before,

$$F^*_j = \sum_{\beta=1}^{q} \lambda^*_{(j-1)q+\beta} \phi^*_{(j-1)q+\beta} + \sum_{r=1}^{m} \lambda^*_{(m-1)q+r} z'_{(m-1)n+r}.$$

The first derivative yields

$$\frac{\partial F^*_j}{\partial z'_{(m-1)n+1}} = \lambda^*_{(m-1)q+1}.$$

Note that a specific value of r has been chosen – namely 1. We shall examine the first of these new equations in detail.

The next derivative to be taken, $\dfrac{\partial F^*_j}{\partial z_k}$, for $k = (m-1)n+1$, requires more work. Recalling that $z_{(m-1)n+1}$ is just another label for t_1 we have

$$\frac{\partial F_j^*}{\partial z_{(m-1)n+1}} = \frac{\partial F_j^*}{\partial t_1}$$

$$= \frac{\partial}{\partial t_1} \left\{ \sum_{\beta=1}^{q} \lambda_{(j-1)q+\beta}^*(\tau)\, \phi_{(j-1)q+\beta}^* \left(\tau(t), z(x,t), z'(\dot{x})\right) \right\} .$$

By definition of τ

$$t = t_j + (t_{j+1} - t_j)\tau ,$$

and the only place we have t_1 is when $j = 1$. In that case

$$t = t_1 + (t_2 - t_1)\tau .$$

Then if we set $j = 1$, F_j^* becomes

$$\frac{\partial F_j^*}{\partial t_1} = \frac{\partial}{\partial t_1} \left\{ \sum_{\beta=1}^{q} \lambda_\beta^* \phi_\beta^* \right\} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial t_1}$$

$$= \sum_{\beta=1}^{q} \lambda_\beta^* \left[\frac{\partial \phi_\beta^*}{\partial t} \frac{\partial t}{\partial t_1} + \sum_{i=1}^{n} \frac{\partial \phi_\beta^*}{\partial x_i} \frac{\partial x_i}{\partial t_1} + \sum_{i=1}^{n} \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial t_1} \right] .$$

From the equation given for t with $j = 1$ we have

$$\frac{\partial t}{\partial t_1} = 1 - \tau .$$

Also, since the x_i's and t_j's are independent variables we have

$$\frac{\partial x_i}{\partial t_1} = 0.$$

Then

$$\frac{\partial F^*}{\partial t_1} = \sum_{\beta=1}^{q} \lambda_\beta^* \left[\frac{\partial \phi_\beta^*}{\partial t} (1 - \tau) + \sum_{i=1}^{n} \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial t_1} \right].$$

As commented previously

$$\dot{x}_i = z_i \frac{d\tau}{dt}$$

and

$$\tau = \frac{t - t_1}{t_2 - t_1} \quad (\text{for } j = 1),$$

so

$$\frac{d\tau}{dt} = \frac{1}{t_2 - t_1}$$

and

$$\frac{\partial \dot{x}_i}{\partial t_1} = \frac{\partial}{\partial t_1} \left(\frac{z_i'}{t_2 - t_1} \right) = \frac{z_i'}{(t_2 - t_1)^2}.$$

Finally,

$$\frac{\partial F^*}{\partial t_1} = \frac{\partial F^*}{\partial z_{(m-1)n+1}} = \sum_{\beta=1}^{q} \lambda_\beta^* \left[\frac{\partial \phi_\beta^*}{\partial t} (1 - \tau) + \sum_{i=1}^{n} \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right].$$

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Inserting the expressions thus derived for $\dfrac{\partial F^*}{\partial z'_{(m-1)n+1}}$ and $\dfrac{\partial F^*}{\partial t_1}$ in the DuBois-Reymond equation, we find

$$\lambda^*_{(m-1)q+1} - \int_{\tau_1}^{\tau} \sum_{\beta=1}^{q} \lambda^*_{\beta} \left[\frac{\partial \phi^*_{\beta}}{\partial t} (1 - \tau) \right.$$

$$\left. + \sum_{i=1}^{n} \frac{\partial \phi^*_{\beta}}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right] d\tau - C_{(m-1)n+1} = 0. \tag{19}$$

This equation, unlike those above, does not restrict the minimizing arc and merely determines the multiplier $\lambda^*_{(m-1)q+1}(\tau)$ which is associated with the equation $z'_{(m-1)n+1} = 0$. The usefulness of this derivation is not immediately apparent but equation (19) will be of value in later work.

First Use of the Transversality Condition: Application to the Coefficients of $dz_{(m-1)n+j}$

The transversality condition, equation (13), must be written in expanded notation. Thus,

$$dg^* [z(\tau_1), z(\tau_2)] = \sum_{s=1}^{N} \frac{\partial g^*}{\partial z_s(\tau_1)} dz_s(\tau_1) + \sum_{s=1}^{N} \frac{\partial g^*}{\partial z_{s(\tau_2)}} dz_s(\tau_2)$$

where the sum is taken over all z variables, $s = 1, \ldots, N(=(m-1)n+m)$. Furthermore

$$d\psi^*_{\nu}[\tau_1, z(\tau_1), \tau_2, z(\tau_2)] = \sum_{s=1}^{N} \frac{\partial \psi^*_{\nu}}{\partial z_s(\tau_1)} dz_s(\tau_1)$$

$$+ \sum_{s=1}^{N} \frac{\partial \psi^*_{\nu}}{\partial z_s(\tau_2)} dz_s(\tau_2) + \frac{\partial \psi^*_{\nu}}{\partial \tau_1} d\tau_1 + \frac{\partial \psi^*_{\nu}}{\partial \tau_2} d\tau_2 .$$

Since transversality must hold irrespectively of the values chosen for the differentials, the individual coefficients must vanish. From equation (13) we gather together the individual pieces of the dz_k coefficients and write expressions at the end points as

$$\frac{\partial F_j^*}{\partial z_k'}\bigg|^{\tau_2} + \lambda_0^* \frac{\partial g^*}{\partial z_k(\tau_2)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_k(\tau_2)} = 0$$

$$-\frac{\partial F_j^*}{\partial z_k'}\bigg|^{\tau_1} + \lambda_0 \frac{\partial g^*}{\partial z_k(\tau_1)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_k(\tau_1)} = 0 \; .$$

Summations on s do not appear since we chose the specific coefficient s = k. Furthermore, the sum on ν extends over all end conditions $1, \ldots, P(=(m-2)n+p+2)$. The above equations are decoupled at the points τ_1, τ_2 since $dz_k(\tau_1)$ and $dz_k(\tau_2)$ are independent insofar as transversality considerations are concerned.

We now restrict our discussion by deciding on a value of k to be investigated. As was done in the derivation of equation (19), we use k = (m-1)q+1. The first specific equation in this derivation was

$$\frac{\partial F_j^*}{\partial z'_{(m-1)n+1}} = \lambda^*_{(m-1)q+1} \; .$$

The above two transversality conditions now take the form

$$\lambda^*_{(m-1)q+1}(\tau_2) + \lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)q+1}(\tau_2)}$$

(20a)

$$+ \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)q+1(\tau_2)}} = 0$$

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$$-\lambda^*_{(m-1)q+1}(\tau_1) + \lambda^*_0 \frac{\partial g^*}{\partial z_{(m-1)q+1}(\tau_1)}$$

<div align="right">(20a)</div>

$$+ \sum_{\nu=1}^{P}{}' \ell_\nu \frac{\partial \psi^*_\nu}{\partial z_{(m-1)q+1}(\tau_1)} = 0 .$$

(Note that our variable $z_{(m-1)q+1}$ cannot enter $\psi^*_{(m-2)n+p+1}$ and $\psi^*_{(m-2)n+p+2}$ so that our sum could run only to P-2, but it makes no difference if we carry it to P.)

We now <u>choose</u> λ^*_0 and ℓ_ν in such a way that

$$\lambda^*_0 \frac{\partial g^*}{\partial z_{(m-1)q+1}(\tau_1)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi^*_\nu}{\partial z_{(m-1)q+1}(\tau_1)} = 0$$

so that our second transversality at τ_1 gives

$$\lambda^*_{(m-1)q+1}(\tau_1) = 0.$$

<div align="right">(20b)</div>

Returning to equation (19) we can choose our upper limit of integration to be τ_1 so that the integral vanishes. This leaves

$$\lambda^*_{(m-1)q+1}(\tau_1) - C_{(m-1)n+1} = 0,$$

but since this λ at τ_1 vanishes we have

$$C_{(m-1)n+1} = 0.$$

Equation (19) then becomes

$$\lambda^*_{(m-1)q+1}(\tau) - \int_{\tau_1}^{\tau} \sum_{\beta=1}^{q} \lambda^*_{\beta} \left[\frac{\partial \phi^*_{\beta}}{\partial t} (1 - \tau) + \sum_{i=1}^{n} \frac{\partial \phi^*_{\beta}}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right] d\tau = 0.$$

If we now choose our upper integration limit as τ_2, we obtain an expression for $\lambda^*_{(m-1)n+1}(\tau_2)$ in terms of the integral. But our first transversality condition gave another expression for $\lambda^*_{(m-1)n+1}(\tau_2)$ and we can eliminate this Lagrange multiplier between them giving

$$\lambda^*_0 \frac{\partial g^*}{\partial z_{(m-1)n+1}(\tau_2)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi^*_\nu}{\partial z_{(m-1)n+1}(\tau_2)}$$

(21)

$$+ \int_{\tau_1}^{\tau_2} \sum_{\beta=1}^{q} \lambda^*_{\beta} \left[\frac{\partial \phi^*_{\beta}}{\partial t} (1 - \tau) + \sum_{i=1}^{n} \frac{\partial \phi^*_{\beta}}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right] d\tau = 0.$$

Equation (21) is essentially a modification of equation (19). The information derived by the first transversality condition has been incorporated into equation (19) to obtain (21). In order to obtain directly useful information, further development of equation (21) is required. This development is initiated by evaluating the integral in this equation in the next derivation.

To initiate this third segment, we note that by definition of τ,

$$\frac{d}{dt} = \frac{1}{t_2 - t_1} \frac{d}{d\tau} \quad \text{(for } j = 1\text{)},$$

which gives

$$\dot{x}_i = \frac{1}{t_2 - t_1} z'_i$$

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and

$$\ddot{x}_i = \frac{1}{(t_2 - t_1)^2}\, z''_i \,.$$

Previously we required that our minimizing arc, $E*$, had no corners between any pair of the points (t_{j+1}, t_j) ; thus we can differentiate the following expression (which is written simply as an inspired guess).

$$\frac{d}{d\tau}\left\{ \sum_{\beta=1}^{q} \left[\frac{\lambda^*_\beta \phi^*_\beta}{t_2 - t_1} - \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right] \right\}$$

$$= \frac{d}{dt}\left\{ \sum_{\beta=1}^{q} \left[\lambda^*_\beta \phi^*_\beta - \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z'_i}{t_2 - t_1} \right] \right\}$$

where we have converted from τ to t via absorption of $t_2 - t_1$.

The right hand side is now expanded – by the chain rule, and we pick up a second summation on the first terms since we apply the chain rule through the x variables:

$$\frac{d}{dt}\left\{ \sum_{\beta=1}^{q} \left[\lambda^*_\beta \phi^*_\beta - \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z'_i}{t_2 - t_1} \right] \right\} = \sum_{\beta=1}^{q}\left\{ \lambda^*_\beta \sum_{i=1}^{n} \frac{\partial \phi^*_\beta}{\partial x_i} \dot{x}_i \right.$$

$$+ \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \ddot{x}_i + \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial t} + \dot{\lambda}^*_\beta \phi^*_\beta - \frac{z'_i}{(t_2 - t_1)} \frac{d}{dt}\left(\lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \right)$$

$$\left. - \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z''_i}{(t_2 - t_1)^2} \right\} = \sum_{\beta=1}^{q} \sum_{i=1}^{n}\left\{ \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial x_i} \dot{x}_i + \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \ddot{x}_i \right.$$

(above equation concluded on following page)

$$+ \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial t} + \dot{\lambda}^*_\beta \phi^*_\beta - \dot{x}_i \frac{d}{dt} \left(\lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \right) - \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \ddot{x}_i \Bigg\}$$

$$= \sum_{\beta=1}^{q} \sum_{i=1}^{n} \left\{ \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial x_i} \dot{x}_i + \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial t} + \dot{\lambda}^*_\beta \phi^*_\beta - \dot{x}_i \frac{d}{dt} \left(\lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \right) \right\}$$

where the relationship between z''_i and $\ddot{x}_i$ derived above was used. By differentiation of equation (15), with $j = 1$, we have

$$\frac{d}{d\tau} \left[\sum_{\beta=1}^{q} \frac{\lambda^*_\beta}{t_2 - t_1} \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \right] = \sum_{\beta=1}^{q} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial x_i}$$

or

$$\frac{d}{dt} \left[\sum_{\beta=1}^{q} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \right] = \sum_{\beta=1}^{q} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial x_i} \quad .$$

Insertion of this expression into our above equivalence gives

$$\frac{d}{d\tau} \left\{ \sum_{\beta=1}^{q} \left[\frac{\lambda^*_\beta \phi^*_\beta}{t_2 - t_1} - \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right] \right\}$$

$$= \sum_{\beta=1}^{q} \left[\lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial t} + \dot{\lambda}^*_\beta \phi^*_\beta \right] = \sum_{\beta=1}^{q} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial t}$$

where the last step follows, since ϕ^*_β vanishes along an extremal. We now set

$$A = \sum_{\beta=1}^{q} \left[\frac{\lambda^*_\beta \phi^*_\beta}{t_2 - t_1} - \sum_{i=1}^{n} \lambda^*_\beta \frac{\partial \phi^*_\beta}{\partial \dot{x}_i} \frac{z'_i}{(t_2 - t_1)^2} \right]$$

for algebraic convenience.

Then

$$\frac{dA}{d\tau} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial t} .$$

We are now in a position to begin evaluation of the integral contained in equation (21). The last equation in (21) gives

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}(\tau_2)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}(\tau_2)}$$

$$+ \int_{\tau_1}^{\tau_2} \left[\frac{dA}{d\tau}(1-\tau) + \sum_{\beta=1}^{q} \sum_{i=1}^{n} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right] d\tau = 0.$$

The first portion of this integral can be written as

$$\int_{\tau_1}^{\tau_2} \frac{dA}{d\tau}(1-\tau) d\tau = A \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \tau \, dA .$$

Setting $u = \tau$, $dv = dA$, we obtain via integration by parts

$$\int_{\tau_1}^{\tau_2} \frac{dA}{d\tau}(1-\tau) d\tau = A \Big|_{\tau_1}^{\tau_2} - \tau A \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} A d\tau .$$

The total expression for equation (21) is now

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}(\tau_2)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}(\tau_2)} + [A(1-\tau)] \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} A d\tau$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \sum_{\beta=1}^{q} \sum_{i=1}^{n} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right\} d\tau = 0 .$$

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Substituting the value of A back in the integral where it appears gives

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}}(\tau_2) + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}}(\tau_2) + [A(1-\tau)]\Big|_{\tau_1}^{\tau_2}$$

$$+ \int_{\tau_1}^{\tau_2} \sum_{\beta=1}^{q} \frac{\lambda_\beta^* \phi_\beta^*}{t_2 - t_1} \, d\tau - \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n} \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \, d\tau$$

$$+ \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n} \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \, d\tau = \lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}}(\tau_2)$$

$$+ \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}}(\tau_2) + [A(1-\tau)]\Big|_{\tau_1}^{\tau_2}$$

$$+ \int_{\tau_1}^{\tau_2} \sum_{\beta=1}^{q} \lambda_\beta^* \phi_\beta^* \frac{1}{(t_2 - t_1)} \, d\tau = 0 \, .$$

Again using the fact that our ϕ_β^*'s are satisfied along an extremal, i.e., $\phi_\beta^* = 0$, this becomes

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}}(\tau_2) + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}}(\tau_2) + [A(1-\tau)]\Big|_{\tau_1}^{\tau_2} = 0 \, .$$

Inserting A (as was done above during the integration) gives

$$\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\lambda_\beta^* \phi_\beta^* \frac{1}{(t_2 - t_1)} - \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right] (1-\tau)\Big|_{\tau_1}^{\tau_2}$$

$$+ \lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}}(\tau_2) + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}}(\tau_2) = 0 \, .$$

The only portion of this equation evaluated at τ_1 gives rise to

$$-\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\lambda_\beta^* \phi_\beta^* \frac{1}{(t_2 - t_1)} - \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i}{(t_2 - t_1)^2} \right]_{}^{\tau_1} = 0 \; .$$

Previously we chose λ_0^*, ℓ_ν in such a way that

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)q+1}(\tau_1)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)q+1}(\tau_1)} = 0 \; .$$

We can add the last two equations to give

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)q+1}(\tau_1)} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)q+1}(\tau_1)}$$

$$-\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\lambda_\beta^* \phi_\beta^* \frac{1}{(t_2 - t_1)} - \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right]_{}^{\tau_1} = 0 \; .$$

We can now make the identifications

$$z_{(m-1)q+1}(\tau_1) = t_1 \; ,$$

$$\lambda_0^* = \lambda_0 \; ,$$

$$\frac{\lambda_\beta^*}{t_2 - t_1} = \lambda_\beta^{(1)} \; , \quad \text{and}$$

$$\frac{z_i'}{t_2 - t_1} = \dot{x}_i$$

and obtain

$$\lambda_0 \frac{\partial g}{\partial t_1} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu}{\partial t_1} - \sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\lambda_\beta^{(1)} \phi_\beta^{(1)} - \lambda_\beta \frac{\partial \phi_\beta}{\partial \dot{x}_i} \dot{x}_i \right] = 0 .$$

We had defined for our first arc

$$F = \sum_{\beta=1}^{q} \lambda_\beta^{(1)} \phi_\beta^{(1)} \; ;$$

so the above equation becomes

$$\lambda_0 \frac{\partial g}{\partial t_1} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu}{\partial t_1} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right]^{t_1} = 0 . \tag{22}$$

This equation applies at our initial time point. The right hand end point, t_m, has a similar relationship (derived in the same way) which is

$$\lambda_0 \frac{\partial g}{\partial t_m} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu}{\partial t_m} + \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right]^{t_m} = 0 . \tag{23a}$$

(The evaluation at upper and lower points as in the last two equations indicates whether the term is to be added or subtracted.)

The rest of the equations come from the observation that the evaluation of our basic equation at τ_2 yields

$$\lambda_0^* \frac{\partial g^*}{\partial z_{(m-1)n+1}} (\tau_2) + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}} (\tau_2) = 0 ,$$

since the first term vanishes identically at $\tau_2(\tau_2 = 1)$. This term would vanish in any case since our x variables are defined on the interval $t_j \leq t < t_{j+1}$ or $\tau_1 \leq \tau < \tau_2$ and $\phi_\beta^* = 0$ along an extremal. Then

$$\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\frac{\lambda_\beta^*}{(t_2 - t_1)} \phi_\beta^* - \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right] \Bigg|^{\tau_2} = 0 .$$

Now the next equation for the point t_2, i.e., $z_{(m-1)q+2}$ will yield the equation

$$-\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\lambda_\beta^* \phi_\beta^* \frac{1}{t_3 - t_2} - \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_3 - t_2)^2} \right]^{\tau_1} = 0 .$$

Addition of the last three equations yields

$$\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\frac{\lambda_\beta^*}{(t_2 - t_1)} \phi_\beta^* - \frac{\lambda_\beta^*}{(t_2 - t_1)^2} \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{(t_2 - t_1)^2} \right] \Bigg|^{\tau_2}$$

$$-\sum_{\beta=1}^{q} \sum_{i=1}^{n} \left[\frac{\lambda_\beta^*}{t_3 - t_2} \phi_\beta^* - \frac{\lambda_\beta^*}{t_3 - t_2} \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{z_i'}{t_3 - t_2} \right] \Bigg|^{\tau_1}$$

$$+ \lambda_0 \frac{\partial g^*}{\partial z_{(m-1)n+1}}(\tau_2) + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_{(m-1)n+1}}(\tau_2) = 0 .$$

Making the same identifications between the z representation and x representation, as was done above, and recalling the definition of F, we have

$$\lambda_0 \frac{\partial g}{\partial t_2} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu}{\partial t_2} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|_{t_2^-}^{t_2^+} = 0 ,$$

The above equation could just as well have been derived in terms of any other point τ_j, so

$$\lambda_0 \frac{\partial g}{\partial t_j} + \sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu}{\partial t_j} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|_{t_j^-}^{t_j^+} = 0 \quad (j = 2, \ldots, m-1). \quad (23b)$$

Second Use of the Transversality Condition: Application to the Coefficients of $d\tau_1$ and $d\tau_2$

The transversality condition, equation (13), has not yet been played out. We derived our first set of conditions by equating to zero the coefficients of $dz_{(m-1)n+i}(\tau_2)$ and $dz_{(m-1)n+j}(\tau_1)$, $(i = 1, \ldots, n)$. Another set of conditions could be derived by equating to zero the coefficients of $d\tau_1$ and $d\tau_2$. These are associated with the end conditions $\psi^*_{(m-2)n+p+1} = \tau_1 = 0$ and $\psi^*_{(m-2)n+p+2} = \tau_2 - 1 = 0$. Such a procedure does yield values of the multipliers $\ell_{(m-2)n+p+1}$ and $\ell_{(m-2)n+p+2}$, but since these do not come into our problem at any other point we really do not require them.

Third Use of the Transversality Condition: Application to the Coefficients of $dz_{(j-1)n+i}$

Another important set of conditions, however, does come from setting to zero the coefficients of $dz_{(j-1)n+i}(\tau_2)$ and $dz_{(j-1)n+i}(\tau_1)$ $(j = 1, \ldots, m-1; i = 1, \ldots, n)$. We shall investigate the case of $j = 1$ (i. e., the first arc) in detail. From equation (13)

$$\left[\left(F^* - \sum_{k=1}^{N} z'_k \frac{\partial F^*}{\partial z'_k} \right) d\tau + \sum_{k=1}^{N} \frac{\partial F^*}{\partial z'_k} dz_k \right]_1^2$$

$$+ \lambda_0^* \, dg^* + \ell_\nu \, d\psi^*_\nu = 0 .$$

As before,

$$dg* = \sum_{s=1}^{N} \frac{\partial g*}{\partial z_s(\tau_1)} dz_s(\tau_1) + \sum_{s=1}^{N} \frac{\partial g*}{\partial z_s(\tau_2)} dz_s(\tau_2)$$

and

$$d\psi_\nu^* = \sum_{s=1}^{N} \frac{\partial \psi_\nu^*}{\partial z_s(\tau_1)} dz_s(\tau_1) + \sum_{s=1}^{N} \frac{\partial \psi_\nu^*}{\partial z_s(\tau_2)} dz_s(\tau_2)$$

$$+ \frac{\partial \psi_\nu^*}{\partial \tau_1} d\tau_1 + \frac{\partial \psi_\nu^*}{\partial \tau_2} d\tau_2$$

and

$$F_j^* = \sum_{\beta=1}^{q} \lambda_{(j-1)q+\beta}^* \phi_{(j-1)q+\beta}^* + \sum_{r=1}^{m} \lambda_{(m-1)q+r}^* z_{(m-1)n+r}'$$

which, for $j = 1$, gives

$$F_j^* = \sum_{\beta=1}^{q} \lambda_\beta^* \phi_\beta^* + \sum_{r=1}^{m} \lambda_{(m-1)q+r}^* z_{(m-1)n+r}' \cdot$$

Thus

$$\frac{\partial F^*}{\partial z_i'} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial z_i'} = \sum_{\beta=1}^{q} \sum_{s=1}^{n} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_s} \frac{\partial \dot{x}_s}{\partial z_i'}$$

$$= \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial z_i'} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{d\tau}{dt}$$

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For j = 1 we have

$$\tau = \frac{t - t_1}{t_2 - t_1}$$

$$\frac{d\tau}{dt} = \frac{1}{t_2 - t_1} ;$$

then

$$\frac{\partial F^*}{\partial z_i'} = \sum_{\beta=1}^{q} {}' \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_2 - t_1} \quad .$$

Transversality now reads

$$\sum_{\beta=1}^{q} {}' \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_2 - t_1} \Bigg]_1^2 + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_1)} + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_2)}$$

$$+ \sum_{\nu=1}^{P} {}' \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_1)} + \sum_{\nu=1}^{P} {}' \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_2)} = 0 \quad .$$

Now equation (11) with j = 1 gives

$$\psi_{i+p}^* = z_i(\tau_2) + \Delta x_{i1} - z_{n+i}(\tau_1) = 0.$$

In our summation on ψ_ν^* above, the last two terms (i.e., $\tau_1 = \psi_{(m-2)n+p+1}^* = 0$ and $\tau_2 - 1 = \psi_{(m-2)n+p+2} = 0$) contribute nothing. If we write our summations as

$$\sum_{\nu=1}^{P} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i} = \sum_{\nu=1}^{P-2} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i} = \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i} + \sum_{\nu=p+1}^{P-2} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i}$$

then the last sum yields simply

$$\sum_{\nu=p+1}^{P-2} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_2)} = \ell_{i+p} .$$

Note that we do not obtain the multiplier ℓ_{n+i+p} here since our derivative is with respect to z_i. Thus, transversality now gives

$$\sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_2 - t_1}\Bigg|_1^2 + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_2)} + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_1)}$$

$$+ \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_2)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_1)} + \ell_{i+p} = 0$$

(remember the last term is associated with τ_2 although it is a constant) giving rise to the two equations

$$\sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_2 - t_1}\Bigg|^2 + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_2)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_2)} + \ell_{i+p} = 0, \qquad (24)$$

and

$$- \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_2 - t_1}\Bigg|^1 + \lambda_0^* \frac{\partial g^*}{\partial z_i(\tau_1)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu^*}{\partial z_i(\tau_1)} = 0 . \qquad (25)$$

The last equation is unique in that it occurs at the left hand end point.

Just as we transformed the λ_β^*, ϕ_β^*, z_i, and λ_0^* before we now do so again to equation (25) giving

$$-\left.\frac{\partial F}{\partial \dot{x}_i}\right|^{t_1} + \lambda_0 \frac{\partial g}{\partial x_i(t_1)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_1)} = 0. \tag{26}$$

Equations (24) and (26) are the first two equations from the transversality condition applied to z_i. Equation (26) is in final form but (24) may be modified. This is done by combining it with the "next" equation – i.e., since equation (24) is valid at the right hand end of the first arc, it would seem that we would be able to combine it with an equation which holds at the left end of the second arc (for the same x variable) giving an equation valid across the point t_2.

The next equation we need can be derived from equation (25). The first point to be noted is that two of the terms in equation (24) vanish identically. Earlier it was commented that, for our x variables, the range of definition was $t_j \le t \le t_{j+1}$. For our present transversality application we are dealing with the variables $z_{(j-1)n+i}$ for $j = 1, \dots, m$; $i = 1, \dots, n$. Thus, these z's are defined for $\tau_1 \le \tau < \tau_2$ and our variables are not defined at τ_2. So our g* is a function of τ_2 only at the point t_m (i.e., right hand end point of the entire interval). We may now write equation (24) as

$$\sum_{\beta=1}^{q} \frac{\lambda_\beta^*}{t_2 - t_1} \left.\frac{\partial \phi_\beta^*}{\partial \dot{x}_i}\right|^{2} + \ell_{i+p} = 0 .$$

To obtain the equation which corresponds to the variable x_i, but on the next arc, we use a similar procedure. For the first term we have

$$\frac{\partial F^*}{\partial z'_{i+n}} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{d\tau}{dt} = \sum_{\beta=1}^{q} \lambda_\beta^* \frac{\partial \phi_\beta^*}{\partial \dot{x}_i} \frac{1}{t_3 - t_2}$$

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for $j = 2$. So

$$\sum_{\beta=1}^{q} \lambda_{\beta}^{*} \frac{\partial \phi_{\beta}^{*}}{\partial \dot{x}_{i}} \frac{1}{t_{3} - t_{2}} \Bigg|_{1}^{2} + \lambda_{0}^{*} \frac{\partial g^{*}}{\partial z_{n+i}(\tau_{2})} + \lambda_{0}^{*} \frac{\partial g^{*}}{\partial z_{n+i}(\tau_{1})}$$

$$+ \sum_{\nu=1}^{p} \ell_{\nu} \frac{\partial \psi_{\nu}^{*}}{\partial z_{n+i}(\tau_{2})} + \sum_{\nu=1}^{p} \ell_{\nu} \frac{\partial \psi_{\nu}^{*}}{\partial z_{n+i}(\tau_{1})}$$

$$+ \frac{\partial}{\partial z_{n+1}} \ell_{i+p} [z_{i}(\tau_{2}) + \Delta x_{ij} - z_{n+i}(\tau_{1})] = 0.$$

Then for the point τ_{1} we have

$$- \sum_{\beta=1}^{q} \lambda_{\beta}^{*} \frac{\partial \phi_{\beta}^{*}}{\partial \dot{x}_{i}} \frac{1}{t_{3} - t_{2}} \Bigg|^{1} + \lambda_{0}^{*} \frac{\partial g^{*}}{\partial z_{n+1}(\tau_{1})} + \sum_{\nu=1}^{p} \ell_{\nu} \frac{\partial \psi_{\nu}^{*}}{\partial z_{n+i}(\tau_{1})} - \ell_{i+p} = 0.$$

Addition of this equation to the modified form of equation (24) cancels the constant ℓ_{i+p}, leaving:

$$\sum_{\beta=1}^{q} \frac{\lambda_{\beta}^{*}}{t_{2} - t_{1}} \frac{\partial \phi_{\beta}^{*}}{\partial \dot{x}_{i}} \Bigg|^{2} - \sum_{\beta=1}^{q} \frac{\lambda_{\beta}^{*}}{t_{3} - t_{2}} \frac{\partial \phi_{\beta}^{*}}{\partial \dot{x}_{i}} \Bigg|^{1} + \lambda_{0}^{*} \frac{\partial g^{*}}{\partial z_{n+i}(\tau_{1})}$$

$$+ \sum_{\nu=1}^{p} \ell_{\nu} \frac{\partial \psi_{\nu}^{*}}{\partial z_{n+i}(\tau_{1})} = 0.$$

Converting this to the original coordinates we have

$$\sum_{\beta=1}^{q} \lambda_{\beta} \frac{\partial \phi_{\beta}}{\partial \dot{x}_{i}} \Bigg|^{t_{2}^{-}} - \sum_{\beta=1}^{q} \lambda_{\beta} \frac{\partial \phi_{\beta}}{\partial \dot{x}_{i}} \Bigg|^{t_{2}^{+}} + \lambda_{0} \frac{\partial g}{\partial x_{i}(t_{2})} + \sum_{\nu=1}^{p} \ell_{\nu} \frac{\partial \psi_{\nu}}{\partial x_{i}(t_{2})} = 0,$$

or

$$\lambda_0 \frac{\partial g}{\partial x_i(t_2)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_2)} - \left[\frac{\partial F}{\partial \dot{x}_i}\right]_{t_2^-}^{t_2^+} = 0 .$$

Now the point t_2 is not unique; we can apply the same argument to any other point t_j $(j = 2, \ldots, m-1)$. Thus,

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j)} - \left[\frac{\partial F}{\partial \dot{x}_i}\right]_{t_j^-}^{t_j^+} = 0 . \tag{28a}$$

As a final datum from the transversality applied to the set of variables under discussion, we obtain a condition at the right hand end of the final arc. As was done above, we obtain

$$\lambda_0 \frac{\partial g}{\partial x_i(t_m)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_m)} + \left[\frac{\partial F}{\partial \dot{x}_i}\right]^{t_m} = 0 . \tag{28b}$$

Summary of Information Derived from the Transversality Condition

Equations (22), (23a), (23b), (26), (28a), and (28b) are the set of transversality conditions we have now derived for the problem. These are as follows:

$$\lambda_0 \frac{\partial g}{\partial t_1} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_1} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}\right]\Bigg|^{t_1} = 0 \tag{22}$$

$$\lambda_0 \frac{\partial g}{\partial t_m} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_m} + \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}\right]\Bigg|^{t_m} = 0 \tag{23a}$$

$$\lambda_0 \frac{\partial g}{\partial t_j} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_j} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|_{t_j^-}^{t_j^+} = 0 \qquad (23b)$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_1)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_1)} - \frac{\partial F}{\partial \dot{x}_i} \Bigg|^{t_1} = 0 \qquad (26)$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j)} - \frac{\partial F}{\partial \dot{x}_i} \Bigg|_{t_j^-}^{t_j^+} = 0 \qquad (28a)$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_m)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_m)} + \frac{\partial F}{\partial \dot{x}_i} \Bigg|^{t_m} = 0 \qquad (28b)$$

The conditions can be counted to determine the exact number derived. Only one each is obtained from equations (22) and (23a)', and, since $j = 2, \ldots$, $m-1$, $m-2$ equations are derived from (23b). Thus, our first set yields m conditions.

Equation (26) with $i = 1, \ldots, n$ yields n conditions as does equation (28b). In equation (28a), we have $i = 1, \ldots, n$ and $j = 2, \ldots, m-1$, giving $(m-2)n$ equalities. Our total from the second set is now mn, so we obtain a final total of $m + mn = m(n+1)$ conditions.

The next obvious question is the utilization of these equations. They are used for obtaining the values of the end point multipliers $\lambda_0, \ell_1, \ldots, \ell_p$, a total of $p+1$ unknowns. Notice that this entire set of equations may be written in matrix notation as

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$$
\begin{pmatrix}
\left[F - \dot{x}_i \dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|^{t_1} & \dfrac{\partial g}{\partial t_1} & \dfrac{\partial \psi_\nu}{\partial t_1} \\[2em]
\left[F - \dot{x}_i \dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|_{t_j^-}^{t_j^+} & \dfrac{\partial g}{\partial t_j} & \dfrac{\partial \psi_\nu}{\partial t_j} \\[2em]
\left[F - \dot{x}_i \dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|_{t_m} & \dfrac{\partial g}{\partial t_m} & \dfrac{\partial \psi_\nu}{\partial t_m} \\[2em]
\left[\dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|^{t_1} & \dfrac{\partial g}{\partial x_i(t_1)} & \dfrac{\partial \psi_\nu}{\partial x_i(t_1)} \\[2em]
\left[\dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|_{t_j^-}^{t_j^+} & \dfrac{\partial g}{\partial x_i(t_j)} & \dfrac{\partial \psi_\nu}{\partial x_i(t_j)} \\[2em]
\left[\dfrac{\partial F}{\partial \dot{x}_i}\right]\Big|_{t_m} & \dfrac{\partial g}{\partial x_i(t_m)} & \dfrac{\partial \psi_\nu}{\partial x_i(t_m)}
\end{pmatrix}
\begin{pmatrix} 1 \\[1em] \lambda_0 \\[1em] \ell_\nu \end{pmatrix} = 0 \qquad (29)
$$

(where a summation is implied when a repeated index occurs). Now since we have $p+1$ unknowns (with $p \leq m(n+1)$) and $m(n+1)$ equations we have more equations that unknowns (unless $p = m(n+1)$) and some equations had better be dependent; otherwise we have an inconsistency. That is, each $(p+2) \times (p+2)$ submatrix must be singular; or, better, the augmented coefficient matrix should have rank $< p+2$.

(At this point Hunt argues that λ_0, $\lambda_\beta^{(j)}$ do not simultaneously vanish and that λ_0^*, ℓ_ν do not all vanish. His argument is primarily referenced to Bliss and will not be reproduced here since inclusion of the details would be a rather lengthy process.)

To proceed to a few definitions, we state first that an extremal for the problem of Bolza is an arc without corners in R_1^* together with a set of multipliers $z_j(\tau), \lambda_0^*, \lambda_\alpha^*(\tau)$ (with $j = 1,\ldots,N; \alpha = 1,\ldots,M; 0 \leq \tau \leq 1$) such that the functions $z_j'(\tau), \lambda_\alpha^*(\tau)$ have continuous first order derivatives and satisfy the constraints $\phi_\alpha^* = 0$. An extremal is called non-singular if the determinant

$$
R^* = \begin{vmatrix} \dfrac{\partial^2 F^*}{\partial z_j' \partial z_k'} & \dfrac{\partial \phi_{\alpha_1}^*}{\partial z_j'} \\[2em] \dfrac{\partial \phi_\alpha}{\partial z_k'} & 0 \end{vmatrix} \cdot \begin{pmatrix} j,k = 1,\ldots,N \\ \alpha, \alpha_1, = 1,\ldots,Q \end{pmatrix} \tag{30}
$$

is non-zero along the extremal. An extremal is normal if it satisfies the multiplier rule with a unique set of multipliers having $\lambda_0 = t$.

These definitions for the transformed problem can also be restated for the original problem we attacked. Thus, an extremal is defined to be an arc in R_1 and a set of multipliers, $x_i(t), \lambda_0, \lambda_\beta^{(j)}(t)$ (where $x_i(t)$ is the arc) with $i = 1,\ldots,n; j = 1,\ldots,m-1; \beta = 1,\ldots,q; t_j \leq t < t_{j+1}$ which have corners only at t_j ($j = 1,\ldots,m$). Between corners, $\dot{x}_i(t)$ and $\lambda_\beta^{(j)}(t)$ have continuous first order derivatives, and the x_i's satisfy the constraints $\phi_\beta^{(j)} = 0$. An extremal is non-singular if the determinant

$$
R = \begin{vmatrix} \dfrac{\partial^2 F}{\partial \dot{x}_h \partial \dot{x}_i} & \dfrac{\partial \phi_\gamma}{\partial \dot{x}_h} \\[2em] \dfrac{\partial \phi_\beta}{\partial \dot{x}_i} & 0 \end{vmatrix} \begin{pmatrix} i,h = 1,\ldots,n \\ \beta, \gamma = 1,\ldots,q \end{pmatrix} \tag{31}
$$

is non-zero along the extremal. An extremal is normal if it satisfies the multiplier rule with a unique set of multipliers having $\lambda_0 = 1$. Normality for the original problem implies normality for the transformed problem, but this will not be proven here. Only the normal case will be considered in what is to follow.

THE NECESSARY CONDITIONS OF WEIERSTRASS AND CLEBSCH

The Weierstrass function, E, is defined by

$$E^* (\tau, z, z', \lambda^*, Z') = F^* (\tau, z, Z', \lambda^*) - F^* (\tau, z, z', \lambda^*)$$

$$- \sum_{k=1}^{N} (Z'_k - z'_k) \frac{\partial F^*}{\partial z'_k} (\tau, z, z', \lambda^*) . \tag{32}$$

An arc is said to satisfy the Weierstrass necessary condition (for a minimum) if

$$E^* (\tau, z, z', \lambda^*, Z') \geq 0. \tag{33}$$

at every element (τ, z, z', λ^*) of the arc for all sets $(\tau, z, Z') \neq (\tau, z, z')$ interior to the region R_1^* and satisfying the equations $\phi_\alpha^* = 0$. Every normal minimizing arc for the problem of Mayer or Bolza must necessarily satisfy this condition.

Using (14) (which defines F^*) this condition may be written as

$$E^* (\tau, z, z', \lambda^*, Z') = \sum_{\beta=1}^{q} \lambda^*_{(j-1)q+\beta} [\phi^*_{(j-1)q+\beta} (t, x, \dot{x})$$

$$- \phi^*_{(j-1)q+\beta} (t, x, \dot{x})] \tag{34}$$

$$- \sum_{\beta=1}^{q} \sum_{k=1}^{n} (Z'_k - z'_k) \lambda^*_{(j-1)q+\beta} \left(\frac{\partial \phi^*_{(j-1)q+\beta}}{\partial \dot{x}_k} \right) \left(\frac{1}{t_{j+1} - t_j} \right) .$$

As has been done before, we indicate the derivative of a starred quantity with respect to the $\dot{x}_k$. This is done with the understanding that when we write $\phi^*_{(j-1)q+\beta}(x, \dot{x}, t)$ we imply $\phi^*_{(j-1)q+\beta}[x(\tau, z), \dot{x}(\tau, z, z'), t(\tau)]$, so that even though we can indicate the derivative of ϕ^* with respect to $\dot{x}_i$, we really have the independent variable as τ, not t. Thus, in $\phi^*_{(j-1)q+\beta}$ and

$$\frac{\partial \phi^*_{(j-1)q+\beta}}{\partial \dot{x}_k}$$ we substitute $t = t_j + (t_{j-1} - t_j)\tau$, $x_i(t) = z_{(j-1)n+i}(\tau)$,

$$\dot{x}_i = \frac{z'_{(j-1)n+i}}{t_{j+1} - t_j} \quad \text{and } t_j = z_{(m-1)n+j}.$$

To comment on the application of this test, we first note that we can consider all possible sets like (z, Z', τ) in equation (32), and then we pick out those which satisfy equation (32). Note that those we pick must satisfy the equations $z'_{(m-1)n+j} = 0$ and $\phi^*_{(j-1)q+\beta} = 0$. Those we select are a subset (of all possible choices). Those we vary to check the Weierstrass condition need only be members of the set of variables which correspond to the x's, namely $z'_{(j-1)n+i}$. We vary these one at a time; that is, we choose one index where $Z'_{(j-1)n+i} \neq z_{(j-1)n+i}$ and choose $Z'_{(j-1)n+i} = z_{(j-1)n+i}$ for all other values of j. Then the sets which satisfy equation (32) in z notation must satisfy equation (34) when we change to an x representation.

In the other direction, if each of the m-1 expressions (j = 1, ..., m-1, one condition on each arc) is non-negative, then Weierstrass' condition is satisfied.

We can transform these (m-1) equations to a complete representation in our original notation to obtain the condition that

$$\sum_{\beta=1}^{q} (t_{j+1} - t_j) \left\{ \lambda_\beta^{(j)} [\phi_\beta^{(j)}(t, x, \dot{X}) - \phi_\beta^{(j)}(t, x, \dot{x})] \right.$$

$$\left. - \sum_{i=1}^{n} (\dot{X}_i - \dot{x}_i) \frac{\partial F(t, x, \dot{x}, \lambda)}{\partial \dot{x}_i} \right\} \geq 0.$$

This equation must hold at every point $(t, x, \dot{x}, \lambda)$ of E on the m-1 intervals $t_j \leq t < t_{j+1}$ for all sets $(t, x, \dot{X}) \neq (t, x, \dot{x})$ which satisfy $\phi_\beta = 0$ and are interior to our region R_1.

By definition of our F function and the requirement that $t_{j+1} > t_j$ we must have

$$E(t, x, \dot{x}, \lambda, \dot{X}) = F(t, x, \dot{X}, \lambda) - F(t, x, \dot{x}, \lambda)$$

$$-\sum_{i=1}^{n} (\dot{X}_i - \dot{x}_i) \frac{\partial F(t, x, \dot{x}, \lambda)}{\partial \dot{x}_i} \geq 0.$$

(35)

This is the Weierstrass condition for the original problem and must hold for every minimizing arc.

To delineate the Clebsch condition briefly, suppose we find numbers $\pi_1, \ldots, \pi_n$ which are not all zero and which satisfy

$$\sum_{i=1}^{n} \frac{\partial \phi_{\beta}^{(j)}(t, x, \dot{x})}{\partial \dot{x}_i} \pi_i = 0$$

for all $j = 1, \ldots, m-1$. Then every normal minimizing arc must satisfy the condition that

$$\sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial^2 F(t, x, \dot{x})}{\partial \dot{x}_i \partial \dot{x}_h} \pi_i \pi_h \geq 0$$

(36)

as is shown in Bliss (Ref. 4) using the Weierstrass necessary condition.

THE PROBLEM OF UNSPECIFIED DISCONTINUITIES

We have thus far determined a set of four necessary conditions which treat the problem of an optimal trajectory with specified discontinuities at unspecified points. (Note, however, that we have not derived sufficient conditions.) It is now possible to generalize the above formulation with little additional work to the point at which problems having unknown discontinuities at unknown time points can be treated.

In the case of a rocket vehicle which has constant thrust for each stage, prespecifying our corner points (i.e., staging times) automatically prespecifies the mass that must be dropped. In order to determine the optimal weight staging we need a formulation which neither requires the staging times nor the drop weights to be specified.

The difference may be illustrated as follows. For a fixed discontinuity we have an equation of the form

$$x_i(t_j^+) - x_i(t_j^-) = C,$$

where C is a given constant. For a variable discontinuity we could write

$$x_i(t_j^+) - x_i(t_j^-) = C(t_j - t_{j-1}).$$

In the second case the discontinuity depends upon the length of the segment preceding the point t_j.

It is worthwhile to recall at this time that the discontinuity values assumed in the earlier part of this report, Δx_{ij}, did not enter into our boundary conditions. For this reason, equations (11) were specifically built into the transformed problem to account for the discontinuities.

To proceed to the present problem, we now try to minimize

$$J = g[t_1, \ldots, t_m, x_1(t_1^+), \ldots, x_n(t_1^+), x_1(t_2^-), \ldots,$$

$$x_n(t_2^-), \ldots, x_1(t_m^-), \ldots, x_n(t_m^-)]$$

$$= g[t_1, \ldots, t_m, x(t_1^+), x(t_2^-), \ldots, x(t_m^-)]$$

$$= g[t_1, \ldots, t_m, x(t_j^+), x(t_k^-)]$$

(37)

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where $j = 1, \ldots, m-1$ and $k = 2, \ldots, m$. The ϕ_β constraint equations remain the same as they were previously. We can have three types of boundary conditions. The first, for no discontinuity is

$$\psi_\nu = x_i(t_j^-) - x_i(t_j^+) = 0, \tag{38}$$

which is stated at every corner where the x_i's are continuous. The second type of boundary condition is

$$\psi_\nu = x_i(t_j^-) - x_i(t_j^+) - \text{Constant} = 0, \tag{39}$$

which is the type considered in the first part of this paper. Third, we have

$$\psi_\nu = x_i(t_j^-) - x_i(t_j^+) - D[t_1, \ldots, t_m, x(t_j^+), x(t_j^-)] = 0, \tag{40}$$

where the D's are specified functions (which, for example, give a relationship between the burning time of a stage and the drop weights).

The basic difference between the present formulation and the previous work was that we had to include the constraints (11) which related to the discontinuity Δx_{ij}. If we explicitly include the constraints given above by equations (38), (39), and (40), then we can exclude the constraints of equations (11). (What we have really done is to convert all constraints to the form of equations (11) and then renumber them from the beginning as ψ_ν, $\nu = 1, \ldots, p \leq m(n+1)$.

The modification due to excluding the constraints of equations (11) is rather minor. In the development of equation (24), an end point multiplier ℓ_{i+p} turned up which was then eliminated by the same term in a subsequent equation. This time we do not get the term ℓ_{i+p} to occur in either case and there is no necessity to add the equations in question to eliminate it. Notice that we have both the points t_j^+ and t_j^- in the constraints ψ_ν since our discontinuity is not fixed. Previously it was argued that the terms of the form $\lambda_0^* \dfrac{\partial g}{\partial z_i(\tau_2)}$ and $\dfrac{\partial \psi_\nu}{\partial z_i(\tau_2)}$ occurred only at the point t_m. In the present case this is not true since the discontinuity is not fixed.

Without further ado, we write the two equations that replace equation (28a) as

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j^+)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j^+)} - \left[\frac{\partial F}{\partial \dot{x}_i}\right]\Bigg|^{t_j^+} = 0$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j^-)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j^-)} + \left[\frac{\partial F}{\partial \dot{x}_i}\right]\Bigg|^{t_j^-} = 0 \ .$$

The D functions occurring in equation (40) modify the problem by their occurrence in the ψ_ν constraints.

RESULTS AND CONCLUSIONS

Our first set of variables were $x_1, \ldots, x_n$. These are the coordinates of the point in phase space. After transformation these become $z_1, \ldots, z_{(m-1)n}$.

The second set of variables were $t_1, \ldots, t_m$. These are the "times" (i.e., values of the independent variables) at which discontinuities occur. After transformation these become $z_{(m-1)n+1}, \ldots, z_{(m-1)n+m}$. Between these two, we now have variables $z_1, \ldots, z_{(m-1)n+m}$ after the transformation, and we set $(m-1)n+m = N$.

The first set of constraints we labeled as $\phi_\beta^{(j)} = 0$, where the j indicated the particular "arc" (or interval between two values of t_j) on which the constraint applies and the β indicates the particular constraint on this arc. After transformation these become $\phi_1^*, \ldots, \phi_{(m-1)q}^*$ $(q<n)$.

The second set of constraints were $z'_{(m-1)n+j} = 0$, which were introduced to account for the time points $t_1, \ldots, t_m$ being constant. Our total set of constraints were $\phi_1^*, \ldots, \phi_{(m-1)q+m}^*$, and we set $(m-1)q+m = Q$.

The first set of end conditions were the $\psi_\nu = 0$ constraints, or $\psi_1, \ldots, \psi_p$ $(p \leq m(n+1))$. After transformation these became $\psi_1^*, \ldots, \psi_p^*$.

The second set of end conditions accounts for the discontinuities of the x_i's at the t_j points which are, after transformation, $z_{(j-1)n+i}(1) - \Delta x_{ij} - z_{(j-1)n+n+i} = 0$, which were written as $\psi^*_{(j-1)n+i+p} = 0$.

The third set were $\tau_1 = \psi_{(m-2)n+p+1} = 0$ and $\tau_2 - 1 = \psi_{(m-2)n+p+2} = 0$. This gives the total set as $\psi^*_1, \ldots, \psi_{(m-2)n+p+1}$, and we label $(m-2)n+p+1 = P$.

Next, certain Lagrange multipliers λ^*, ℓ_ν, λ_0 were introduced and the function F^* formed. Application of the multiplier rule yielded the following results:

The Euler equations for the original set of x variables are

$$\frac{\partial F}{\partial \dot{x}_i} - \int_{t_j}^{t} \frac{\partial F}{\partial x_i} \, dt - C_{(j-1)n+1} = 0 \qquad (t_j \le t < t_{j-1}).$$

Transversality conditions:

$$\lambda_0 \frac{\partial g}{\partial t_1} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_1} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|^{t_1} = 0$$

$$\lambda_0 \frac{\partial g}{\partial t_m} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_1} + \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|^{t_m} = 0$$

$$\lambda_0 \frac{\partial g}{\partial t_j} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial t_j} - \left[F - \sum_{i=1}^{n} \dot{x}_i \frac{\partial F}{\partial \dot{x}_i} \right] \Bigg|^{t_j^+}_{t_j^-} = 0$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_1)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_1)} - \frac{\partial F}{\partial \dot{x}_i} \Bigg|^{t_1} = 0$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j)} - \frac{\partial F}{\partial \dot{x}_i} \Bigg|_{t_j^-}^{t_j^+} = 0$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_m)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_m)} + \frac{\partial F}{\partial \dot{x}_i} \Bigg|^{t_m} = 0$$

Weierstrass condition:

$$F(t, x, \dot{X}, \lambda) - F(t, x, \dot{x}, \lambda) - \sum_{i=1}^{n} (\dot{X}_i - \dot{x}_i) \frac{\partial F(t, x, \dot{x}, \lambda)}{\partial \dot{x}_i} \geq 0.$$

Clebsch condition:

$$\sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial^2 F(t, x, \dot{x})}{\partial \dot{x}_i \partial \dot{x}_h} \pi_i \pi_h \geq 0.$$

The problem with unspecified discontinuities is similar to equation (40) in which the D functions appear, modifying the ψ_ν constraints. Aside from this difference, the only modifications occur in the fourth equation listed above under transversality. This becomes the two equations

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j^+)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j^+)} - \left[\frac{\partial F}{\partial \dot{x}_i}\right] \Bigg|^{t_j^+} = 0, \text{ and}$$

$$\lambda_0 \frac{\partial g}{\partial x_i(t_j^-)} + \sum_{\nu=1}^{p} \ell_\nu \frac{\partial \psi_\nu}{\partial x_i(t_j^-)} + \left[\frac{\partial F}{\partial \dot{x}_i}\right] \Bigg|^{t_j^-} = 0.$$

The Weierstrass and Clebsch conditions for this case have not been investigated here.

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