NASA NSG-545

Emory University
Atlanta Ga

GENERAL AND ALMOST GENERAL PERIODIC FUNCTIONS

by

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GPO PRICE $ ________

CFSTI PRICE(S) $ ________

Hard copy (HC) 2.00

Microfiche (MF) .50

ff 653 July 65

1. Basic Facts

Definition: If

(1) $$f(z + h(z)) = f(z)$$

whenever z and $z + h(z)$ belong to a domain R, then $f(z)$ is called general periodic over R with period $h(z)$.

An example of a general periodic function is $\sin z^2$. Here $h(z) = -z + \sqrt{z^2 + 2\pi}$. Any determination can be given to $\sqrt{z^2 + 2\pi}$. A second example is $\sin e^z$ where $h(z) = -z + \ln(e^z + 2\pi)$.

The following theorem is easy to prove.

Theorem I: If $F(z)$ is general periodic over R with period $q(z)$ and if $f(z)$ is a solution over R of the equation

(2) $$f(z + h(z)) = f(z) + q(z)$$

then $F(f(z))$ is general periodic over R with period $h(z)$.

Corollary: If $f(z)$ is a solution of

(3) $$f(z + h(z)) = f(z) + 2\pi/c$$

then $e^{-icf(z)}$ is general periodic with period $h(z)$.

An important problem is the establishment of the existence of general periodic functions with given period, $h(z)$. This problem has been previously discussed [1] by the author and an existence theorem proved with certain restrictions on $h(z)$. No further discussion will be given here. A second problem is: Given $f(z)$ to determine $h(z)$. This is a problem in implicit functions. In many cases it can readily be solved by means of the methods of elementary mathematics.

2. Multiple Periods

Let
$$z_1 = z$$
$$z_2(z) = z_1 + h(z_1)$$
$$z_3(z) = z_2 + h(z_2)$$
$$\dots\dots\dots\dots\dots\dots$$

(4) $$z_n(z) = z_{n-1} + h(z_{n-1})$$

Let $h_1(z) = h(z)$, $\quad h_j(z) = 0$ if $j < 1$.

$$\dots\dots\dots\dots\dots\dots$$

(5) $$h_n(z) = \sum_{j=0}^{n-1} h(z + h_j(z))$$

Consequently

(6) $$z_n = z + h_{n-1}(z)$$

Theorem II: If $h(z)$ is a period of $f(z)$ then so is $h_n(z)$ over a domain to which $z + h_j(z)$, $j = 0, 1, \dots, (n)$ belong.

Proof: $f(z + h_n(z)) = f(z_n + h(z_n)) = f(z_n) =$
$f(z_{n-1} + h(z_{n-1})) = f(z_{n-1}) = \dots\dots = f(z)$

Theorem III:

(7) $$z_m(z_n(z)) = z_{m+n}(z)$$

Proof: $z_m(z_n) = z_n + h(z_n) + h(z_{n+1}) + \dots + h(z_{m+n-1})$

$= z + h(z) + h(z_2) + \dots + h(z_{n-1}) + h(z_n) + h(z_{n+1}) + \dots + h(z_{m+n-1})$

$= z_{m+n}$.

Corollary: If $z = 0$ then $z_n(z) = z_n(0) = h_{n-1}(0)$. Consequently $z_m(z_n(0)) = z_m(h_{n-1}(0)) = h_{m+n-1}(0)$.

3. The Fundamental Difference Equation

Equation (2) is an example of a difference equation with varying difference interval. It is non-homogeneous and of the first order. So far as the author knows such equations have never been studied prior to the author's own work. We let

(8) $$\Delta\, y(z) = y(z + h(z)) - y(z).$$

We consider (2) where the complex variable z is replaced by the real variable x and replace $q(x)/h(x)$ by $f(x)$ and f by y. We write the equation

(9) $$\frac{\Delta\, y(x)}{h(x)} = f(x)$$

and cite the paper (1) above referred to. By simply replacing $\ln x$ by $f(x)$ we have the following theorem. In this theorem and hereafter superscripts denote differentiation:

Theorem IV: If $f^{(j)}(x)$ retains the same sign when $x \geq a$, $j = 2, 3, \dots$ and $f^{(2j)}(x) \cdot f^{(2j-2)}(x) > 0$ and $\sum_{i=1}^{\infty} f(x_i)\, \Delta\, h(x_i)$

converges uniformly in x when $x \geq a$ and $f^{(v)}(x)\, x^{v-1} \to 0$ when $x \to \infty$, $v = 1, 2, \ldots$ then (7) has a solution when $x \geq a$ which is differentiable. Any two solutions differ at most by a general periodic function with period $h(x)$. We assume $h(x) > 0$ and differentiable.

This theorem establishes the existence, under the conditions stated, of a general periodic function with period $h(x)$ which is differentiable and which can be written in the form $Q(y(x))$ where Q is periodic and $y(x)$ is a solution of (9). It is quite possible to write asymptotic forms for our solutions of (9) and by means of them to establish the existence of solutions over a right hand half of the complex plane. The author does not do this in order to avoid repetition of work in the paper already alluded to [1] and in order not to increase the length of the present paper. The existence of analytic general periodic functions is, however, immediately inferred.

4. Orthogonality and Fourier Series

A discussion of this topic is postponed to Part II of the present paper.

PART II

ALMOST-GENERAL PERIODIC FUNCTIONS

5. General

We have remarked that if $h(x)$ is a period of $F(x)$ so is $h_n(x)$. Now let $F_1(x)$ and $F_2(x)$ be distinct functions with respective periods ${}_1h(x)$ and ${}_2h(x)$. If there exist integers, n_1 and n_2 such that ${}_1h_{n_1}(x) = {}_2h_{n_2}$ for all x in R, then this function is a period for $F_1(x)$ and $F_2(x)$. Consequently $F_1(x) + F_2(x)$ is general periodic. In case there are no integers n_1 and n_2 which make ${}_1h_{n_1}(x) = {}_2h_{n_2}(x)$ for all values of x in R then the function $F_1(x) + F_2(x)$ suggests a definition of almost general periodic functions analogous to that given by Harold Bohr for almost periodic functions [2].

In part II we demand that the independent variable x be real and that $h(x)$ satisfy a relation such as (3). Usually $h(x)$ will depend upon a parameter and to emphasize this we may write $h(x, c)$. We give certain restrictions on $h(x)$ and $f(x)$. These are more than required for all theorems but will nevertheless be assumed. We require that $x + h(x)$ be strictly increasing, $x \geqq a$ and that $h'(x)$ and $f'(x) \geqq 1$ exist and be continuous when $x > a$. We denote the inverse of $x + h(x)$ by $x - H(x)$. From its definition $H(x)$ is defined when $x \geqq a + h(a)$. We require that when $x \geqq a$ then $0 < \eta < h(x) < M$. It results that

$0 < \bar{h} < H(x) < \bar{H}$. We also require that $h_n(x)$ become infinite with n. We choose $f^{-1}(x)$ single-valued and increasing as ~~if~~ f(x) is increasing. Thus if $f(x) = x + 1/x$ then $f^{-1}(x) = \frac{x + \sqrt{x^2 - h}}{2}$. We also assume $f'(x')/f'(x)$ and consequently $(f^{-1}(x'))'/(f^{-1}(x))'$ bounded above and away from zero if $|x' - x|$ is bounded. It results also that $h_n'(x')/h_n'(x)$ is bounded away from zero. We use equation (3) to prove that

(10) $$h(x) = -x + f^{-1}\left[f(x) + \frac{2\Pi}{c}\right]$$

. . . .

(11) $$h_n(x) = -x + f^{-1}\left[f(x) + \frac{2n\Pi}{c}\right]$$

(12) $$x_n(x) = f^{-1}\left[f(x) + \frac{2(n-1)\Pi}{c}\right]$$

In order to prove the mean value theorem which is of fundamental importance the author has been compelled to require that $h_m'(x)$ approach unity uniformly in m. The author doubts the necessity of this restriction on $h_m(x)$. Consequently it is not made until the proof of theorem XV. We also assume $h(x, c)$ uniformly continuous in c when $c > \eta > 0$.

6. Definitions and Some Simple Theorems

We shall assume that we deal with a function depending upon a parameter h(x, c).

Definition: By a category of general periodic functions we mean the set of functions having the period, h(x, c) where h satisfies an equation of the type $f(x + h_{\ell}(x, c)) - f(x) = \frac{2\Pi}{c}$. Certain restrictions have been placed on f and h. We require that $c > 0$.

Definition: A continuous function, F(x) is said to be almost general periodic of category h(x, c) when $x \geq a$ if given any $\varepsilon > 0$ there exist $L > 0$ and $c > 0$ both in general dependent upon ε such that on every interval of length L as $a + g \leq x \leq a + g + L$, $g \geq 0$ there is an integer m such that

$$\left| F(x + h_m(x, c)) - F(x) \right| < \varepsilon$$

whenever $x \geq a$.

We call $h_m(x, c)$ a translation function.

Theorem VI: General periodic functions which are continuous are almost general periodic if h(x) obeys the restrictions placed on h(x) of this part of the paper.

Proof: Let $L = 1$. Then

$$\left| F(x + h_n(x)) - F(x) \right| = 0 < \varepsilon.$$

Theorem VII: The almost periodic functions of Bohr(2) are almost-general periodic functions with $h(x, c) = \frac{1}{c}$ and $f(x) = 2\Pi x$.

Proof: Consider an almost periodic function F(x). Given ε suppose that ℓ is the corresponding length such that on every interval of length ℓ there lies a translation number τ. Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ be translation numbers lying respectively on the intervals $[0, \ell]$,

$[\ell, 2\ell]$, ... $[(n-1)\ell, n\ell]$, ... Due to the uniform continuity of f(x) each τ can be changed by an amount which does not exceed δ where δ is a uniform continuity constant corresponding to $\epsilon/2$. We then choose each τ an integral multiple of h where $0 < h < \delta$. Let $h(x, c) = h = \frac{1}{c}$. Then $h_m(x, c) = \sum_{j=0}^{m-1} h(x + h_j(x, c), c) = \frac{m}{c} = \tau_m$. We now note that $\tau_{m_1} - \tau_{m_2} < 2\ell$. Hence $\frac{m_1}{c} - \frac{m_2}{c} < 2\ell$ or $m_1 - m_2 < 2\ell c$. We choose $L \geqq 2\ell c$. We also note that $f(x) = 2\pi cx$.

<u>Remark: Not all almost-general periodic functions are uniformly continuous over their domain of definition.</u>

We have considered $\sin x^2$. This function is general periodic. It oscillates between -1 and 1. However the period of oscillation approaches zero when x becomes infinite. For this function $h_n'(x)$ approaches zero as x becomes infinite but does not do so uniformly in n. ~~Also $h_n'(x)$ does not approach zero uniformly in~~ n. This is to be noted when reading "The Mean Value Theorem". (Theorem XV).

Given an almost general periodic function F(x), choose a and then h(x, c). Let $x' < x''$ be two points on the interval $[0, h_{m_1}(0)]$. Let the images of these points made by any transformation $\bar{x} = x + h_{jm_1}(x, c)$ be $\bar{x}'$ and $\bar{x}''$.

<u>Theorem VIII: If $x'' - x' \leqq \theta(\bar{x}'' - \bar{x}')$ where $\theta > a$ is fixed then F(x) is uniformly continuous.</u>

Proof: Let δ be a uniform continuity constant of F(x) on the interval $[0, h_{m_1}(0, c)]$ corresponding to η. If x" - x' then x" - x' $< \delta$ and $/F(x") - F(x')/$ $|F(x" - F(x')|$ + 2ϵ $\eta + 2\epsilon$. Now if η + 2ϵ = ξ is given we choose ϵ and η. Having chosen η, then δ is determined and hence $\frac{\delta}{\theta}$.

Theorem IX: If F(x) is almost-general periodic, x a, then F(x) is bounded when $x \geq a$.

Proof: We assume for simplicity and without loss of generality that a = 0. There exist positive integers m_1, m_2, ... as described in the definition. Mark the points 0, $h_{m_1}(0)$, $h_{m_2}(0)$, ... on the axis. We know that $x_{m+1}(x) = x + h_m(x)$. For convenience we let $m_1 + 1 = n_1$, $m_2 + 1 = n_2$, ... Perform the transformation $x' = x_{n_1}(x)$ on the points of the interval $[0, x_{n_1}(0)]$. This interval is carried into the interval $[x_{n_1}(0), x_{n_1}2(0)]$. Suppose that on $[0, x_{n_1}(0)]$ we have $|F(x)| \leq A$, then on $[x_{n_1}(0), x_{n_1}2(0)]$ we know that $|F(x)| \leq A + 2\epsilon$. Now perform the transformation $x" = x_{n_1}(x)$ on $[x_{n_1}(0), x_{n_1}2(0)]$. This interval is carried into $[x_{n_1}2(0), x_{n_1}3(0)]$ and on this interval $|F(x)| \leq A4\epsilon$. We continue this process until $x_{n_1}j(0)$ $x_{n_2}(0)$. We know that $n_2 - n_1 \leq 2L$. Consequently to this point $/F(x)/ \leq A + 4L\epsilon$. But $F(x + h_{m_2}(x)) - F(x) < \epsilon$. We now proceed from m_2 until m_3 is reached precisely as we did from m_1 to m_2, then from m_1 to m_2. We find that $|F(x)| \leq A + \cancel{4}L\varepsilon$. We then go from m_3 to m_4 etc. finding that $|F(x)| \leq A + \cancel{4}L\varepsilon$ always.

7. Invariance under Fundamental Operations

We denote a category of almost general periodic functions by $h(x, c)$ and denote two particular members of this category by $F_1(x)$ and $F_2(x)$ where $F_1(x)$ corresponds to c_1 and $F_2(x)$ to c_2. Now $F_1(x)$ and $F_2(x)$ are each continuous, $a \leq x < \infty$. Consequently given any $\varepsilon > 0$ there is a $\delta(x)$ such that when $|x - x_1| < \delta(x)$ then both $|F_1(x) - F_1(x_1)| \leq \varepsilon$ and $|F_2(x) - F_2(x_1)| \leq \varepsilon$. Of all possible choices for $\delta(x)$ we shall assume that we have the superior limit for all x. Instead of writing $(h_{m_p}(x, c_1))_j$ and $(h_{m_q}(x, c_2))_i$ we write $h_j(x, c_1)$ and $h_i(x, c_2)$ respectively.

Theorem X: If there exist positive integers, n_1 and n_2 such that $h_{n_1}(x, c_1) - h_{n_2}(x, c_2) < \delta(x)$ for all x then $h_{n_2}(x, c_2)$ is a translation function for $F_1(x, c_1)$.

Proof: It is immediate on account of continuity that

$$| F_1(x + h_{n_1}(x, c_1)) - F_1(x + h_{n_2}(x, c_2)) | < \varepsilon.$$

Consequently

$$|F_1(x + h_{n_2}(x, c_2)) - F_1(x)| \leq |F_1(x + h_{n_2}(x, c_2)) - F_1(x + h_{n_1}(x, c_1))| + |F_1(x + h_{n_1}(x, c_1)) - F_1(x)| \leq 2\varepsilon.$$

This inequality tells us that $F_1(x)$ is almost general periodic with $h_{n_2}(x, c_2)$ a translation function corresponding to 2ε.

Corollary: Under the conditions of the theorem $F_1(x) + F_2(x)$ is an almost-general periodic function with translation function $h_{n_2}(x, c_2)$ corresponding to 3ε.

Proof: $\left| F_1(x + h_{n_2}(x, c_2)) + F_2(x + h_{n_2}(x, c_2)) - (F_1(x) + F_2(x)) \right.$

$$\leq \left| F_1(x + h_{n_2}(x, c_2)) - F_1(x) \right| + \left| F_2(x + h_{n_2}(x, c_2)) - F_2(x) \right| \leq 3\varepsilon.$$

The following theorem is a special case of the more general theorem to follow. However, due to its interest and to the brevity of the proof, it is stated and proved independently. This proof should be compared with that given by Bohr.

Theorem XI: The sum of two almost periodic functions as defined by Bohr is again almost periodic.

Proof: Suppose $h_1(x, c_1) = \frac{1}{c_1}$ and $h_2(x, c_2) = \frac{1}{c_2}$. We can choose m_1 and m_2 so that $\left| m_1 \frac{1}{c_1} - m_2 \frac{1}{c_2} \right| < \delta$ where δ is the superior limit of the smaller of the two largest uniform continuity constants possible for F_1 and F_2 respectively.

Theorem XII: If $F_1(x)$ and $F_2(x)$ are almost-general periodic functions belonging to the same category and corresponding to c_1 and c_2 respectively then $F_1(x) + F_2(x)$ is an almost general periodic function of this same category.

Proof: We retain the notation of recent previous theorems.

$$h_{n_2}(x, c_2) - h_{n_1}(x, c_1) = f^{-1}[f(x) + \frac{2n_2\Pi}{c_2}] - f^{-1}[f(x) + \frac{2n_1\Pi}{c_1}].$$

We apply the law of the mean.

$$h_{n_2}(x, c_2) - h_{n_1}(x, c_1) = 2\Pi [\frac{n_2}{c_2} - \frac{n_1}{c_1}]\{f^{-1}[f(x) + k]\}'$$

where $\frac{2\Pi n_1}{c_1} < k < 2\Pi \frac{n_2}{c_2}$. But $[f^{-1}(x)]'$ is bounded if we assume as we do that $f'(x) \geq 1$. Consequently since we can choose $\frac{n_2}{c_2} - \frac{n_1}{c_1}$ as small as we please, we can write

$$\left| h_{n_2}(x, c_2) - h_{n_1}(x, c_1) \right| < \zeta \text{ where } \zeta \text{ is arbitrarily small.}$$

Now let δ be the upper limit of the continuity constant corresponding to ε for both $F_1(x)$ and $F_2(x)$ over the smaller of the two intervals $[0, h_1(0, c_1)]$, $[0, h_1(0, c_2)]$. We assume the interval $[0, h_1(0, c)]$ to be the smaller.

We perform the transformations $\bar{x} = x + h_{n_1}(x, c_1)$ and $\bar{\bar{x}} = x + h_{n_2}(x, c_2)$ on this smaller interval. Let x'' and x' be two points on this interval such that $x'' - x' = \delta$. We assume that x'' and x' are so chosen that $\bar{\bar{x}}'' - \bar{x}'$ is as small as would be obtained by choosing any other two points distant δ on $[0, h_1(0, c)]$. Let $\bar{\delta} = \bar{x}'' - \bar{x}'$.

$$\bar{\delta} = f^{-1}[f(x'') + \frac{2n_1\Pi}{c_1}] - f^{-1}[f(x') + \frac{2n_1\Pi}{c_1}]$$

$$= \{f^{-1}[f(\xi) + \frac{2n_1\Pi}{c_1}]\}'(x'' - x') = \{f^{-1}[f(\xi) + \frac{2n_1\Pi}{c_1}\}'\delta$$

Now x and ξ both lie on $[0, h(0, c_1)]$ consequently

$\{f^{-1}[f(\xi) + \frac{2n_1\Pi}{c_1}]\}'/\{f^{-1}[f(x) + k]\}'$ is bounded above and away from zero. We notice that $\left| \frac{2n_1\Pi}{c_1} - k \right| < 2\Pi \left[\frac{n_2}{c_2} - \frac{n_1}{c_1}\right]$.

As a result

$\frac{1}{\delta'}[h_{n_2}(x, c_2) - h_{n_1}(x, c_1)] < 1$ if

$\frac{n_2}{c_2} - \frac{n_1}{c_1}$ is small enough.

This completes the proof ~~by~~ of the theorem with reference to theorem VII.

Theorem XIII: If F(x) is almost general periodic of category h(x, c) then so is $(F(x))^2$.

Proof: Following Bohr

$\left| F^2(x + h(x, c) - F^2(x) \right| = \left| F(x + h(x, c) - F(x) \right| . \left| F(x + h_n(x, c) + F(x) \right|$

$\leq 2C\varepsilon$ where C is an upper bound of F(x).

Theorem: If F(x) and G(x) are almost general periodic of category h(x, c) then so is F(x).G(x).

Following Bohr we remark that the theorem follows from the equality

$$F(x).G(x) = \frac{1}{2}\{[F(x) + G(x)]^2 - [F(x) - G(x)]^2\}$$

Theorem XIV: The limit function F(x) of a uniformly converging sequence of almost general periodic functions $\{F_n(x)\}$ of category h(x, c) is almost general periodic of this same category.

Proof: Let $F_N(x, c_1)$ be a member of the sequence and let $h_n^N(x, c)$ be a translation function for $F_N(x, c_1)$ corresponding to $\varepsilon/3$. We follow Bohr and write the inequalities

$$\left| F(x + h_n^N(x, c)) - F(x) \right| \leq \left| F(x + h_n^N(x, c)) - F_N(x + h_n^N(x, c)) \right|$$
$$+ \left| F_N(x + h_n^N(x, c)) - F_N(x) \right| + \left| F_N(x) - F(x) \right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

if N is so chosen that $\left| F_N(x) - F(x) \right| < \varepsilon/3$ for all x.

8. Mean Value

Theorem XV:

$$\frac{1}{T}\int_a^{a+T} F(x)dx$$

has a limit when T becomes infinite.

In the proof of this theorem we assume for the first time that $h_n'(x)$ approaches zero uniformly in n.

Proof: We let a = 0 without loss of generality. Remember that $x_j(0)$ becomes infinite with j and that $x_{j+1}(0) - x_j(0)$ is bounded. We choose m_1 of the definition. For brevity in writing we shall not write $(x_{m_1}(x))_j$ but simply $x_j(0)$. With this understanding we shall consider $\frac{1}{x_j}\int_0^{x_j} F(x)dx$, instead of $\frac{1}{T}\int_0^T F(x)dx$. Here T will differ from an x_j by an amount which is bounded. Under this arrangement when T becomes infinite so does j and vice versa. Since F(x) is bounded $\frac{1}{T}\int_0^T F(x)dx$ and $\frac{1}{x_j}\int_0^{x_j} F(x)dx$ differ by an amount which approaches zero. We consequently shall only consider $\frac{1}{x_j}\int_0^{x_j} F(x)dx$.

We shall prove that this has a limit by establishing the existence of a number j_0 corresponding to ε such that when $j_1 > j_0$ and $j_2 > j_0$ simultaneously then

$$\left| \frac{1}{x_{j_1}}\int_0^{x_{j_1}} F(x)dx - \frac{1}{x_{j_2}}\int_0^{x_{j_2}} F(x)dx \right| < \varepsilon.$$

We now consider

$$\frac{1}{x_j}\int_0^{x_j(0)} F(x)dx - \int_{x_{(\nu-1)_j}(0)}^{x_{\nu_j}(0)} F(x)dx .$$

We perform the transformation $x' = x + h_{(\nu-1)_j}(x)$ on the integral $\int_0^{x_j(0)} F(x)dx$, getting

$$\int_{x_{(\nu-1)_j}}^{x_{\nu_j}(0)} F(x + H_{(\nu-1)_j}(x))(1 - H'_{(\nu-1)_j}(x))dx = \int_{x_{(\nu-1)_j}(0)}^{x_{\nu_j}(0)} (F(x)+\eta)(1-H'_{(\nu-1)_j}(x))dx$$

where $|\eta| < \varepsilon$. We now choose $\nu > 1$ an integer and $j > j_0$ where j_0 is so large that $H_{(\nu-1)_j}(x) < \zeta$, where ζ is arbitrarily small, so long as $j > j_0$. Now

$$\frac{1}{x_j(0)}\int_0^{x_j(0)} F(x)dx = \frac{1}{\int_{x_{(\nu-1)_j}(0)}^{x_{\nu_j}(0)} (1 - H_{(\nu-1)_j}(x))dx}\int_0^{x_j(0)} F(x)dx$$

Consequently

$$\frac{1}{x_j(0)}\int_0^{x_j(0)} F(x)dx - \frac{1}{x_{\nu_j}(0) - x_{(\nu-1)_j}(0)}\int_{x_{(\nu-1)_j}(0)}^{x_{\nu_j}(0)} F(x)dx < \eta$$

We sum this from $\nu = 1$ to n_1 and we have

$$\frac{1}{x_j}\int_0^{x_j} F(x)dx - \frac{1}{n_1 j}\int_0^{x_{n_1 j}} F(x)dx < \epsilon_1$$

We now choose two integers $j_1 > j_0$ and $j_2 > j_0$ and two integers n_1 and n_2 such that $n_1 j_1 = n_2 j_2 = \delta$. Then

$$\left| \frac{1}{x_{j_1}(0)}\int_0^{x_j(0)} F(x)dx - \frac{1}{x_{j_2}(0)}\int_0^{x_{j_2}(0)} F(x)dx \right|$$

$$= \left| \frac{1}{x_{j_1}(0)}\int_0^{x_{j_1}(0)} F(x)dx - \frac{1}{x_{n_1 j}(0)}\int_0^{x_{n_1 j}(0)} F(x)dx - \frac{1}{x_{j_2}(0)}\int_0^{x_{j_2}(0)} F(x)dx \right.$$

$$\left. + \frac{1}{x_{n_2 j_2}(0)}\int_0^{x_{n_2 j_2}(0)} F(x)dx \right| < 2\epsilon_1$$

and the theorem is proved.

<u>Theorem XVI:</u> $\frac{1}{T}\int_a^{a+T} F(x)dx$ <u>approaches its limit uniformly in a.</u>

Proof: We retain the notation of the previous theorem.

It is clear that we need only consider

$$\frac{1}{x_j(0)} \int_{x_q(0)}^{x_{q+j}(0)} F(x)dx .$$

We shall show that this integral approaches its limit uniformly in q.

$$\frac{1}{x_j(0)} \int_0^{x_j(0)} F(x)dx = \frac{1}{x_j(0)} \int_{x_q(0)}^{x_{q+j}(0)} (F(x) + h)(1 - H_q'(x))dx.$$

Since $|h|$ is uniformly small, also $H_q'(x)$, if x is great enough. The theorem follows.

<u>Theorem XVII:</u> <u>If</u> $\underline{M[(F(x))^2] = 0}$ <u>then</u> $\underline{F(x) \equiv 0}$.

Proof: Choose ε for $(F(x))^2$, then c and h(x, c) all for $(F(x))^2$ which is almost general periodic. Let L, m_1, m_2, ... also correspond to $(F(x))^2$ and be as described in the definition of almost-general periodic functions. For convenience in writing we let $n_1 = m_1 + 1$, $n_2 = m_2 + 1$, ...

Now assume that $(F(x))^2 > h > 0$ at some point. Since F(x) is continuous there is an interval $[x', x'']$ over which $(F(x))^2 > \frac{h}{2}$. We assume this interval to lie on the interval $[0, x_{n_1}(0)]$. We recall that $h_m(0) = x_{m+1}(0)$. There is no loss of generality in this assumption. We could simply replace F(x) by $F_1(x-a)$. Let

$$\delta = x'' - x'.$$

Now perform the transformation $\bar{x} = x_{n_1}(x)$ on the interval $[0, x_{n_1}(0)]$. This interval is carried into $[x_{n_1}(0), x_{n_1^2}(0)]$. By the law of the mean the length of this interval is $x'_{n_1}(\xi) \cdot x_{n_1}(0)$ where $0 < \xi < x_{n_1}(0)$. The interval $[x', x'']$ is carried into an interval of length $x'_{n_1}(\theta)\delta$ where $0 < x' < \theta < x'' < x_{n_1}(0)$. We note that over this interval $F(x)^2 > \frac{h}{2} - 2\varepsilon$. We assume ε to be so chosen that this is positive. We now make the transformation $\bar{\bar{x}} = x_{n_1}(\bar{x})$ on the interval $[x_{n_1}(0), x_{n_1^2}(0)]$. This transformation is equivalent to $\bar{\bar{x}} = x_{n_1^2}(x)$ on $[0, x_{n_1}(0)]$. The new interval is of length $x_{n_1}(0) \cdot x'_{n_1^2}(\xi_2)$ where $0 < \xi_2 < x_{n_1}(0)$. The interval $[x', x'']$ is carried into one of length $\delta \cdot x_{m_1^2}(\theta_2)$ where $0 < \theta_2 < x_{n_1}(0)$. Over this interval $(F(x))^2 > \frac{h}{2} - 4\varepsilon$ which again we take to be positive. We can choose ε so as to make this the case. We continue this process until we reach $[x_{n_1^j}(0), x_{n_1^{j+1}}(0)]$ where $m_1^j \leqq n_2 \leqq n_1^{j+1}$. This interval is of length $x_{n_1}(0) \cdot x'_{n_1^j}(\xi)$. The interval $[x', x'']$ is carried into one of length $\delta \cdot x_{n_1^j}(\theta)$ where both ξ and θ lie on the interval $(0, x_{n_1}(0))$. We now note that over the interval $[x_{m_2}, x_{n_2 n_1}(0)]$ we have $(F(x))^2$ differing from $(F(x))^2$ over $[0, x_{n_1}(0)]$ by an amount in absolute value less than ε. Consequently over the transform of $[x', x'']$ we have $(F(x))^2 > \frac{h}{2} - 2\varepsilon$. We now begin all over again remembering that $n_{j+1} - n_j < 2L$. It results that over any transformation of $[x', x'']$ we have $(F(x))^2 > \frac{\lambda}{2} - 4L\varepsilon$. We choose $\varepsilon < \frac{\lambda}{8L}$. Now the ratio of the length of any transform of $[x', x'']$ to the length of the corresponding transform of $[0, x_{n_1}(0)]$ is $\delta\, x_n'(\theta_n)/x_{n_1}(\xi_n)$. But $x_n'(\theta_n)/x_n'(\xi_n)$ is bounded away from zero since $|\theta_n - \xi_n| < x_{m_1}(0)$ [See § 5]. It results that over any interval $[0, x_n(0)]$ the ratio of the length where $(F(x))^2 > \frac{\lambda}{2} - 4L\varepsilon$ to the total length is greater than some positive constant. It results that $M[(F(x))^2] > 0$. This proves the theorem.

9. Fourier Constants

We have assumed that $f'(x)$ exists and is continuous and is positive and that $f(x)$ becomes infinite when x becomes infinite. We have also stated that $f'(x + h(x))/f'(x + p(x))$ approaches 1 if $h(x)$ and $p(x)$ are bounded. Now $f(x)$ is strictly increasing and $f(x + h(x)) - f(x) = \frac{2\Pi}{c}$. Hence there exists a unique $p(x)$ such that $f(x + p(x)) - f(x) = \frac{\Pi}{C}$, $C > 0$. Subtraction yields $f(x + h(x)) - f(x + p(x)) = \frac{2\Pi}{c} - \frac{\Pi}{C}$. Since $h(x)$ is bounded $p(x)$ is also bounded. Differentiation yields

$$f'(x + h(x)(1 + h'(x)) - f'(x + p(x))(1 + p'(x)) = 0$$

Since $h'(x)$ approaches zero so does $p'(x)$ and since $f(x)$ is increasing $p'(x) < 0$.

Now consider $\cos C f(x)$. Denote its zeros by x_i, $i = 1, 2, \ldots$

Theorem XVIII: $\lim_{j \to \infty} \frac{1}{x_j} \int_0^{x_j} \cos C\, f(x)\,dx = 0$

Proof: Mark the points x_i on the x-axis and plot the graph of $y = \cos C\, f(x)$. We have a curve consisting of successive arches above and below the x-axis. Now $x_n - x_{n-1}$ decreases as n increases and the area of the region bounded by an arch and the x-axis is larger than the area of the region bounded by the arch immediately succeeding it and the x-axis. To see this we proceed as follows. We note that $\cos C\, f(x)$ is general periodic. Hence assume, to fix the ideas that

$$\int_{x_{n-1}}^{x_n} \cos C\, f(x)\,dx > 0$$

Then

$$\int_{x_n}^{x_{n+1}} \cos C\, f(x)\,dx = -\int_{x_{n-1}}^{x_n} (\cos C\, f(x))(1 + p'(x))\,dx$$

$$= -\int_{x_{n-1}}^{x_n} \cos C\, f(x)\,dx - \int_{x_{n-1}}^{x_n} (\cos C\, f(x))\,p'(x)\,dx$$

But $p'(x) < 0$. Hence

$$\int_{x_{n-1}}^{x_n} \cos C\, f(x)\,dx = -\int_{x_n}^{x_{n+1}} \cos C\, f(x)\,dx + h$$

where $h > 0$.

$$\cancel{\int_{x_n}^{x_{n+1}} \cos C\, f(x)\,dx = -\int_{x_{n-1}}^{x_n} (\cos C\, f(x))(1 + p'(x))\,dx}$$

Now

$$\sum_{n=1}^{\infty}\int_{x_n}^{x_{n+1}} \cos C\, f(x)dx \;=\; \int_{x_1}^{\infty} \cos C\, f(x)dx$$

is a series of alternating sign each term of which is in absolute value less than the absolute value of the term immediately preceding it. Also since $x_n - x_{n-1}$ approaches zero the last term approaches zero. The series consequently converges. Since x_j becomes infinite the theorem is proved.

<u>Corollary I.</u> $\lim\limits_{j \to \infty} \frac{1}{x_j} \int_0^{x_j} \sin C\, f(x)dx = 0;$

$$\lim_{j \to \infty} \frac{1}{x_j} \int_0^{x_j} e^{-iCf(x)}dx = 0$$

<u>Corollary II.</u> The functions $\sin C\, f(x)$ and $\cos C\, f(x)$ for varying C form an orthogonal set, so do $e^{-Cf(x)}$.

Bohr's work is a generalization of classical theory. If we consider our functions as a generalization of Bohr's we find that in some cases the proofs given by Bohr go over with only trivial changes to our more general case, precisely as many of Bohr's results come with but little change from classical work.

We now proceed to state without proof certain very important theorems but which are of the type just referred to.

<u>Theorem XIV:</u> <u>The functions $a(C) = M\,[F(x)\,e^{-iCf(x)}]$ are zero for all values of C with the exception of an enumerable set of numbers.</u>

Here, of course, $F(x)$ is almost general periodic and $f(x)$ is the function determining the category.

<u>Corollary.</u> $M[F(x) \sin C\, f(x)]$ is likewise zero for all but an enumerable set of values for C. Similarly $M[F(x) \cos C\, f(x)]$.

This theorem coupled with the orthogonality corollary previously proved provide us with the notion of Fourier constants and Fourier series. If the set of numbers of the theorem when enumerated in any order are $C_1, C_2, \ldots,$ of course this may be an empty set, we let

$$A_n = M[e^{-i\,C_n f(x)}\,F(x)]$$

and refer to

$$\sum_{n=1}^{\infty} A_n\, e^{-i\,C_n f(x)}$$

as the Fourier series corresponding to $F(x)$.

Theorem XV: Let the series $\sum_{n=1}^{\infty} A_n e^{-i C_n f(x)}$ with distinct real C_n's be uniformly convergent $-\infty < x < \infty$ then this series is the Fourier series of its sum function.

Theorem

$$\sum_1^{\infty} |A_n^2| \leq M [\,|F(x)|^2]$$

10. Uniqueness Theorem

Lemma I: Given any $\zeta > 0$ there exists a $\bar{C}$ and an $\bar{j}$ such that when $C > \bar{C}$ and $j > \bar{j}$.

$$\left| \frac{1}{x_j} \int_0^{x_j} F(x) \cos C\, f(x)\,dx \right| < \zeta.$$

Proof: We assume $F(x) > 0$. This is brought about by simply adding and subtracting a constant. We have already treated

$$\frac{1}{x_j} \int_0^{x_j} \cos C\, f(x)\,dx.$$

Let $\alpha_1, \alpha_2, \ldots$ be the zeros of cos C f(x). Given ε choose c and consequently h(x, c). Draw the graph of y = F(x)cos C f(x). Let C be large and assume C an integral multiple of c. We can vary c to bring this about We know that h(x, c) is uniformly continuous in c when c > a. Now, $h_m(x, c)$ is a period of cos C f(x)

Make the transformation $x' = x + h_{m_1}(x)$.

$$\int_{\alpha_1}^{\alpha_2} F(x) \cos C\, f(x)\,dx = -\int_{\alpha_1 + h_{m_1}(\alpha_1)}^{\alpha_2 + h_{m_1}(\alpha_2)} F(x) \cos C\, f(x)\,dx$$

$$+\int_{\alpha_1 + h_{m_1}(\alpha_1)}^{\alpha_2 + h_{m_1}(\alpha_2)} F(x).\ H'(x) \cos C f(x) dx$$

$$-\int_{\alpha_1 + h_{m_1}(\alpha_1)}^{\alpha_2 + h_{m_1}(\alpha_2)} \eta\ (1 - H'(x) \cos C f(x) dx, \text{ where } |\eta| < \epsilon .$$

The signs here are determined under the assumption that cos C f(x) has an even number of zeros on the interval $[\alpha_1, \alpha_1 + h_{m_1}(\alpha_1)]$. This can be brought about with a sufficiently large C. However, in case the number is assumed odd changes in the proof are trivial. Let $\beta_1 = \alpha_1 + h_{m_1}(\alpha_1)$, $\beta_2 = \alpha_2 + h_{m_1}(\alpha_2)$

We now take X so large that when $x > X$, then $|H'(x)| \cdot F(x) < \zeta_2$. Now choose ϵ so that $|\eta(1 + h'(x)| < \frac{\zeta}{2}$. Also $H_m'(x)$ approaches zero as x becomes infinite. It results that

$$\left| \int_{\alpha_1}^{\alpha_2} F(x) \cos C f(x) dx + \int_{\alpha_1 + h_{m_1}(\alpha_1)}^{\alpha_2 + h_{m_1}(\alpha_2)} F(x) \cos C f(x) dx \right| <$$

$$\zeta \quad 2(\alpha_2 - \alpha_1) \quad + \quad (\alpha_2) - H_{m_1}(\alpha_1) \quad .$$

Replace α_1 by α_i and α by α_{i+1}. Sum, noting cancellations just illustrated.

Let k be any large positive integer.

$$\int_{\alpha_1}^{\alpha_k} F(x) \cos C f(x) dx \quad 2\zeta\ (\alpha_k - \alpha_1) + \zeta\ (H_{m_1}(\alpha_k) - H_{m_1}(\alpha_1)$$

Hence

$$\frac{1}{x_k} \int_X^{\alpha_k} F(x) \cos C f(x)\ dx < 2\zeta + \zeta M$$

Hold X fast, We can choose $\bar{j}$ so that when $j > \bar{j}$

Hold X fast. We can choose $\bar{J}$ so that when $j > \bar{J}$

$$\frac{1}{x_j} \int_0^{x_j} F(x)\cos C\, f(x)dx < \frac{\epsilon}{4}$$

The proof of the lemma is now complete.

<u>Corollary. In the statement of the lemma x_j can be replaced by $\bar{x}$.</u>

<u>Corollary. In the statement of the lemma we can replace cos C f(x) by sin C f(x) or $e^{-i\,C\,f(x)}$</u>.

<u>Lemma II.</u> If

$$\lim_{j} \frac{1}{x_j(0)} \int_0^{x_j(0)} F(x)\, e^{-iC_0 f(x)} dx = 0$$

<u>then given a $\epsilon > 0$ there exists a δ and a $\bar{j}$ such that when $|C-C_0| < \delta$ and $j > \bar{j}$ then</u>

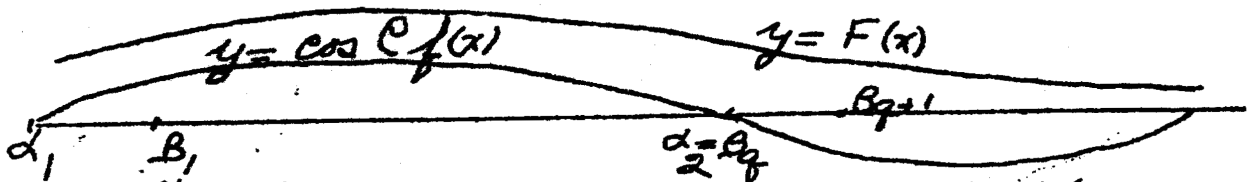
$$\left| \frac{1}{x_j(0)} \int_0^{x_j(0)} F(x)\, e^{-iCf(x)} dx \right| < \epsilon$$

Proof: Without loss of generality we assume $C_0 = 0$. Otherwise we need only consider $F(x)\, e^{-i\,C_0 x}$.

We shall as in the previous lemma consider

$$\frac{1}{x_j} \int_0^{x_j} F(x)\cos C\, f(x)dx$$

Given ϵ, choose c and hence $h(x, c) = h(x)$. We now adjust c so that c is an integral multiple of C. Notice that this is exactly the reverse of what was done in the proof of the previous lemma. Suppose $\frac{C}{c} = \ell$. Then there is an integer k such that $2k\pi/c = 2n\pi/C$ We draw graphs.



Suppose $\alpha_2 = \alpha_1 + h_{m_1+k}(\alpha_1)$ then $|F(x) - F(x+h_{m_1+k}(x))| \leq 4L\epsilon$.

Let $\alpha_1 = \beta_0$, $\alpha_1 + h_{m_1}(\alpha_1) = \beta_1$, $\beta_1 + h_{m_1}(\beta_1) = \beta_2 \ldots\ldots, \alpha_2 = \beta_q$.

In brief:

$$\int_{\beta_0}^{\beta_1} F(x) \cos C f(x) dx = -\int_{\beta_q}^{\beta_{q+1}} (F(x) + \eta(x))(\cos C f(x)(1 - H'_{m_1}(x)) \; dx$$

Hence

$$\int_{\beta_0}^{\beta_1} F(x) \cos C f(x) ds + \int_{\beta_q}^{\beta_{q+1}} F(x) \cos C f(x) \, dx \; \zeta(\beta_{q+1} - \beta_q) \quad (m - 0)$$

if $m = q+1$. We sum this from $m = 0$ to $m = \bar{j}$. We find $\left|\frac{1}{x_j}\int_0^{x_j} F(x) \cos C f(x) \, dx\right| < \zeta$. We disregard the fact that $\alpha_{\bar{j}}$ may not coincide with x_j since j can always be chosen so that $\alpha_{\bar{j}} - x_j < M$ a constant. Hence the integral over such an interval devided by x_j is disregarded in the limit.

<u>Lemma III.</u> Let $a(C) = M(F(x)e^{-Cf(x)}) = 0$ for all C. Then

$$\lim_{n \to \infty} \frac{1}{x_n}\int_0^{x_n} F(x) \; e^{-i\, C\, f(x)} dx = 0$$

uniformly,

Proof: We follow Bohr exactly.

11. Remarks:

The paper is suspended at this point. With the results of this paper the remaining work in Bohr's book[2] follows with only trivial changes. This includes "The Uniqueness Theorem", "The Multiplication Theorem", "Parseval's Theorem" and "The Fundamental Theorem". The author does not think that a repetition of this work or the development of new proofs worth an enlargement of this paper.

References

(1) Fort, Tomlinson: The Gamma Function with varying difference interval: The Amer. Math. Monthly ________ .

(2) Bohr, Harold: Almost Periodic Functions, Chelsea Publishing Co., 1951

(3) Bohr, Harold: Zur Theorie der fast-periodischen Funktionen: Acta Math. 1925, Bd. 45, S.29-127.

Emory University, May 1966.