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Exact Solution to the Scattering of Waves From a Rough Surface

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ABSTRACT

An exact solution is obtained for the scattering of an arbitrary scalar wave from a rough surface in terms of a series of functions, by means of a complete set of functions orthogonal over the illuminated area. The method used is that of undetermined coefficients and is adopted here to the solution of either the Neumann problem or the Dirichlet problem.

Introduction

The Huygens formulation [Beckmann, 1963; Daniels, 1961; Davies, 1954; Feinstein, 1952; Hoffman, 1955, etc.] of a solution to the wave equation and the method of small perturbation [Al' pert et al, 1953; Katsenelenbaum, 1955; Miles, 1953; Parker, 1956; Lysanov, 1955, etc.] have been the two main approaches to solving the scattering problem. Other methods that rely on a single principle such as the geometric optics [Muhleman, 1964; Mitzner, 1964], the image method [Twersky, 1950, 1951, 1953 the principle of reciprocity Ament, 1960], the method of matching fields [Deringen, 1954], the brilliant point method [Du Castel and Spizzichino, 1959] etc., are either restricted to rather special problems for a practical solution or limited in applicability by the principle itself. The only general and exact approach then is the Huygens formulation. However, the requirement of the boundary conditions leads to an integral equation which in general cannot be readily solved in a closed form. Thus, various approximations [Lysanov, 1956; Meecham, 1956] have been made in order to affect a solution. To the author's knowledge no exact solution has been obtained for the said integral equation for the case of a rough surface boundary, although the same boundary condition formulated in terms of a continuous spectrum of plane waves has been solved by Marsh [1961]. In what follows, we shall present an exact solution in series form to the integral equation and conse uently the scattering problem.

The Scattered Field

Through the use of Huygen's Principle and Green's second theorem, the total field at a point, P, in the space above the rough surface due to an incident field ϕ_i can be shown to be [Bergman and Schiffer, p. 258, 1953]

$$\phi(P) = \phi_i(P) + \frac{1}{4\pi}\iint_{S'} \left[U(S') \frac{\partial G(P,S')}{\partial n'} - G(P,S') \frac{\partial U(S')}{\partial n'} \right] dS' \tag{1}$$

where $G(P,S') = e^{ikR}/R$ is the free space Green's function. U (S') and ϕ(P) are the wave potential functions on the surface and at the point P in space respectively. n' is the local unit normal on S'. For a free surface, (1) reduces on the surface to

$$\phi_i(S) = \frac{1}{4\pi}\iint_{S'} G(S,S') \frac{\partial U(S')}{\partial n'} dS' \tag{2}$$

where S represents the observation point on the scattering surface (ξ_1, ξ_2, ξ_3) and S' the source point on the scattering surface. Let the mean surface fit into a constant surface of some orthogonal coordinate system. Then dS' can be written as

$$dS' = h_1 h_2 \left[\left(\frac{h_3}{h_1} \frac{\partial S'}{\partial \xi_1'} \right)^2 + \left(\frac{h_3}{h_2} \frac{\partial S'}{\partial \xi_2'} \right)^2 + 1 \right]^{1/2} d\xi_1' \, d\xi_2' \tag{3}$$

where the h_i's are the scale factors. Hence (2) can be written as

$$\phi_i(S) = \frac{1}{4\pi}\iint_{S'} G(S,S') \Psi(S') h_1 h_2 \, d\xi_1' \, d\xi_2' \tag{4}$$

where $$\Psi(S') = \frac{\partial U(S')}{\partial n'} \left[\left(\frac{h_3}{h_1} \frac{\partial S'}{\partial \xi_1'} \right)^2 + \left(\frac{h_3}{h_2} \frac{\partial S'}{\partial \xi_2'} \right)^2 + 1 \right]^{1/2}$$

Let $g_n(\xi_1, \xi_2)$ be a complete set of normalized functions orthogonal on S'. Then the following expansions are possible

$$G(S,S') = \sum_n b_n(\xi_1, \xi_2) \, g_n(\xi_1', \xi_2') \tag{5a}$$

$$\Psi(S') = \sum_m c_m \, g_m(\xi_1', \xi_2') \tag{5b}$$

By substituting (5) into (4) and integrating over S' the following expression is obtained

$$\phi_i(S) = \frac{1}{4\pi} \sum_n C_n \, b_n(\xi_1, \xi_2) \tag{6}$$

The problem now is to determine Cn's. If $b_n(\xi_1, \xi_2)$ is an orthogonal set of functions, the Cn's can be easily found by quadratures. If not, let U_q be the orthogonal set of functions constructed from the set b_n by the Gram-Schmidt procedure [Conrant and Hilbert, vol I]. Then $\phi_i(S)$ can be expressed in terms of U_q. Thus,

$$\phi_i(S) = \sum_q a_q U_q = \sum_q a_q \left(\sum_{n=0}^{q} \alpha_{qn} \, b_n \right)$$

$$= \frac{1}{4\pi} \sum_n C_n \, b_n \tag{7}$$

where α_{qn} are coefficients obtained from the Gram-Schmidt procedure. Hence, $C_n = 4\pi \sum_q a_q \alpha_{qn}$. Theoretically, the problem is solved, but for numerical calculation an expression for α_{qn} is needed. This expression is given in the appendix.

Conclusions

From the method of approach, it is clear that this method works for the Neumann problem also. It has the advantage of being the most direct and straightforward as an exact method and simpler conceptually and in its final form than Marsh's method. The usefulness of exact methods, of course, is that it works where approximate methods fail and it serves to check the validity of such methods.

Appendix

To find an expression for the α_{qn}'s, observe that for $p < q$

$$\iint b_p U_q h_1 h_2 \, d\xi_1 \, d\xi_2 = 0 \tag{A-1}$$

i.e.
$$\sum_n^q \iint b_p \alpha_{qn} b_n h_1 h_2 \, d\xi_1 \, d\xi_2 \equiv \sum_n^q \alpha_{qn} d_{np} = 0 \tag{A-2}$$

where
$$d_{np} = \iint b_p b_n h_1 h_2 \, d\xi_1 \, d\xi_2$$

For $p = q$, denote the sum, $\sum_n \alpha_{qn} d_{nq}$ by N_q.

Consider now the determinants, D_q

$$D_0 = |d_{00}|$$

$$D_1 = \begin{vmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} d_{00} & d_{01} & d_{02} \\ d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{vmatrix}$$

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It is known that a determinant can be expressed in terms of its minors, M_{nq}, i.e.

$$\sum_{n=0}^{q} M_{nq} d_{np} = \delta_{pq} D_q \qquad ; \quad p \leq q \tag{A-3}$$

Comparison of (A-2) and (A-3) shows that α_{qn} is proportional to the minor M_{nq} of the element d_{nq} in D_q. Since the coefficient of b_n in the expression for U_n is supposed to be unity, we set

$$\alpha_{qn} = M_{nq} / M_{qq} \tag{A-4}$$

Thus, an explicit expression for U_q in terms of the b_n's is also obtained

$$U_q = \sum_{n=0}^{q} \left(\frac{M_{nq}}{M_{qq}} \right) b_n \quad .$$

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