ANALYSIS OF AN F. M. DISCRIMINATOR WITH FADING SIGNAL PLUS ADDITIVE GAUSSIAN NOISE

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ABSTRACT

A fading signal plus additive Gaussian noise are applied to an F. M. discriminator. It is desired to compute the output signal-to-noise ratio in terms of that at the input in order to determine the effects of fading on the threshold.

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SUMMARY

I - INTRODUCTION

A signal, characterized by non-selective fading, is added to white stationary Gaussian noise of zero mean and passed through a bandpass i. f. amplifier which is symmetric about the carrier frequency, f_o. The bandwidth, 2B, is assumed wide enough so that the signal passes undistorted. The i. f. output which is now in the form of signal plus bandlimited Gaussian noise is applied to an ideal limiter and then frequency demodulated. This is illustrated in Figure 1.

It is desired to investigate the signal-to-noise ratio at the output to that at the input of the ideal limiter in order to determine the effects of fading on the threshold. To do this one must first compute the power spectrum at $\underset{\sim}{\dot{\phi}}$. Hence, one needs the correlation function $R_{\dot{\phi}}(\tau)$.

The modulation $\underset{\sim}{D}(t)$ will be considered as $\beta \sin(2\pi f_m t + \underset{\sim}{\gamma})$ where $\underset{\sim}{\gamma}$ is uniformly distributed in $(0, 2\pi)$. In practice one never knows the phase of the signal exactly. The inclusion of $\underset{\sim}{\gamma}$ makes the process stationary.

With the modulation set equal to zero $(\beta = 0)$, the fading signal is usually taken to be a complex Gaussian stationary random process with its power spectrum assumed to be symmetric about f_o. Specifically, the one-sided power spectrum $W_c(f)$ is:

$$W_c(f) = \frac{\sigma_c^2}{\sqrt{2\pi}\, f_{H_1}} e^{-(f-f_o)^2/2 f_{H_1}^2} \tag{1}$$

The envelope $\underset{\sim}{r}(t)$ is then Rayleigh distributed and the carrier power is

$$\overline{c^2} = \frac{\overline{r^2}}{2} = \sigma_c^2 \tag{2}$$

The statistics of $\underset{\sim}{r}$ agree approximately with the measured statistics, in some cases, for periods of the order of several minutes on tropospheric and ionospheric scatter systems as well as for an ionospheric reflection transmission. The fading rate, or average number of times per second that the envelope crosses its median value with positive slope is equal to $1.48\, f_{H_1}$, a quantity which experimentally varies from a few tenths to a few cycles per second depending on the carrier frequency.

Using the Rayleigh model one finds that the noise at $\dot{\underset{\sim}{\phi}}$ above threshold is proportional to $\frac{1}{\underset{\sim}{r}}$ and that the resulting output noise power is infinite. Physically this is not true, for when the signal fades below a minimum level r_o, the discriminator is detecting only noise and $\dot{\underset{\sim}{\phi}}$ is not proportional to $\frac{1}{\underset{\sim}{r}}$.

The CCIR[1] has assumed that it is permissible to truncate the Rayleigh distribution at some point since when the fade drops below a certain level, the circuit will be completely cut off, or switched to a better path. Rather than ignore this small percentage of time, Pearson[2] states that it is more realistic to assume the envelope $\underset{\sim}{r}(t)$ to fade to a minimum level r_o and that no switching occurs. It is this approach that we take so that the statistics of $\underset{\sim}{r}$ now follow the form of a truncated Rayleigh density:

$$p(r) = \left\{ 1 - e^{-r_o^2/2\sigma_c^2} \right\} \delta(r - r_o) + {}^{r}/_{\sigma_c^2}\; e^{-r^2/2\sigma_c^2}\; u_{-1}(r - r_o) \tag{3}$$

where $\delta(r-r_o)$ is an impulse function, and $u_{-1}(r-r_o)$ a unit step function. The ratio of $({}^{r_o}/_{\sigma_c})^2$ will be defined as the fading depth (e.g. ${}^{r_o}/_{\sigma_c} = .1$ corresponds to a 20db fade). Assuming $(r_o/\sigma_c)^2 << 1$, as is always the case, it has been shown that the carrier power is the same as indicated by equation 2. The fading rate, obtained from a knowledge of the joint statistics of the truncated envelope and its derivative has been shown to be approximately $1.48\, f_{H_1}$. These statistics are obtained from the true Rayleigh statistics by means of the non-linear transformation

$$r^* = \begin{cases} r & r \geq r_o \\ 0 & r < r_o \end{cases} \tag{4}$$

The statistics of the phase $\underset{\sim}{\theta}(t)$ for which there is no experimental evidence would be uniformly distributed in $(0, 2\pi)$ if the signal were truly Gaussian. It is assumed that these statistics apply here, the joint statistics of $\underset{\sim}{r}$ and $\underset{\sim}{\theta}$ being obtained from the Rayleigh case by means of the above non-linear transformation on $\underset{\sim}{r}$ and the linear relation $\theta^* = \theta$.

II - THEORETICAL RESULTS

To compute the power spectrum at $\dot{\phi}$ one first considers the expression for ϕ.

$$\phi = D + \theta + \tan^{-1} \frac{\rho \sin(\lambda - D - \theta)}{r + \rho \cos(\lambda - D - \theta)} \tag{5}$$

Assuming the fading signal to be much larger than the additive noise $(\sigma_c^2 >> \sigma_n^2)$ so that with high probability $\rho/r << 1$

$$\phi \approx \phi_K = D + \theta + \frac{\rho \sin(\lambda - D - \theta)}{r} \tag{6}$$

Following the procedure of Rice[3], $\dot{\phi}$ is written as

$$\dot{\phi} = \dot{\phi}_K + 2\pi (Z_+ - Z_-) \tag{7}$$

where the $2\pi (Z_+ - Z_-)$ term accounts for the positive and negative clicks that are present at and above threshold. These clicks are due to rapid changes in ϕ by 2π radians when $\rho > r$. It is essentially a correction factor since at threshold $\tan^{-1} u$ does not equal u. The Z_+, Z_- processes are sums of positive and negative impulses, respectively, which occur at random times and are assumed to be independent of one another. They are regarded as Poisson processes with average click rates of N_+ and N_-. Furthermore, they are also assumed to be independent of $\dot{\phi}_K$.

With these assumptions, the correlation function at $\dot{\phi}$ is the sum of $R_{\dot{\phi}_K}$ and $R_{Z_\pm}$. Now R_{ϕ_K} can be written as the sum of R_D, R_θ, and R_s since the modulation, fading signal, and additive noise are independent of one another. This result is easily verified by expanding $\rho \sin(\lambda - D - \theta)$ in equation (6) in terms of the quadrature noise components $\xi = \rho \cos \lambda$ and $\eta = \rho \sin \lambda$. The correlation function at $\dot{\phi}$ is:

$$R_{\dot{\phi}} = R_{\dot{D}} + R_{\dot{\theta}} + R_{\dot{s}} + R_{Z_\pm} \tag{8}$$

The power spectrum is obtained by Fourier transforming equation (8).

$$W_{\dot{D}}(f) = \frac{\beta^2 \omega_m^2}{2} \delta(f) \tag{9}$$

$$W_{Z\pm} = 8\pi^2 (N_+ + N_-) \tag{10}$$

The R_θ term is determined by averaging the product of θ_1 and θ_2. The joint statistics of θ_1 and θ_2, obtained by integrating $p(r_1 r_2 \theta_1 \theta_2)$ over r_1 and r_2, the truncated Rayleigh variables, are the same as if r_1 and r_2 were Rayleigh (the fading signal with no modulation being Gaussian). Thus, $R_\theta(\tau)$ is the same as in the Gaussian signal case and hence, $R_{\dot{\theta}}(\tau) = -\frac{d^2}{d\tau^2} R_\theta(\tau)$ is the same. In particular:

$$R_{\dot{\theta}}(\tau) = \frac{1}{2} \left\{ \frac{\ddot{R}(\tau)}{R(\tau)} - \left[\frac{\dot{R}(\tau)}{R(\tau)}\right]^2 \right\} \ln \left(1 - \left[\frac{R(\tau)}{\sigma_c^2}\right]^2\right) \tag{11}$$

$W_{\theta}(f)$ has been computed by Rice[4] for the case when $R(\tau) = \sigma_c^2 \, e^{-(2\pi f_{H_1})^2 \tau^2/2}$ (For a Gaussian signal this means that power spectrum would be of the Gaussian shape described by equation 1.)

The $R_s(\tau)$ term can be shown to be

$$R_s(\tau) = R_{\xi}(\tau) \; \overline{\cos\left[D(t+\tau) - D(t)\right]} \; \overline{\frac{\cos(\theta_2 - \theta_1)}{r_1 r_2}} \tag{12}$$

where $R_{\xi}(\tau)$ is the correlation function of the quadrature term of the additive noise. $R_{\xi}(\tau)$ decreases very rapidly, decaying to zero in a time of the order of the reciprocal of the noise bandwidth. Since the fading is very slow compared with noise one writes

$$R_s(\tau) = R_{\xi}(\tau) \; \overline{\cos\left[D(t+\tau) - D(t)\right]} \; \overline{\frac{1}{r^2}} \tag{13}$$

for $r_1 \simeq r_2$ and $\theta_2 \simeq \theta_1$ when $R_{\xi}(\tau)$ is different from zero. Hence, $W_s(f)$ is found, from which $W_{\dot{s}}(f) = \omega^2 W_s(f)$. (Note, that if $\underset{\sim}{r}$ were Rayleigh, $r_o = 0$, then $\overline{\frac{1}{r^2}} = \infty$.)

The output signal-to-noise ratio is found by passing $\dot{\phi}$ through an ideal low pass filter of cutoff frequency f_m. The output power is:

$$P_o = \frac{\beta \omega_m^2}{2} + N_{o_{\dot{\theta}}} + N_{o_{\dot{s}}} + N_{o_{Z\pm}} \tag{14}$$

where the last three terms are noise and signal mixed with noise. We define the output signal-to-noise ratio as:

$$\frac{S_o}{N_o} = \frac{\frac{(\beta \omega_m^2)^2}{2}}{N_{o_{\dot{\theta}}} + N_{o_{\dot{s}}} + N_{o_{Z\pm}}} \tag{15}$$

A rectangular shape for the i. f. amplifier is assumed with $B = \beta f_m$, where

$\beta = 10$. The ratio of f_m/f_{H_1} is taken as 10^4. With an i. f. bandwidth of 200 KC , this yields a fading rate of 1. 48 cps.

The signal-to-noise ratio curves, shown in Figure 2 for 20 and 40db fades, are computed by neglecting the modulation interaction with the noise. (i. e. in finding $N_{o_{\dot{s}}}$ and $N_{o_{Z\pm}}$, $\beta = 0$) Inclusion of the modulation, as shown in Figure 3, shifts the threshold to the right by approximately one db.

III CONCLUSIONS

With no fading and β = 5 or 10 the threshold has been shown to occur at input signal-to-noise ratios of approximately 10db[3, 5]. The effect of fading is to raise the threshold to about 32db for a 20db fade and 52db for a 40db fade. In contrast with the no fading case, the output noise power is present even with infinite input signal-to-noise ratio $\left(\frac{S_o}{N_o} \rightarrow \frac{\beta^2\omega_m^2}{2 N_{o_{\dot{\theta}}}}\right)$.

Acknowledgement

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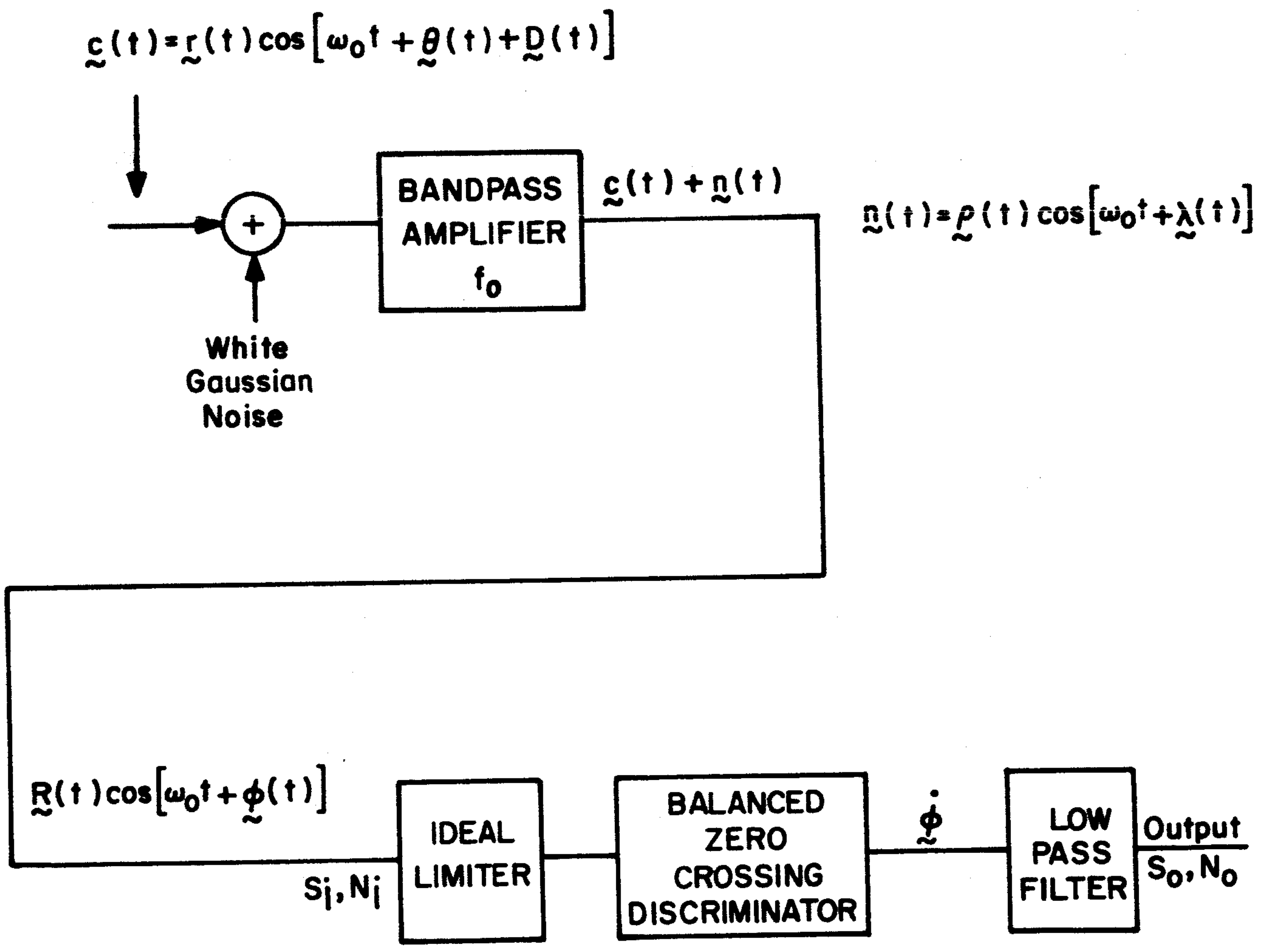

c(t)=r(t)cos[ω0t+θ(t)+D(t)]
+
BANDPASS AMPLIFIER f0
c(t)+n(t)
n(t)=ρ(t)cos[ω0t+λ(t)]
White Gaussian Noise
R(t)cos[ω0t+φ(t)]
Si, Ni
IDEAL LIMITER
BALANCED ZERO CROSSING DISCRIMINATOR
φ̇
LOW PASS FILTER
Output
So, No

FIGURE 1

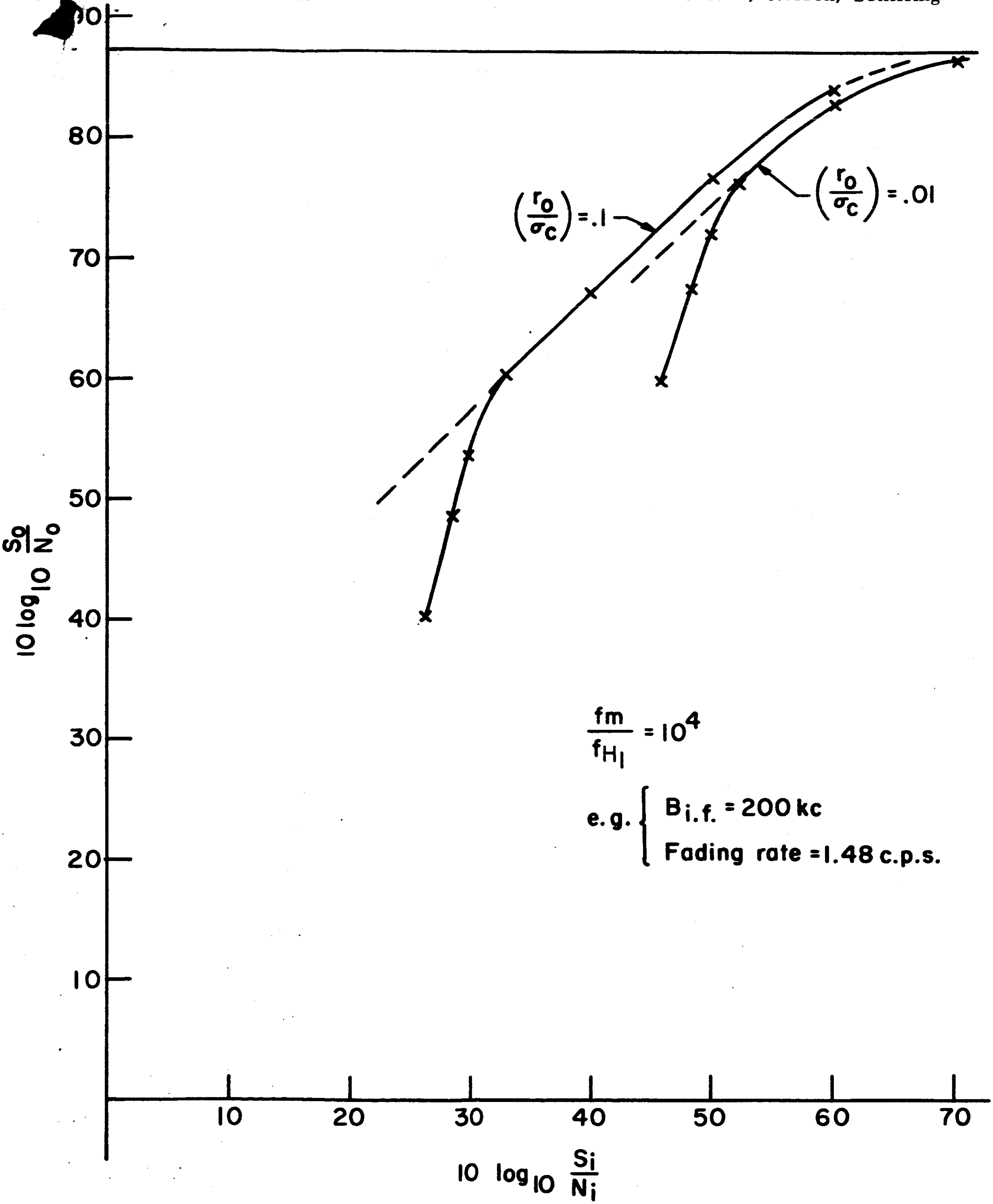
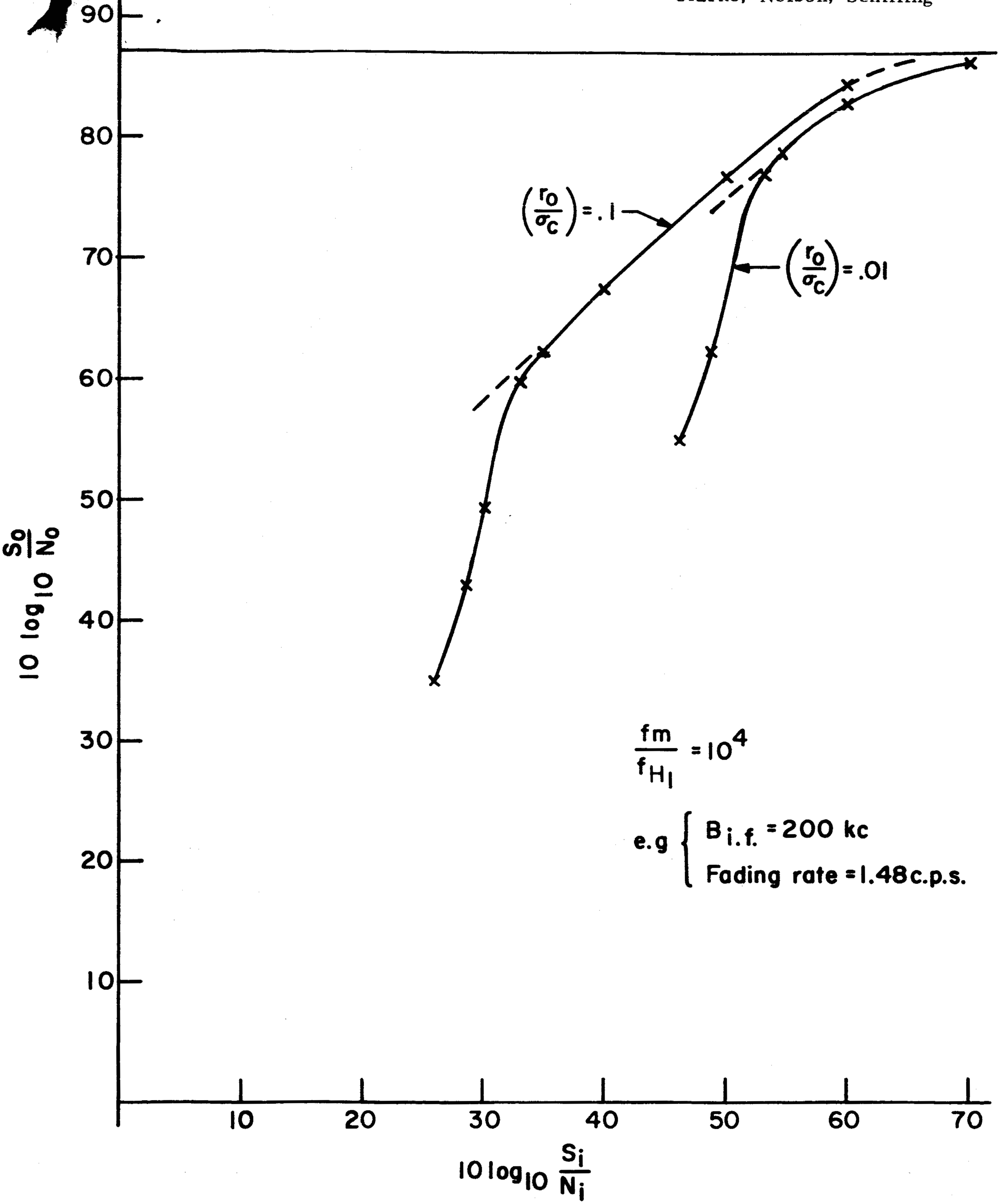

$10 \log_{10} \frac{S_o}{N_o}$
$10 \log_{10} \frac{S_i}{N_i}$
$\left(\frac{r_o}{\sigma_c}\right) = .1$
$\left(\frac{r_o}{\sigma_c}\right) = .01$
$\frac{f_m}{f_{H_1}} = 10^4$
e.g. $B_{i.f.}$ = 200 kc
Fading rate = 1.48 c.p.s.
10
20
30
40
50
60
70
80
90

FIGURE 2



90
80
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$10 \log_{10} \frac{S_o}{N_o}$
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40
50
60
70
$10 \log_{10} \frac{S_i}{N_i}$
$\left(\frac{r_o}{\sigma_c}\right) = .1$
$\left(\frac{r_o}{\sigma_c}\right) = .01$
$\frac{fm}{f_{H_1}} = 10^4$
e.g $\begin{cases} B_{i.f.} = 200 \text{ kc} \\ \text{Fading rate} = 1.48 \text{ c.p.s.} \end{cases}$

FIGURE 3