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A Development of the Describing Functions for Two Nonlinearities Separated by a Linear Function

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A Development of the Describing Function for Two Nonlinearities Separated by a Linear Function

Frequency response techniques are a valuable tool in the analysis and synthesis of linear as well as nonlinear systems. The describing function analysis of a nonlinear system often clearly indicates the existance of a possible limit cycle frequency and amplitude as well as the effect of system gain on the amplitude and frequency of the limit cycle.

The use of describing functions in analyzing a system containing a single nonlinearity has been investigated in many papers and books in the past. When the describing function of a nonlinearity is amplitude but not frequency sensitive the analysis of a system containing only one such nonlinearity can be carried out by standard procedure. If the nonlinearity is frequency sensitive, the resultant analysis is considerably more involved.

In many practical problems there exist two or more nonlinearities which are separated by one or two linear blocks. In this case the overall describing function is almost always frequency and amplitude dependent. This is so because even though the two nonlinearities may be frequency insensitive the intervening linear block is usually frequency dependent.

The often encountered nonlinearities of dead zone and blacklash appear many times in such a way that the separation of linear and non-linear portions cannot be achieved. This report deals in analyzing such type of problem.

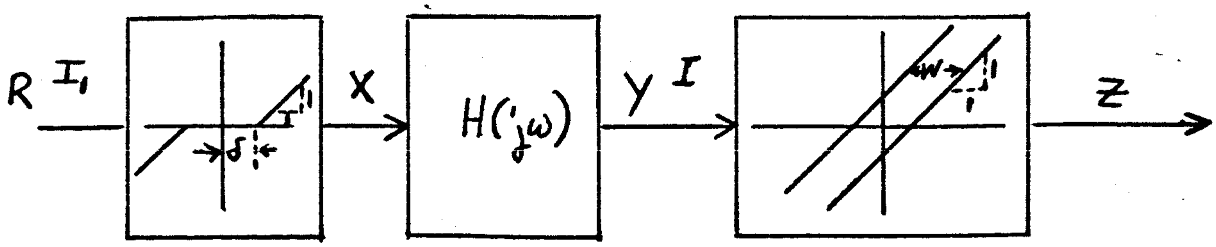

Fig. 1.

The combined nonlinear function of [deadzone .H. backlash].

Fig. 1 indicates a combination of deadzone and backlash nonlinearity separated by a linear block $H(j\omega)$. It is apparent that, in general, the describing function of the combined nonlinearity will be sensitive to the amplitude as well as the frequency of R. It is known that if $H(j\omega)$ is a sufficiently good filter so that a sinusoidal input may be assumed to the backlash nonlinearity, then it is permissible to multiply the describing functions obtaining $G_D{}'(I_1,\omega_o) \triangleq G_{DZ}(I_1)\, G_{BL}(I)$

where $R = I_1 \sin \omega_o t$

and $I = I_1 G_{DZ}(I_1) H(\omega_o)$

where the describing function G_{BL} of the backlast nonlinearity is seen to be a function of I_1, G_{DZ}, and H.

When $H(j\omega)$ does not act as a good filter then the problem is much more complicated due to the presence of higher harmonics in Y. In the general case let

$$H(s) = \frac{K}{\tau s + 1}$$

$$R = I_1 \sin \omega_o t$$

Then X(t) = (See Fig. 2)

$$\beta = \sin^{-1} \frac{\delta}{I_1}$$

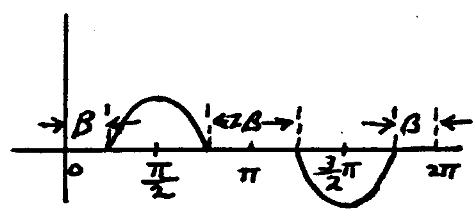

Fig. 2. Deadzone Output

$X(t) = 0 \ . \ . \ . \ 0 \leq t \leq \beta$

$X(t) = (I_1 \sin \omega_o t - \delta) \ . \ . \ . \ \beta \leq \omega_o t \leq \pi - \beta$

$X(t) = 0 \qquad (\pi - \beta) \le \omega_o t \le (\pi + \beta)$

$X(t) = (I_1 \sin \omega_o t + \delta) \; . \; . \; . \; (\pi + \beta) \le \omega_o t \le (2\pi - \beta)$

$\quad = 0 \qquad (2\pi - \beta) \le \omega_o t \le 2\pi$

$X(t)$ is a periodic function of period $\frac{2\pi}{\omega_o}$.

Now let

$$X_1(t) = X(t) \qquad 0 \le t \le \frac{2\pi}{\omega_o}$$

$$= o \qquad t > \frac{2\pi}{\omega_o}$$

Note that $X_1(t) = X(t)$ over one interval and is zero elsewhere.
1

It can be shown that where Y_{ss} is the desired steady state solution, Y_1 is the total solution and Y_t is the transient solution.

$$Y_{ss}(t) = Y_1(t) - Y_t(t)$$

and $\quad Y_1(t) = \mathcal{L}^{-1}\left[X_1(s)\ H(s)\right]$

and $\quad Y_t(t) =$ Residue of $\dfrac{X_1(s)\ H(s)}{1 - \exp\left(\frac{-2\pi}{\omega_o} s\right)}\ e^{st}$ at the poles of $H(s)$.

$$H(s) = \frac{k}{\tau s + 1} = \frac{k}{\tau}\left(\frac{1}{s + 1/\tau}\right)$$

Now

$$X_1(s) = \frac{I_1(\cos\beta)\ \omega_o}{s^2 + \omega_o^2}\left[e^{-as} + e^{-bs} - e^{-cs} - e^{-ds}\right]$$
$$+ \frac{\delta}{s}\left[-e^{-as} + e^{-bs} + e^{-cs} - e^{-ds}\right]$$
$$+ \frac{\delta s}{s^2 + \omega_o^2}\left[e^{-as} - e^{-bs} - e^{-cs} + e^{-ds}\right]$$

where $\quad a = \beta/\omega_o$

$b = \dfrac{\pi - \beta}{\omega_o}$

$c = \dfrac{\pi + \beta}{\omega_o}$

$d = \dfrac{2\pi - \beta}{\omega_o}$

[1]Seshu & Balbanian, LINEAR NETWORK ANALYSIS, Wiley, pp 168-173.

Now

$$Y_1(t) = \mathcal{L}^{-1}\left[X_1(s)\,H(s)\right]$$

$$= \mathcal{L}^{-1}\left[X_1(s)\cdot\frac{1}{(\tau s + 1)}\right]$$

$$Y_1(t) = \frac{I_1\omega_o\cos\beta}{\left(\frac{1}{\tau^2} + \omega_o^2\right)}\left[e^{-t/\tau} - \cos\omega_o t + \frac{\sin\omega_o t}{\tau\omega_o}\right]\left\{\left(e^{-as} + e^{-bs} - e^{-cs} - e^{-ds}\right)\right\}$$

$$+ \frac{\delta}{\tau^2\left(\frac{1}{\tau^2} + \omega_o^2\right)}\left[\cos\omega_o t + \omega_o\sin\omega_o t - e^{-t/\tau}\right]\left\{e^{-as} - e^{-bs} - e^{-cs} + e^{-ds}\right\}$$

$$-\delta\left[1 - e^{-t/\tau}\right]\left\{e^{-as} - e^{-bs} - e^{-cs} + e^{-ds}\right\}$$

and

$$Y_t(t) = \mathrm{Res}\,\frac{X_1(s)\,H(s)}{1 - \exp\left(\frac{-2\pi}{\omega_o}s\right)}$$

$$= \left\{\frac{I_1\omega_o\cos\beta}{\tau\left(\frac{1}{\tau^2} + \omega_o^2\right)}\right\}\left(e^{a/\tau} + e^{b/\tau} - e^{c/\tau} - e^{d/\tau}\right)$$

$$- \left\{\frac{\delta}{\tau^2\left(\frac{1}{\tau^2} + \omega_o^2\right)}\right\}\left(e^{a/\tau} - e^{b/\tau} - e^{c/\tau} + e^{d/\tau}\right)$$

$$- \left(-e^{a/\tau} + e^{c/\tau} - e^{d/\tau} + e^{b/\tau}\right)\left\{\frac{e^{-t/\tau}}{1 - \exp\left(\frac{2\pi}{\omega_o\tau}\right)}\right\}$$

$$Y(t) = Y_1(t) - Y_t(t) \qquad (1)$$

Equation 1 was solved on a digital computer and $Y(t)$ was obtained for various values of I_1. Because $Y(t)$ is periodic and symmetric $Z(t)$ can be easily plotted. $Z(t)$ was then expanded in a Fourier series and the amplitude as well as phase shift of the fundamental term, Z_o, was determined. The overall describing function is

$$G_D(I_1,\omega_o) = \frac{Z_o\,\angle\phi}{I_1}$$

[1]The delay terms e^{-as}, e^{-bs}, e^{-cs}, and e^{-ds} should be incorporated into the time functions; i.e., $F(s)e^{-as} = f(t-a)\,U(t-a)$. This is not done here because it would result in a lengthy, cumbersome expression.

From this a family of describing functions is generated.

For the analysis of any system we have to solve the characteristic equation, $G_L(j\omega_o)\ G_D(I_1, j\omega_o) = -1$ where G_L is any linear transfer function and G_D is the describing function of the combined nonlinearities.

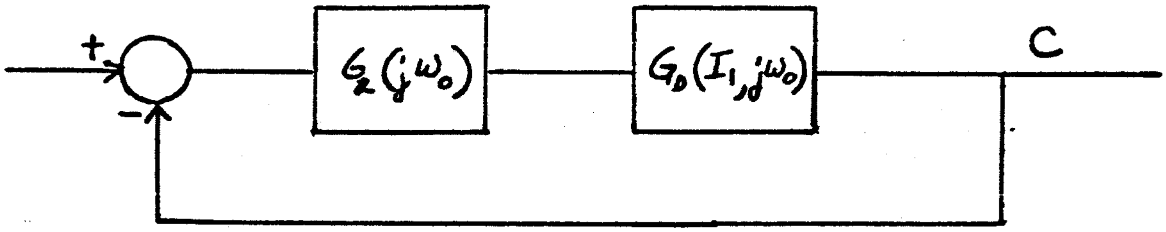

Fig. 3. A block diagram of a typical control system.

So for the constant ratio of $\frac{(\delta K)}{W}$ a family of $\left|\frac{1}{G_D}\right|_{dbs}$ vs. (-180°+ phase shift) curves are plotted from which the amplitude and frequency of the limit cycle can be determined. The describing function plots are shown on Figures 4 through 10 for the various parameter variations. As an example of how to use these curves consider the following problem.

Example: Referring to Fig. 3 let the linear transfer function G_L be

$$G_L(j\omega) = \frac{12.6K_a}{(j\omega+1)\ (j0.05\omega+1)}$$

and

$$H(j\omega) \text{ in } G_D \text{ be } \frac{K}{(j0.1\omega+1)}$$

Assume that the deadzone half-width δ and the hysteresis width W are each 0.8. Also assume that the gain factor K in $H(j\omega)$ of the combined nonlinearity is 1. For $K_a = 1$ and $K_a = 2.5$ determine the limit cycle frequency and amplitude, if any.

Solution: A plot of $G_L(j\omega) = |G_L|\underline{/\theta_L}$ for $K_a = 1$ is shown in Fig. 11. Since $\frac{\delta K}{W} = \frac{0.8 \text{x} 1}{0.8} = 1$ choose Fig. 4 which shows a family of describing function curves for $\frac{\delta K}{W} = 1$. If Fig. 11 is superimposed on Fig. 4 note that there are no interceptions between G_L and $\frac{1}{G_D}$ plots. Therefore for K_a=1 the system exhibits no limit cycle oscillation.

If the G_L curve is moved up 8 db (for K_a=2.5) it can be seen that the ω=10 point on Fig. 13 coincides with the $\tau\omega=1$ (or $\omega=\frac{1}{0.1}=10$) curve on Fig. 4. This intercept occurs at a $\frac{\delta}{I_1}$ ratio of 0.32. Since $\delta=0.8$ $I_1=\frac{0.8}{0.32}=2.5$. Thus the limit cycle frequency is 10 radians/second and its amplitude is 2.5.

Conclusions. The families of describing function plots shown in Figures 4 through 10 should be of definite aid to the control system engineer who is confronted with analyzing a system containing the particular nonlinearity combination of deadzone and backlash separated by a linear block. Although only one sample problem is given in this report, there are several ways in which the curves can be used in a problem analysis. For instance, the internal gain K of the linear block between the two nonlinearities exerts considerable influence over the effective backlash or deadzone widths. Thus in addition to controlling the overall system gain with K an additional factor must be considered in using K. Also the time constant τ in H can be used as a design parameter.

The technique presented here for obtaining the describing function of a particular nonlinear-linear function can be used in obtaining the describing function of many other nonlinear-linear combinations. It is anticipated that work of this nature will be done in the future.

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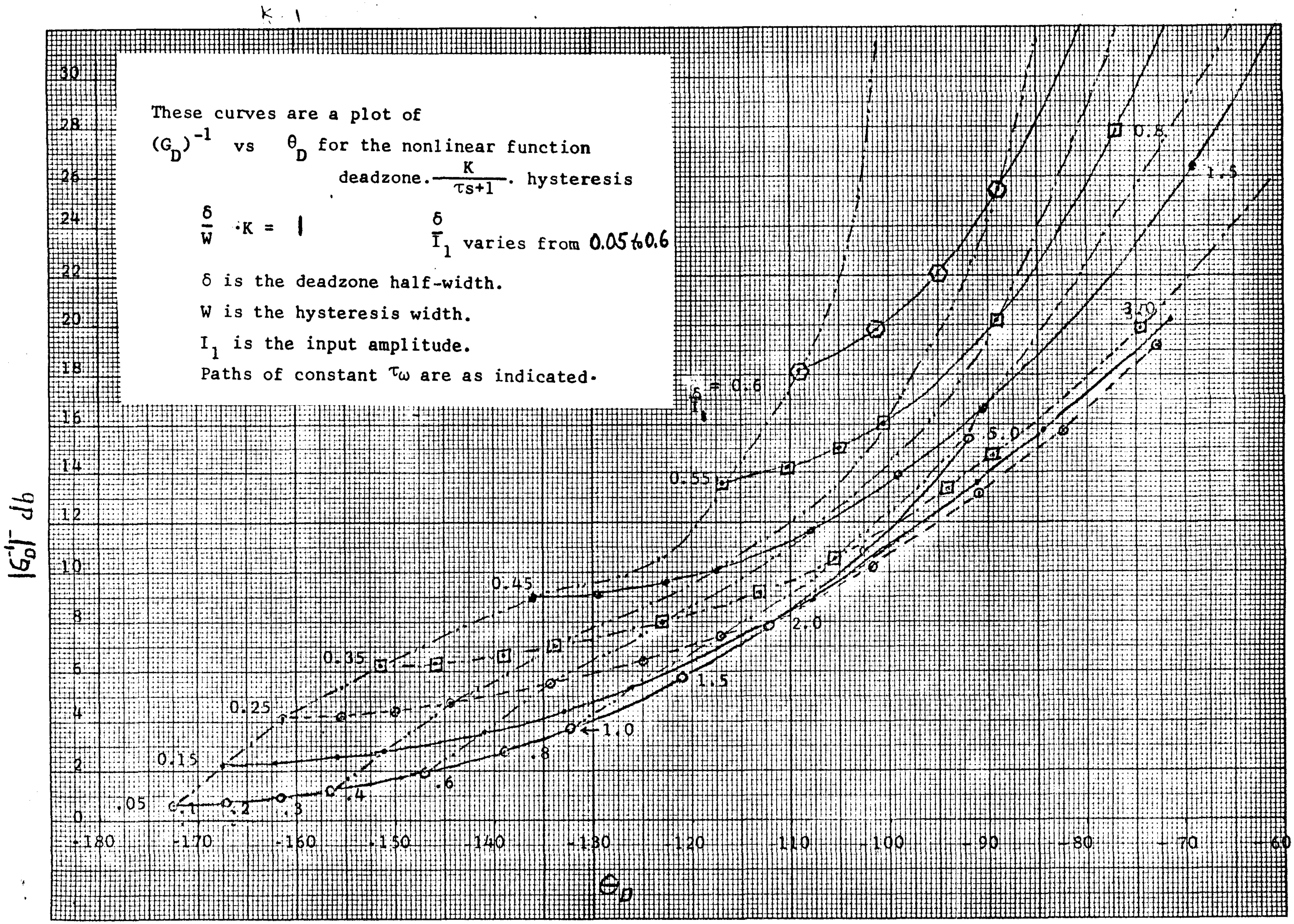


K · 1
These curves are a plot of
$(G_D)^{-1}$ vs θ_D for the nonlinear function
deadzone. $\frac{K}{\tau s+1}$. hysteresis
$\frac{\delta}{W} \cdot K = 1$
$\frac{\delta}{I_1}$ varies from 0.05 to 0.6
δ is the deadzone half-width.
W is the hysteresis width.
I_1 is the input amplitude.
Paths of constant $\tau\omega$ are as indicated.
$|G_D^{-1}|$ db
Θ_D
$\frac{\delta}{I_1} = 0.6$
0.55
0.45
0.35
0.25
0.15
.05
0.8
1.5
3.0
5.0
2.0
1.5
1.0
.8
.6
.4
.3
.2
.1
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
-180
-170
-160
-150
-140
-130
-120
-110
-100
-90
-80
-70
-60

Figure 4

$\frac{\tau}{\omega} = 2$

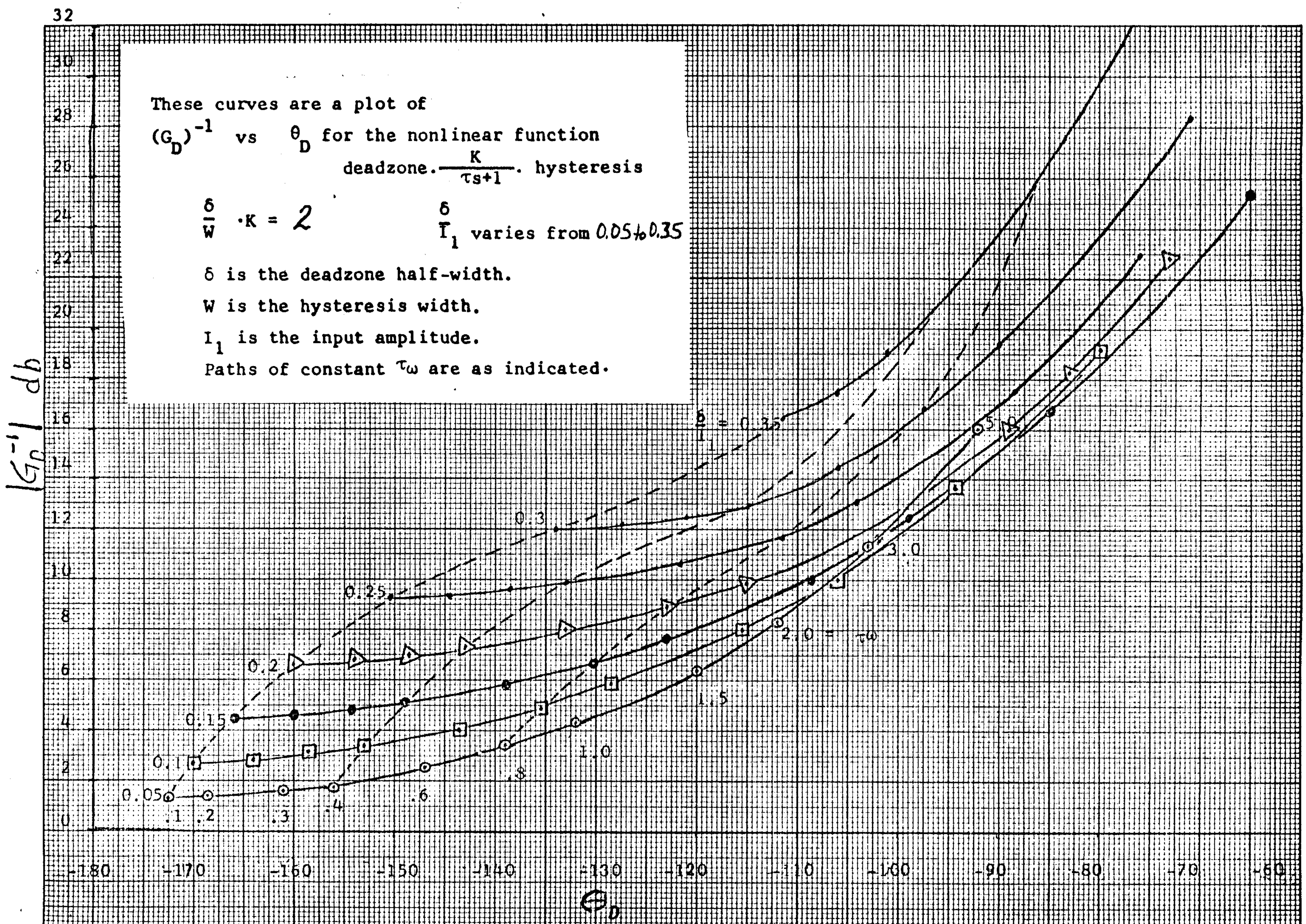


Figure 5

$\frac{\delta}{\omega} = 3$, $\frac{5K}{\omega} = 3$

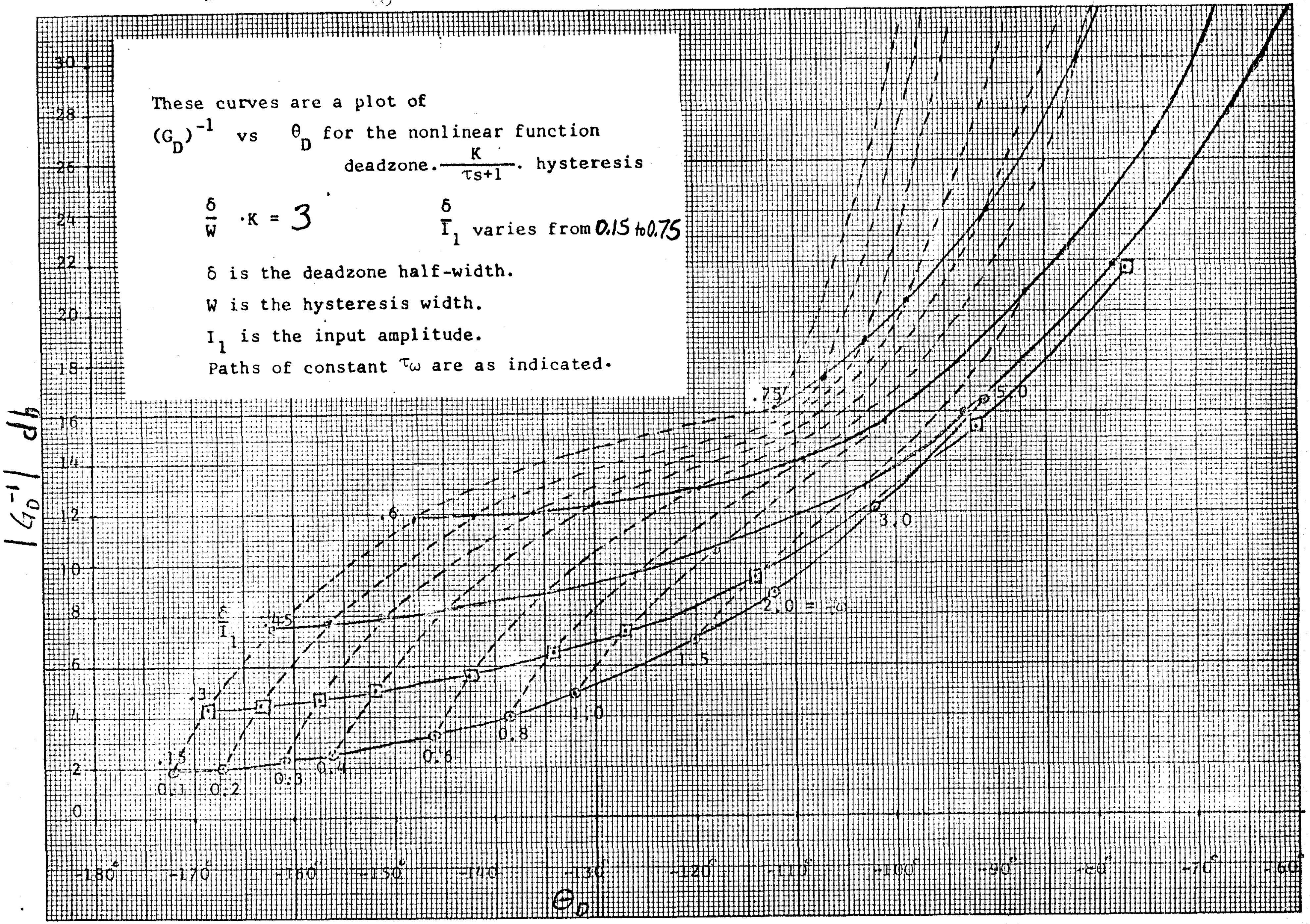


Figure 6

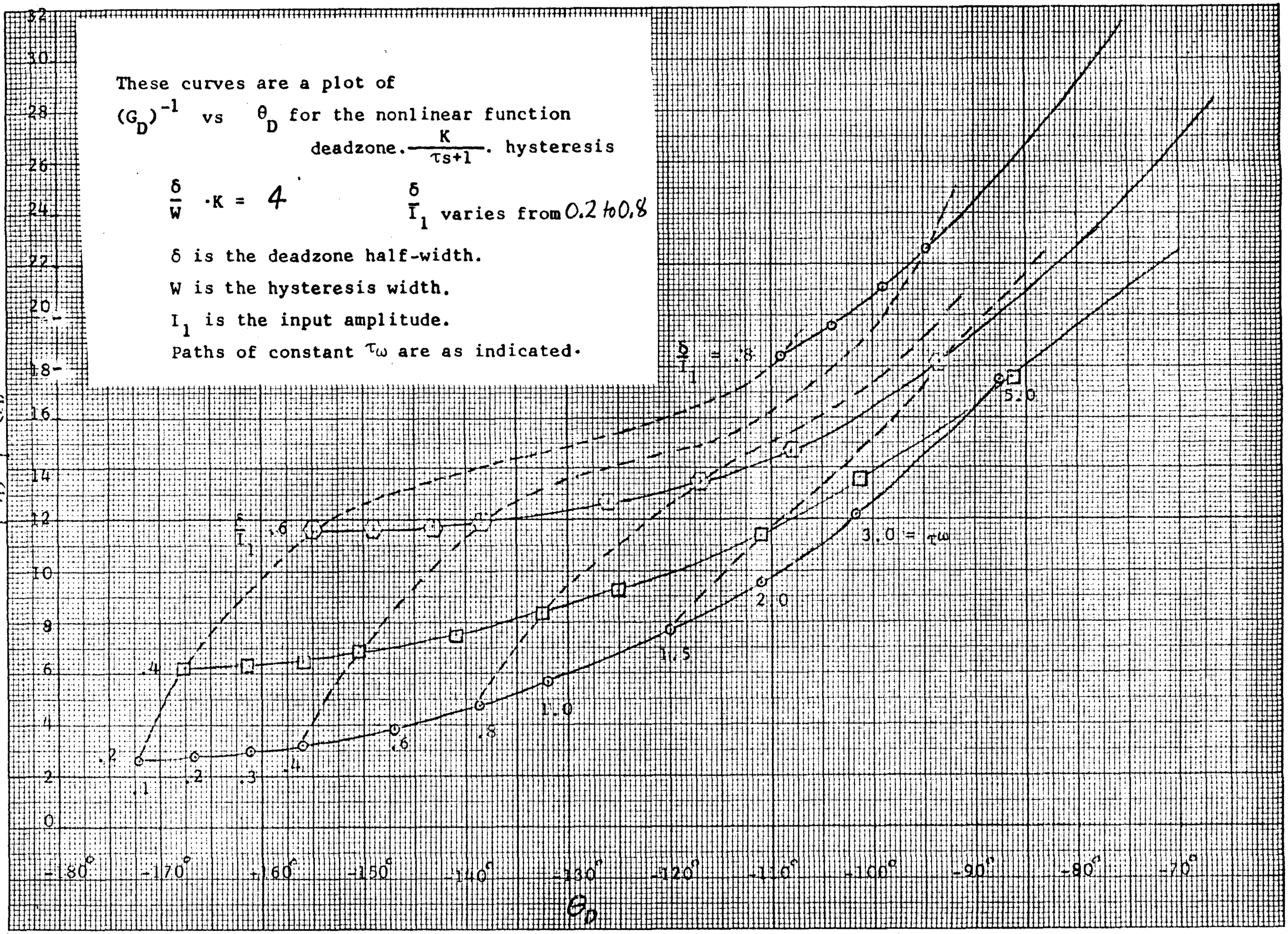
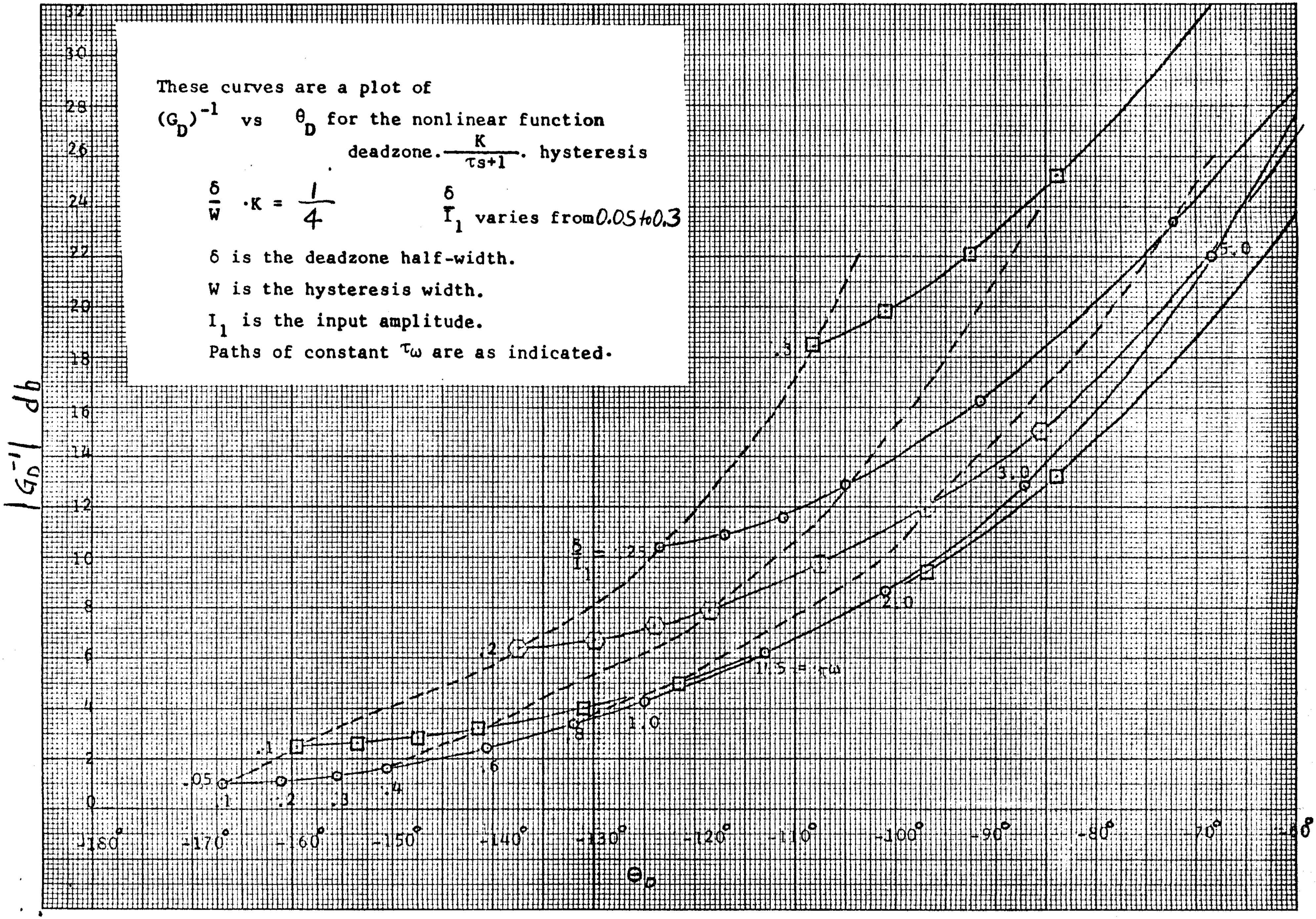
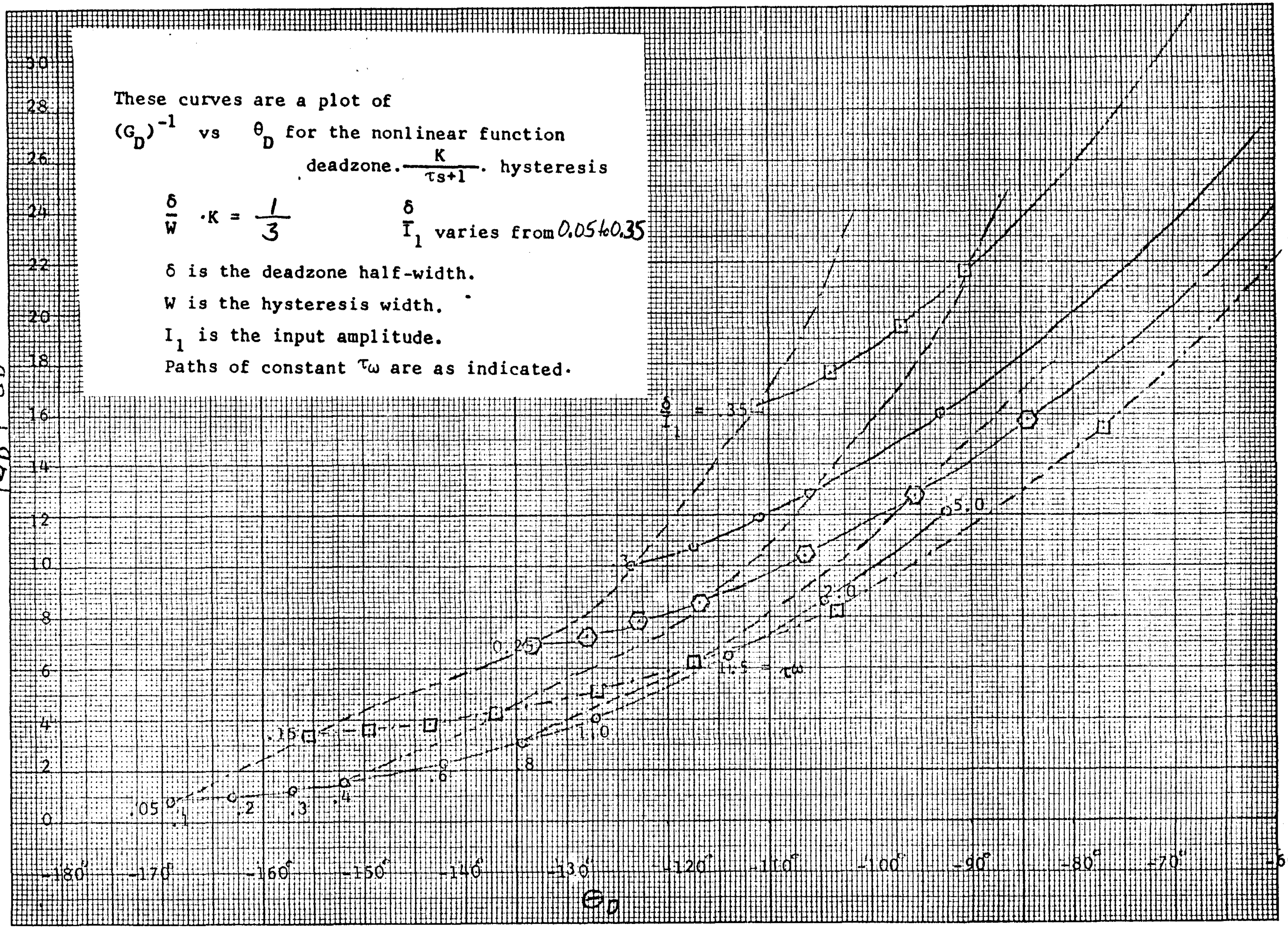

These curves are a plot of
$(G_D)^{-1}$ vs θ_D for the nonlinear function
deadzone. $\frac{K}{\tau s+1}$. hysteresis
$\frac{\delta}{W} \cdot K = 4$
$\frac{\delta}{I_1}$ varies from 0.2 to 0.8
δ is the deadzone half-width.
W is the hysteresis width.
I_1 is the input amplitude.
Paths of constant $\tau\omega$ are as indicated.
$\frac{\delta}{I_1} = .8$
$\frac{\delta}{I_1}$ .6
.4
.2
.1
.2
.3
.4
.6
.8
1.0
1.5
2.0
$3.0 = \tau\omega$
5.0
$|G_D^{-1}|$ db
θ_D
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
-180°
-170°
-160°
-150°
-140°
-130°
-120°
-110°
-100°
-90°
-80°
-70°

Figure 7



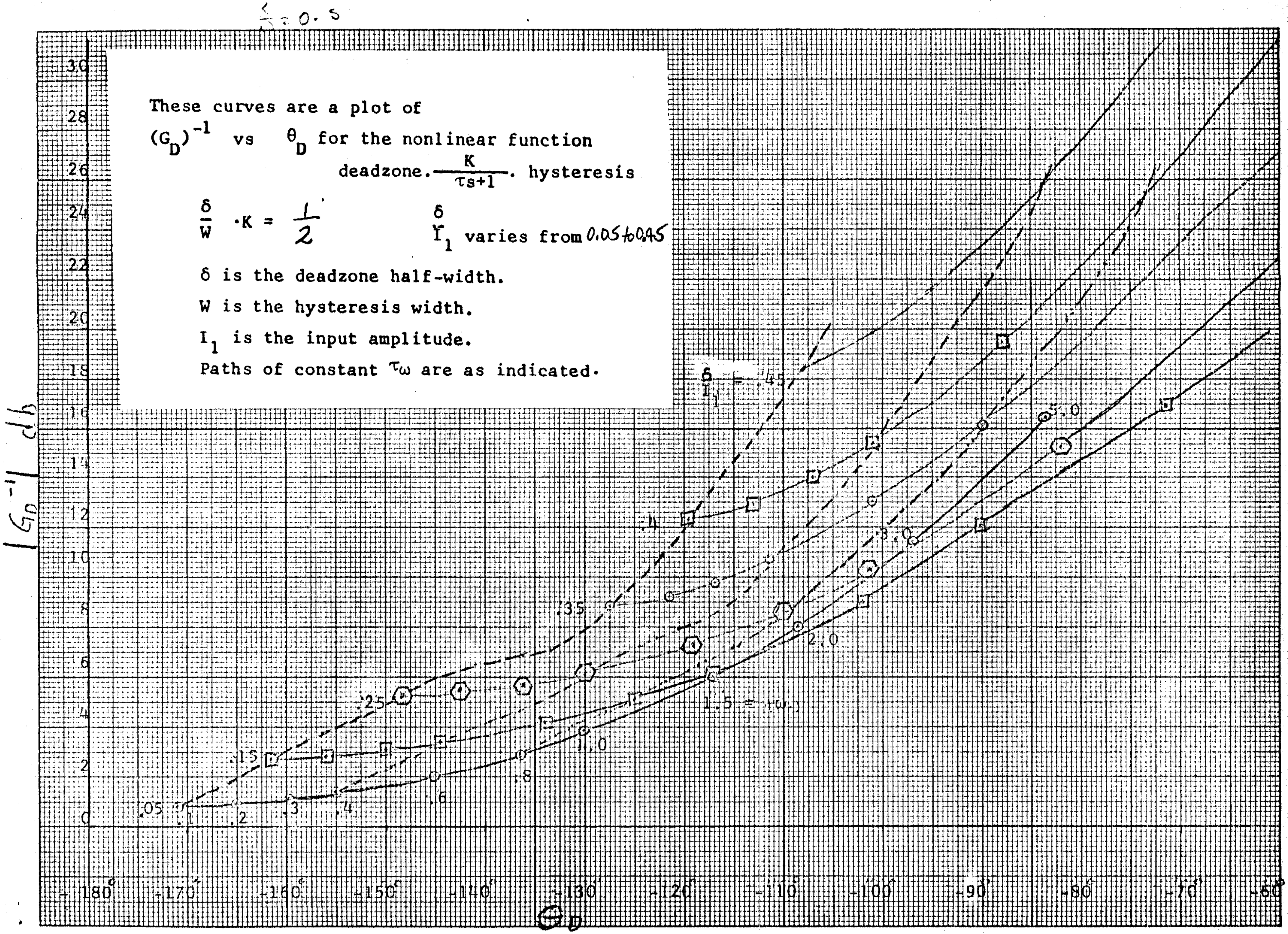
These curves are a plot of
$(G_D)^{-1}$ vs θ_D for the nonlinear function
deadzone. $\frac{K}{\tau s+1}$. hysteresis
$\frac{\delta}{W} \cdot K = \frac{1}{4}$
$\frac{\delta}{I_1}$ varies from 0.05 to 0.3
δ is the deadzone half-width.
W is the hysteresis width.
I_1 is the input amplitude.
Paths of constant $\tau\omega$ are as indicated.
$|G_D^{-1}|$ db
θ_D
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
-180°
-170°
-160°
-150°
-140°
-130°
-120°
-110°
-100°
-90°
-80°
-70°
-60°
$\frac{\delta}{I_1}$ = .25
.3
.2
.1
.05
.1
.2
.3
.4
.6
.8
1.0
1.5 = $\tau\omega$
2.0
3.0
5.0

Figure 8



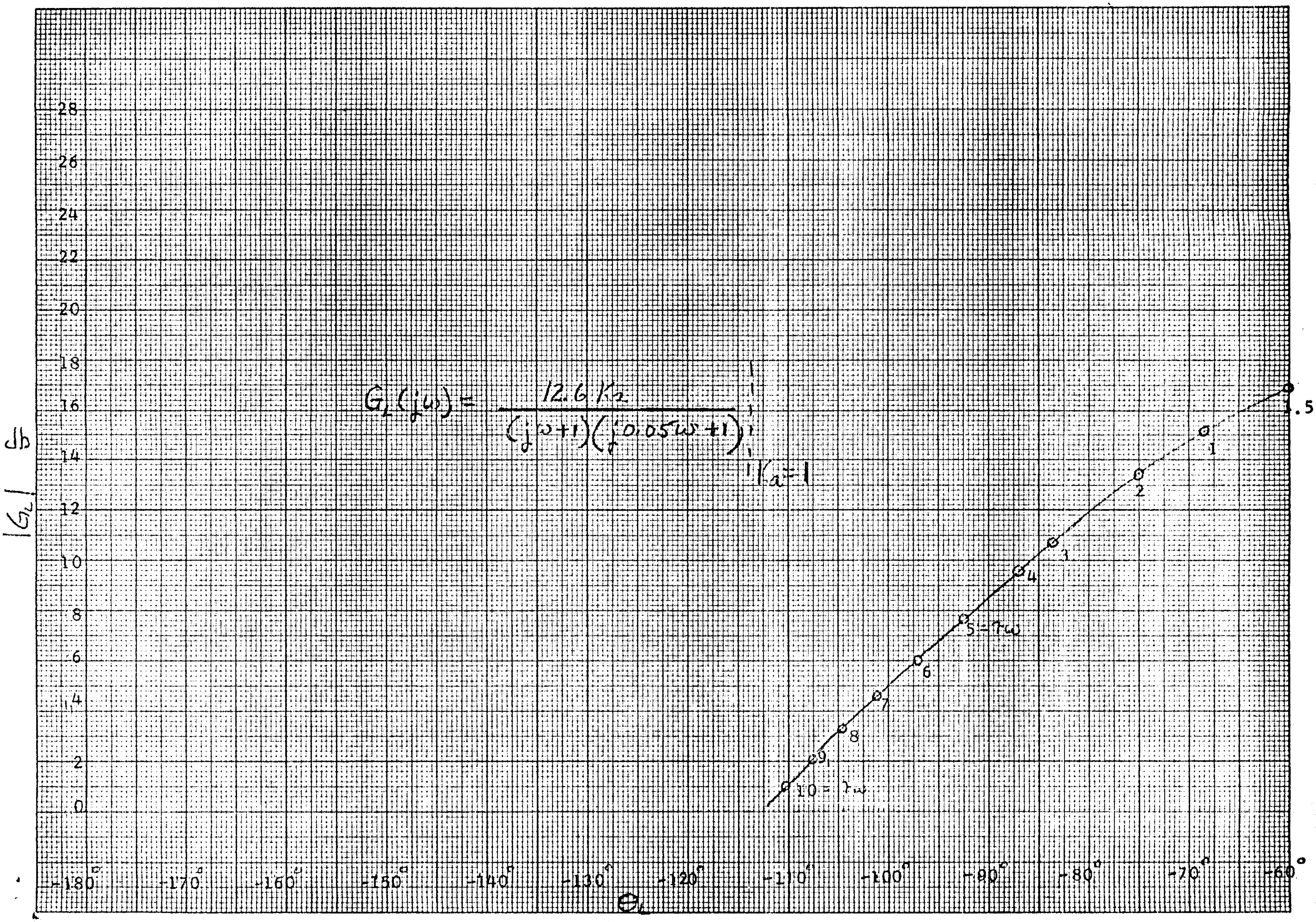
These curves are a plot of
$(G_D)^{-1}$ vs θ_D for the nonlinear function
deadzone. $\frac{K}{\tau s+1}$. hysteresis
$\frac{\delta}{W} \cdot K = \frac{1}{3}$
$\frac{\delta}{I_1}$ varies from 0.05 to 0.35
δ is the deadzone half-width.
W is the hysteresis width.
I_1 is the input amplitude.
Paths of constant $\tau\omega$ are as indicated.
$|G_D^{-1}|$ db
Θ_D
$\frac{\delta}{I_1}$ = .35
0.25
.15
.05
.1
.2
.3
.4
.6
.8
1.0
1.5 = $\tau\omega$
2.0
5.0
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
-180°
-170°
-160°
-150°
-140°
-130°
-120°
-110°
-100°
-90°
-80°
-70°
-60°

Figure 9



$\frac{\delta}{W} = 0.5$
These curves are a plot of
$(G_D)^{-1}$ vs θ_D for the nonlinear function
deadzone. $\frac{K}{\tau s+1}$. hysteresis
$\frac{\delta}{W} \cdot K = \frac{1}{2}$
$\frac{\delta}{I_1}$ varies from 0.05 to 0.45
δ is the deadzone half-width.
W is the hysteresis width.
I_1 is the input amplitude.
Paths of constant $\tau\omega$ are as indicated.
$\frac{\delta}{I_1} = .45$
.4
.35
.25
.15
.05
.1
.2
.3
.4
.6
.8
1.0
1.5
2.0
3.0
5.0
$|G_D^{-1}|$ db
θ_D

Figure 10



$G_L(j\omega) = \frac{12.6\,K_2}{(j\omega+1)(j0.05\omega+1)}\Big|_{K_a=1}$
db
$|G_L|$
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
-180°
-170°
-160°
-150°
-140°
-130°
-120°
-110°
-100°
-90°
-80°
-70°
-60°
θ_L
1.5
1
2
3
4
5 = ω
6
7
8
9
10 = ω

Figure 11