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ADIABATIC INVARIANT TRAPPED RADIATION SHELLS AND COSMIC RAY CUTOFFS IN MODELS OF THE MAGNETOSPHERE DISTORTED BY THE SOLAR WIND PLASMA

by

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ABSTRACT

Ray has found an approximate first integral for the motion of charged particles in static magnetic fields not necessarily displaying axial symmetry. This involves a canonical transformation to a new set of coordinates, one of which is approximately ignorable.

In field models which approximately preserve this new kind of symmetry, of which the geomagnetic field is an example, a perturbation technique is here employed to derive an improved version Ray's first integral. In the technique, conservation of energy, adiabatic invariance of the magnetic moment, Ray's first integral, and the slow drift motion of the particles under consideration are used as the zeroth order solution. The resulting quadrature predicts the splitting of magnetic shells in field models which do not display axial symmetry. This result was first suggested by Stone but is unsatisfactorily described by the McIlwain "L" parameter for models of the earth's field which are severely distorted because of the "Solar Wind Plasma". The improved first integral is further used to derive a new method for calculating Cosmic Ray cutoffs at the surface of the earth. This method replaces the traditional Størmer theory and its refinements put forward by other authors.

INTRODUCTION

A-Charged Particle Motion

It is a most elementary fact that the motion of a charged particle in a uniform magnetostatic field is a helix. The component of the velocity parallel to the field remains constant while the transverse velocity components gyrate harmonically with (in cgs units) a "Larmour Frequency" $\frac{qB}{mc}$, q being the charge, B the magnetic field, m the relativistic mass $m_o(1 - \frac{v^2}{c^2})^{-1/2}$ and c the speed of light. The speed and, therefore, the energy of such a particle is constant. The radius of gyration of "Larmour Radius Vector" is given by $\vec{R}_\ell = \frac{mc}{q}\frac{\vec{V} \times \vec{B}}{B^2}$ and points <u>from</u> the particle to a ficticious point called the "guiding center of motion". In this case, it is simply the center of the circle. As the field model becomes more complicated, such as the addition of transverse $\vec{E}$ field, or the addition of a slight transverse magnetic field gradient or gravitational force, or a curvature of the field lines or slight time variation in the fields, we find that the mathematical[33] solution of the trajectories still suggests the facility of a "guiding center concept" provided that a "drift velocity" is now introduced. This drift is in a direction mutually perpendicular to both the magnetic field direction and

$\overline{E}$ (or ∇B, $\overline{g}$, ... etc.). The necessary and sufficient conditions for the continued validity of such a concept is that the energy of the charged particle be sufficiently low. When we consider a field whose lines of force converge, the larmour radius and hence the transverse velocity components increase (as well as the "Larmour Frequency") at the expense of the velocity parallel to the field since the energy is conserved. When all the parallel velocity is expired, the particle is said to "mirror" or reverse its spiraling motion then moving toward more sparse fields while its guiding center approximately remains on the same line of force as where it mirrored.

B-Adiabatic Invariants

Associated with this so-called "Alfen Motion" are a few approximate first integrals or adiabatic invariants of motion.[33]

The most important, and hardest to destroy, (as the particle's energy is increased) is the "Magnetic Moment", $\mu = \frac{W}{B}$, W being the energy of the transverse motion. This invariance rests on the fact that the gyrating particle produces a dipole (or magnitude current times area of larmour circle) which moves with the guiding center of the trajectory.

The second approximate invariant, which is related to the "action" of the parallel motion along the magnetic field, is given by $J = \oint p \, d\ell$. It states that the guiding center will be restricted to that set of lines of force where this "Integral Invariant" is conserved. The integral is performed along a line of force between "conjugate mirror points". This gives rise to the magnetic shells which will be discussed in the next section.

There exists another approximate "flux" invariant which will not be used in this thesis.

C-Magnetic Shells

Shortly after the first ionospheric rockets were sent aloft, it was discovered that the earth's field caused "trapping" of energetic charged particles. The mechanism of such trapping was clear from the trajectories of charged particles in, what was thought to be, an approximate dipole field.[1] Because of the above mentioned adiabatic invariants, the motion of trapped particles was suggested[34] to be Alfvén motion. This consists of the particle gyrating about the guiding center while it "bounces" between "conjugate mirror points" and slowly drifts carrying the guiding center from line of force to line of force such that its integral invariant, J, is conserved. The surface generated by this set of lines of force is called a "magnetic shell". McIlwain realized that a convenient parameter for labeling these shells is the equatorial radius of a dipole line of force having the same integral invariant, J, and magnetic moment, μ, that the actual field gave. This analysis, however, rested heavily on the fact that the earth's field (from surface measurements[7]) was close to that of a dipole. While this is approximately true near the earth, more recent satellite measurements have undoubtedly disclosed that the earth's magnetic field is severely distorted by its interaction with Solar Corpuscular Radiation in the form of the "Solar Wind Plasma". This necessitates a much more accurate label for such shells in these distorted magnetosphere models.

In this thesis we will give a method for labeling these Adiabatic Invariant Magnetic Shells with a new parameter "α" and show how particles with different energies change the shape of the shells. The analysis <u>does not</u> depend on the approximate dipole nature of the earth's field. The resulting surfaces (equation IV-(83)) and their high and low latitude mirror point limits (equation IV-(84) and IV-(85)) are derived. These equations are then applied to a simple analytical model in Section IV, and to numerical models of the magnetosphere put forward by Hones[15] and Mead.[26] In that section plots are made comparing the shells predicted by J and μ, by equation IV-(83), (84), and (85), and by the poor approximation of the McIlwain parameter.

D-Cosmic Ray Cutoffs

By a similar error, Støermer[51] assumed the earth's field is dipole in nature. He worked out the motion of a charged particle in such a field and discovered a first integral of charged particle motion. From this he was able to predict cosmic ray cutoffs (the rigidity below which particles are not energetic enough to reach the earth from infinity). Further corrections were made to his theory[17, 18, 32] producing better but still not completely accurate predictions.

In Section V of this thesis we work out a further correction to a new cutoff theory first put forward by Ray.[38] The resulting "Vertical Cutoff Rigidty" is given by the equation V-(41). This result is then applied to the Finch and Leaton[7] model of the earth's field in section VII. The numerical results are then compared to those previously predicted by Ray, Shea, and Dropkin at the same observation points.

E-Expansion Parameters

In the following thesis perturbation theory computations will be made on the assumption that (a) The geomagnetic field is approximately independent of β; (b) The Magnetic Moment, μ, is approximately an adiabatic invariant of motion; (c) Trapped Radiation moves in such a way that the additional "Integral Invariant-J", is an approximate adiabatic invariant of motion while the actual trajectory is closely approximated by Alfen motion.

That the assumption (a) is correct, in a rather broad sense, is phenominological in that the theory, worked out in the zeroth order,[38] predicts, reasonably accurately, the measurable results of trapped particle-magnetic shells and cosmic ray cutoffs. The higher order correction performed in this thesis predicts, even more accurately, these same phenomena. The validity of (b) is very accurate in the light of Northrup and Teller's[34] famous paper on adiabatic invariants. It is pointed out in that paper that this is the most difficult invariant to destroy for motion of charged particles in the geomagnetic field. Again, from Northrup and Teller[34] and Alfen[1] the motion of trapped radiation, when the energy of the particles is sufficiently low, is very well approximated by a "spiraling" around a line of force, and a bouncing and drift motion of the particles guiding center. Only when we get to energies of the order of "bev" does the Integral Invariant and guiding center concept no longer have meaning.

Expansion parameters would have the form of: (the β-dependent portions of field variables)/(the total field variable); so that as the field becomes symmetric in β as this ratio approaches zero.

(II) REVIEW OF PREVIOUS WORK

A- Description of the Field

A magnetostatic field may be described in the following manner (see Brand[2] for instance). One may always add the gradient of a scalar function to the vector potential describing the magnetic field without destroying any properties which are measurable. If this scalar function is selected properly, we may construct a new vector potential which satisfies the gauge (see Appendix III)

$$\vec{A} \cdot \vec{B} = \vec{A} \cdot (\nabla \times \vec{A}) = 0 \quad . \tag{1}$$

This is the well known condition that the differential form $\vec{A} \cdot \vec{dl}$ (where $\vec{dl} \equiv \frac{\vec{dB}}{B}$) possesses an integrating factor α. It then follows

$$\vec{A} = \alpha \nabla \beta \tag{2}$$

where α and β are independent scalar functions.

Since $\nabla \times (\nabla \beta) = 0$, the magnetic field may now be expressed as

$$\vec{B} = \nabla \alpha \times \nabla \beta \quad . \tag{3}$$

Furthermore, if we consider only those fields in which spacial currents are everywhere perpendicular to $\vec{B}$, we have

$$\vec{B} \cdot \vec{J} = \vec{B} \cdot (\nabla \times \vec{B}) = 0 \tag{4}$$

and by the same argument leading to (2), we may express $\vec{B}$ as

$$\vec{B} = \bar{\mu} \nabla V \tag{5}$$

In harmonic fields, we may select $\bar{\mu}$ = constant, with no loss of generality, in which case "V" may be interpreted as the magnetic scalar potential.

Furthermore, from (3) we have

$$\vec{B} \cdot \nabla\alpha = \vec{B} \cdot \nabla\beta = 0 \quad . \tag{6}$$

In differential form, this may be written as

$$\nabla\alpha \cdot \vec{dl} = \nabla\beta \cdot \vec{dl} = 0 \tag{7}$$

which implies

$$\frac{d\alpha}{dl} = \frac{d\beta}{dl} = 0 \quad . \tag{8}$$

That is, the functions α and β are constant along lines of force of the magnetic field. Furthermore, (6) implies that ∇V is perpendicular to both $\nabla\alpha$ and $\nabla\beta$, ergo, V is a scalar function independent of both α and β.

The scalar functions α, β, V describing the field are not unique. For instance, one description of the field α, β, V may give a non-vanishing dot product $\nabla\alpha \cdot \nabla\beta \neq 0$, while a point transformation

$$\alpha' = \alpha'(\alpha, \beta, V)$$
$$\beta' = \beta'(\alpha, \beta, V) \qquad (9)$$
$$V' = V'(\alpha, \beta, V)$$

to a new set of functions α', β', V', which describe the same magnetic field, may give a vanishing dot product $\nabla\alpha' \cdot \nabla\beta' = 0$. A proof of this, derived by the author, for all fields which vary in two directions only, can be found in Appendix VI.

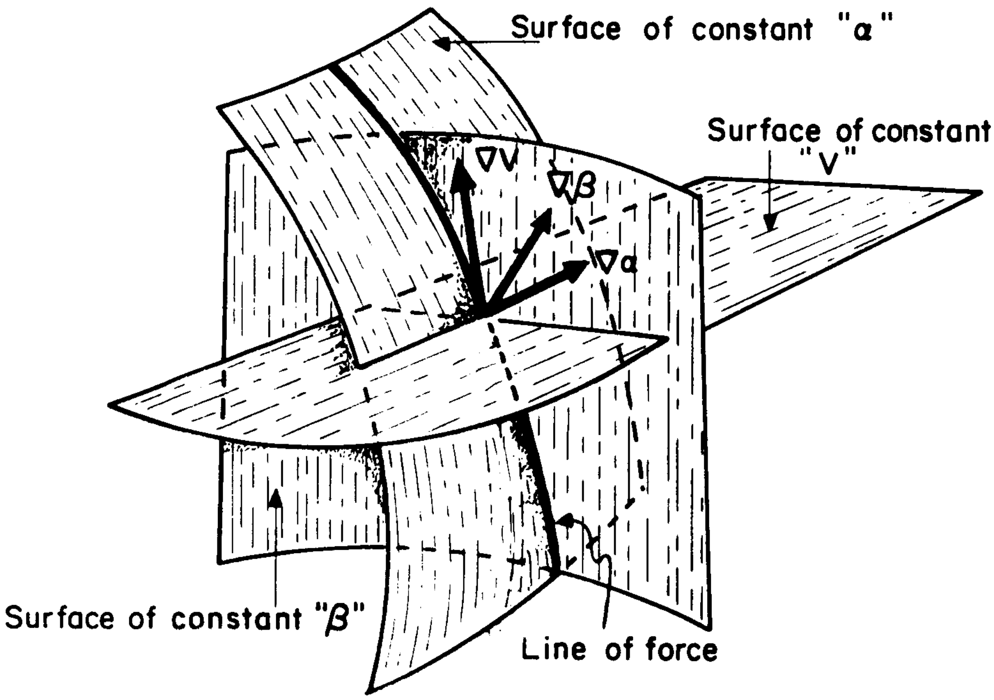
Geometrically α, β, and V surfaces would appear as shown in figure 1. "V" is a measure of "distance" along lines of force. We may select "α" to be a measure of distance from the source of the field (eg. a dipole at the origin) in which case "β" will be related to a measure of azimuth.

B- A Spherical Harmonic Description of the Earth's Field

The magnetic scalar potential describing the geomagnetic field may be expressed a spherical harmonic expansion[6,7,26,28]

$$V(\bar{r},\theta,\phi) = r_e \sum_{n=1}^{\infty} \left[\left(\frac{1}{\bar{r}}\right)^{n+1} T_n(\theta,\phi)+(\bar{r}^n)\bar{T}_n(\theta,\phi)\right] \quad (10)$$

where r_e is the radius of the earth in kilometers, $\bar{r}$ is the radial distance from the center of the earth expressed



Surface of constant "α"
Surface of constant "V"
∇V
∇β
∇α
Surface of constant "β"
Line of force

figure 1

in earth-radii, θ is the colatitude, ϕ is the azimuth and $V(r,\theta,\phi)$ is the magnetic scalar potential expressed in gauss-kilometers. T_n and $\overline{T}_n$ in (10) represent the components of the field due to internal and external sources of the field, respectively. They are given by

$$\left.\begin{aligned} T_n(\theta,\phi) &= \sum_{m=0}^{n} \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] P_{n,m}(\cos\theta) \\ \overline{T}_n(\theta,\phi) &= \sum_{m=0}^{n} \left[\overline{g}_n^m \cos(m\phi) + \overline{h}_n^m \sin(m\phi)\right] P_{n,m}(\cos\theta) \end{aligned}\right. \qquad (11)$$

where $P_{n,m}(\cos\theta)$ are the Schmidt-Normalized associated Legendre Polymonials and are defined in terms of the conventional Legendre Functions as follows

$$\left.\begin{aligned} P_{n,m}(\cos\theta) &= \left[2\,\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\cos\theta) \qquad m > 0 \\ P_{n,0}(\cos\theta) &= P_n(\cos\theta) \end{aligned}\right\} \qquad (12)$$

where $P_n(\cos\theta)$ are the Legendre Functions defined by[31]

$$P_n(\cos\theta) = \frac{1}{2^n n!} \frac{d^n (\cos^2\theta - 1)^n}{d(\cos\theta)^n} \qquad (13)$$

while $P_n^m(\cos\theta)$ are the associated Legendre Functions defined by[31]

$$P_n^m(\cos\theta) = \frac{\sin^m\theta}{2^n n!} \frac{d^{n+m}(\cos^2\theta - 1)^n}{d(\cos\theta)^{n+m}} \quad . \qquad (14)$$

The coefficients g_n^m , h_n^m , $\bar{g}_n^m$, $\bar{h}_n^m$ are known as the "Gauss Coefficients". g_n^m , h_n^m are a measure of the sources which reside inside the earth while $\bar{g}_n^m$, $\bar{h}_n^m$ arise from sources external to the earth's surface. All coefficients are measured in gauss.

The magnetic field is obtained from (10) by taking its positive gradient.

$$\vec{B}(r,\theta,\phi) = \nabla V(r,\theta,\phi) \quad . \tag{15}$$

Hence the components of the magnetic field become

$$\left.\begin{aligned}
B_r(\bar{r},\theta,\phi) &= \sum_{n=1}^{\infty} \left[- \frac{(n+1)}{\bar{r}^{n+2}} T_n(\theta,\phi) + n\, \bar{r}^{n-1}\, \overline{T}_n(\theta,\phi) \right] \\
B_\theta(\bar{r},\theta,\phi) &= \sum_{n=1}^{\infty} \left[\frac{1}{\bar{r}^{n+2}} U_n(\theta,\phi) + \bar{r}^{n-1}\, \overline{U}_n(\theta,\phi) \right] \\
B_\phi(\bar{r},\theta,\phi) &= \sum_{n=1}^{\infty} \left[\frac{1}{\bar{r}^{n+2}} V_n(\theta,\phi) + \bar{r}^{n-1}\, \overline{V}_n(\theta,\phi) \right]
\end{aligned}\right\} \tag{16}$$

where

$$U_n(\theta,\phi) = \sum_{m=0}^{n} \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] \frac{dP_{n,m}(\cos\theta)}{d\theta}$$

$$\bar{U}_n(\theta,\phi) = \sum_{m=0}^{n} \left[\bar{g}_n^m \cos(m\phi) + \bar{h}_n^m \sin(m\phi)\right] \frac{dP_{n,m}(\cos\theta)}{d\theta}$$

$$V_n(\theta,\phi) = \sum_{m=0}^{n} \left[-m\ g_n^m \sin(m\phi) + h_n^m \cos(m\phi)\right] \frac{P_{n,m}(\cos\theta)}{\sin\theta} \tag{17}$$

$$\bar{V}_n(\theta,\phi) = \sum_{m=0}^{n} \left[-m\ \bar{g}_n^m \sin(m\phi) + m\ \bar{h}_n^m \cos(m\phi)\right] \frac{P_{n,m}(\cos\theta)}{\sin\theta} \quad .$$

The $P_{n,m}(\cos\theta)$ and the $\frac{dP_{n,m}(\cos\theta)}{d\theta}$ have been calculated for the first 48 functions and appear in Appendix VII. In addition the coefficients, g_n^m , h_n^m , $\bar{g}_n^m$, $\bar{h}_n^m$ are tabulated in Appendix VIII for the field models of Hones[15], Mead[26], Finch and Leaton[7] (geomagnetic coordinates) and the Finch and Leaton field (geographic coordinates).

Since it is convenient to have the Finch and Leaton field in geomagnetic coordinates also, the following transformation was performed by the author. The rotation relating geographic (unprimed) and geomagnetic (primed) coordinates are given by $\bar{r}' = \bar{r}$, $\theta' = \theta'(\theta,\phi)$, and $\phi' = \phi'(\theta,\phi)$. Placing this into equation (10) we have, by the theorem of "completeness"

$$V' \equiv V'(\bar{r}',\theta',\phi) = V(\bar{r},\theta'(\theta,\phi),\phi'(\theta,\phi)) = V(\bar{r},\theta,\phi) \equiv V \quad . \tag{18}$$

Equating coefficients of equal powers of $\bar{r}$ in (18) we obtain six sets of linear equations relating the g'^m_n, h'^m_n to the g^m_n, h^m_n which may be solved for the new set of coefficients. The calculations were performed on the Control Data 1604 computer giving the resulting Gaussian coefficients appearing in Appendix VIII.

C- The Lagrangian

The Lagrangian of the motion of a charged particle under the influence of a static magnetic field is given by

$$\mathcal{L} = \frac{m}{2} v^2 + \frac{q}{c} \vec{v} \cdot \vec{A}$$

where

$$m = m_o(1 - \frac{v^2}{c^2})^{-1/2} \tag{19}$$

is the relativistic mass, $\vec{v}$ is the velocity and $\vec{A}$ is the vector potential of the field. For the static magnetic field, the validity of (19) for relativistic particles is proven in Appendix I. The only proviso that must be kept in mind when deriving the Lagrange equations from (19) is that the mass m must be considered a constant and not differentiated.

D- The Transformed Lagrangian and Associated Equations of Motion

The independence of α, β, V has been utilized by Ray[38] who has suggested that these coordinates be chosen as a new set of spatial coordinates for describing charged particle motion in a static magnetic field. The utility of such a transformation will become evident shortly.

Following Ray[38], we make a point transformation on the spherical (or cartesian) to the α, β, V system, ie:

$$\begin{aligned} x &= x(\alpha, \beta, V) \\ y &= y(\alpha, \beta, V) \qquad (20) \\ z &= z(\alpha, \beta, V) \quad . \end{aligned}$$

The canonically conjugate velocities are then (see Appendix II) expressible as

$$\left.\begin{aligned} \dot{x} &= \dot{x}(\alpha, \beta, V; \dot{\alpha}, \dot{\beta}, \dot{V}) \\ \dot{y} &= \dot{y}(\alpha, \beta, V; \dot{\alpha}, \dot{\beta}, \dot{V}) \\ \dot{z} &= \dot{z}(\alpha, \beta, V; \dot{\alpha}, \dot{\beta}, \dot{V}) \quad . \end{aligned}\right\} \qquad (21)$$

The Lagrangian (19) may then be transformed to the new canonical coordinates (see Appendix II).

$$\mathcal{L}(\alpha,\beta,V;\dot{\alpha},\dot{\beta},\dot{V}) = \frac{m}{2}\left[\frac{|\nabla\beta|^2}{B^2}\dot{\alpha}^2 + \frac{|\nabla\alpha|^2}{B^2}\dot{\beta}^2 + \frac{\mu^2}{B^2}\dot{V}^2 - \frac{2(\nabla\alpha\cdot\nabla\beta)}{B^2}\dot{\alpha}\dot{\beta}\right] + \frac{q}{c}\alpha\dot{\beta} \qquad (22)$$

where, again "m" is the constant relativistic mass $m_0\gamma$, while $\nabla\alpha$, $\nabla\beta$, and B are all functions of α, β, and V only.

The equations of motion corresponding to (22) are now the set of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \qquad (23)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = 0 \qquad (24)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{V}}\right) - \frac{\partial \mathcal{L}}{\partial V} = 0 \qquad (25)$$

subject to the initial conditions α_o, β_o, V_o, $\dot{\alpha}_o$, $\dot{\beta}_o$, $\dot{V}_o$.

E- First Integrals

One obvious first integral of (23) - (25) is the conservation of energy. This is clear from the fact that the Lagrangian, and thus the Hamiltonian, does not explicitly depend on the time and is therefore conserved, ie:

$$E = \text{constant} \quad . \qquad (26)$$

We may now replace any one of the equations (23) - (25) by the first integral (26). The set of equations determining the motion of the charged particle then becomes

$$E = (p^2c^2 + m_o^2c^4)^{1/2} - m_oc^2 = \text{constant} \qquad (27)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \qquad (28)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = 0 \quad . \qquad (29)$$

Whenever the rigidity is sufficiently low (the rigidity is defined as $R \equiv \frac{mvc}{q}$) equations (27) - (29) may be combined to yield another approximate constant of the motion known as the magnetic moment "μ". Once again we may replace (28) by this first integral yielding the description of the particle motion as

$$E = (p^2c^2 + m_o^2c^4)^{1/2} - m_oc^2 = \text{constant} \tag{30}$$

$$\mu = \frac{p_\perp^2}{2m_o B} \equiv \frac{W_\perp}{B} = \text{constant} \tag{31}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = 0 \; . \tag{32}$$

note: $\begin{bmatrix} \vec{p} = m\vec{v} \\ \vec{p}_\perp = m\vec{v}_\perp \\ W_\perp = \dfrac{p_\perp^2}{2m_o} \end{bmatrix}$

F- A New First Integral

Consider equation (32). First let us calculate $\frac{\partial \mathcal{L}}{\partial \dot{\beta}}$ and $\frac{\partial \mathcal{L}}{\partial \beta}$. We have from (22)

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{m}{2}\left[\frac{\partial}{\partial \beta}\left(\frac{|\nabla\beta|^2}{B^2}\right)\dot{\alpha}^2 + \frac{\partial}{\partial \beta}\left(\frac{\overline{\mu}^2}{B^2}\right)\dot{v}^2 + \frac{\partial}{\partial \beta}\left(\frac{|\nabla\alpha|^2}{B^2}\right)\dot{\beta}^2 - 2\frac{\partial}{\partial \beta}\left(\frac{\nabla\alpha \cdot \nabla\beta}{B^2}\right)\dot{\alpha}\,\dot{\beta}\right] \tag{33}$$

and

$$\frac{\partial \mathcal{L}}{\partial \beta} \equiv p_\beta = m\left[\frac{|\nabla\alpha|^2}{B^2}\ \dot{\beta} - \frac{(\nabla\alpha\cdot\nabla\beta)}{B^2}\dot{\alpha}\right] + \frac{q}{c}\alpha \quad . \qquad (34)$$

Clearly, a sufficient set of conditions for $\frac{\partial \mathcal{L}}{\partial \beta}$ to vanish in (33) is the independence from β of $|\nabla\alpha|$, $|\nabla\beta|$, $\nabla\alpha \cdot \nabla\beta$, $|\nabla\alpha \times \nabla\beta|$. The restrictions this places on the form of the magnetic field is derived by the author in Section VIII.

When, in fact, $\frac{\partial \mathcal{L}}{\partial \beta} = 0$ we have a new first integral of the form derived by Størmer.[51]

Suppose that we consider a magnetic field of such a form that, when we define α as a particular function (constant along lines of force) and construct the corresponding β from the solution of the P.D.E. (3), we find $|\nabla\alpha|$, $|\nabla\beta|$, $(\nabla\alpha \cdot \nabla\beta)$, $|\nabla\alpha \times \nabla\beta|$, are approximately independent of β. It follows, from (32), that p_β is correspondingly "approximately" conserved. It was in this way that Ray[38] showed that, for the earth's field, we have a new approximate Størmer first integral

$$m\left[\frac{|\nabla\alpha|^2}{B^2}\ \dot{\beta} - \frac{(\nabla\alpha\ .\ \nabla\beta)}{B^2}\dot{\alpha}\right] + \frac{q}{c}\alpha = 2\frac{q}{c}\,\overline{\gamma} = \text{constant} \quad . \qquad (35)$$

The special case of the set of all axial symmetric harmonic fields give rise to the Størmer Integral (35), exactly. For such fields we may always choose

$$\alpha = r \sin\theta\, A_\phi(r,\theta) \qquad (36)$$

$$\beta = \phi \qquad (37)$$

Taking the gradient of (36) and (37) we find, in fact, that $|\nabla\alpha|$, $|\nabla\beta|$, $(\nabla\alpha \cdot \nabla\beta)$, B are all β-independent, ergo, we have an exact first integral.

It is intuitively obvious that fields which display a close to axial symmetry, also exhibit an approximate Størmer Integral. The geomagnetic field is such a case. The model determined by surface measurements of the magnetic field lead to a spherical harmonic expansion involving only inverse powers of the radial distance from the center of the earth. The "Gaussian" coefficients of such an expansion have been determined by several investigators (see, for example, Finch and Leaton[7]). The most important property of such an expansion is that the leading term, the dipole term, is axially symmetric and very large compared to subsequent terms. This naturally suggests the field displays an approximate axial symmetry, a sufficient condition for the existence of an approximate Størmer Integral.

Other models of the earth's field[34, 15, 26, 27, 28, 30] determined from satellite measurements and theoretical considerations, give rise to a spherical harmonic expansion which includes positive powers of the radial distance from the earth (due to external sources, eg: ring currents, solar wind interface currents) in addition to the inverse powers of radial distance which arise from internal sources (molten iron motion) within the earth.

These models demonstrate a highly non-axial symmetric field in the shape of, crudely, a paraboloid with its axis of symmetry passing through the sun. This cavity, formed by the interaction of the "solar wind's" charged particle nature and the earth's magnetic field forms a cavity which is generally referred to as the magnetosphere. Even though these models display a vast divergence from axial symmetry, Ray[38] has pointed out that corrections may be added on to the approximate first integral (35) to make it applicable to trapped radiation and cosmic ray motion in the magnetosphere.

G- Trapped Radiation

(i) Adiabatic Invariants of Motion

Trapped radiation of sufficiently low rigidity will display two important adiabatic invariants of motion. The first is known as the "magnetic moment", μ, and the second is known as the "Integral Invariant", J.

The magnetic moment is given by[33]

$$\mu = \frac{p_\perp^2}{2m_o B} \tag{38}$$

where $p_\perp$ is the particle's transverse relativistic momentum, m_o is its rest mass, and B is the magnitude of the magnetic field at its guiding center of motion. The magnetic field is assumed constant over the Larmour radius of the gyrating particle. The trapped particle therefore behaves like a tiny dipole of magnitude "μ" moving in a stronger magnetic field.

In a static magnetic field (p = constant) the following integral is conserved at the guiding center of trapped radiation.[33]

$$I \equiv \frac{J}{P} = \oint \frac{p_{\parallel}}{p}\, dl = 2 \int_{l_1(B_m)}^{l_2(B_m)} (1 - B/B_m)^{1/2} dl \qquad (39)$$

where p is the relativistic momentum, B is the magnetic field, B_m is the magnetic field at the "mirror point", and the integral is to be performed along a line of force, upon which the trapped particle's guiding center resides, from the lower mirror point, $l_1(B_m)$, to mirror upper point, $l_2(B_m)$.

(ii) A New Adiabatic Invariant

All models of the geomagnetic field are assumed independent of time. This implies that the functions α, β and V do not explicitly depend on the time, ie:

$$\frac{\partial\alpha}{\partial t} = \frac{\partial\beta}{\partial t} = \frac{\partial V}{\partial t} = 0 \qquad (40)$$

so that the total time derivatives of α, β and V become

$$\dot{\alpha} = \vec{v} \cdot \nabla\alpha$$

$$\dot{\beta} = \vec{v} \cdot \nabla\beta$$

$$\dot{V} = \vec{v} \cdot \nabla V = \frac{\vec{v} \cdot \vec{B}}{\overline{\mu}} \qquad (41)$$

Placing (41) in (35), following Ray[38], we have

$$\frac{mc}{q}\left[\left(\frac{|\nabla\alpha|^2}{B^2}\right)(\vec{v}\cdot\nabla\beta)-\frac{(\nabla\alpha\cdot\nabla\beta)}{B^2}(\vec{v}\cdot\nabla\alpha)\right]+\alpha=2\overline{\gamma} \qquad (42)$$

Using some vector algebra and (3), we may transform (42) to the following form

$$\alpha+\vec{a}_c\cdot\nabla\alpha=2\overline{\gamma} \qquad (43)$$

where "$\vec{a}_c$" is the Larmous radius of the particle pointing from the particle to the guiding center, (see figure 2) ie:

$$\vec{a}_c\equiv\frac{mc}{q}\frac{(\vec{v}\times\vec{B})}{B^2} \qquad (44)$$

and all lengths in (43) have been expressed in Størmer units which is the unit length $c\equiv(M/R)^{1/2}$; "M" being the dipole moment of the earth (in gauss-kilometers^3) and "R" is the particle's rigidity expressed in gauss-kilometers.

If the rigidity of the particle is sufficiently low (ie: low energy), then "a_c" is also small as a consequence. Under these circumstances, we may interpret (43) as the first two terms of a Taylor expansion of α about the position of the particle and determined at the particle's guiding center. If we call the value of α

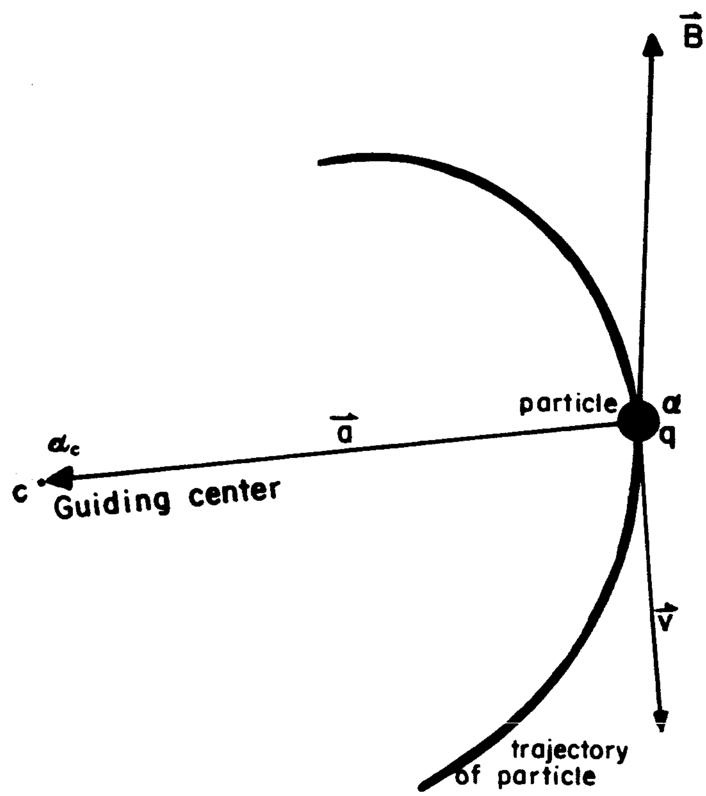

B
particle
α
q
a
α_c
c
Guiding center
v
trajectory
of particle

figure 2

at the particle's guiding center α_c, then (43) implies, for these low-rigidity particles, that

$$\alpha_c \cong 2\bar{\gamma} = \text{constant} \quad . \tag{45}$$

That is, the guiding center of such a particle will approximately travel on a surface of constant "α".

Since α is defined constant along lines of force and it is an approximate constant of motion at the guiding center of particles of low rigidity, it serves the same function of describing the "invariant shells" of trapped particle radiation as does the McIlwain[25] "L" parameter. In fact, in the case of a dipole field it is easily shown[38] that "α" is related to "L" by

$$\alpha = \frac{1}{L} \quad . \tag{46}$$

The advantage of using "α" instead of "L" is that it may be chosen as a function of the magnetic field only, and does not depend in any way on the properties of charged particle motion. For any specific field model we may proceed to map α - constant surfaces never making mention of the particle trajectories. In fact, just such a mapping has been done for the Finch and Leaton model of the magnetic field of the earth by Stern.[49]

(45) was arrived at by assuming $\frac{\partial \mathcal{X}}{\partial \beta} = 0$ which, in fact, is not the case for models of the earth's field. As a manifestation of this, the phenomena of "shell splitting"

(discussed in next section) has been neglected in (45). This necessitates carrying out the new first integral to one higher order of approximation.

(iii) (I, B_m) Invariant Magnetic Shells

Particles with sufficiently low rigidities satisfying the conditions for use of the Adiabatic Invariants, (38), (39), can be described as having their guiding centers lie on shells described by the label (I, B_m). The description in terms of "B_m" arises from the invariance of the magnetic moment during the entire motion of the particle and, in particular, at the mirror point where $p_{\perp}^2 = p^2$.

Physically, the Invariant Shells arise from the fact that the motion of a trapped particle is a spiral about its guiding center (of magnitude equal to the Larmour Radius) while the guiding center "bounces" along a line of force from mirror point to mirror point while slowly drifting perpendicular to both the direction of the magnetic field and its gradient.[1]

If we had assumed that the earth was a pure dipole, two particles having different mirror points but starting on the same line of force would both remain on the same invariant shell[21] (see figure 3). The surfaces in this dipole field are generated by sweeping out the surface generated by swinging a line of force 2π radians in azimuth.

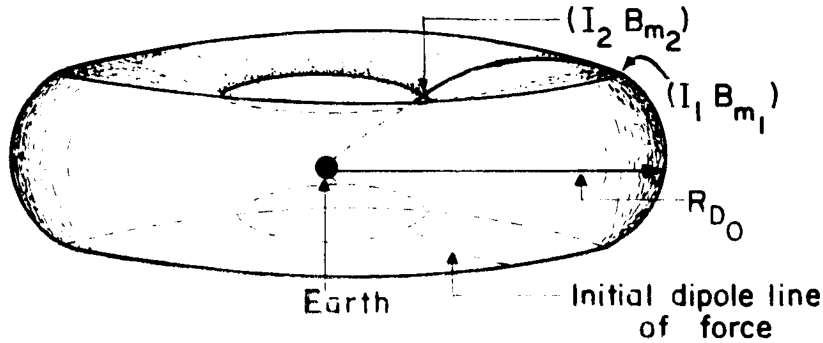


figure 3

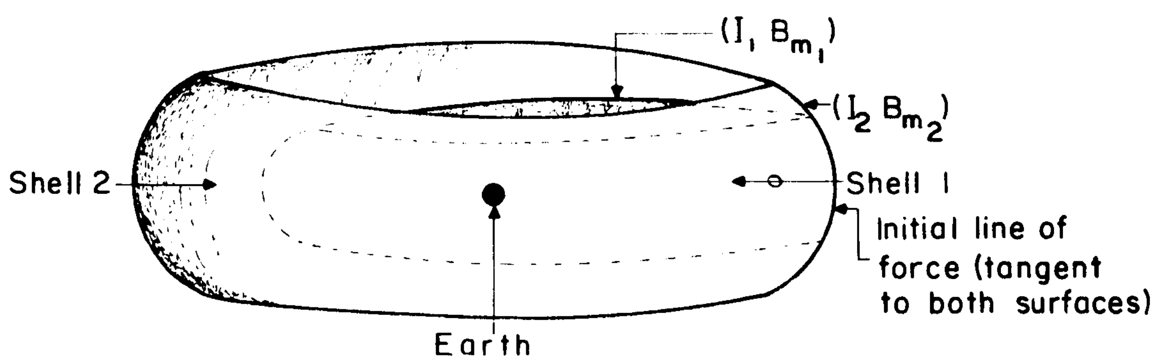


figure 4

Although both guiding centers of the particles (I_1, B_{m_1}) and (I_2, B_{m_2}) reside on the same surface, one sweeps out a larger area of the surface than the other.

The (I, B_m) labelling of shells hence has an immediate disadvantage of in no way indicating that two obviously topologically equal shells indeed lie on the same surface. That is, given (I_1, B_{m_1}) and (I_2, B_{m_2}) we cannot tell, without actually plotting them for the particular field model, that they lie on the same surface.

If a small perturbation is now applied to the dipole field so that it loses its axial symmetry, the previously "degenerate" shells now "split" as was pointed out by Stone.[50] Referring to figure 4, we see that, in this case, the two particles whose guiding centers start on the same line drift in such a way that their invariant surfaces diverge only meeting along the original line on which the two started. The "degeneracy" has now been removed.

(iv) The McIlwain Parameter

Another way of labelling the invariant shells has been proposed by McIlwain.[25] It has the advantage of labelling degenerate shells uniquely.

If we refer to (39) placing the dipole field in the right side of the equation, ie:

$$\vec{B} = -\frac{M}{r^3}\,(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta) \tag{47}$$

and integrate it along the dipole line of force

$$R = R_{D_o} \sin^2\theta \qquad (48\_$$

we may find an expression relating "I" to B_m and R_{D_o}.

Ie: $$I = I(B_m, R_{D_o}) \quad . \qquad (49)$$

This is a rather complicated expression which must be performed on a computer. However, in principle, we may "invert" (49) to obtain, if only as a numerical table, the relationship

$$R_{D_o} = R_{D_o}(B_m, I) \quad . \qquad (50)$$

We then assume that some function "L" exists which satisfies (50) for "I" computed in the actual earth's field (eg: the Finch and Leaton model)

$$L = R_{D_o}(B_m, I) \qquad (51)$$

where R_{D_o} is the same numerical function as in (50) calculated using the dipole field. Hence "L" is defined such that it would be the equitorial radius of a shell with (I, B_m) had the particle been moving in a dipole field. (51) defines "L" to be a constant of the motion, and, further, through McIlwain's use of the computer he showed that, indeed, "L" is very close to constant along lines of force of the actual earth's field.

This pleasing method of labelling has two shortcomings, however. First, the functional relationship (51) is numerical, not analytic, and second, more important, there is no guarantee that "L" so defined would be constant along line of force of a field as distorted as the solar wind cavity.

H- Cosmic Ray Cutoffs

(i) Definition

The "cosmic ray cutoff" at a particular point on the surface of the earth in a particular direction is defined as the lowest rigidity particle that may arrive at that point, in that direction, from a source outside the earth. The direction of arrival is usually measured with respect to the zenith. The minimum rigidity arriving vertically downward along the zenith is known as the "vertical cutoff rigidity."

(ii) Experiments

Mappings of the cutoffs obtained by experimental techniques have been tabulated by several authors (see, for example, References 8,9,10,20,23,24,35,41,44,45,46,52).

The discrepancies between the cosmic ray cutoffs expected in a pure dipole field (Størmer Theory) and the actual field of the earth were evident from neutron latitude surveys (Kodama et.al.,[20] Skorka[45]) and at higher altitudes by Simpson.[44] Further evidence arising from measurements of alpha particles was demonstrated by McDonald,[24] and Waddington.[51]

(iii) Computer Simulation

Several investigators have calculated the vertical cutoff rigidities at points at the surface of the earth by simulating the trajectories of charged particles in a model of the earth's field more detailed than the dipole (eg. Finch and Leaton).[7] The method is simply the ejection of a particle, with a given rigidity, along the zenith of that particular point in the simulated earth's field. . The orbit is then traced. If it is found to return to the surface of the earth, the computation is repeated with a particle of higher rigidity. This is continued until the particle just escapes to infinity. This is the vertical cutoff rigidity for that point on the earth's surface.[17,18,32]

(iv) Størmer Theory of Cosmic Ray Cutoffs

Størmer[51] treated a charged particle in a dipole (47) field. After the equations of motion were set up and properly transformed to a useful set of coordinates, in addition to the conservation of energy first integral, another first integral of motion was found. This was expressed as, in Størmer units,

$$\sin\omega = \frac{2\gamma}{R'} + \frac{R'}{r^3} \tag{52}$$

where $R' = r \sin \theta$, "γ" is a constant and "ω" is the angle between $\vec{v}$ and the "azimuthally sweeping plane's normal" which is pointed in the $\vec{v}_\phi$ direction (see figure 5). Since the energy is conserved for this static magnetic field, so is the speed "v". Therefore, (52) is a constraint on the ϕ component of the velocity in terms of the position of the particle. From this first

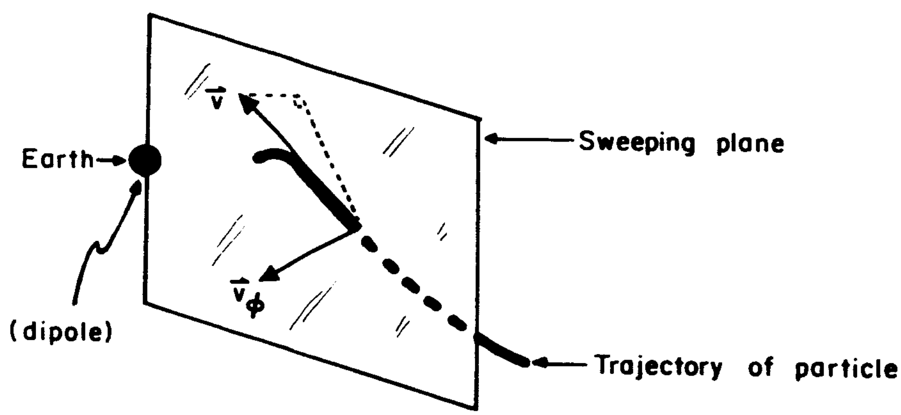

$\vec{v}$
Sweeping plane
Earth
$\vec{v}_\phi$
(dipole)
Trajectory of particle

figure 5

integral Størmer was able to show that for a given γ, certain regions of this azimuthally sweeping plane (rotating with v_ϕ) could not be reached for the values $|\sin\omega| > 1$. This separates the plane into "allowed" and "forbidden" regions. The bounding curve between the regions is determined by setting $|\sin\omega| = 1$ in (52), while the allowed region is found when $|\sin\omega| < 1$. Three typical "Størmer" plots are shown in figure 6. The value of γ changes the characteristic shape of the plots. When γ is smaller than 1, the "jaws" of the figure are open leaving an allowed region which stretches from ∞ , up the "horns" of the plot to the origin. For γ just equal to 1 the jaws just close, allowing no trajectory to pass from the "outer" allowed region to the "inner" allowed region and, hence, not to the origin. When γ is greater than 1 the two allowed regions are separated by a forbidden region. The case $\gamma = 1$ is called the "critical" value of γ because this is the condition under which cosmic rays are "cut-off", ie: cannot hit the surface of the earth. Since the plots in figure 6 are in Størmer units, the radius of the earth is a circle about the origin whose radius depends on the particle's rigidity. A superimposed circle is shown in figure 6b.

Placing the value of $\gamma = 1$ in (52), restoring cgs units from the Størmer units, we obtain an expression

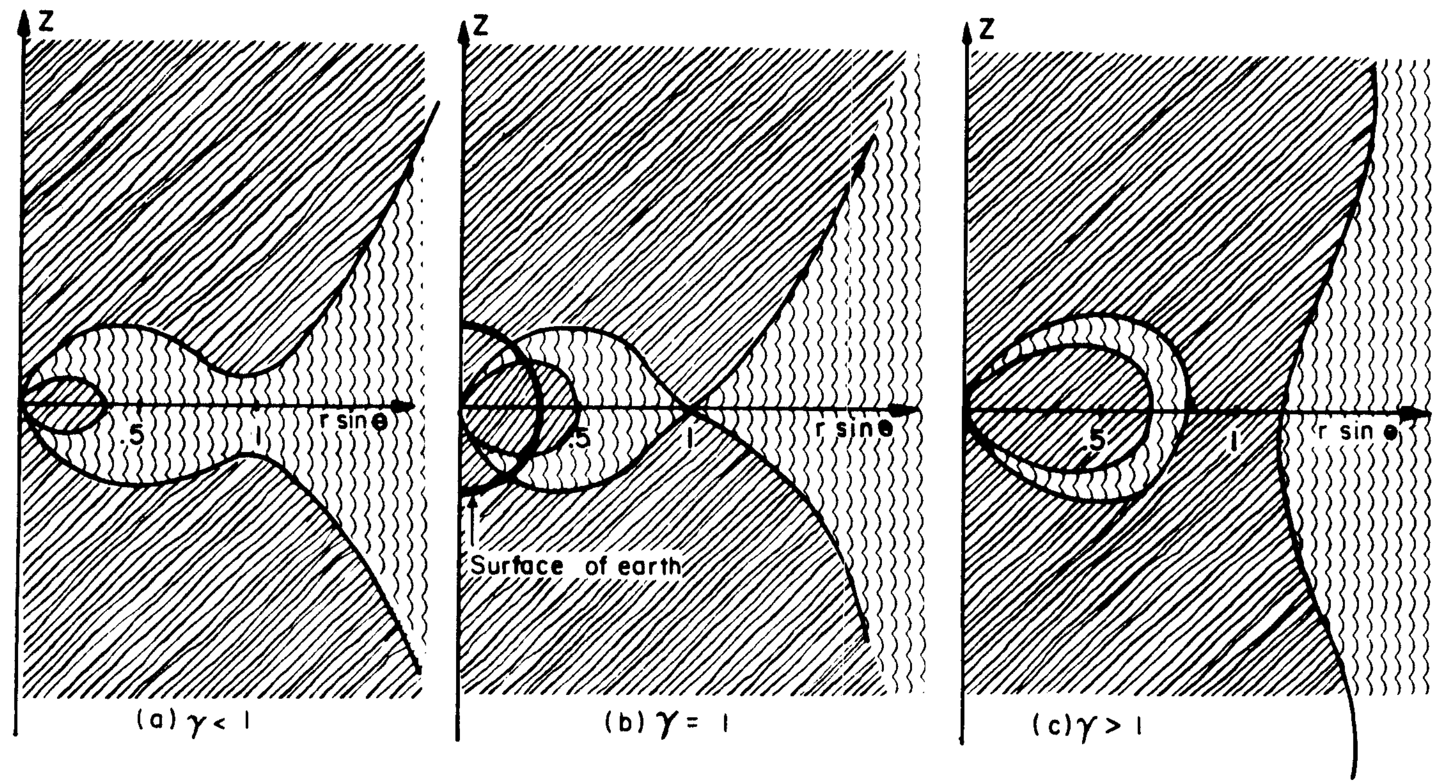


figure 6

Forbidden ////
Allowed ⟩⟩⟩⟩⟩

for the cut-off rigidity at the surface of the earth ($r = r_e$) as a function of the colatitude (Θ) and the impact direction (with respect to the zenith) of the particle, (ω). This is

$$R_{cut} = \frac{M}{r_e^2} \left[\frac{1 - (1 - \sin\omega \sin^3\Theta)^{\frac{1}{2}}}{\sin\Theta \sin\omega} \right]^2 \qquad (53)$$

where $\frac{M}{r_e^2} = 1.971 \times 10^3$ gauss-kilometers.

(v) Refinements on Størmer Theory

Since the dipole approximation of the earth's field is poor the vertical cutoff rigidities predicted by Størmer theory (eq.(53) with $\omega = 0$) are not completely accurate. Several methods have therefore been devised to predict more accurate results.

One technique put forth by Ray and Sauer[42,43] rested on the fact that the sixth order harmonic expansion model of the geomagnetic field had the property that it became, approximately, that of a dipole beyond 4 earth radii. A particle emitted along the zenith, with a given rigidity, was then traced using the properties of Alfvén motion in this "near zone" until it reached 4 earth radii. At this point its velocity and angle of incidence was computed. This now served as the initial conditions of a charged particle in a dipole field, thus allowing the use of Størmer theory to predict the cutoff.

Another method suggested by Quenby and Webber[36] with further refinements given by Quenby and Wenk[37] was to treat high and low latitude zones of cosmic ray impact separately. In the high regions, the actual field line is traced to the equator (now in the region where the field is approximately that of a dipole). At this point the dipole line is found thus giving a latitude where it intersects the earth, which may now be used to give an equivalent cutoff rigidity when this is placed back in (53). On the other hand, at low latitudes, a more complicated analysis is given[36] with the result of an expression for the cutoff that depends on the actual field components at the point, the dip angle, and the latitude. The region of validity is a band of 20° about the equator. The region between the high and low approximations are extrapolated from the previous results. A further investigation along this line was carried out by Makino.[22]

(vi) A New Approach to Cutoff Theory

Equation (35) may be written in still another way[38]

$$\frac{|\nabla\alpha|}{B}\cos\psi + \alpha = 2\bar{\gamma} \tag{54}$$

where everything is expressed in Størmer units, and "ψ" is the angle between $\vec{v}$ and $\vec{B} \times \nabla\alpha$. This equation may be restored to cgs units whence we may then solve for the cutoff rigidity (with $\overline{\gamma}_c = 1$), yielding

$$R_{cut.} = \frac{1}{M\left[\frac{\overline{\gamma}_c}{\alpha} \pm \sqrt{\frac{\overline{\gamma}_c^2}{\alpha^2} - \frac{|\nabla\alpha|}{MB}\,\frac{\cos\psi}{\alpha}}\right]^2} \qquad (55)$$

Whenever the field is approximately independent of "β" (55) is a good first approximation to the cutoff rigidity. However, because of the slight perturbing effects of a field with "β" dependence, the "$\overline{\gamma}_c$" given in (55), instead of being 1, is somewhat greater or smaller than 1 depending on the second order correction that will be worked out in section IV. The field variables in (55) are all evaluated at the position of the particle as it impacts the earth's surface. The small correction in "$\gamma_{cut.}$" will just be sufficient to close the "jaws" of the equivalent Størmer plot (see figure 6b).

(III) THE SECOND APPROXIMATION TO THE LAGRANGE'S EQUATION

A- The New Form of the Lagrange's Equation

Consider a particle of charge "q" and rest mass "m_o" moving in a static magnetic field which is almost "β" independent. We showed in section (II) that its entire motion is described by equations II -(31, 32, 33) which are repeated below

$$E = (p^2c^2 + m_o^2c^4)^{\frac{1}{2}} - m_oc^2 = \text{constant} \tag{1}$$

$$\mu = \frac{p_\perp^2}{2m_oB} = \frac{W_\perp}{B} = \text{constant} \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = 0 \tag{3}$$

where "$\vec{p}$" is the relativistic momentum of the particle, "W" is the total energy, "$\vec{p}_\perp$" is momentum component perpendicular to the magnetic field at that point, and "B" is the magnetic field at that point. We will now proceed to correct the first integral II-(35) by the inclusion of the $\frac{\partial \mathcal{L}}{\partial \beta}$ term in (3) where, previously, it was assumed a completely β-independent field. Since we are considering current-free regions of space $\bar{\mu} = 1$ in II-(22). Using this fact, we may now take the partial derivative of II-(22) with respect to β, yielding

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{m_o \gamma}{2} \left[\frac{\partial}{\partial \beta}\left(\frac{|\nabla\beta|^2}{B^2}\right) \dot{\alpha}^2 + \frac{\partial}{\partial \beta}\left(\frac{|\nabla\alpha|^2}{B^2}\right) \dot{\beta}^2 + \frac{\partial}{\partial \beta}\left(\frac{1}{B^2}\right) \dot{V}^2 - 2 \frac{\partial}{\partial \beta} \left(\frac{\nabla\alpha \cdot \nabla\beta}{B^2}\right) \dot{\alpha}\dot{\beta} \right] . \qquad (4)$$

Our method of calculating corrections to the first integral will rest on the application of perturbation theory. That is, assuming the $\frac{\partial \mathcal{L}}{\partial \beta}$ term is small, we will inject the "first order motion" into this correction term thus yielding higher order corrections to the resulting equation. This is no more than an iteration process.

Let us begin by placing II-(41) in (4). We then have, placing the result in (3),

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \beta}\right) - \frac{m}{2} \left[\frac{\partial}{\partial \beta}\left(\frac{|\nabla\beta|^2}{B^2}\right)(\vec{v} \cdot \nabla\alpha)^2 + \frac{\partial}{\partial \beta}\left(\frac{|\nabla\alpha|^2}{B^2}\right)(\vec{v} \cdot \nabla\beta)^2 + \frac{\partial}{\partial \beta}\left(\frac{1}{B^2}\right)(\vec{v} \cdot \vec{B})^2 - 2\frac{\partial}{\partial \beta}\left(\frac{\nabla\alpha \cdot \nabla\beta}{B^2}\right)(\vec{v} \cdot \nabla\alpha)(\vec{v} \cdot \nabla\beta) \right] . \qquad (5)$$

B- The Perturbation Theory Approach

(i) The First Order Alfén Motion

The first order motion that we will insert in the right hand side of equation (5) is the so-called Alfén motion of a charged particle in a magnetic field. We are tacitly assuming that the energy of particles under consideration is sufficiently small that the motion of these charged particles is a spiralling motion about a line of force in addition to a "bouncing" motion from mirror point to mirror point and a drift motion due to gradients in the magnetic field (Alfén[1], Chandrasechar[5]). A sufficient condition that must be satisfied for the Alfén regime to be applicable is that

$$\left| \frac{\vec{a}_c \cdot \nabla B}{B} \right| \tag{6}$$

be sufficiently small compared to the nominal value of "B" over a Larmour circle. $\vec{a}_c$ is the Larmour radius defined by II-(44). (Note: "m" in this equation is the relativistic mass, $m_o\gamma$).

Since we are treating particles that satisfy the Alfén regime, we may, therefore, express the instantaneous velocity appearing in the perturbation (the second term of the right hand side) term of equation (5) as made up of a parallel and perpendicular component to the instantaneous magnetic field

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp} \tag{7}$$

where the perpendicular component alone may be broken into two components, the part due to spiralling and another part due to the drifting of the particle

$$\vec{v}_{\perp} = \vec{v}_D + \vec{v}_{rot} \quad . \tag{8}$$

During Alfén motion, the guiding center does not move significantly during the time associated with a Larmour cycle so that the rotational speed in (8) is much larger than that due to the combination of parallel and drift motion. A typical Larmour cycle of this particle is then pictured as the path shown in figure 7. The total guiding center motion and rotational velocity is then given by, respectively, (as shown in the figure)

$$\dot{\vec{R}} = \vec{v}_{\parallel} + \vec{v}_D \tag{9}$$

$$\dot{\vec{a}} = \vec{v}_{rot} \quad . \tag{10}$$

The time interval considered in the figure is

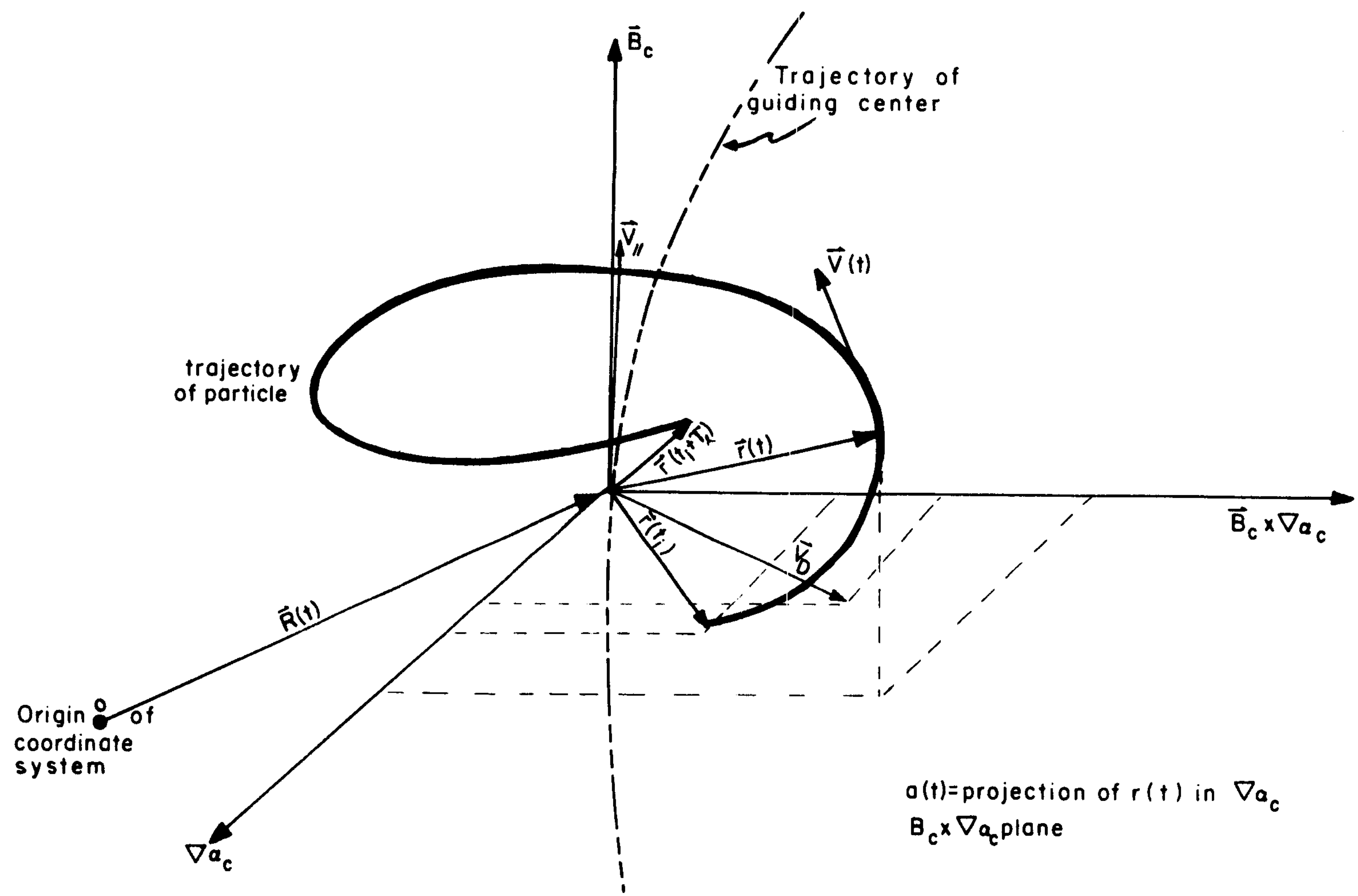
$$t_i \leq t \leq t_i + T_l \tag{11}$$

where the Larmour period is

$$T_{\ell} = \frac{2\pi mc}{qB} \quad . \tag{12}$$

(ii) Averaging the Lagrange's Equation Over a Larmour Period

We will now proceed to average equation (5) over the characteristic Larmour period which is, in fact, appropriate for invariant motion. Our procedure will be to first expand all field variables about the guiding center and then time average the result over a Larmour Period.



$\vec{B}_c$
Trajectory of guiding center
$\vec{V}_{\parallel}$
$\vec{V}(t)$
trajectory of particle
$\vec{r}(t_i+T_L)$
$\vec{r}(t)$
$\vec{B}_c \times \nabla a_c$
$\vec{r}(t_i)$
$\vec{V}_D$
$\vec{R}(t)$
Origin O of coordinate system
∇a_c
a(t)=projection of r(t) in ∇a_c $B_c \times \nabla a_c$ plane

figure 7

Referring to figure 7 let us now Taylor expand α and β about the guiding center, "c",

$$\alpha(x(t),y(t),z(t)) = \alpha_c + \overrightarrow{r(t)} \cdot \nabla\alpha_c + \sigma(\lambda) \quad (13)$$

$$\beta(x(t),y(t),z(t)) = \beta_c + \overrightarrow{r(t)} \cdot \nabla\beta_c + \sigma(\lambda) \quad (14)$$

where "λ" is a "smallness" parameter and $\vec{r}(t)$ is the instantaneous value of the radius vector. The gradients of (13,), (14) are found, simply by taking derivatives with respect to x,y,z of (13),(14) yielding

$$\nabla\alpha(x(t),y(t),z(t)) = \nabla\alpha_c + \sigma(\lambda) \quad (15)$$

$$\nabla B(x(t),y(t),z(t)) = \nabla\beta_c + \sigma(\lambda) \quad (16)$$

and $\vec{B}$ is obtained by placing (15),(16) in II-(3)

$$\vec{B} = \nabla\alpha \times \nabla\beta = \nabla\alpha_c \times \nabla\beta_c + \sigma(\lambda) = \vec{B}_c + \sigma(\lambda) \ . \quad (17)$$

Since the guiding center does not significantly move during the period of one Larmour cycle, the radius vector measured from the origin of the local coordinate system $\vec{r}(t)$ can be replaced by its projection on the $\nabla\alpha_c$, $\vec{B}_c \times \nabla\alpha_c$ plane, ie:

$$\vec{r}(t) \approx \vec{a}(t) \quad . \quad (18)$$

Placing (18) in (13) and (14) and dropping terms of order "λ" in (13) through (17) we may now proceed to calculate each of the terms in the second term of the right hand side of (5). We have

$$\frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta|^2}{B^2}\right) = \frac{\vec{B} \times \nabla\alpha}{B^2} \cdot \nabla\left(\frac{|\nabla\beta|^2}{B^2}\right) = \frac{(\vec{B}_c + \sigma(\lambda)) \times (\nabla\alpha_c + \sigma(\lambda))}{(\vec{B}_c + \sigma(\lambda))^2} \cdot \nabla\left[\frac{|\nabla\beta_c + \sigma(\lambda)|^2}{|\vec{B}_c + \sigma(\lambda)|^2}\right]$$

$$= \frac{\vec{B}_c \times \nabla\alpha_c}{B_c^2} \cdot \nabla\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right) + \sigma(\lambda)$$

So

$$\frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta|^2}{B^2}\right) = \frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right) + \sigma(\lambda) \quad . \tag{20}$$

In a similar manner we may calculate

$$\frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha|^2}{B^2}\right) = \frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right) + \sigma(\lambda) \tag{21}$$

$$\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha \cdot \nabla\beta}{B^2}\right) = \frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c \cdot \nabla\beta_c}{B_c^2}\right) + \sigma(\lambda) \tag{22}$$

$$\frac{\partial}{\partial\beta}\left(\frac{1}{B^2}\right) = \frac{\partial}{\partial\beta}\left(\frac{1}{B_c^2}\right) + \sigma(\lambda) \quad . \tag{23}$$

Let us now define the symbol $\langle\ \rangle_{t_\ell}$ for the time average over a Larmour cycle as

$$\langle Q \rangle_{T_\ell} \equiv \frac{1}{T_\ell} \int_{t_i}^{t_i + T_\ell} Q\,dt \quad . \tag{24}$$

Placing (13) through (17) and (20) through (23) in (5) in addition to replacing the velocities by (7),(8) we have, in the notation of (24), the Larmour cycle time average of (5)

$$\frac{2}{m}\int_{\frac{\partial\mathcal{L}}{\partial\dot\beta}(t_i)}^{\frac{\partial\mathcal{L}}{\partial\dot\beta}(t_i+T_\ell)} d\left(\frac{\partial\mathcal{L}}{\partial\dot\beta}\right) = \frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)\left\langle\left[\vec{v}_{rot}\cdot\nabla\alpha_c + \vec{v}_{\parallel}\cdot\nabla\alpha_c + \vec{v}_D\cdot\nabla\alpha_c\right]^2\right\rangle_{T_\ell}$$

$$+\frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right)\left\langle\left[\vec{v}_{rot}\cdot\nabla\beta_c + \vec{v}_{\parallel}\cdot\nabla\beta_c + \vec{v}_D\cdot\nabla\beta_c\right]^2\right\rangle_{T_\ell}$$

$$-2\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c\cdot\nabla\beta_c}{B_c^2}\right)\left\langle\left[\vec{v}_{rot}\cdot\nabla\alpha_c + \vec{v}_{\parallel}\cdot\nabla\alpha_c + \vec{v}_D\cdot\nabla\alpha_c\right]\left[\vec{v}_{rot}\cdot\nabla\beta_c + \vec{v}_{\parallel}\cdot\nabla\beta_c + \vec{v}_D\cdot\nabla\beta_c\right]\right\rangle_{T_\ell}$$

$$+\frac{\partial}{\partial\beta}\left(\frac{1}{B_c^2}\right)\left\langle\left[\vec{v}_{rot}\cdot\vec{B}_c + \vec{v}_{\parallel}\cdot\vec{B}_c + \vec{v}_D\cdot\vec{B}_c\right]^2\right\rangle_{T_\ell} \qquad (25)$$

In (25) we have removed all functions which are approximately independent of time from our time averages.

Since "B_c" is perpendicular to both $\nabla\alpha_c$ and $\nabla\beta_c$, we have

$$\vec{v}_{\parallel}\cdot\nabla\alpha_c = \vec{v}_{\parallel}\cdot\nabla\beta_c = 0 \quad . \qquad (26)$$

Furthermore, we have in addition

$$\vec{v}_{\perp}\cdot\vec{B}_c = (\vec{v}_{rot} + \vec{v}_D)\cdot\vec{B}_c = 0 \quad . \qquad (27)$$

Hence, placing (26) and (27) in (25) it simplifies to

$$\left\langle \frac{2}{m}\frac{d}{dt}\left(\frac{\partial \lambda}{\partial \dot{\beta}}\right)\right\rangle_{T_\ell} = \frac{\partial}{\partial \beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)\left[\left\langle\left[\vec{v}_{rot}\cdot\nabla\alpha_c\right]^2\right\rangle_{T_\ell} + \left[\vec{v}_D\cdot\nabla\alpha_c\right]^2 + 2\left\langle\left[\vec{v}_{rot}\cdot\nabla\alpha_c\right]\left[\vec{v}_D\cdot\nabla\alpha_c\right]\right\rangle_{T_\ell}\right]$$

$$+\frac{\partial}{\partial \beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right)\left[\left\langle\left[\vec{v}_{rot}\cdot\nabla\beta_c\right]^2\right\rangle_{T_\ell} + \left[\vec{v}_D\cdot\nabla\beta_c\right]^2 + 2\left\langle\left[\vec{v}_{rot}\cdot\nabla\beta_c\right]\left[\vec{v}_D\cdot\nabla\beta_c\right]\right\rangle_{T_\ell}\right]$$

$$-2\frac{\partial}{\partial \beta}\left(\frac{\nabla\alpha_c\cdot\nabla\beta_c}{B_c^2}\right)\left[\left\langle(\vec{v}_{rot}\cdot\nabla\alpha_c)(\vec{v}_{rot}\cdot\nabla\beta_c)\right\rangle_{T_\ell} + (\vec{v}_D\cdot\nabla\alpha_c)(\vec{v}_D\cdot\nabla\beta_c) + \left\langle(\vec{v}_D\cdot\nabla\alpha_c)(\vec{v}_{rot}\cdot\nabla\beta_c)\right\rangle_{T_\ell} + \left\langle(\vec{v}_D\cdot\nabla\beta_c)(\vec{v}_{rot}\cdot\nabla\alpha_c)\right\rangle_{T_\ell}\right]$$

$$+\frac{\partial}{\partial \beta}\left(\frac{1}{B_c^2}\right) v_\parallel^2 B_c^2 \tag{28}$$

where we have integrated over all functions that are approximately time independent over a Larmour cycle. The remaining rotational velocity which is time dependent ($\vec{v}_{rot}$) may now be expressed, over one Larmour period as

$$\vec{v}_{rot} = v_{rot}\left(\sin(\omega_c(t-t_i))\,\hat{e}_{\nabla\alpha_c} + \cos(\omega_c(t-t_i))\,\hat{e}_{\vec{B}\times\nabla\alpha_c}\right) \tag{29}$$

where ω_c is the Larmour frequency and is given by

$$\omega_c = \frac{qB_c}{mc} \quad . \tag{30}$$

If we now use the identities

$$\left\langle\cos(\omega_c(t-t_i))\right\rangle_{T_\ell} = \left\langle\sin(\omega_c(t-t_i))\right\rangle_{T_\ell} =$$

$$\left\langle\sin(\omega_c(t-t_i))\cos(\omega_c(t-t_i))\right\rangle_{T_\ell} = 0$$

and

$$\left\langle \sin^2(\omega_c(t-t_i)) \right\rangle_{T_\ell} = \left\langle \cos^2(\omega_c(t-t_i)) \right\rangle_{T_\ell} = 1/2 \quad (31)$$

we may reduce the time averages in (28) to

$$\frac{2}{m}\left\langle \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \beta}\right)\right\rangle_{T_\ell} = \left[\frac{v_{rot}^2}{2}|\nabla\alpha_c|^2 + (\vec{v}_D \cdot \nabla\alpha_c)^2\right]\frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)$$

$$+ \left[\frac{v_{rot}^2}{2}|\nabla\beta_c|^2 + (\vec{v}_D \cdot \nabla\beta_c)^2\right]\frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right)$$

$$- \left[2\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c \cdot \nabla\beta_c}{B_c^2}\right)\right]\left[\frac{v_{rot}^2}{2}(\nabla\alpha_c \cdot \nabla\beta_c) + (\vec{v}_D \cdot \nabla\alpha_c)(\vec{v}_D \cdot \nabla\beta_c)\right]$$

$$+ \frac{\partial}{\partial\beta}\left(\frac{1}{B_c^2}\right) v_\parallel^2 B_c^2 \quad (32)$$

We now may further manipulate the terms involving v_{rot}^2 as follows:

$$v_{rot}^2\left[|\nabla\alpha_c|^2 \frac{\partial\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)}{\partial\beta} + |\nabla\beta_c|^2 \frac{\partial\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right)}{\partial\beta} - 2(\nabla\alpha_c \cdot \nabla\beta_c) \times \right.$$

$$\left.\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c \cdot \nabla\beta_c}{B_c^2}\right)\right] = v_{rot}^2\left[\frac{1}{B_c^2}\frac{\partial}{\partial\beta}\left[|\nabla\alpha_c|^2|\nabla\beta_c|^2 - \right.\right.$$

$$\left.\left.(\nabla\alpha_c \cdot \nabla\beta_c)^2\right] + 2\left[|\nabla\beta_c|^2|\nabla\alpha_c|^2 - (\nabla\alpha_c \cdot \nabla\beta_c)^2\right]\frac{\partial}{\partial\beta}\left(\frac{1}{B_c^2}\right)\right]$$

$$= \left(\frac{2}{B_c}\ \frac{\partial B_c}{\partial\beta} - \frac{4}{B_c}\ \frac{\partial B_c}{\partial\beta}\right) v_{rot}^2 = -v_{rot}^2 \left[\frac{2}{B_c}\ \frac{\partial B_c}{\partial\beta}\right] \quad . \tag{33}$$

Furthermore, from (7) and (8) we also have

$$v_{\parallel}^2 = v^2 - v_{\perp}^2 \tag{34}$$

$$v_{\perp}^2 = v_{rot}^2 + v_D^2 \tag{35}$$

which may be placed in (32) in addition to using (33). We then have

$$\frac{c}{q}\left\langle \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\beta}\right)\right\rangle T_{\ell} = \overbrace{\frac{mc}{2q}\left[\frac{v_{\perp}^2}{B_c} - \frac{2v^2}{B_c}\right]\frac{\partial B_c}{\partial\beta}}^{\mathrm{I}} + \frac{mc}{2q}\ \frac{v_D^2}{B_c}\ \frac{\partial B_c}{\partial\beta}$$

$$+ \frac{mc}{2q}(\vec{v}_D \cdot \nabla\beta_c)^2 \frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right) + \frac{mc}{2q}(\vec{v}_D \cdot \nabla\alpha_c)^2 \frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)$$

$$- \frac{mc}{q}(\vec{v}_D \cdot \nabla\beta_c)\left[\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c \cdot \nabla\beta_c}{B_c^2}\right)\right](\vec{v}_D \cdot \nabla\alpha_c) \ . \tag{36}$$

The first term on the right hand side of (36) may be further changed into a simpler form by introducing the magnetic moment equation (2) in the form

$$\mu = \frac{m^2 v_{\perp}^2}{2m_o B_c} = \frac{m^2 v^2}{2m_o B_m} \tag{37}$$

into it. The first term then becomes

$$\text{I} = \frac{\mu c B_m}{q\gamma}\left[\frac{1}{B_m} - \frac{2}{B_c}\right]\frac{\partial B_c}{\partial \beta} = -\frac{\mu c B_m}{\gamma q}\left[\frac{1}{B_m} - \frac{2}{B_c}\right]\frac{(\vec{B}_c \times \nabla\alpha_c)\cdot\nabla B_c}{B_c^2}$$

$$= \vec{v}_D \cdot \nabla\alpha_c \qquad (38)$$

where we have recognized the definition of the drift velocity[19]

$$\vec{v}_D = \frac{\mu c B_m}{\gamma q}\left[\frac{2}{B_c} - \frac{1}{B_m}\right]\frac{(\vec{B}_c \times \nabla B_c)}{B_c^2} \qquad (39)$$

"μ" being the magnetic moment, and "B_m" the mirror point magnetic field of the drifting particle. The important thing to be noticed here is that the drift velocity $\vec{v}_D$ is recognized from the equations of motion as a natural consequence of the Larmour cycle averaging. Placing (38) back in (36) we now have

$$\left\langle \frac{c}{q}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\beta}\right)\right\rangle_{T_\ell} = \overbrace{\vec{v}_D\cdot\nabla\alpha_c}^{(\mathrm{I})} + \frac{v_D^2}{2\omega_c}\Bigg[\overbrace{\frac{\partial B_c}{\partial\beta} + B_c(\hat{e}_D\cdot\hat{e}_{\nabla\beta})^2|\nabla\beta_c|^2\frac{\partial}{\partial\beta}\left(\frac{|\nabla\alpha_c|^2}{B_c^2}\right)}^{(\mathrm{II})}$$

$$\overbrace{+ B_c(\hat{e}_D\cdot\hat{e}_{\nabla\alpha})^2|\nabla\alpha_c|^2\frac{\partial}{\partial\beta}\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right) - 2B_c(\hat{e}_D\cdot\hat{e}_{\nabla\alpha})(\hat{e}_D\cdot\hat{e}_{\nabla\beta})|\nabla\alpha_c||\nabla\beta_c|\frac{\partial}{\partial\beta}\left(\frac{\nabla\alpha_c\cdot\nabla\beta_c}{B_c^2}\right)}^{(\mathrm{III})}\Bigg]$$

$$(40)$$

where all small $\hat{e}$'s denote unit vectors in their respective directions.

(iii) Simplifying Approximations

We will now proceed to show that we may neglect terms (III) and part of (II) in comparison to (I), on the right hand side of (58). First consider the ratio of term (III) divided by term (I). We have

$$\frac{(III)}{(I)} = \frac{v_D}{2\omega_c}\left[\overbrace{B_c(\hat{e}_D\cdot\hat{e}_{\nabla\alpha})\,|\nabla\alpha_c|\,\frac{\partial\left(\frac{|\nabla\beta_c|^2}{B_c^2}\right)}{\partial\beta}}^{(a)}\right.$$

$$\left.\overbrace{-2B_c(\hat{e}_D\cdot\hat{e}_{\nabla\beta})\,|\nabla\beta_c|\,\frac{\partial\left(\frac{\nabla\alpha_c\cdot\nabla\beta_c}{B_c^2}\right)}{\partial\beta}}^{(b)}\right] \tag{41}$$

but we may majorize both terms on the right hand side of (41) as follows. Term (a) is always smaller than

$$(a) \lesssim \left|\frac{B_c}{|\nabla\alpha_c|}\,|\nabla\alpha_c|^2\,\frac{\partial\left(\frac{1}{|\nabla\alpha_c|^2}\right)}{\partial\beta}\right| = 2\,\frac{\left|\nabla|\nabla\alpha_c|\right|}{|\nabla\alpha_c|} \tag{42}$$

while term (b) is always smaller than

$$(b) \lesssim \left|\frac{2B_c|\nabla\beta_c|}{B_c^2}\,\frac{\partial B_c}{\partial B}\right| = \frac{2\,|\nabla B_c|}{B_c}\quad. \tag{43}$$

Now, the ratio $\frac{v_D}{\omega_c}$ is no more than the distance the guiding center moves during the Larmour period. Thus, using (42) and (43) in (41) we see that the right hand side of (41) is smaller than the sum of the change in

"B" at guiding center plus the change in $|\nabla\alpha|$ at the guiding center during the period of a Larmour cycle. This indeed is much smaller than unity. Hence it is clear that we may neglect term (III) in comparison to (I) in (40). That the same is true of part of term (II) is seen from what follows. Neglecting (III) in (40), term (II) may be manipulated so that the resulting equations becomes

$$\frac{c}{q}\left\langle\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\beta}\right)\right\rangle_{T_\ell} = \overbrace{\vec{v}_D \cdot \nabla\alpha_c}^{(\mathrm{I})} + \frac{v_D^2}{2\omega_c}\left[\overbrace{\left(1 - \frac{2\,|\nabla\beta_c|^2\,|\nabla\alpha_c|^2}{B_c^2}\left(\hat{e}_D\cdot\hat{e}_{\nabla\beta}\right)^2\right)\frac{\partial B_c}{\partial\beta}}^{(\mathrm{II}a)} + \overbrace{\frac{2\,|\nabla\alpha_c|\,|\nabla\beta_c|^2}{B_c}\left(\hat{e}_D\cdot\hat{e}_{\nabla\beta}\right)^2\frac{\partial|\nabla\alpha_c|}{\partial\beta}}^{(\mathrm{II}b)}\right]. \tag{44}$$

Again, we may show that in the second term on the right of (44), (IIa), is negligible compared to the first term by noting

$$\frac{v_D^2}{2\omega_c}\,\frac{\partial B_c}{\partial\beta} = \frac{v_D}{2\omega_c}\,v_D\,\frac{(\vec{B}_c \times \nabla\alpha_c)\cdot\nabla B_c}{B_c^2}$$

and, because $\vec{v}_D$ is in the $\vec{B}_c \times \nabla B_c$ direction, this becomes

$$= \frac{v_D\,(\vec{v}_D \cdot \nabla\alpha)}{2\omega_c}\left(\frac{|\nabla B_c|}{B_c}\right) \tag{45}$$

so that the ratio of term (IIa) and term (I) of (44) becomes

$$\frac{(IIa)}{(I)} = \frac{|\vec{v}_D|}{2\omega_c}\frac{|\nabla B_c|}{B_c}\left[1 - \frac{2(\hat{e}_D \cdot \hat{e}_{\nabla\beta_c})^2}{1 - (\hat{e}_{\nabla\alpha_c} \cdot \hat{e}_{\nabla\beta_c})^2}\right] . \tag{46}$$

Since $\nabla\alpha_c$ and $\nabla\beta_c$ are never parallel, the right hand side of (46) is smaller than some majorized constant times (following the previously mentioned argument) a factor which is the percent change in the magnetic field at the guiding center during a Larmour cycle. This again therefore is a very small quantity compared to unity. The resulting terms of (44) may then be written as

$$\frac{c}{q}\left\langle\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right)\right\rangle_{T_L} = \overbrace{\vec{v}_D \cdot \nabla\alpha_c}^{(I)} + \overbrace{\frac{(\vec{v}_D \cdot \nabla\beta_c)^2}{2\omega_c B_c}\frac{\partial(|\nabla\alpha_c|^2)}{\partial\beta}}^{(IIb)} . \tag{47}$$

In equation (47), that the second term on the R.H.S. is small compared to the first in all regions except those near the equator (where $\vec{v}_D \cdot \nabla\alpha_c = 0$) may be seen by writing the second term as

$$(IIb) = \left[\left(\frac{v_D}{\omega_c}\right)\frac{|\nabla\alpha_c|^2|\nabla\beta_c|^2}{B_c^2}\frac{(\hat{e}_{\vec{B}_c\times\nabla\alpha_c}\cdot\nabla|\nabla\alpha_c|)}{|\nabla\alpha_c|}\right]\vec{v}_D\,|\nabla\alpha_c|\,(\hat{e}_D\cdot\hat{e}_{\nabla\beta})^2 . \tag{48}$$

Except for the factor of cosine between $\vec{v}_D$ and $\nabla\beta_c$ (which, of course, is smaller than unity) it is clear that this term is smaller than the first term (R.H.S. of (47)) by the factor in the above bracket. As was previously mentioned, this is approximately the change in $|\nabla\alpha|$, at the guiding center, during the Larmour drift period; which is small compared to unity.

Thus, barring small regions of space around the magnetic equator, the Larmour cycle average of $\frac{\partial \chi}{\partial \beta}$ is given, by the next approximation, as $\vec{v}_D \cdot \nabla\alpha_c$. That this is in fact very reasonable is clear from the realization that, for particles of sufficiently low rigidity, $\frac{c}{q}\ \frac{\partial \chi}{\partial \beta} \approx \alpha_c$ (the value of "α" at the guiding center) and its rate of change, with respect to time, is given by

$$\frac{d\alpha_c}{dt} = \frac{\partial \alpha_c}{\partial t} + \vec{v}_c \cdot \nabla \alpha_c \tag{49}$$

where $\frac{\partial \alpha_c}{\partial t} = 0$ because we are only considering static fields, and $\vec{v}_c$ is the guiding center's velocity. The component of the guiding center velocity which survives in (49), since $\vec{v}_{\parallel} \cdot \nabla\alpha_c = 0$, is no more than the drift velocity, $\vec{v}_D$, and hence the time average of $\frac{d}{dt}\left(\frac{\partial \chi}{\partial \beta}\right)$ should be like $\vec{v}_D \cdot \nabla\alpha_c$; the answer we obtained.

C- The Resulting Time-Averaged Lagrange's Equation

The result we have obtained so far may be summed up as follows. The Lagrange's equation (3) which lead to a first integral in the first approximation of a slightly β-dependent field[38] is now corrected, in the second approximation, to read

$$\left\langle \frac{c}{q} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right)\right\rangle_{T_\ell} = \vec{v}_D \cdot \nabla \alpha_c + \frac{(\vec{v}_D \cdot \nabla \beta_c)^2}{2\,\omega_c B_c} \frac{\partial |\nabla \alpha_c|^2}{\partial \beta} \tag{50}$$

so long as we only consider the equation valid during time intervals large compared to a Larmour period. The terms on the right hand side of (50) are all evaluated at the particle's guiding center, and the second term may be neglected except for particles mirroring near the magnetic equator.

Further, (50) makes clear the fact that as we approach closer and closer to a field which displays β-symmetry the right hand side of (50) approaches zero thus yield the result of Ray.[38] That this is true is seen from the employment of (39) and some algebra to show the first term on the right hand side is proportional to $\frac{\partial B_c}{\partial \beta}$, while the second is proportional to $\frac{\partial |\nabla \alpha_c|^2}{\partial \beta}$ both of which vanish when the arguments of the derivatives become independent of β.

D- Expansions of the Field Variables

Deeper insight into the problem and a path for further calculations is made by adopting a perturbation theory approach and expanding the field variables in Fourier Series in β. Since it is only this divergence from β-independence which destroys the exactness of the first integral of Ray,[38] the correcting term may now be made to display its dependence on the non-axial symmetric components of the field.

In Appendix X we show that β is a measure of azimuth and is therefore functionally related to the angle ϕ. Since α will be chosen as a function of the minimum value of the magnetic field along a line of force, it becomes a unique label for topological shells in space. Since these surfaces of constant α are concentric and generate the space it follows that $|\nabla\alpha|$ is a single-valued function in space. The magnetic field, B, is also a single-valued space function in addition to being continuous. We may therefore define a period of β as[13] (in general a function of α) (see Appendix IV)

$$P_\beta(\alpha) \equiv \oint_{1\ \text{cycle}} d\beta = \oint_{1\ \text{cycle}} \frac{B}{|\nabla\alpha|}\, dx \qquad (51)$$

where

$$\vec{dx} = \frac{\vec{B} \times \nabla\alpha}{B\,|\nabla\alpha|}\, dx \quad . \qquad (52)$$

The integral is to be performed once around the curve defined by a given constant α surface and any constant V surface intersecting it.

We may therefore expand B and $|\nabla\alpha|$ as Fourier Series' periodic in $P_\beta(\alpha)$. The leading terms of these expansions will capture the approximate independence of β, while the remaining terms will give higher order β-dependent corrections. We start with

$$\nabla\alpha = \left|\nabla\alpha(\alpha,\beta,V)\right| \hat{e}_{\nabla\alpha}(\alpha,\beta,V) \tag{53}$$

$$\vec{B} = B(\alpha,\beta,V)\ \hat{e}_{\vec{B}}\ (\alpha,\beta,V) \tag{54}$$

and make the expansions

$$|\nabla\alpha| = a_o(\alpha,V) + \sum_{\substack{n\neq 0\\ n=-\infty}}^{n=\infty} a_n(\alpha,V)\, e^{\frac{in\beta}{P_\beta(\alpha)}} \tag{55}$$

$$B = b_o(\alpha,V) + \sum_{\substack{m\neq 0\\ m=-\infty}}^{m=\infty} b_m(\alpha,V)\, e^{\frac{im\beta}{P_\beta(\alpha)}} \tag{56}$$

where a_o and b_o are given, in terms of the actual field variables, as

$$a_o(\alpha,V) = \left\langle |\nabla\alpha| \right\rangle_\beta \equiv \frac{1}{P_\beta(\alpha)}\oint |\nabla\alpha|\, d\beta = \frac{1}{P_\beta(\alpha)}\oint B\, dx \tag{57}$$

$$b_o(\alpha,V) = \left\langle B \right\rangle_\beta \equiv \frac{1}{P_\beta(\alpha)}\oint B\, d\beta = \frac{1}{P_\beta(\alpha)}\oint \frac{B^2}{|\nabla\alpha|}\, dx\ . \tag{58}$$

Furthermore, in the spirit of perturbation theory, the fields having only slight β-dependence implies that the leading Fourier Series component is much larger than subsequent β-dependent terms, ie:

$$|a_o(\alpha,V)| >> \left| \sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty} a_n(\alpha,V)\, e^{\frac{in\beta}{P_\beta(\alpha)}} \right| \tag{59}$$

$$|b_o(\alpha,V)| >> \left| \sum_{\substack{m\neq 0\\ m=-\infty}}^{\infty} b_n(\alpha,V)\, e^{\frac{im\beta}{P_\beta(\alpha)}} \right| \tag{60}$$

This allows us to express the actual fields (53) and (54) by the approximate perturbation expansion

$$\nabla\alpha \cong \left(\left\langle |\nabla\alpha| \right\rangle_\beta + |\nabla\alpha|^{(1)} \right) \hat{e}_{\nabla\alpha}(\alpha,\beta,V) \tag{61}$$

$$\vec{B} \cong \left(\left\langle B \right\rangle_\beta + B^{(1)} \right) \hat{e}_{\vec{B}}\,(\alpha,\beta,V) \tag{62}$$

where

$$|\nabla\alpha|^{(1)}(\alpha,\beta,V) \equiv \sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty} a_n(\alpha,V)\, e^{\frac{in\beta}{P_\beta(\alpha)}} \tag{63}$$

$$B^{(1)}(\alpha,\beta,V) \equiv \sum_{\substack{m\neq 0\\ m=-\infty}}^{\infty} b_n(\alpha,V)\, e^{\frac{im\beta}{P_\beta(\alpha)}} \tag{64}$$

Return now to the result obtained in equation (50). As is pointed out in the paragraphs following this equation, the terms on the right hand side are proportional to $\frac{\partial B_c}{\partial \beta}$ and $\frac{\partial |\nabla \alpha_c|^2}{\partial \beta}$ respectively. In the light of (61) and (62) it is now clear that, if we only consider the leading β-independent terms of our expansions the correction terms on the right hand side of (50) are zero giving Ray's result.[38] It is now further clear that the only things that destroy this first integral are inherent in the magnetic field itself and not in the particle's motion.

(IV) TRAPPED RADIATION

A- Guiding Center Motion

We will first consider particles which have sufficiently low rigidity that they are trapped by the magnetic field. The naturally formed Van-Allen Belts and the injected charged particles of the Argus experiments are examples of trapped radiation.

It is important to realize the second order correction (the right hand side of (50)) need only be computed along the trajectory that the charged particle would have taken in a field which was β-independent; ie: the β-independent field assumed, to obtain the first approximation. That is, we must substitute into the right hand side of (50) the guiding center trajectory of a charged particle in the field made up of the leading terms of the Fourier expansions of the actual field (given by the first terms on the right hand side of III-(61,62). According to Alfvén,[1] this is given by

$$\vec{v}_D^{(o)} = \frac{\mu c B_m}{q\gamma}\left[\frac{2}{B_c^{(o)}} - \frac{1}{B_m}\right] \frac{\hat{e}_{\vec{B}_c} \times \nabla(B_c^{(o)})}{B_c^{(o)}} \qquad (1)$$

$$\vec{v}_{/\!/}^{(o)} = v\sqrt{1 - \frac{B_c^{(o)}}{B_m}}\;\hat{e}_{\vec{B}} \qquad (2)$$

where "v" is the speed of the particle (a constant of the motion), $v_{/\!/}^{(o)}$ is the component of the guiding center speed parallel to the magnetic field and $\vec{v}_D^{(o)}$ its corresponding perpendicular component (as if the motion had taken place in a field made up of the first term of the Fourier series of the actual field). For the remainder of the paper we will use the symbol $B_c^{(o)}$ for $\langle B_c \rangle_\beta$. The mirror point, B_m, is defined as the point where the component of the parallel velocity, $v_{/\!/}^{(o)}$ is zero. Note that, from the chain rule, we have

$$\nabla B_c^{(o)} = \frac{\partial B_c^{(o)}}{\partial \alpha} \nabla \alpha_c + \frac{\partial B_c^{(o)}}{\partial V} \vec{B}_c \qquad (3)$$

in (1).

<u>B- Trapped Particles Mirroring at a "Flat" Magnetic Equator</u>

Along each line of force there exists a point of minimum magnetic field magnitude. The totality of such points make up a topological surface in space which is called the "Magnetic Equator." If this surface is flat, then the totality of points must lie on a plane. Examples of such field models which satisfy this condition are given by Mead,[26] or Hones.[15]

Since a particle mirroring along such an equator does not bounce, (ie: it is caught in a magnetic potential well) its entire motion may be analyzed by a two dimensional model of the actual field. That is, we orient our z-axis along the direction of the magnetic field vector (which is coparallel throughout space in accordance with our assumption of a flat magnetic equator) and assume that its extension in space is straight lines of force along which the magnetic field is constant and equal in magnitude to the value at the actual magnetic equator.

Hence we have

$$\vec{B}(x,y,z) = B(x,y)\hat{e}_z \quad . \tag{4}$$

This suggests that we choose α as a function of $B(x,y)$ since it is constant along lines of force (see Appendix V). Furthermore, from II-(5) we have

$$\vec{B}(x,y,z) = \bar{\mu}\nabla V(x,y) \tag{5}$$

and we may always choose $V = z$ in this type of field thus implying

$$\nabla V(x,y) = \hat{e}_z$$

$$\bar{\mu}(x,y) = B(x,y) \quad . \tag{6}$$

β is then the solution to the partial differential equation

$$B(x,y) = \frac{\partial B}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial B}{\partial y}\frac{\partial \beta}{\partial x} \quad . \tag{7}$$

To establish such a field as (4), however, it is necessary to assume that we have perpendicular (to magnetic field) current densities in space. This means that we no longer can choose $\bar{\mu} = 1$ as was done in arriving at the result III-(50). Indeed to be correct, we must return to equation II-(22) and rederive the result including $\bar{\mu}$ in our calculation. However, because α_c may be chosen as $B(x,y)$ in this particular example of the flat equator, the $\frac{\partial(\bar{\mu}^2/B^2)}{\partial\beta} \cdot \dot{V}^2$ term that would be an additional term in our rederivation vanishes, thus leaving us with the consequence that, indeed, our result III-(50) is valid.

Since we may choose α as an arbitrary function of $B(x,y)$ it follows that $\nabla\alpha_c$ and ∇B_c are in the same direction so that the $\vec{v}_D \cdot \nabla\alpha_c = 0$ in III-(30). As a result the equation becomes, using the previously noted fact that $\frac{c}{q}\ \frac{\partial\alpha}{\partial\beta} \sim \alpha_c$

$$\frac{d\alpha_c}{dt} = \frac{(\vec{v}_D \cdot \nabla\beta_c)^2}{2\omega_c B_c} \quad \frac{\partial|\nabla\alpha_c|^2}{\partial\beta} \quad . \tag{8}$$

Furthhremore, since we are at the equator

$$\nabla B_c = \nabla f(\alpha_c) = f'(\alpha_c)\nabla\alpha_c = g(B_c)\nabla\alpha_c \tag{9}$$

so that

$$\frac{\vec{v}_D \cdot \nabla\beta_c}{\omega_c B_c} = \frac{\mu c B_m}{q\gamma}\left[\frac{2}{B_c} - \frac{1}{B_m}\right]\frac{[g(B_c)]B_c^2}{B_c^2} = h(B_c). \tag{10}$$

But $\frac{\partial h(B_c)}{\partial \beta} = 0$ because B_c is only a function of α_c. In addition, $\vec{v}_D \cdot \nabla\beta_c$ is no more than $\frac{d\beta}{dt}$ at the guiding center so that (8) becomes

$$d[\alpha_c] = \frac{\partial}{\partial\beta}\left[\frac{(\vec{v}_D \cdot \nabla\beta_c)\,|\nabla\alpha_c|^2}{2\,\omega_c\,B_c}\right] d\beta \quad . \tag{11}$$

The right hand side of (11) is an exact differential in light of the fact that the first order motion predicts that $\vec{v}_D \cdot \nabla\alpha_c = 0$ which means that the path of integration of the first order trajectory is on an α_c = constant surface. We may therefore integrate (11) to obtain our corrected first integral

$$\alpha_c - \frac{\vec{v}_D \cdot \nabla\beta_c}{2\omega_c B_c}\,|\nabla\alpha_c|^2 = 2\bar{\gamma} \quad . \tag{12}$$

If the field were axially symmetric then the correction term in (12) would not vary in azimuth so that the correction term would be constant and might be brought to the right hand side of (12) thus retrieving the result that α_c = constant are the invariant surfaces in space. That the correcting term in (12) is small may be seen as follows. We define the new length

$$\vec{a}\,'_c = \frac{\vec{v}_D \times \vec{B}_c}{B_c^2}\;\frac{mc}{q} \tag{13}$$

in anology to the Larmour radius. Clearly this is much smaller than the Larmour radius itself, by virtue of the fact that $|\vec{v}_D| \ll |\vec{v}|$. Using (13) and a little algebra, we may change the correction term in (12) to $-\vec{a}_c' \cdot \nabla\alpha_c$, so that (12) may be simply written as

$$\alpha_c - \vec{a}_c' \cdot \nabla\alpha_c = 2\overline{\gamma} \quad . \tag{14}$$

The second term on the left hand side of (14) accounts for the splitting of invariant shells as a function of particle energy. An example of the effect of such a term on the motion of trapped radiation at the equator of the Mead[26] model of the magnetosphere is worked out in Section VI.

C- Particles Mirroring at Latitudes Above the Magnetic Equator

(i) The First Order Trajectory

Next consider the more general case of a particle which now mirrors at higher latitudes than the magnetic equator. We will now proceed to derive an expression that is more general than (12).

Consider the motion of the guiding center of trapped radiation. Qualitatively, equation (1) and (2) describes the average motion of the guiding center as a "fast" bouncing motion from mirror point to mirror point in addition to a "slow" drifting motion perpendicular to the direction of the magnetic field and its gradient. Consider now in detail the motion of a charged particle in the field made up of the β-independent component of the magnetic field. This motion is quantitatively described by equations (1) and (2) and pictured in figure 8. Depicted is a section of a constant "α" surface. The curves in the horizontal direction are constant V curves, while those in the vertical direction are constant β curves.

Suppose the guiding center of a particle starts at (α_o, β_o, V_o) "bounces" down to the lower mirror point, V_{m-} (where $B_c^{(o)} = B_m$), is reflected to the upper mirror point, V_{m+} (again where $B_c^{(o)} = B_m$), is reflected again and finally reaches (α_o, β_2, V_1) one cycle later. Notice that the first order motion dictates that at every point of the path $\vec{v}_D^{(o)} \cdot \nabla\alpha_c = 0$ so that the entire trajectory lies on the constant "α_o" surface.

(ii) The Averaged Drift Velocity

Let us now define the velocity, $\vec{\mathcal{V}}_c(V_1, V)$, (see Northrup and Teller[34]) such that the following two conditions are satisfied

$$\vec{\mathcal{V}}_c(V_1, V) \cdot \nabla\alpha(V_1) = 0 \tag{15}$$

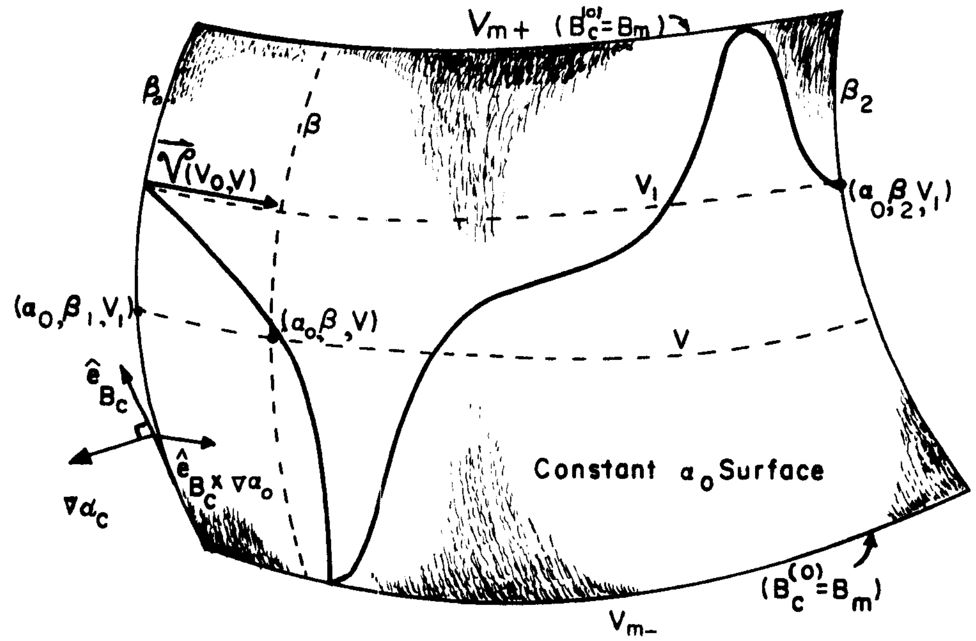

V_{m+} $(B_c^{(o)}=B_m)$
β_o
β
β_2
$\vec{V}(V_o,V)$
V_1
(α_o,β_2,V_1)
(α_o,β_1,V_1)
(α_o,β,V)
V
$\hat{e}_{B_c}$
$\hat{e}_{B_c}\times\nabla\alpha_o$
$\nabla\alpha_c$
Constant α_o Surface
$(B_c^{(o)}=B_m)$
V_{m-}

figure 8

$$\vec{\mathcal{V}}_c(V_1, V) \cdot \nabla\beta_c(V_1) = \vec{v}_D^{(o)}(V) \cdot \nabla\beta_c(V) \quad . \tag{16}$$

$\vec{\mathcal{V}}_c(V_1, V)$ is defined in such a way that it carries a "ficticious" particle from (α_o, β_1, V_1) to (α_o, β_2, V_1) (shown in figure 8) in the same time as it takes the actual guiding center to move to the point (α_o, β, V). Since all motion will now only concern the guiding center of the trapped particle, we will drop the subscript "c" but remember that the motion will be that of the guiding center.

The solution of equations (15) and (16) is given by

$$\vec{\mathcal{V}}_c(V_1, V) = -\dot{\beta}_c^{(o)}(V) \frac{\nabla\alpha_c(V_1) \times \hat{e}_{\vec{B}}(V_1)}{B(V_1)} \quad . \tag{17}$$

That (17) is the solution of (15) and (16) may be demonstrated by substituting it into the equations and using II-(3).

We now define the "bounce average" of the function "Q" as

$$\langle Q \rangle_{T_c} \equiv \frac{1}{T_c} \oint Q\, dt = \frac{2}{T_c} \oint_{V_{m-}}^{V_{m+}} Q \frac{dl}{v_{\parallel}^{(o)}} = \frac{2}{v T_c} \int_{V_{m-}}^{V_{m+}} \frac{Q\, dV}{B_c \left(1 - \frac{B_c^{(o)}}{B_m}\right)^{\frac{1}{2}}} \tag{18}$$

where the "period" is given by

$$T_c \equiv 2 \int_{V_{m-}}^{V_{m+}} \frac{dV}{v B_c \left(1 - \frac{B_c^{(o)}}{B_m}\right)^{\frac{1}{2}}} \quad . \tag{19}$$

"dl" is the differential length along a line of force, and we have used (2) in (18) and (19).

Using the definitions (18), (19) we may now obtain the "bounce-averaged drift velocity" as

$$\vec{\mathcal{V}}_c(V_1) \equiv \left\langle \vec{\mathcal{V}}_c(V_1, V) \right\rangle_{T_c} = \frac{\hat{e}_{\vec{B}}(V_1) \times \nabla\alpha(V_1)}{B(V_1)} \cdot \left\langle \dot{\beta}_c^{(o)} \right\rangle_{T_c} \tag{20}$$

so that the motion of the guiding center may be replaced by average drifting motion, given by equation (20), along a curve made up of a constant α_o, constant V_1 surface. Furthermore, any function to be evaluated along the actual first order trajectory may now be replaced by its average value over a bounce period. Therefore we have for the average values of $\dot{\alpha}_c^{(o)}$ and $\dot{\beta}_c^{(o)}$

$$\left\langle \dot{\alpha}_c^{(o)} \right\rangle_{T_c} = 0 \tag{21}$$

$$\left\langle \dot{\beta}_c^{(o)} \right\rangle = \frac{2}{T_c} \int_{V_{m-}}^{V_{m+}} \left\{ \frac{\mu c B_m}{q\gamma v} \left[\frac{2}{B_c^{(o)}} - \frac{1}{B_m} \right] \frac{\partial B_c^{(o)}}{\partial \alpha} \times \frac{(\vec{B}_c^{(o)} \times \nabla\alpha_c) \cdot \nabla\beta_c}{(B_c^{(o)})^2} \right\} \frac{dV}{B_c\left(1 - \frac{B_c^{(o)}}{B_m}\right)^{1/2}} \ . \tag{22}$$

Since $\nabla\alpha \times \nabla\beta = \vec{B}$ and $\vec{B} \cdot \vec{B}^{(o)} = B^{(o)} B$, equation (22) may be written more briefly as, after some algebra,

$$\left\langle \vec{v}_D^{(o)} \cdot \nabla\beta_c \right\rangle_{T_c} = \frac{-c}{qT_c} \frac{\partial J^{(o)}}{\partial\alpha} \tag{23}$$

where

$$J^{(o)}(\alpha_o) \equiv 2mv \int_{Vm-}^{Vm+} \left(1 - \frac{B_c^{(o)}}{B_m}\right)^{1/2} \frac{dV}{B_c^{(o)}} \, . \tag{24}$$

The similarity between $J^{(o)}$ and the regular definition of the integral invariant[5] is obvious. Furthermore, equation (22) also checks with Northrup and Teller's result[34] which was obtained in a completely different manner. We may further note that the partial derivative of (24) with respect to α again checks with the Northrup-Teller result; ie:

$$\frac{\partial J^{(o)}}{\partial\beta} \propto \dot{\alpha}_c^{(o)} = 0 \quad . \tag{25}$$

(iii) The Bounce-Averaged First Integral

Let us now return to equation III-(50) and average the entire equation over a bounce period. We have, noting $\frac{c}{q}\frac{\partial\chi}{\partial\beta} \approx \alpha_c$

$$\left\langle \frac{d\alpha_c}{dt} \right\rangle_{T_c} = \overbrace{\left\langle \vec{V}_D \cdot \nabla\alpha_c \right\rangle_{T_c}}^{(i)} + \overbrace{\left\langle \frac{(\vec{V}_D \cdot \nabla\beta_c)^2}{2\omega_c B_c} \frac{\partial|\nabla\alpha_c|^2}{\partial\beta} \right\rangle_{T_c}}^{(ii)} \, . \tag{26}$$

The left hand side of this equation may be interpreted as the time rate of change of α_c for times large compared to a bounce period, T_c. With this restriction understood, we may interpret the left hand side as

$$\text{L.H.S.} = \frac{d}{dt}(\alpha_c) \quad . \tag{27}$$

Next consider the first term, (i), of the right hand side of (26). This term is <u>not</u> zero because we now have $\vec{v}_D$ and <u>not</u> $\vec{v}_D^{(o)}$. Its magnitude is, in fact,

$$\left\langle \vec{v}_D \cdot \nabla\alpha_c \right\rangle_{T_c} = \frac{2\mu c B_m}{T_c q \gamma v} \int_{V_{m-}}^{V_{m+}} \left[\frac{2}{B_c} - \frac{1}{B_m} \right] \frac{\vec{B}_c \times \nabla B_c \cdot \nabla\alpha_c}{B_c^2} \frac{dV}{B_c\left(1 - \frac{B_c^*}{B_m}\right)^{\frac{1}{2}}} \tag{28}$$

where we have used the definition of $\vec{v}_D$ given in III-(39). We may write (28) in another more useful form as follows. From the chain rule we have

$$\nabla B = \frac{\partial B}{\partial \alpha} \nabla\alpha + \frac{\partial B}{\partial \beta} \nabla\beta + \frac{\partial B}{\partial V} \vec{B} \tag{29}$$

so that the term

$$\frac{(\vec{B}_c \times \nabla B_c) \cdot \nabla\alpha_c}{B_c^2}$$

in the integral of (38) becomes $-\frac{\partial B_c}{\partial \beta}$. Placing this result back in (28) we may remove a "$\frac{\partial}{\partial \beta}$" from the inside of the integral (for the guiding center moving under the influence of the β-independent field under the conditions of the first order motion V_{m+} and V_{m-} do not vary with β) we have

$$\left\langle \vec{V}_D \cdot \nabla \alpha_c \right\rangle_{T_c} = \frac{2}{T_c} \frac{\mu c}{q^{v} \gamma} \frac{\partial}{\partial \beta} \left\{ \int_{V_{m-}}^{V_{m+}} \left[\ln(\eta) + \frac{2}{\eta} \right] \frac{dV}{(1-\eta^{(0)})^{\frac{1}{2}}} \right. \tag{30}$$

where "η", "$\eta^{(0)}$", "$\eta^{(1)}$" are defined as

$$\eta \equiv \frac{B_e}{B_m} ; \quad \eta^{(0)} \equiv \frac{B_c^{(0)}}{B_m} ; \quad \eta^{(1)} \equiv \frac{B_c^{(1)}}{B_m} ; \ \eta \equiv \eta^{(0)} + \eta^{(1)}. \tag{31}$$

We may now approximate the integrand of (30) be using our Fourier Series expansion III-(62). We have, since $\frac{B^{(1)}}{B^{(0)}} \ll 1$,

$$\ln(\eta) = \ln\left(\eta^{(0)}\left(1 + \frac{\eta^{(1)}}{\eta^{(0)}}\right)\right) \approx \ln \eta^{(0)} + \ln\left(1 + \frac{\eta^{(1)}}{\eta^{(0)}}\right) \approx \frac{\eta^{(1)}}{\eta^{(0)}} + \ln(\eta^{(0)}) \tag{32}$$

and

$$\frac{2}{\eta} = \frac{2}{\eta^{(0)}\left(1 + \frac{\eta^{(1)}}{\eta^{(0)}}\right)} \approx \frac{2}{\eta^{(0)}} - \frac{2\eta^{(1)}}{(\eta^{(0)})^2} . \tag{33}$$

Since $\frac{\partial}{\partial \beta}$ of any function of $B^{(0)}$ is zero, (30) becomes

$$\left\langle \vec{V}_D \cdot \nabla \alpha_c \right\rangle_{T_c} = \frac{2\mu c}{q v \Upsilon T_c} \frac{\partial}{\partial \beta} \left\{ \int_{V_{m-}}^{V_{m+}} \left[1 - \frac{2}{\eta^{(0)}} \right] \frac{\eta^{(1)} \, dV}{\eta^{(0)} (1 - \eta^{(0)})^{\frac{1}{2}}} \right\} \tag{34}$$

Next define the function $J^{(1)}$ as follows

$$J^{(1)}(\alpha_0, \beta) \approx \frac{2mv}{B_m} \int_{V_{m-}}^{V_{m+}} \frac{(1 - \eta\ )^{\frac{1}{2}}}{\eta} \, dV \ . \tag{35}$$

which again resembles the integral invariant. The integrand of (35) may be broken into its Fourier components and binomially expanded as follows

$$\frac{(1-\eta)^{\frac{1}{2}}}{\eta} = \frac{(1 - \eta^{(0)} - \eta^{(1)})^{\frac{1}{2}}}{\eta^{(0)} + \eta^{(1)}} = \frac{(1 - \eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}} \frac{\left(1 - \frac{\eta^{(1)}}{1 - \eta^{(0)}}\right)^{\frac{1}{2}}}{1 + \frac{\eta^{(1)}}{\eta^{(0)}}}$$

$$\approx \frac{(1 - \eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}} \left[1 - \frac{\eta^{(1)}}{2(1 - \eta^{(0)})} \right] \left[1 - \frac{\eta^{(1)}}{\eta^{(0)}} \right] . \tag{36}$$

Placing this in (35) it becomes, approximately,

$$J^{(1)} \approx \frac{2mv}{B_m} \int_{V_{m-}}^{V_{m+}} \frac{(1 - \eta^{(0)})^{\frac{1}{2}} dV}{\eta^{(0)}} + \frac{2mv}{B_m} \int_{V_{m-}}^{V_{m+}} \frac{\eta^{(1)}}{\eta^{(0)}} \left(\frac{1}{2} - \frac{1}{\eta^{(0)}} \right) \frac{dV}{(1 - \eta^{(0)})^{\frac{1}{2}}} \tag{37}$$

or, using the definition in equation (24) this becomes

$$J^{(1)} - J^{(0)} = \frac{2mv}{B_m} \int_{V_{m-}}^{V_{m+}} \frac{\eta^{(1)}}{\eta^{(0)}} \left[\frac{1}{2} - \frac{1}{\eta^{(0)}}\right] \frac{dV}{\left(1 - \eta^{(0)}\right)^{\frac{1}{2}}} \quad . \tag{38}$$

Taking the partial derivative of both sides of (38) with respect to β gives

$$\frac{\partial \left(J^{(1)} - J^{(0)}\right)}{\partial \beta} = \frac{2mv}{B_m} \frac{\partial}{\partial \beta} \left\{ \int_{V_{m-}}^{V_{m+}} \frac{\eta^{(1)}}{\eta^{(0)}} \left[\frac{1}{2} - \frac{1}{\eta^{(0)}}\right] \frac{dV}{\left(1 - \eta^{(0)}\right)^{\frac{1}{2}}} \right\} . \tag{39}$$

Upon comparison of (39) to (34) we see that we may express (34) as

$$\left\langle \vec{v}_D \cdot \nabla \alpha_c \right\rangle_{T_c} = \frac{1}{T_c} \quad \frac{c}{q} \frac{\partial \left(J^{(1)} - J^{(0)}\right)}{\partial \beta} \quad . \tag{40}$$

Finally consider term (ii) of equation (26). Since the term has significance only for particles whose mirror points are near the equator, we may proceed to compute its bounce average for low latitude mirror points only. So we start by expanding the integrand in a Taylor Series about the equator. The equator is defined as that value of "V_o" such that $B_c^{(o)}$ is a minimum value. The value of the magnetic potential

"V_o" is very near the value corresponding to a minimum in B_c, the error being of second order. We now choose the function α as $(B_{min})^{1/3}$. The integrand of (ii) then becomes

$$\vec{v}_D \cdot \nabla\beta_c = \frac{\mu c B_m}{q\gamma}\left[\frac{2}{\alpha_c^3} - \frac{1}{B_m}\right]\frac{(\hat{e}_{B_{equator}} \times \nabla\alpha^3)\cdot\nabla\beta_c}{\alpha_c^3}$$

$$+ \text{ higher order terms} \tag{41}$$

where we have Taylor expanded about the equator. (41) may be further simplified to read

$$\vec{v}_D \cdot \nabla\beta_c = \frac{\mu c B_m}{q\gamma}\left[\frac{2}{\alpha_c^3} - \frac{1}{B_m}\right] 3\alpha_c^2 \quad + \text{ higher order terms.} \tag{42}$$

It is clear from (42) that

$$\frac{\vec{v}_D \cdot \nabla\beta_c}{2\omega_c B_c} = f(\alpha_c) \tag{43}$$

and, hence, we may write the term (ii) of (26) as

$$(ii) \cong \left\langle \frac{\partial}{\partial\beta}\left\{\frac{(\overline{v}_D \cdot \nabla\beta_c)^2}{2\omega_c B_c} \quad |\nabla\alpha_c|^2\right\}\right\rangle_{T_c} \tag{44}$$

which is only to be averaged for low mirror points. However, the average value in (44) over these low mirror points is no more than the equatorial value of the integrand, so that we have approximately,

$$(ii) \cong \left\{ \frac{\partial}{\partial \beta} \left[\frac{(\vec{v}_D \cdot \nabla \beta_c)^2 \ |\nabla \alpha_c|^2}{2\omega_c B_c} \right] \right\}_{\alpha_o, V_o} \tag{45}$$

ie: evaluated at the equator. We may extend the validity of (45) to higher mirror points by virtue of the fact that it becomes swamped by the term (i) of the same equation thus making it inconsequential whether it be included or not. We conclude that the bounce average of equation (26) becomes

$$\left\langle \frac{d\alpha_c}{dt} \right\rangle_{T_c} \approx \frac{1}{T_c} \frac{c}{q} \frac{\partial (J^{(1)} - J^{(0)})}{\partial \beta} + \left\{ \frac{\partial}{\partial \beta} \left[\frac{(\vec{v}_D \cdot \nabla \beta_c)^2 |\nabla \alpha_c|^2}{2\,\omega_c B_c} \right] \right\}_{\alpha_o, V_o} . \tag{46}$$

Equation (46) describes the time rate of change in α_c as the guiding center has an average drift $\vec{\mathcal{V}}_c(V_o)$ which carries it from line of force to line of force on a shell of constant "α_c" predicted by the first order theory. To compute the entire change in α_c in going from some initial to some final "β", we simply integrate equation (46).

Consider the average drift motion. We have, from (22) - (24),

$$dt = \frac{d\beta}{\left\langle \dot{\beta}_c^{(0)} \right\rangle_{T_c}} = -T_c \left[\frac{q}{c} \frac{d\beta}{\frac{\partial J^{(0)}(\alpha_0)}{\partial \alpha}} \right] . \tag{47}$$

Since $\left.\frac{\partial J^{(0)}}{\partial \alpha}\right|_{\alpha_0}$ is independent of β, and is evaluated on the initial α_0 shell we have, placing (47) in (46) and integrating

$$\alpha_c(\beta_2) - \alpha_c(\beta_1) = -\int_{\beta_1}^{\beta_2} \frac{\partial}{\partial \beta}\left(\frac{J^{(1)} - J^{(0)}}{\frac{\partial J^{(0)}(\alpha_0)}{\partial \alpha}} \right) d\beta + \int_{\beta_1}^{\beta_2} \frac{\partial}{\partial \beta}\left(\frac{(\vec{V}_D \cdot \nabla\beta)^2 \, |\nabla\alpha|^2}{2\omega_c B_c} \right) \frac{d\beta}{\left\langle \dot{\beta}_c^{(0)} \right\rangle_{T_c}} . \tag{48}$$

In the second term on the right hand side of (48) we need only calculate $\left\langle \dot{\beta}_c^{(0)} \right\rangle_{T_c}$ for low mirror point particles since this is the only place this term has significance. In this case we have, approximately

$$\left\langle \dot{\beta}_c^{(0)} \right\rangle_{T_c} = \left[\vec{V}_D \cdot \nabla\beta_c \right]_{\alpha_0, V_0} . \tag{49}$$

That is, the average drift velocity, $\left\langle \dot{\beta}_c^{(0)} \right\rangle_{T_c}$, is approximately given by its value at the equator. As was previously mentioned in equation (10), (49) is not a function of β. Hence, we finally obtain the result

$$\langle \alpha_c \rangle_{T_c} + \frac{I^{(1)}}{\left(\frac{\partial I^{(0)}(\alpha_0)}{\partial \alpha}\right)} - \left[\frac{(\vec{V}_D \cdot \nabla \beta_c)|\nabla \alpha_c|^2}{2\omega_c B_c}\right]_{\alpha_0, V_0} = 2k(MR)^{\frac{1}{2}} = \text{constant}. \tag{50}$$

upon integrating (48). In (50) we have defined the "unitless" constant "k" and the two functions $I^{(o)}$ and $I^{(1)}$ as

$$I^{(o)}(\alpha_o) \equiv \frac{J^{(o)}(\alpha_o)}{mv} \tag{51}$$

$$I^{(1)}(\alpha_o,\beta) \equiv \frac{J^{(1)}(\alpha_o,\beta)-J^{(o)}(\alpha_o)}{mv} \tag{52}$$

where we must keep in mind that all quantities in the expression apply to the guiding center of the trapped particle. The second and third terms on the left hand side of (50) are second order corrections to the first term. The "unitless" constant "k" is therefore not far from its value for the β-independent field case.

(iv) Simplification of the New First Integral

We will now proceed to evaluate the integrals in the corrected first integral equation, (50). As has been previously mentioned, we may replace $\frac{\partial \chi}{\partial \beta}$ by α_c (at the guiding center). Furthermore, define ϵ such that

$$\epsilon \equiv \frac{\eta^{(1)}}{\eta^{(o)}} \tag{53}$$

so that (50) becomes (using (24), (38), III-(39) and some algebra)

$$\alpha_c + \left[\frac{\displaystyle\int_{V_{m-}}^{V_{m+}} \epsilon \left[\frac{1}{2} - \frac{1}{\eta^{(o)}} \right] \frac{dV}{(1-\eta^{(o)})^{\frac{1}{2}}}}{\displaystyle\frac{\partial}{\partial \alpha} \left\{ \int_{V_{m-}}^{V_{m+}} (1-\eta^{(o)})^{1/2} \frac{dV}{\eta^{(o)}} \right\}} \right]_{\alpha_o} - \left[\frac{3R^2 M^4 |\nabla \alpha|^2}{4 \alpha_o^7} \right]_{\alpha_o, V_o} = 2k\sqrt{MR} \tag{54}$$

where "R" is the rigidity of the trapped particle, "M" the magnetic moment of the earth, the second term on the left hand side is evaluated on a surface of constant $\alpha = \alpha_o$, and the third term is evaluated on the same surface, but at the equator "V_o", and "k" is a unitless constant.

In order to evaluate the integrals in (54), we use the numerical fact that, over a large range of mirror points in both the Finch and Leaton and Mead fields,[26] we find that the magnetic field varies approximately as V^2 along a line of force. We may therefore "quadratically fit" the magnetic field along the line of force. Thus we choose

$$\eta^{(o)} - \eta_{V_o}^{(o)} = (1-\eta_{V_o}^{(o)}) \left[\frac{V-V_o}{V_{m-}-V_o}\right]^2 \qquad V \leq V_o \tag{55}$$

$$\eta^{(o)} - \eta_{V_o}^{(o)} = (1-\eta_{V_o}^{(o)}) \left[\frac{V-V_o}{V_{m+}-V_o}\right]^2 \qquad V \geq V_o \tag{56}$$

$$\eta - \eta_{V_o} = \left[\frac{(\eta_{V_{m-}} - \eta_{V_o})(V_{m+}-V_o)^2 - (\eta_{V_{m+}} - \eta_{V_o})(V_{m-}-V_o)^2}{(V_{m+}-V_o)(V_{m-}-V_o)(V_{m+}-V_{m-})}\right](V-V_o)$$
$$+ \left[\frac{(\eta_{V_{m-}} - \eta_{V_o})(V_{m+}-V_o) + (\eta_{V_{m+}} - \eta_{V_o})(V_o-V_{m-})}{(V_{m+}-V_o)(V_o-V_{m-})(V_{m+}-V_{m-})}\right](V-V_o)^2 \tag{57}$$

where V_o is V at $B_c^{(o)}$ = minimum, V_{m+} and V_{m-}, (functions of α_c only) are the magnetic scalar potentials at the upper and lower mirror points respectively (where $B^{(o)} = B_m$).

First let us evaluate the integral in the denominator of (54). Placing (55) and (56) in this term we have

$$\int_{V_{m-}}^{V_{m+}} \frac{(1-\eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}}\, dV = \int_{V_{m-}}^{V_0} \frac{\left[(1-\eta^{(0)}_{V_0}) - (1-\eta^{(0)}_{V_0})\left(\frac{V-V_0}{V_{m-}-V_0}\right)^2\right]^{\frac{1}{2}} dV}{\eta^{(0)}_{V_0} + (1-\eta_{V_0})\left(\frac{V-V_0}{V_{m-}-V_0}\right)^2}$$

$$+ \int_{V_0}^{V_{m+}} \frac{\left[(1-\eta^{(0)}_{V_0}) - (1-\eta^{(0)}_{V_0})\left(\frac{V-V_0}{V_{m+}-V_0}\right)^2\right]^{\frac{1}{2}} dV}{\eta^{(0)}_{V_0} + (1-\eta^{(0)}_{V_0})\left(\frac{V-V_0}{V_{m+}-V_0}\right)^2} \tag{58}$$

or, more simply

$$\int_{V_{m-}}^{V_{m+}} \frac{(1-\eta^{(0)})^{\frac{1}{2}} dV}{\eta^{(0)}} = (1-\eta^{(0)}_{V_0})^{\frac{1}{2}} \left\{ \frac{(V_0-V_{m-})}{1-\eta^{(0)}_{V_0}} \int_0^1 \frac{(1-U^2)^{\frac{1}{2}}\, dU}{\left(\frac{\eta^{(0)}}{1-\eta^{(0)}}\right) + U^2} \right.$$

$$\left. + \frac{(V_{m+}-V_0)}{1-\eta^{(0)}_{V_0}} \int_0^1 \frac{(1-U^2)^{\frac{1}{2}}\, dU}{\left(\frac{\eta^{(0)}}{1-\eta^{(0)}}\right) + U^2} \right\} \tag{59}$$

where $U \equiv \frac{V - V_0}{V_{m-} - V_0}$ in the first integral on the right hand side of (58) and, again $U \equiv \frac{V - V_0}{V_{m+} - V_0}$ in the second term on the right hand side of (58).

We may still further accumulate both integrals on the right hand side of (59) so that it reads

$$\int_{V_{m-}}^{V_{m+}} \frac{(1-\eta^{(o)})^{1/2}}{\eta^{(o)}}\, dV = \frac{\Delta V}{(1-\eta^{(o)})^{1/2}} \int_0^1 \frac{(1-U^2)^{1/2}}{U^2+a^2}\, dU \equiv$$

$$\frac{\Delta V}{(1-\eta^{(o)})^{1/2}} \; F_1^1(a^2,1) \tag{60}$$

where we have defined

$$a^2 \equiv \frac{\eta^{(o)}_{V_o}}{1-\eta^{(o)}_{V_o}} \tag{61}$$

$$\Delta V \equiv V_{m+} - V_{m-} \tag{62}$$

and $F_1^1(a^2,1)$ is tabulated in Appendix IX. The result is

$$\int_{V_{m-}}^{V_{m+}} \frac{(1-\eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}}\, dV = \frac{\pi \Delta V}{2(1-\eta^{(0)}_{V_o})^{1/2}} \left[\frac{\sqrt{\eta^{(0)}_{V_o}} - \eta^{(0)}_{V_o}}{\eta^{(0)}_{V_o}} \right] \tag{63}$$

so that the denominator of the second term on the left hand side of (54) becomes

$$\frac{\partial}{\partial \alpha} \int_{V_{m-}}^{V_{m+}} \frac{(1-\eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}}\, dV = \frac{\pi \Delta V}{4} \left[\frac{2\eta^{(0)}_{V_o} - 1 - (\eta^{(0)}_{V_o})^{3/2}}{(\eta^{(0)}_{V_o})^{3/2} (1-\eta^{(0)}_{V_o})^{3/2}} \right] \frac{\partial \eta^{(0)}_{V_o}}{\partial \alpha}$$

$$+ \frac{\pi\, (1-(\eta^{(0)}_{V_o})^{\frac{1}{2}})}{2(\eta^{(0)}_{V_o})^{\frac{1}{2}}(1-\eta^{(0)}_{V_o})^{\frac{1}{2}}} \frac{\partial \Delta V}{\partial \alpha}\, . \tag{64}$$

Next consider the numerator of this same second term. We may, with some simple algebra, rewrite the term as

$$-\int_{V_{m-}}^{V_{m+}} \epsilon \left[\frac{(1-\eta^{(0)})^{\frac{1}{2}}}{\eta^{(0)}} + \frac{\frac{1}{2}}{(1-\eta^{(0)})^{1/2}} \right] dV \tag{65}$$

where the function ϵ is given by

$$\epsilon_{\pm} \equiv \frac{1}{\left(1-\eta^{(0)}_{V_0}\right)} \left[\frac{\left(\eta_{V_0} - \eta^{(0)}_{V_0}\right) + C_1^{\pm}\left(\frac{V-V_0}{V_{m\pm}-V_0}\right) + C_2^{\pm}\left(\frac{V-V_0}{V_{m\pm}-V_0}\right)^2}{a^2 + \left(\frac{V-V_0}{V_{m\pm}-V_0}\right)^2} \right] \tag{66}$$

where the upper sign is for $V_{m+} \geqslant V \geqslant V_0$ while the lower sign holds over the interval $V_{m-} \leqslant V \leqslant V_0$ and the constants $C_1^{\pm}$, $C_2^{\pm}$ are defined by

$$C_1^{\pm} \equiv \frac{(\eta_{V_{m-}} - \eta_{V_0})(V_{m+}-V_0)^2 - (\eta_{V_{m+}} - \eta_{V_0})(V_{m-}-V_0)^2}{(V_{m\mp} - V_0)\,\Delta V} \tag{67}$$

$$C_2^{\pm} \equiv \eta_{V_0}^{(0)} - 1 - \frac{(V_{m\pm} - V_0)\left[(\eta_{V_{m-}} - \eta_{V_0})(V_{m+} - V_0) + (\eta_{V_{m+}} - \eta_{V_0})(V_0 - V_{m-})\right]}{(V_{m\mp} - V_0)\,\Delta V}$$

(68)

where we have used (55) through (57) in arriving at (66). Again using these equations in (65) plus the substitution following equation (59), it becomes

$$(65) = -\frac{(V_0 - V_{m-})}{(1-\eta_{V_0}^{(0)})^{1/2}}\int_0^1 \epsilon_- \frac{(1-U^2)^{1/2}\,dU}{(a^2+U^2)} - \frac{(V_{m+} - V_0)}{(1-\eta_{V_0}^{(0)})^{1/2}}\int_0^1 \epsilon_+ \frac{(1-U^2)^{1/2}\,dU}{(a^2+U^2)}$$

$$-\frac{(V_0 - V_{m-})}{2(1-\eta_{V_0}^{(0)})^{1/2}}\int_0^1 \frac{\epsilon_-\,dU}{(1-U^2)^{1/2}} - \frac{(V_{m+} - V_0)}{2(1-\eta_{V_0}^{(0)})^{1/2}}\int_0^1 \frac{\epsilon_+\,dU}{(1-U^2)^{1/2}} .$$

(69)

(69) may be simplified by making the approximations (appropriate to the field models we will be considering). We may write

$$V_0 - V_{m-} = \frac{V_{m+} - V_{m-}}{2} - \frac{V_{m+} - V_{m-}}{2} + \frac{2V_0}{V_0}$$

$$= \frac{\Delta V}{2} + \frac{V_0 - (V_{m+} + V_{m-})}{2} .$$

(70)

Since V_o is going to be near the magnetic equator, it is going to be approximately 0. Furthermore, $V_{m+} + V_{m-}$ (which is really their difference since they always have opposite signs) is also approximately 0 because of the approximate equitorial symmetry. We see, therefore, that the first term on the right hand side of (62) is all that we need. In a similar way we may show the same about $V_{m+} - V_o \approx \frac{\Delta V}{2}$. That is, we assume

$$V_o - V_{m-} \approx \frac{\Delta V}{2} \tag{71}$$

$$V_{m+} - V_o \approx \frac{\Delta V}{2} \quad . \tag{72}$$

Placing this in (61) it becomes

$$(65) = -\frac{\Delta V}{2(1-\eta_{V_o}^{(o)})^{1/2}} \left\{ \int_o^1 (\epsilon_+ + \epsilon_-) \frac{(1-U^2)^{1/2}}{a^2+U^2} dU \right.$$

$$\left. + (1/2) \int_o^1 \frac{(\epsilon_+ + \epsilon_-)}{(1-U^2)^{1/2}} dU \right\} \quad . \tag{73}$$

Using (66) through (68) and some algebra we obtain

$$\epsilon_+ + \epsilon_- = \frac{1}{(1-\eta_{V_o})} \left[\frac{C_3}{U^2+a^2} + C_4 \right] \tag{74}$$

where

$$C_3 = \frac{1}{(1-\eta_{V_o}^{(o)})} \left\{ 2(\eta_{V_o} - \eta_{V_o}^{(o)}) - \eta_{V_o}^{(o)} (\eta_{V_{m+}} - 1) + (\eta_{V_{m-}} - 1) \right\}$$

$$C_4 = (\eta_{V_{m+}} - 1) + (\eta_{V_{m-}} - 1) - 2(\eta_{V_o} - \eta_{V_o}^{(o)}) \tag{75}$$

which may now be placed in (73) to obtain

$$(65) = -\frac{\Delta V}{2(1-\eta_{V_0}^{(0)})^{3/2}}\left[c_3\left\{ F_2^1(a^2,1) + \frac{F_1^{-1}(a^2,1)}{2}\right\}\right.$$

$$\left. + c_4\left\{ F_1^1(a^2,1) + \frac{F_0^{-1}(a^2,1)}{2}\right\}\right] . \qquad (76)$$

where we have defined

$$F_n^m(a^2,q^2) \equiv \int_0^1 \frac{(q^2-U^2)^{m/2}}{(a^2+U^2)^n}\, dU \quad . \qquad (77)$$

The evaluation of these integrals are tabulated in the Appendix IX, whence upon their substitution in (76) (in addition to using (75)) the integral in (65) becomes

$$(65) = -\frac{\pi \Delta V}{4(1-\eta_{V_0}^{(0)})^{3/2}(\eta_{V_0}^{(0)})^{3/2}}\left\{\left(1-2\eta_{V_0}^{(0)}+(\eta_{V_0}^{(0)})^{3/2}\right)\left(\eta_{V_0}-\eta_{V_0}^{(0)}\right)\right.$$

$$\left. + \frac{\eta_{V_0}^{(0)}(1-(\eta_{V_0}^{(0)})^{1/2})}{2}\left[(\eta_{V_{m+}}-1)+(\eta_{V_{m-}}-1)\right]\right\} .$$

(78)

Recall that

$$\eta^{(o)}_{V_o} \equiv \frac{B^{(o)}_{V_o}}{B_m} \approx \frac{\alpha^3}{M^2 B_m} \tag{79}$$

so that

$$\frac{\partial \eta^{(o)}_{V_o}}{\partial \alpha} = \frac{3}{\alpha_o}\ \eta^{(o)}_{V_o} \quad . \tag{80}$$

Using (79) and (80) in (64) and (78) we have the result that the second term on the left hand side of (54) becomes

$$(ii) = \frac{\alpha_o \left[\left(1 - 2\eta^{(0)}_{V_o} + (\eta^{(0)}_{V_o})^{3/2}\right)\left(\frac{B_{V_o} - B^{(0)}_{V_o}}{B^{(0)}_{V_o}}\right) + \left(\frac{1-(\eta^{(0)}_{V_o})^{\frac{1}{2}}}{2}\right)\left((\eta_{V_{m+}} - 1) + (\eta_{V_{m-}} - 1)\right)\right]}{3\left(1 - 2\eta^{(0)}_{V_o} + (\eta^{(0)}_{V_o})^{3/2}\right) - 2\alpha_o \left(1 - (\eta^{(0)}_{V_o})^{1/2} - (\eta^{(0)}_{V_o}) + (\eta^{(0)}_{V_o})^{3/2}\right)\left[\frac{\partial \ln (\Delta V)}{\partial \alpha}\right]_{\alpha_o}} . \tag{81}$$

Putting all the terms together (54) is evaluated to be

$$\alpha_c - \frac{3}{4}\frac{R^2 M^2}{\alpha_o^2}\left(|\nabla\alpha|^2\right)_{\alpha_o, V_o} - 2k\sqrt{MR}$$

$$+ \frac{\alpha_o \left[\left(1 - 2\eta^{(0)}_{V_o} + (\eta^{(0)}_{V_o})^{3/2}\right)\left(\frac{B_{V_o} - B^{(0)}_{V_o}}{B^{(0)}_{V_o}}\right) + \left(\frac{1-(\eta^{(0)}_{V_o})^{1/2}}{2}\right)\left((\eta_{V_{m+}} - 1) + (\eta_{V_{m-}} - 1)\right)\right]}{3\left[1 - 2\eta^{(0)}_{V_o} + (\eta^{(0)}_{V_o})^{3/2}\right] - 2\alpha_o\left[1 - (\eta^{(0)}_{V_o})^{1/2} - \eta^{(0)}_{V_o} + (\eta^{(0)}_{V_o})^{3/2}\right]\left[\frac{\partial \ln \Delta V}{\partial \alpha}\right]_{\alpha_o}}$$

$$= 0 \, . \tag{82}$$

or, since we previously showed that we may neglect the third term on the left hand side of (82) in comparison to the second term, when the mirror point is not close to the equator, more simply as

$$\alpha_c \left\{ 1 + \frac{\left(1-2\eta_{V_o}^{(o)}+(\eta_{V_o}^{(o)})^{3/2}\right)\left(\frac{B_{V_o}-B_{V_o}^{(o)}}{B_{V_o}^{(o)}}\right)+\left(\frac{1-(\eta_{V_o}^{(o)})^{1/2}}{2}\right)\left((\eta_{V_{m+}}-1)+(\eta_{V_{m-}}-1)\right)}{3\left(1-2\eta_{V_o}^{(o)}+(\eta_{V_o}^{(o)})^{3/2}\right)-2\alpha_o\left(1-(\eta_{V_o}^{(o)})^{\frac{1}{2}}-\eta_{V_o}^{(o)}+(\eta_{V_o}^{(o)})^{3/2}\right)\left(\frac{\partial \ln \Delta V}{\partial \alpha}\right)_{\alpha_o}} \right\}$$

$$= 2k\sqrt{MR} \qquad (83)$$

(v) Interpretation and Discussion of Corrected First Integral

Consider the result (83). We must recall that all terms refer to the guiding center of the trapped particle. If the field were independent of β the correction term would be identically zero since $B_{V_o} \equiv B_{V_o}^{(o)}$ and $\eta_{V_{m+}} \equiv 1$, $\eta_{V_{m-}} \equiv 1$, and therefore we would recapture the result that α_c = constant, the result of the first order theory. In fact, it is due to the fact that these terms are not zero that causes the splitting of the invariant shells. This is no more than saying that the correction term is proportional to the β-dependent terms of the Fourier expansion of the magnetic field. The correction term becomes larger as the field becomes more β dependent, a result that seems intuitively obvious. Referring to (83) we see that the splitting is only a function of mirror point, B_m. This equation may be interpreted as follows. Given a starting point for the guiding center of a trapped particle with a certain mirror point, that is, an initial starting line of force "α_o", the magnetic shell upon which the guiding center of this particle will remain is not that of the topological surface α_c = constant = α_o but instead, will change surfaces as the particle drifts in azimuth. This drift in azimuth (ie: the β direction) depends on the inherent β-dependence of the field and the mirror point of the particle. As the mirror point is changed (changing the particles magnetic moment "μ"), for the same initial starting line, "α_o" the invariant shell will change shape, but always be a closed surface, so that the drifting particle returns to its initial starting line. Again this is observed from the result. The shells are now said to be non-degenerate.[50] We may look at equation (83) then as $h_c(\alpha_o, \beta; B_m) = 2k(MR)^{1/2}$ = constant, where α_o is the initial starting line, β is the azimuth coordinate, and B_m the particles mirror magnetic field. The constant "k" is determined by the initial conditions of the particle (on which line it starts).

(vi) Special Cases

(a) Particle mirroring on the equator:

Returning to (82) we set $\eta_{V_o}^{(o)} = 1$ (taking the appropriate limiting process we find $\left.\frac{\partial \Delta V}{\partial \alpha}\right|_{\alpha_o}$ always remains finite) to find

$$\alpha_c + \frac{\alpha_o}{3}\left[\frac{B_{V_o} - B_{V_o}^{(o)}}{B_{V_o}^{(o)}}\right] - \frac{3}{4}\,\frac{R^2M^2}{\alpha_o^2}\,\left|\nabla\alpha\right|^2_{\alpha_o,V_o} = 2k\sqrt{MR}\,. \tag{84}$$

The second term on the left hand side arises from the fact that the magnetic equator does not necessarily remain flat in space. If it did, B_{V_o} would equal $B_{V_o}^{(o)}$ and this term would be zero thus recapturing the result we obtained in equation (12). For the case of the Mead[26] and Hones[15] models this is just the case. However, in the Finch and Leaton Model[7] the equator is not flat, but, in fact, is some topologically warped surface in space.

(b) Particle mirroring at high latitude:

In this case we may start with equation (83) and set $\eta_{V_o}^{(o)} \to 0$ (since the mirror magnetic field is now much much larger than the magnetic field at the equator). Under these conditions, (83) becomes

$$\alpha_c + \frac{\dfrac{B_{V_o} - B_{V_o}^{(o)}}{B_{V_o}} + \dfrac{B_{V_{m+}} - B_m}{2B_m} + \dfrac{B_{V_{m-}} - B_m}{2B_m}}{3 - 2\alpha_o\left[\dfrac{\partial \ln \Delta V}{\partial\alpha}\right]_{\alpha_o}} = 2k\sqrt{MR}\,. \tag{85}$$

The correction term (the second term on the left) becomes larger with increasing mirror point. This arises from the fact that the separation of constant "V" — constant

"B" curves in space (on a surface of constant α_o)(which, in fact, is a manifestation of the fact that the field is not β-independent) becomes more acute as we travel up a line of force away from the magnetic equator.

(vii) An Approximation Useful for Computing $B^{(o)}$

In the case of a model of the magnetosphere determined from measurements of the magnetic field near the earth's surface (ie: a Finch and Leaton model) which leads to a representation which is very close to that of a dipole, we may approximate β as

$$\beta \cong \phi + G(r,\theta,\phi) \tag{86}$$

where $G(r,\theta,\phi)$ is a small correction to ϕ. Because of this, we may replace an arbitrary function $F(r,\theta,\phi)$ by $F^{(o)}(r,\theta)$

$$F^{(o)} \simeq \frac{1}{2\pi} \int_0^{2\pi} F \, d\phi \equiv F_{(axial)} \tag{87}$$

so that the β-average is no more than the axially symmetric portion of the spherical harmonic expansion of the magnetic field. As a consequence of (87) we have

$$\frac{B_{V_o} - B_{V_o}^{(o)}}{B_{V_o}^{(o)}} \cong \frac{B_{V_o}(\text{non-axial})}{B_{V_o}(\text{axial})} \tag{88}$$

evaluated at α_o, V_o. In addition

$$\eta_{V_{m\pm}} - 1 \equiv \frac{B_{V_{m\pm}} - B_m}{B_m} \cong \frac{B_{V_{m\pm}}(\text{non-axial})}{B_{V_{m\pm}}(\text{axial})} \tag{89}$$

so that in this case we may compute the corrected first integral from the lines of force and the axial and non-axial symmetric portions of the spherical harmonic expansions of the field.

In the case of models which are based on satellite measurements and which give a spherical harmonic expansion which includes currents on the solar wind interface, the field is sufficiently distorted so that we may no longer make the approximation (87). However, in such models (see figure 9) we may make other approximations which replace it. For instance, the Mead field, for $r \leqslant 10$ earth radii, is reasonably close enough to the dipole field that we can make the approximation (87) but in the tail of the magnetosphere we must do the following. Since the shells of constant α_o (their intersection with the equatorial plane is shown in figure 9) run into the walls of the solar wind interface the invariant shell, which is close to this curve, only has significance over this same domain. In other words it only makes sense to talk about this section of the invariant shell on the night side of the cavity, because the particle will never drift out to the day side. Thus for the

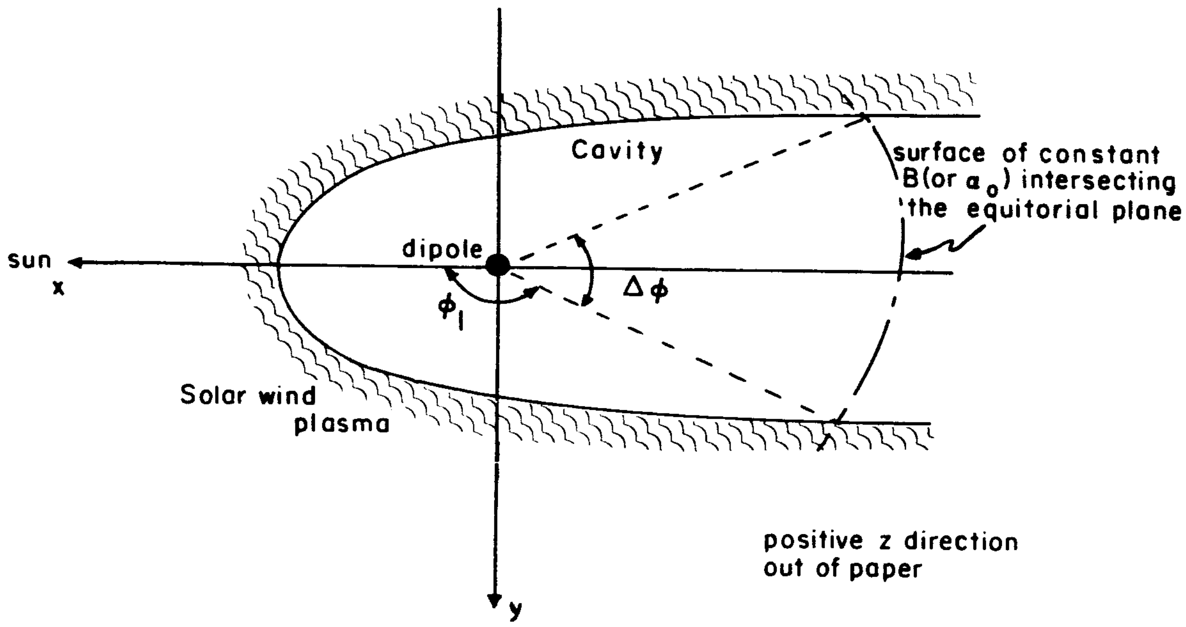

Cavity
surface of constant
B(or α_0) intersecting
the equitorial plane
sun
x
dipole
$\Delta\phi$
ϕ_1
Solar wind
plasma
positive z direction
out of paper
y

figure 9

section of the invariant surface which it does make contact with, we may replace

$$F^{(o)} = \frac{1}{\Delta\phi} \int_{\phi_1}^{\phi_1 + \Delta\phi} F \, d\phi \qquad (90)$$

where the $\Delta\phi$ is the azimuthal spread in the shell from wall to wall of the cavity. What (90) really does is capture the average component of the field "F" over the "α_o" surface in the tail of the magnetosphere. Given the shape of the magnetosphere, we may determine $\Delta\phi$ and thereby compute (90) from the spherical harmonic expansion of the field.

(V) COSMIC RAY CUTOFFS

A- The Equation of Motion

Next we consider particles with sufficient rigidity so that they may arrive at the earth's surface from infinity. As was mentioned previously in Section II, the critical velocity of "escape", ie: where the particle just becomes "untrapped", is known as the cosmic ray cutoff in that particular direction (with respect to the zenith) and at that point on the earth's surface. In this case the particle does not bounce from mirror point to mirror point since the energy of the particle and its magnetic moment are of such magnitude as to locate the mirror point below the earth's surface.

We now return to equation III-(50) and neglect the second term on the right hand side since, as previously mentioned, this is certainly of second order compared to the first term on the same side. According to Sauer and Ray[43] and Størmer theory[14,51] the radiation arriving from infinity enters the "allowed region" through the "jaws" of the Størmer plot, and then proceeds to spiral around the field line which originates in the horns of the inner allowed region. We therefore need to calculate the correction to the first integral over this path to the surface of the earth.

Hence we start with

$$\frac{c}{q}\frac{d}{dt}\left(\frac{\partial \chi}{\partial \beta}\right) = \vec{v}_D \cdot \nabla\alpha_c \quad . \tag{1}$$

We now proceed to evaluate this equation in a similar manner as was done in Section IV, except now we no longer take the bounce average but must consider the variation of (1) over the individual bounce.

B- Evaluating the Equation of Motion

We return to equations IV-(1) and IV-(2) and follow the first order guiding center from the equatorial "jaws" to some arbitrary final position. For the differential relationship between the time and magnetic scalar potential we have

$$dt = \frac{dl}{v_{/\!/}^{(o)}} = \frac{d\mathbf{V}}{v\ B_m(1-\eta^{(o)})^{1/2}\eta} \tag{2}$$

so that the time integral of (1) becomes

$$\frac{c}{q}\frac{\partial \chi}{\partial \beta}(V) - \frac{c}{q}\frac{\partial \chi}{\partial \beta}(V_o) = \frac{1}{B_m}\int_{V_o}^{V}\left(\frac{\vec{v}_D \cdot \nabla\alpha_c}{v}\right)\frac{dV}{\eta(1-\eta^{(o)})^{1/2}} \quad . \tag{3}$$

Again recall from III-(39)

$$\frac{\vec{v}_D \cdot \nabla\alpha_c}{v\ B_m} = \frac{\mu c}{vq\gamma}\left[\frac{2}{B_c} - \frac{1}{B_m}\right]\frac{\vec{B}_c \times \nabla B_c \cdot \nabla\alpha_c}{B_c^2} = -\frac{\mu c}{vq\gamma}\left[\frac{2}{\eta} - 1\right]\frac{\partial \eta}{\partial \beta} \tag{4}$$

so that (3) becomes

$$\frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot{\beta}}(V) - \frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot{\beta}}(V_o) = \frac{\mu c}{vq\gamma}\int_{V_o}^{V} \frac{\partial}{\partial \beta}\left[\frac{\frac{2}{\eta} + \ln(\eta)}{(1-\eta^{(o)})^{1/2}}\right] dV \quad . \qquad (5)$$

We may now reperform the analysis in Section IV equations IV-(30) to IV-(39) to arrive at an equivalent expression for (3) (good up to and including terms of second order) which is

$$\frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot{\beta}}(V) - \frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot{\beta}}(V_o) \approx \frac{R}{B_m}\frac{\partial}{\partial \beta}\left\{ \int_{V_o}^{V_\pm} \epsilon\left[\frac{1}{2} - \frac{1}{\eta^{(o)}}\right]\frac{dV}{(1-\eta^{(o)})^{1/2}} \right\} \qquad (6)$$

where the rigidity "R" has been defined previously. In addition, we have removed the partial derivative with respect to β from the inside to the outside of the integral on the right hand side of (6) since the drift velocity is small and therefore the scalar potential "V" is approximately independent of β.

We will now proceed to calculate the integral on the right hand side of (6). Whereas in section IV (equations IV-(55),(56),(57)) we chose a parabolic fit of the magnetic scalar potential between the equator and the mirror point of the particle, we now choose a parabolic fit between the equator and the point at at which we would like to compute the cutoff. We replace the equations IV-(55),(56),(57) by the following

$$\eta^{(o)} = \eta^{(o)}_{V_o} + (\eta^{(o)}_{V\pm} - \eta^{(o)}_{V_o})\left(\frac{V-V_o}{V_\pm - V_o}\right)^2 \tag{7}$$

and

$$\eta - \eta_{V_o} = \left[\frac{(\eta_{V_-} - \eta_{V_o})(V_+ - V_o)^2 - (\eta_{V_+} - \eta_{V_o})(V_- - V_o)^2}{(V_+ - V_o)(V_- - V_o)(V_+ - V_-)}\right](V - V_o)$$

$$+ \left[\frac{(\eta_{V_-} - \eta_{V_o})(V_+ - V_o) + (\eta_{V_+} - \eta_{V_o})(V_o - V_-)}{(V_+ - V_o)(V_o - V_-)(V_+ - V_-)}\right](V - V_o)^2 \tag{8}$$

where if "V_+" corresponds to the point at which we compute the cutoff (if it is in the northern hemisphere) then "V_-" corresponds to that value of V such that $B^{(o)}_{V_-} = B^{(o)}_{V_+}$ (in the southern hemisphere). If the cutoff is to be computed in the southern hemisphere then we interchange the roles of V_+ and V_- .

For all practical purposes, in computations taking place in fields like that of Finch and Leaton[7] or cavity fields like that of Mead,[26] we may make the following approximations

$$V_+ - V_o \cong V_o - V_- \equiv \delta V$$

$$\eta_{V_+} \cong \eta_{V_-} \tag{9}$$

so that (7) and (8) would simplify to read

$$\eta^{(o)} = \eta_{V_o}^{(o)} + (\eta_{V_+}^{(o)} - \eta_{V_o}^{(o)})\left(\frac{V-V_o}{\delta V}\right)^2 \tag{10}$$

$$\eta = \eta_{V_o} + (\eta_{V_+} - \eta_{V_o})\left(\frac{V-V_o}{\delta V}\right)^2 \tag{11}$$

Let

$$\left.\begin{aligned} m^2 &\equiv \frac{\eta_{V_o}^{(o)}}{\eta_{V_+}^{(o)} - \eta_{V_o}^{(o)}} & n^2 &\equiv \frac{1-\eta_{V_o}^{(o)}}{\eta_{V_o}^{(o)}} \\ p^2 &\equiv \frac{1-\eta_{V_o}^{(o)}}{\eta_{V_+}^{(o)} - \eta_{V_o}^{(o)}} & \zeta_V^{(o)} &\equiv \frac{B_V^{(o)}}{B_{V_o}^{(o)}} \\ \epsilon_{V_o} &\equiv \frac{\eta_{V_o} - \eta_{V_o}^{(o)}}{\eta_{V_o}^{(o)}} & \epsilon_{V_+} &\equiv \frac{\eta_{V_+} - \eta_{V_+}^{(o)}}{\eta_{V_+}^{(o)}} . \end{aligned}\right\} \tag{12}$$

Using (10),(11) and (12) in the definition of $\epsilon_\pm$ (equation IV-(66)) we find that in this case $\epsilon_+ = \epsilon_- \equiv \epsilon$ so that we have

$$\epsilon = m^2 \left\{ \frac{\epsilon_{V_o} + \left(\epsilon_{V_+}\left(\frac{\eta_{V_+}^{(o)}}{\eta_{V_o}^{(o)}}\right) - \epsilon_{V_o}\right)\left(\frac{V-V_o}{\delta V}\right)^2}{m^2 + \left(\frac{V-V_o}{\delta V}\right)^2} \right\} . \tag{13}$$

In addition,

$$\frac{\left(1-\eta^{(0)}\right)^{1/2}}{\eta^{(0)}} = \frac{1}{\left(\eta^{(0)}_{V_0}\right)^{1/2}} \frac{m\left[\rho^2-\left(\frac{V-V_0}{\delta V}\right)^2\right]^{1/2}}{m^2+\left(\frac{V-V_0}{\delta V}\right)^2}$$

(14)

$$(1/2)\frac{1}{(1-\eta^{(0)})^{1/2}} = \frac{1}{2(\eta^{(0)}_{V_0})^{1/2}} \frac{m}{\left[m^2-\left(\frac{V-V_0}{\delta V}\right)^2\right]^{1/2}} \quad . \tag{15}$$

If we define

$$U \equiv \frac{V-V_0}{\delta V} \tag{16}$$

then (6) becomes

$$\frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot\beta}(V) - \frac{c}{q}\frac{\partial \mathcal{L}}{\partial \dot\beta}(V_0) = \frac{R}{B_m} \frac{\delta V}{(\eta^{(0)}_{V_+}-\eta^{(0)}_{V_0})^{1/2}} \frac{\partial \Lambda}{\partial \beta}$$

(17)

where

$$\Lambda \equiv \int_0^1 \frac{\epsilon(\rho^2-U^2)^{1/2}\,dU}{m^2+U^2} + \frac{1}{2}\int_0^1 \frac{\epsilon\,dU}{(m^2-U^2)^{1/2}} \quad .$$

(18)

The evaluation of Λ is straight forward as was a similar calculation in Section IV. Placing (16) in (13) and the result in (18) we may express it as

$$\Lambda = \epsilon_{V_o} m^2 F_2^1(m^2, p^2) + \left(\epsilon_{V_+} \frac{\eta_{V+}^{(0)}}{\eta_{V_o}^{(0)}} - \epsilon_{V_o}\right) m^2 \left[F_1^1(m^2, p^2) - m^2 F_2^1(m^2, p^2)\right]$$

$$+ \frac{1}{2} \epsilon_{V_o} m^2 F_1^{-1}(m^2, p^2)$$

$$+ \frac{1}{2}\left(\epsilon_{V_+} \frac{\eta_{V+}^{(0)}}{\eta_{V_o}^{(0)}} - \epsilon_{V_o}\right) m^2 \left[F_0^{-1}(m^2, p^2) - m^2 F_1^{-1}(m^2, p^2)\right] \tag{19}$$

using the notation of III-(59). Using the tabulations of Appendix IX, we may quickly evaluate (19) to give

$$\Lambda = \left[\left[\frac{\epsilon_{V_o}(1+n^2)^{1/2}}{2}\right] \cot^{-1} \left\{\frac{(p^2-1)^{1/2}}{(1+n^2)^{1/2}}\right\}\right] + \left[\epsilon_{V_+} \frac{\eta_{V_+}^{(o)}}{\eta_{V_o}^{(o)}} - \epsilon_{V_o}\right] \times$$

$$\left[\frac{(1+n^2)^{1/2}}{2} m^2 \cot^{-1} \left\{\frac{(p^2-1)^{1/2}}{(1+n^2)^{1/2}}\right\} - \frac{m^2}{2} \sin^{-1}\left(\frac{1}{p}\right)\right.$$

$$\left. - \frac{m^2(p^2-1)^{1/2}}{2(1+m^2)}\right] . \tag{20}$$

The partial derivative of (20) with respect to β may now be calculated. Notice that the entire β-dependence resides in the ϵ_{V_o} and ϵ_{V_+} terms. The "n" and "p" terms are functions only of the first (β-independent) terms of the Fourier series of the magnetic field. Therefore, we must first compute, from equation (12), the partial derivatives

$$\frac{\partial \epsilon_{V_o}}{\partial \beta} = \frac{1}{B^{(o)}_{V_o}} \left[\frac{\partial B}{\partial \beta}\right]_{V_o} = \frac{1}{B^{(o)}_{V_o}} \left[\frac{(\vec{B} \times \nabla\alpha)\cdot\nabla B}{B^2}\right]_{V_o} = \frac{1}{B_{V_o}} \left[\frac{(\vec{B} \times \nabla B)\cdot\nabla\alpha}{B^2}\right]_{V_o} \quad (21)$$

$$\frac{\partial \epsilon_{V_+}}{\partial \beta} = \frac{1}{B^{(o)}_{V_+}} \left[\frac{\partial B}{\partial B}\right]_{V_+} = \frac{1}{B^{(o)}_{V_+}} \left[\frac{(\vec{B} \times \nabla\alpha)\cdot\nabla B}{B^2}\right]_{V_+} = \frac{1}{B_{V_+}} \left[\frac{(\vec{B} \times \nabla B)\cdot\nabla\alpha}{B^2}\right]_{V_+} \quad (22)$$

where we have used equation (20) of Appendix II. Placing this in equation (20) after taking the derivative with respect to β we obtain, after some algebra

$$Q \equiv -\frac{\delta V}{(B_{V_o})^{1/2}(B_m)^{1/2}(\varsigma^{(o)}_{V_+}-1)^{1/2}} \frac{\partial \Lambda}{\partial \beta}$$

$$= \frac{\delta V}{B^{(o)}_{V_o}\sqrt{\varsigma^{(o)}_{V_+}-1}} \left\{ \frac{1}{2B^{(o)}_{V_o}} \left[\cot^{-1}\left(\frac{1-\eta^{(o)}_{V_+}}{\varsigma^{(o)}_{V_+}-1}\right)^{1/2}\right] \left[\frac{\vec{B}\times\nabla B\cdot\nabla\alpha}{B^2}\right]_{V_o} \right.$$

$$+ \left[\left[\frac{\vec{B}\times\nabla B\cdot\nabla\alpha}{B^2}\right]_{V_+} - \left[\frac{\vec{B}\times\nabla B\cdot\nabla\alpha}{B^2}\right]_{V_o}\right] \left[\frac{\cot^{-1}\left(\frac{1-\eta^{(o)}_{V_+}}{\varsigma^{(o)}_{V_+}-1}\right)^{1/2}}{2B^{(o)}_{V_o}\left[\varsigma^{(o)}_{V_+}-1\right]} \right.$$

$$\left.\left. - \frac{\sin^{-1}\left(\frac{\varsigma^{(o)}_{V_+}-1}{\frac{B_m}{B^{(o)}_{V_o}}-1}\right)}{2\left(B_m B^{(o)}_{V_o}\right)^{1/2}\left[\varsigma^{(o)}_{V_+}-1\right]} - \frac{\left(\frac{1-\eta^{(o)}_{V_+}}{\varsigma^{(o)}_{V_+}-1}\right)^{1/2}}{2B^{(o)}_{V_+}} \right] \right\} \quad (23)$$

where we have defined the function "Q" in such a way that it will make further calculations less clumsy. Placing the result (23) in (17) we have for the first integral

$$\alpha + \frac{R|\nabla\alpha|\cos\psi}{B} + R\,Q = 2k\,(M\,R)^{1/2} \qquad (24)$$

where we have replaced the value of $\frac{\partial \mathcal{L}}{\partial \beta}$ by the expression previously quoted in equation II-(54) (the left side), "k" is a unitless constant defined in the same way as was defined in IV-(82) determined by the initial conditions of the problem, ψ is the angle between $\vec{v}$ and $\vec{B} \times \nabla\alpha$ evaluated at the actual particle position. Equation (24) is the corrected first integral applicable for high rigidity particles.

C- Cutoffs Near the Equator

It is interesting to show that the correction term "Q" goes to zero as the observation point approaches the equator. This is intuitively as it should be since the particle never gets the opportunity to "drift" off the constant α shell upon which it starts. Referring back to equation (23) we would like to take the limit as $V_+ \to V_o$ (ie; as $\delta v \to 0$, $\zeta^{(o)}_{V_+} \to 1$, $\left[\frac{\vec{B} \times \nabla B \cdot \nabla\alpha}{B^2}\right]_{V_+} \to \left[\frac{\vec{B} \times \nabla B \cdot \nabla\alpha}{B^2}\right]_{V_o}$).

First notice that for small values of δV, $B_{V_+}^{(o)} \rightarrow B_{V_o}^{(o)}$ and we may approximate

$$\sqrt{\zeta_{V_+}^{(o)} - 1} = \frac{1}{\sqrt{B_{V_o}^{(o)}}} \sqrt{B_{V_+}^{(o)} - B_{V_o}^{(o)}} \approx \frac{\sqrt{k'(V_+ - V_o)^2}}{\sqrt{B_{V_o}^{(o)}}} = \frac{\sqrt{k'}\,(V_+ - V_o)}{\sqrt{B_{V_o}^{(o)}}}$$

$$\approx \frac{\sqrt{k'}}{\sqrt{B_{V_o}^{(o)}}}\,\delta V \qquad (25)$$

since the magnetic field $B^{(o)}$ is a minimum at V_o and therefore varies with the square of the magnetic potential in this vicinity. As a consequence of this, the term $\frac{\delta V}{(\zeta_{V_+}^{(o)} - 1)^{\frac{1}{2}}}$ in Q becomes

$$\frac{\delta V}{(\zeta_{V_+}^{(o)} - 1)^{\frac{1}{2}}} \rightarrow \frac{\sqrt{B_{V_o}^{(o)}}}{\sqrt{k'}} \qquad (26)$$

as we become arbitrarily close to making $\delta V \rightarrow 0$. This part of the expression therefore approaches a finite limit. All that remains to be shown is that the remaining terms (multiplying this) approach zero as $\delta V \rightarrow 0$.

If we Taylor expand $\cot^{-1} X$ for large values of X it is easy to show that it approaches $\frac{1}{X}$. Hence, it is clear that

$$\lim_{\delta V \to 0} \left\{ \cot^{-1} \left(\frac{1-\eta_{V_+}^{(0)}}{\varsigma_{V_+}^{(0)}-1} \right)^{\frac{1}{2}} \right\}$$

$$= \lim_{\varsigma_{V_+}^{(0)} \to 1} \left\{ \frac{(\varsigma_{V_+}^{(0)} - 1)^{\frac{1}{2}}}{(1-\eta_{V_+}^{(0)})^{1/2}} \right\} = 0 \qquad (27)$$

and therefore the first term on the right hand side of (23) is zero. Next consider the terms in the second bracket of the second term of the right hand side of (23). Again taking the same limit (noting that $\sin^{-1} X \to X$ for small values of X) we have for this bracketed term

$$\lim_{\varsigma_{V_+}^{(0)} \to 1} \left\{ \frac{\cot^{-1}\left(\frac{1-\eta_{V_+}^{(0)}}{\varsigma_{V_+}^{(0)}-1} \right)^{\frac{1}{2}}}{2B_{V_0}^{(0)}(\varsigma_{V_+}^{(0)}-1)} - \frac{\sin^{-1}\left(\frac{\varsigma_{V_+}^{(0)}-1}{\frac{B_m}{B_{V_0}^{(0)}}-1} \right)^{\frac{1}{2}}}{2\sqrt{B_m B_{V_0}^{(0)}}(\varsigma_{V_+}^{(0)}-1)} - \frac{(1-\eta_{V_+}^{(0)})^{\frac{1}{2}}}{2B_{V_+}^{(0)}(\varsigma_{V_+}^{(0)}-1)^{1/2}} \right\} \qquad (28)$$

$$= \lim_{\varsigma_{V_+}^{(0)} \to 1} \left\{ \frac{1}{2B_{V_0}^{(0)}(\varsigma_{V_+}^{(0)}-1)^{1/2}(1-\eta_{V_+}^{(0)})^{1/2}} - \frac{1}{2\sqrt{B_m B_{V_0}^{(0)}}(\varsigma_{V_+}^{(0)}-1)^{1/2}\left(\frac{B_m}{B_{V_0}^{(0)}}-1\right)^{1/2}} - \frac{(1-\eta_{V_+}^{(0)})^{\frac{1}{2}}}{2B_{V_+}^{(0)}(\varsigma_{V_+}^{(0)}-1)^{1/2}} \right\}. \qquad (29)$$

Using the definitions in equation (12) this may be written

$$= \lim_{\substack{m\to\infty \\ p\to\infty}} \frac{1}{(B_m B_{V_o}^{(o)})^{\frac{1}{2}}} \left\{ \frac{(1+n^2)m^2}{2(p^2-1)^{\frac{1}{2}}} - \frac{m^2}{2p} - \frac{m^2(p^2-1)^{\frac{1}{2}}}{2(1+m^2)} \right\} \qquad (30)$$

combining the third and the first term this becomes

$$= \lim_{\substack{m\to\infty \\ p\to\infty}} \left\{ \frac{1}{(B_m B_{V_o}^{(o)})^{\frac{1}{2}}} \right\} \left\{ \frac{(1+n^2)(1+m^2)-p^2+1}{(p^2-1)^{\frac{1}{2}}(1+\frac{1}{m^2})} - \frac{m^2}{p} \right\} \qquad (31)$$

or

$$= \lim_{\substack{m\to\infty \\ p\to\infty}} \left\{ \frac{1}{(B_m B_{V_o}^{(o)})^{\frac{1}{2}}} \right\} \left\{ \frac{2+n^2+m^2}{p(1+\frac{1}{m^2})(1-\frac{1}{p^2})^{\frac{1}{2}}} - \frac{m^2}{p} \right\} . \qquad (32)$$

Binonially expanding the terms $(1+\frac{1}{m^2})^{-1}$ and $(1-\frac{1}{p^2})^{-\frac{1}{2}}$, keeping the lowest order terms, (32) becomes

$$= \lim_{\substack{m\to\infty \\ p\to\infty}} \left\{ \frac{1}{(B_m B_{V_o}^{(o)})^{\frac{1}{2}}} \right\} \left\{ \left(\frac{2+n^2+m^2}{p}\right)(1-\frac{1}{m^2})(1+\frac{1}{2p^2}) - \frac{m^2}{p} \right\} \qquad (33)$$

or, multiplying out the terms and cancelling we are left with

$$= \lim_{\substack{m\to\infty \\ p\to\infty}} \left\{ \frac{1}{(B_m B_{V_o}^{(o)})^{\frac{1}{2}}} \right\} \left\{ \frac{2+n^2}{p} - \frac{2+n^2+m^2}{p\ m^2} - \frac{\frac{2+n^2}{p^2}+n^2}{2p} - \frac{2+n^2+m^2}{2p^3m^2} \right\} \qquad (34)$$

where we have used the fact that $p = mn$. Equation (34) obviously becomes zero ("n" always remains finite) and therefore we have shown that, indeed $Q \to 0$ as $\delta V \to 0$.

D- The Cosmic Ray Cutoff Rigidity

Returning the equation (24), we now have a quadratic equation in "R" that may be solved yielding the corrected version of equation II-(55). Solving we have

$$R = \frac{\dfrac{\alpha^2}{Mk^2}}{\left[1 + \left\{1 - \dfrac{\alpha}{Mk^2}\left(\dfrac{|\nabla\alpha|\cos\psi}{B} + Q\right)\right\}^{1/2}\right]^2} \quad . (35)$$

The cutoff rigidity in the direction "ψ" is then obtained from (35) by setting $k = 1$.[38] We have

$$R_{cut} = \frac{\dfrac{\alpha^2}{M}}{\left[1 + \left\{1 - \dfrac{\alpha}{M}\left(\dfrac{|\nabla\alpha|\cos\psi}{B} + Q\right)\right\}^{1/2}\right]^2} \quad . \quad (36)$$

The cutoff rigidity is therefore seen to be altered by the fact that the field is not independent of β. This correction is manifest in the term "Q". The vertical cutoff rigidity is obtained from (36) by evaluating "Q" at the earth's surface, and choosing "ψ" as the angle between the zenith (vertical arrival) and the $\vec{B} \times \nabla\alpha$ direction. For fields like that of Finch and Leaton[7] which are basically dipole in nature we may make the following approximations. The "exact" vertical cutoff rigidity in the dipole field is found (as can be shown by simple geometry) by setting $\cos\psi = 0$ and $Q = 0$ in

(36). This yields

$$R_{Cut.}(\text{vertical, dipole}) = \frac{\alpha^2}{4M} \quad . \tag{37}$$

Since the fields which display a close to dipole symmetry are not far from β-independent (since in this case, $\beta \approx \phi$, and the field is reasonably close to axially symmetric) it is clear that both "Q" and $\cos\psi$ are small thus indicating that

$$\frac{|\nabla\alpha| \cos\psi}{M\,k^2\,B} \ll 1 \tag{38}$$

$$\frac{\alpha\,Q}{Mk^2} \ll 1 \quad . \tag{39}$$

We may further simplify the form of (36) by defining several unitless quantities. Define

$$\left.\begin{array}{ll} R_e = 6.3712 \times 10^6 m & \overline{\alpha} = \alpha/(B_e R_e^2) \\ B_e = 0.3120 \text{ gauss} & \overline{Q} = Q\alpha/M \\ & \overline{\nabla\alpha} = R_e \nabla\overline{\alpha} \\ & \overline{\nabla B} = R_e \nabla\overline{B} \\ & \overline{B} = B/B_e \\ & \overline{V} = V/B_e R_e \quad . \end{array}\right\} \tag{40}$$

We may therefore express (36) as

$$R_{Cut} = \frac{\overline{\alpha}^2_{V_+} B_e R_e}{\left[1 + \left\{1 - \left(\frac{\overline{\alpha}_{V_+} \left|\overline{\nabla\alpha}\right|_{V_+} \cos\psi}{B_{V_+}} + \overline{Q}\right)\right\}^{\frac{1}{2}}\right]^2} \text{ (gauss-m)} \tag{41}$$

where the subscript "V_+" means that the quantity should be evaluated at the point on the surface of the earth.

We may further make the approximation that

$$B_{V_o} \approx B^{(o)}_{V_o} \approx \frac{\alpha_c^3}{M^2} \approx \frac{\alpha^3}{M^2} \tag{42}$$

$$\zeta^{(o)}_{V_+} = \frac{B^{(o)}_{V_+}}{B^{(o)}_{V_o}} \approx \frac{M^2 B_{V_+}}{\alpha^3} \quad . \tag{43}$$

In addition, we may compute the mirror point as follows. At the point of impact on the surface of the earth, (coming in along the zenith) we compute the magnetic moment of the particle

$$\mu = \frac{\left(p_\perp^2\right)_{V_+}}{2m_o B_{V_+}} = \frac{p^2 \left|\frac{\hat{e}_\eta \times \vec{B}_{V_+}}{B_{V_+}}\right|^2}{2m_o B_{V_+}} \tag{44}$$

which is the same value of magnetic moment that would be computed at the mirror point

$$\mu = \frac{p^2}{2m_o B_m} \quad . \tag{45}$$

Equating (44) and (45) we may solve for B_m as a function of B_{V_+} as

$$B_m = \frac{B_{V_+}}{\left|\hat{e}_n \times \hat{e}_{\vec{B}_{V_+}}\right|^2} \tag{46}$$

where $\hat{e}_n$ is the unit normal to the earth (in the direction of the zenith) and $e_{\vec{B}_{V_+}}$ is the unit vector at the observation point in the direction of the magnetic field. In unitless form we may therefore express $\overline{Q}$ as

$$\overline{Q} = \frac{(\overline{V}_+ - \overline{V}_o)}{\overline{\alpha}^2\left(\frac{\overline{B}_{V+}}{\overline{\alpha}^3} - 1\right)^{1/2}} \Bigg\{ \frac{\left(\hat{e}_B \times \overline{\nabla B} \cdot \overline{\nabla\alpha}\right)_{V_o}}{\overline{\alpha}^6} \cot^{-1}\left(\frac{1 - \left|\hat{e}_n \times \hat{e}_{\vec{B}_{V_+}}\right|^2}{\frac{\overline{B}_{V+}}{\overline{\alpha}^3} - 1}\right)^{\frac{1}{2}}$$

$$+ \left[\frac{\left(\hat{e}_B \times \overline{\nabla B} \cdot \nabla\alpha\right)_{V_+}}{\overline{B}_{V+}} - \frac{\left(\hat{e}_B \times \overline{\nabla B} \cdot \overline{\nabla\alpha}\right)_{V_o}}{\overline{B}_{V_o}}\right] \times \tag{47}$$

$$\left[\frac{1}{2\overline{\alpha}^3} \cot^{-1}\left(\frac{1 - \left|\hat{e}_n \times \hat{e}_{\vec{B}_{V_+}}\right|^2}{\frac{\overline{B}_{V+}}{\overline{\alpha}^3}}\right)^{\frac{1}{2}} - \frac{\sin^{-1}\left(\frac{\frac{\overline{B}_{V+}}{\overline{\alpha}^3} - 1}{\frac{\overline{B}_{V+}}{\overline{\alpha}^3\left|\hat{e}_n \times \hat{e}_{\vec{B}_{V_+}}\right|^2} - 1}\right)^{\frac{1}{2}}}{2\sqrt{\frac{\overline{\alpha}^3\,\overline{B}_{V+}}{\left|\hat{e}_n \times \hat{e}_{B_{V+}}\right|^2}}\left[\frac{\overline{B}_{V+}}{\overline{\alpha}^3} - 1\right]} - \frac{\left(\frac{1 - \left|\hat{e}_n \times \hat{e}_{\vec{B}_{V_+}}\right|^2}{\frac{\overline{B}_{V+}}{\overline{\alpha}^3} - 1}\right)^{1/2}}{2\,\overline{B}_{V+}}\right]\Bigg\}$$

Equation (41) with (47) now allows us to compute the new corrected cutoff rigidity in the vertical direction on the earth's surface.

(VI) NUMERICAL EXAMPLE

As an example, we will now work out the exact solution of energy-dependent shell splitting for particles mirroring within 7 earth radii of the origin in a field model proposed by Mead.[26]

$$\vec{B}(r,\theta,\phi) = \frac{M}{\rho^3}\,\hat{e}_z + \left(\left|\bar{g}_1^0\right| + \sqrt{3}\left|\bar{g}_2^1\right| \;\rho\cos\phi\right)\hat{e}_z \qquad (1)$$

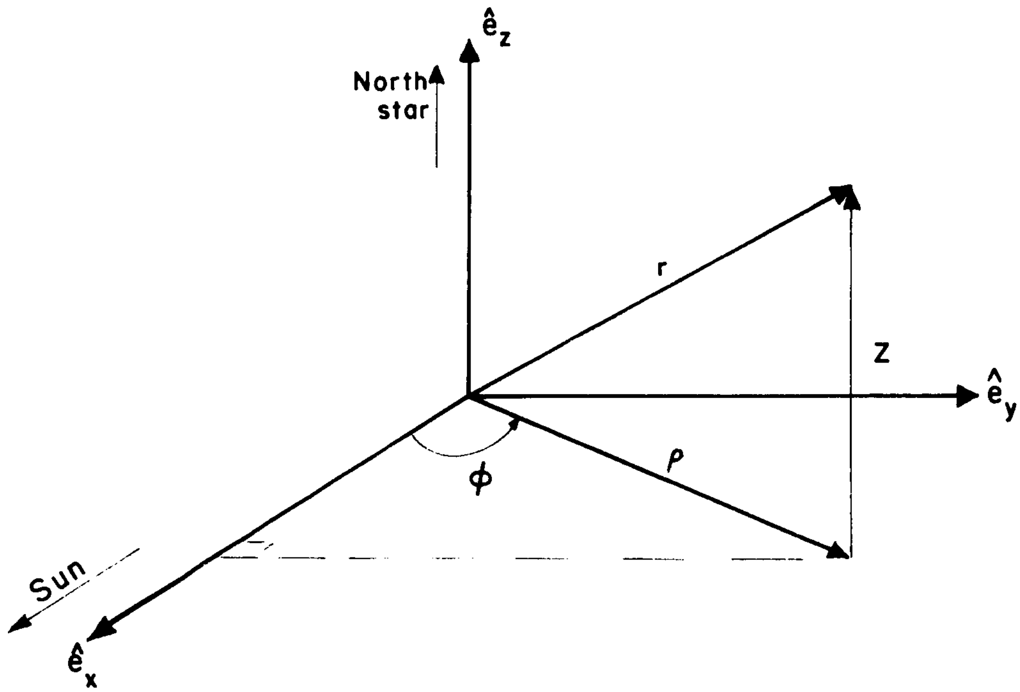
where the first term in the two-dimensional dipole ($M = 7.72 \times 10^{10}$ gauss-km^3 = dipole moment of the earth), and the second term is due to surface currents in the magnetosphere ($\bar{g}_1^0 = 2.515 \times 10^{-4}$ gauss, $\bar{g}_2^1 = 1.215 \times 10^{-5}$ gauss/km). The coordinate system is shown in Figure 10.

We will first proceed to find an α and β describing the field of equation (1). Since the magnetic field is uniform in the "z" direction (along the lines of force) we may choose our α as any function of this constant magnetic field intensity. In particular, let us choose it the particular function

$$\alpha = \left(\frac{B}{M}\right)^{1/3} \qquad (2)$$

We may also choose V to be, simply

$$V = z \qquad (3)$$



<u>figure 10</u>

and

$$\overline{\mu} = B \quad . \tag{4}$$

It is much clearer to write everything that follows in a unitless form. To obtain this result we define

$$\begin{aligned}
\overline{\beta} &\equiv \beta / r_e^3 B_e \\
\overline{\alpha} &\equiv \alpha r_e \\
\overline{\rho} &\equiv \frac{\rho}{r_e} \\
B_e &= \frac{M}{r_e^3} \\
a &= \frac{\left|\overline{g}_1^0\right|}{B_e} \\
b &= \sqrt{3}\, r_e \left|\overline{g}_2^1\right| / B_e
\end{aligned} \tag{5}$$

where "r_e" is the earth radius (6317.2 km). (1) and (2) then become

$$\vec{B}(\overline{\rho},\phi) = B_e \left[1/\overline{\rho}^3 + a + b\overline{\rho} \cos\phi \right] \hat{e}_z \tag{6}$$

$$\overline{\alpha}(\overline{\rho},\phi) = \left[1/\overline{\rho}^3 + a + b\overline{\rho} \cos\phi \right]^{1/3} \quad . \tag{7}$$

"β" may be computed by solving the partial differential equation I-(4) using (6), (7) above. That is, the solution to

$$3\left[\left(1/\,\bar{\rho}^{3}\right)+ a + b\bar{\rho}\ \cos\phi\right]^{5/3} = \left[-\,3/\,\bar{\rho}^{5} + \frac{b\ \cos\ \phi}{\bar{\rho}}\right]\frac{\partial\bar{\beta}}{\partial\phi} +$$

$$\left[b\ \sin\ \phi\right]\ \frac{\partial\bar{\beta}}{\partial\rho} \qquad . \qquad (8)$$

The solution to (8) can be quickly found by the method of characteristics, ie. we must solve the differential equation

$$\frac{d\bar{\rho}}{d\phi} = \frac{b\ \sin\ \phi}{-\,\frac{3}{\bar{\rho}^{5}} + \frac{b\ \cos\ \phi}{\bar{\rho}}} \qquad . \qquad (9)$$

The solution to this equation is not difficult to find and is given by

$$\frac{1}{\bar{\rho}^{3}} + b\bar{\rho}\ \cos\ \phi = k \qquad (10)$$

where "k" is a constant. The solution to (8) is then given by

$$\bar{\beta}(\bar{\rho},\phi) = f(k) + \int^{\phi} \frac{3(k + a)^{5/3}\ \bar{\rho}'^{2}\ d\phi'}{(-4/\ \bar{\rho}'^{3} + k)} \qquad (11)$$

where $\bar{\rho}'$ is the solution of (10), and f(k) is an arbitrary function of "k".

Equations (3), (4), (7), and (11) are thus a complete description of the field (6). The trapped radiation shells, to the first order, are given by α = constant. However, because of the correction term, the invariant shells split as a function of energy, and the corrected expression is given by equation IV-(14). Ie:

$$\left.\begin{aligned} &\overbrace{\alpha + \vec{a}_c \cdot \nabla\alpha}^{(i)}\ \overbrace{- \vec{a}_c^{\,\prime} \cdot \nabla\alpha_c}^{(ii)} = 2\overline{\gamma} \\ &\text{where} \\ &\vec{a}_c = \frac{mc}{q}\ \frac{\vec{v} \times \vec{B}}{B^2} \\ &\vec{a}_c^{\,\prime} = \frac{mc}{2q}\ \frac{\vec{v}_D \times \vec{B}_c}{B_c^2} \\ &\vec{v}_D = \frac{\mu c}{q\gamma}\ \frac{\vec{B}_c \times \nabla B_c}{B_c^2} \end{aligned}\right\} \qquad (12)$$

$\vec{v}_D$ being the drift velocity, $\vec{a}_c$ the Larmour radius, and $\vec{a}_c^{\,\prime}$ an equivalent distance much smaller than the Larmour radius because $|\vec{v}_D| \ll |\vec{v}|$. Whey applying the above to trapped radiation, we will be considering particles of sufficiently low rigidity that we may neglect second and higher derivatives of "α" when expanding it about the instantaneous position of the particle to obtain its value at the guiding center. That is, term (i) of (12) is given approximately as the value of α_c at the guiding center.

Term (ii) of (12), after some algebra, can be shown to be

$$(ii) = -\frac{3}{4}\ \frac{a_c^2\left|\nabla\alpha_c\right|^2}{\alpha_c} \quad . \tag{13}$$

This term over α is of order the square of the second term of (i) of (12) over α and is therefore a small correction to the fact that (i) is the value of α at the guiding center, yet it is larger than the next (second derivative) term in the Taylor Series expansion which is neglected when approximating (i) by α_c. So indeed, the new equation predicting invariant shells becomes

$$\alpha_c\left[1-\frac{3}{4}\ \frac{a_c^2\left|\nabla\alpha\right|_c^2}{\alpha_c^2}\right] = 2\overline{\gamma} \quad . \tag{14}$$

For the given field model, we may now evaluate the space curves that (14) predicts. Placing (6), (7) and (12) in (14) we have the result

$$\overline{\alpha}_c\left[1-\frac{1}{12}\left\{\left(\frac{R/B_e}{r_e}\right)^2\frac{\left(\frac{9}{\overline{\rho}^8}+b^2-\left(\frac{6b}{\overline{\rho}^4}\right)\cos\phi\right)}{\left(\frac{1}{\overline{\rho}^3}+a+b\overline{\rho}\cos\phi\right)^2}\right\}\right]=2k_1 \tag{15}$$

where $k_1 = \frac{\overline{\gamma}}{r_e}$, $\frac{R}{r_e B_e}$ is the ratio of the Larmour radius at the earth's surface and the radius of the earth (very small indeed), $a = 8.09 \times 10^{-4}$, $b = 6.78 \times 10^{-5}$ and r_e = the earth's radius = 6.3172×10^9 cm.

That the second term on the left-hand side of (15) is small compared to unity may be shown as follows. The bracketed part of the second term of (15) is always smaller than about 10 whenever $7 \geqslant \bar{\rho} \geqslant 1$, so that the second term is of order $\frac{R/B_e}{r_e}$. For relativistic particles, the relationship between rigidity and energy is given by

$$\left.\begin{aligned} \eta &= (1 + \xi)^{1/2} - 1 \\ &\text{where} \\ \eta &= \frac{E}{m_o c^2} \\ \xi &= \frac{R}{(m_o c^2)/q} \end{aligned}\right\} \qquad (16)$$

so that if we select an unusually high rigidity particle, say 10 bv, which, using (16), corresponds to about 10 bev electrons or protons, we find $\frac{R/B_e}{r_e} \approx 0.05$, and hence the correction term in (15) is of order 0.0025. It follows that we may safely expand the terms in (15) with the binomial expansion and obtain as a result (keeping the lowest order terms)

$$\bar{\alpha}_c \approx 2k_1 \left[1 + \frac{1}{12} \left\{ \left(\frac{R/B_e}{r_e} \right)^2 \frac{\left(\frac{9}{\bar{\rho}^8} + b^2 - \left(\frac{6b}{\bar{\rho}^4} \right) \cos\phi \right)}{\left(\frac{1}{\bar{\rho}^3} + a + b\bar{\rho}\cos\phi \right)^2} \right] . \qquad (17)$$

Incidentally, that 10 bv is a very large overestimate and therefore overly pessimistic is seen by noting that observed trapped radiation energies are given below[19].

TABLE I

Electrons:	20-600 Kev	inner-Van Allen belts
	40-200 Kev	outer-Van Allen belts
	7-8 Bev	Cosmic Rays
Protons:	40 Mev	inner-Van Allen belts
	30- 60 Mev	outer-Van Allen belts
	7- 8 Bev	Cosmic Rays

Incidentally, note: 1 Bv = 3.3352×10^6 gauss-cm.

To demonstrate the change in invariant surfaces that (17) predicts we will now compute the approximate invariant surfaces for two particles both starting at $\phi = 90^\circ$ and the same equatorial radius but one with $R = 0$ and the other for $R = 10$ bv, an unusually high rigidity for trapped radiation. All other trapped particles will have, respectively, invariant shells which fall within these limits.

First define

$$\bar{\rho}\,(90^\circ) \equiv \bar{\rho}_o \tag{18}$$

and $\epsilon^R(\phi)$ such that

$$\bar{\rho}^R(\phi) = \bar{\rho}_o(1 + \epsilon^R(\phi)) \tag{19}$$

$$\epsilon^R(90^\circ) = 0$$

where $\bar{\rho}^R(\phi)$ is the solution to (17). The facility in such definitions is clear from the fact that, for $7 \geqslant \rho \geqslant 1$ the invariant surfaces are not far from circles of radius $\bar{\rho}_o$.

In the case R = 0, we have

$$\frac{1}{\left[\bar{\rho}^o(\phi)\right]^3} + a + b(\bar{\rho}^o(\phi))\cos\phi = \text{constant} \quad (20)$$

for the invariant shells. The constant is evaluated by placing the initial condition that $\bar{\rho}^o(\phi) = \bar{\rho}_o$ when $\phi = 90^o$. (20) then becomes

$$\frac{1}{\left[\bar{\rho}^o(\phi)\right]^3} + a + b\left[\bar{\rho}^o(\phi)\right] \cos\phi = \frac{1}{(\bar{\rho}_o)^3} + a \; . \quad (21)$$

Placing (19) in (21) expanding binomially and keeping the lowest order terms in $\epsilon^o(\phi)$ we find

$$\epsilon^o(\phi) = \frac{b \cos\phi}{\frac{3}{\bar{\rho}_o^4} - b \cos\phi} \quad (22)$$

which predicts the invariant surfaces

$$\bar{\rho}^o(\phi) = \bar{\rho}_o\left[\frac{1}{1 + \frac{b\bar{\rho}_o^4}{3}\cos\phi}\right] \approx \bar{\rho}_o\left(1 + \frac{b\bar{\rho}_o^4}{3}\cos\phi\right) . \quad (23)$$

On the other hand, we may return to (17), utilizing the result (23) to derive an expression for $\epsilon^R(\phi)$ corresponding to a particle of rigidity "R" having its guiding center start at the same position as the particle $R = 0$. Under the circumstances of this boundary condition, we may evaluate the constant $2k_1$ in (17) to be

$$2k_1 = \frac{\left[\dfrac{1}{\bar{\rho}_o^3} + a \right]^{1/3}}{1 + \dfrac{1}{12}\left(\dfrac{R/B_e}{r_e} \right)^2 \left[\dfrac{\dfrac{9}{\bar{\rho}_o^8} + b^2}{\left(\dfrac{1}{\bar{\rho}_o^3} + a \right)^2} \right]} \quad . \qquad (24)$$

Placing (19) and (24) in (17) we have, putting the equation in unitless form

$$\frac{1}{\left[1 + \epsilon^R(\phi)\right]^3} + \bar{a} + \bar{b}\left(1 + \epsilon^R(\phi)\right)\cos\phi$$

$$= \frac{1 + \bar{a}}{\left[1 + \left(\dfrac{3\ell^2}{4} \right) \dfrac{\left(1 + \left(\frac{\bar{b}}{3}\right)^2\right)}{(1 + \bar{a})^2} \right]^3} \left[1 + \left(\frac{3}{4}\ell^2\right) \frac{\left(1 + \left(\frac{\bar{b}}{3}\right)^2 - \frac{2}{3}\bar{b}\cos\phi\right)}{(1 + \bar{a} + \bar{b}\cos\phi)^2} \right]^3 \qquad (25)$$

where

$$\bar{a} = a\bar{\rho}_o^{-3}$$

$$\bar{b} = b\bar{\rho}_o^{-4}$$

$$\ell = \frac{R}{B_e r_e \bar{\rho}_o}$$

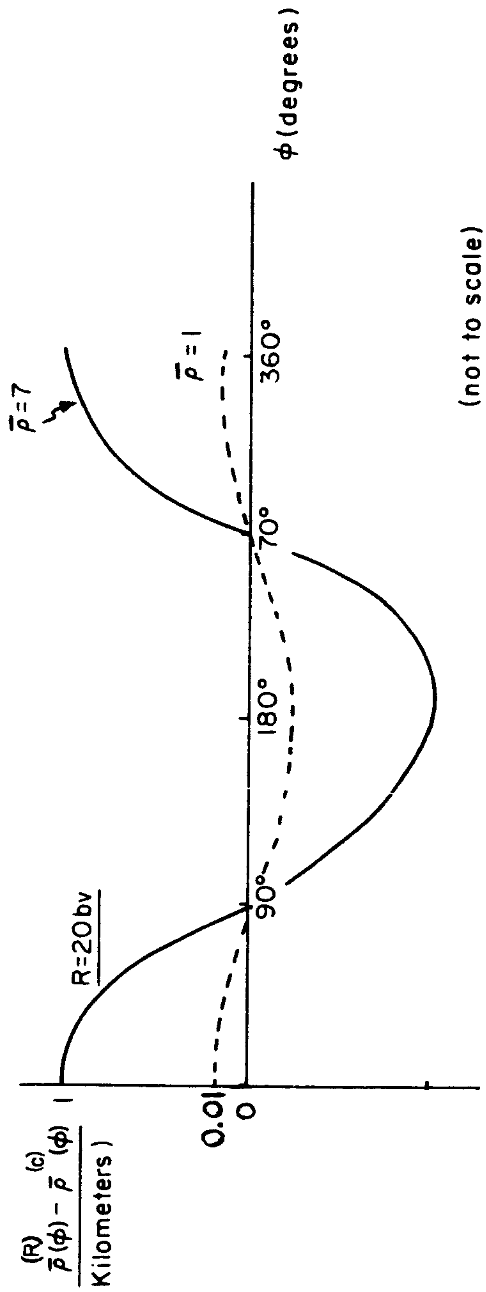
Since "ℓ^2" and "$\epsilon^R(\phi)$" are going to always be much smaller than unity, we may binomially expand (25) in both $\epsilon^R(\phi)$ and "ℓ^2". As a result, after some tedious algebra, we obtain keeping terms linear in $\epsilon^R(\phi)$ and "ℓ^2"

$$\epsilon^R(\phi) - \epsilon^0(\phi) = \frac{-\frac{3}{4}\ell^2\left[1+\left(\frac{\bar{b}}{3}\right)^2\right]}{\left[1+\bar{a}\right]\left[1-\frac{\bar{b}}{3}\cos\phi\right]}\left[\frac{1-\frac{2}{3}\frac{\bar{b}\cos\phi}{1+(b/3)^2}}{\left(1+\frac{\bar{b}\cos\phi}{1+\bar{a}}\right)^2} - 1\right]. \quad (26)$$

We may still further simplify this expression by noting that over the range $7 \geq \rho \geq 1$ the maximum value of $\bar{a}$ is 0.28 and that of $\bar{b}$ is 0.16 thus allowing us to make further binomial expansions in the parameters $\bar{a}$ and $\bar{b}$. However, in this case we must keep more than the linear terms and arbitrarily cut off the expansions when we make a 5% error or less. The result of more tedious algebra is then

$$\bar{\rho}^R(\phi) - \bar{\rho}^0(\phi) = \bar{\rho}_o^3 \left[(3b)(1-\frac{3}{2}\,\bar{a} + 2\bar{a}^2)\right] \left[\frac{R}{B_e r_e}\right]^2 \cos\phi \ . \quad (27)$$

We may infer from (27) that the energy dependence of the shells is small because of the multiplicative $(\frac{R}{B_e r_e})^2$ term. Furthermore, the term "$\bar{\rho}_o^3$" becomes smaller and smaller as the initial starting point on the shell is reduced (closer to the earth's surface) so that the splitting goes something like the third power of the distance from the center of the earth. To further get an understanding of the orders of magnitude involved see the plots in figures 11 and 12. It is clear that the splitting due to energy is very small. Comparing the two figures we see that the first order correction of Ray[1] gives invariant surfaces which diverge from the circular dipole invariant surfaces by about 3000 km for a 20 bv particle at $\overline{\rho}_o = 7$. However, the splitting of this shell is less than 1 kilometer at 7 earth radii for the most energetic particles considered.



$\bar{\rho}^{(R)}(\phi) - \bar{\rho}^{(c)}(\phi)$
(Kilometers)
1
0.01
0
R=20bv
90°
180°
70°
360°
$\bar{\rho}=7$
$\bar{\rho}=1$
ϕ (degrees)
(not to scale)

figure 11

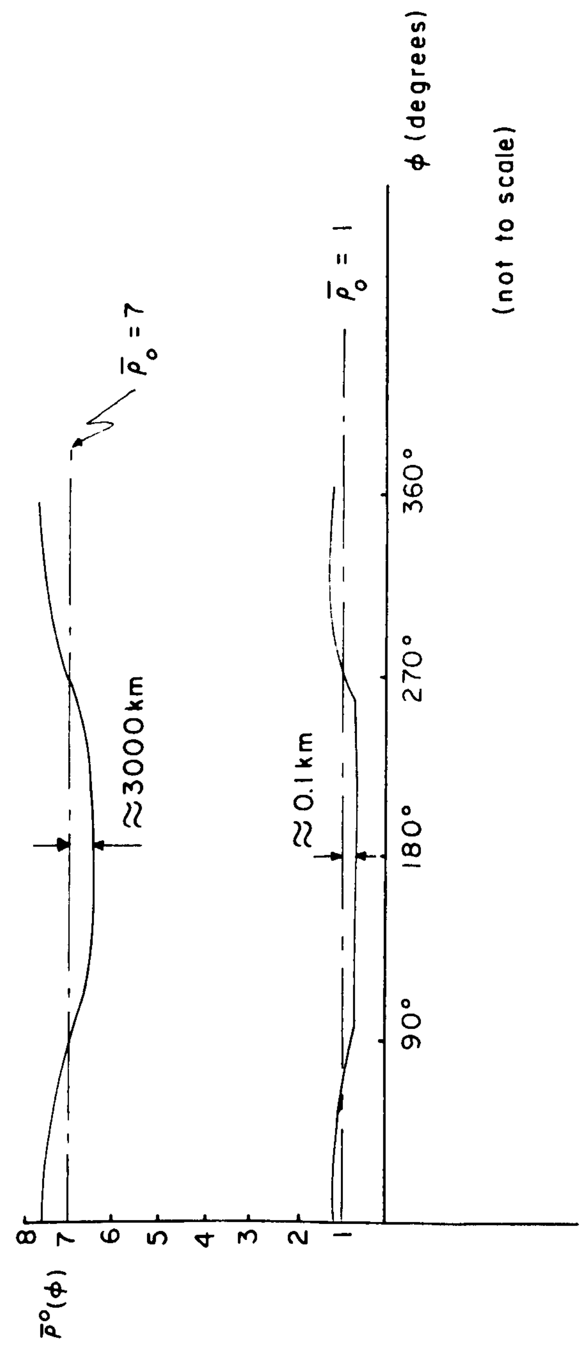

$\bar{\rho}^o(\phi)$
8
7
6
5
4
3
2
1
90°
180°
270°
360°
ϕ (degrees)
$\approx$ 3000 km
$\approx$ 0.1 km
$\bar{\rho}_o = 7$
$\bar{\rho}_o = 1$
(not to scale)

figure 12

(VII) NUMERICAL CALCULATIONS

The Cornell Control-Data computer was employed to map several Invariant Shells in the Hones[15] and Mead[26] models of the distorted solar wind cavity (using equation III-(83)) as well as correcting several predicted Cosmic-Ray cut-off rigidities (using equation V-(41)) in a Finch and Leaton[7] model.

We will now describe, in detail, the computer code used to obtain the numerical results. We will start by explaining the function of the subroutines and then go into detail about the main programs.

A- The Subroutines

(i) MAGNET:

The subroutine MAGNET includes all the magnetic field models of the earth we will be dealing with. While the models of the Dipole and that of Hones are explicitly represented, because they may be expressed simply, those of Mead and Finch and Leaton are tabulated as Spherical Harmonic expansions. The parameter "KONST" selects the field model desired. The input is the position in space (REQ, TEQ, PEQ) and the corresponding output, for the chosen field model, is the magnetic scalar potential "X", the components of the magnetic field at the same point BR, BT, BP and its magnitude BB. The spherical harmonic expansions are taken from II-(10) and II-(16). Tabulations of the terms of these equations are given in Appendix VII while the corresponding Gaussian Coefficients have been tabulated in Appendix VIII.

(ii) RUNKT and DERITV:

The subroutine RUNKT, as its name implies, employs a Runge-Kutta[48] technique for the solution of differential equations. The equations given by

$$\frac{dr}{ds} = \frac{B_r(r,\theta,\phi)}{B(r,\theta,\phi)} \tag{1}$$

$$\frac{d\theta}{ds} = \frac{B_\theta(r,\theta,\phi)}{r\ B(r,\theta,\phi)} \tag{2}$$

$$\frac{d\phi}{ds} = \frac{B_\phi(r,\theta,\phi)}{r\ \sin\theta\ B(r,\theta,\phi)} \tag{3}$$

for the lines of force are housed in the subroutine DERITV which is called from RUNKT. Given the input point (YY(1),YY(2),YY(3)) and a running length along a line of force (denoted by "s" in (1) to (3)) a neighboring point "ds" away on the same line of force is then computed using the Runge-Kutta Taylor Series expansion technique. The output of RUNKT is given by the same (YY(1),YY(2),YY(3)) which now have the new values of the neighboring point on the same line. The input parameter KONST in this subroutine selects the field model which is called in DERITV.

(iii) LINESB :

The subroutine LINESB is designed to trace out lines of force from the magnetic equator to a given mirror point. The equatorial input point (RIN,TIN,PIN) given, the subroutine will trace out a line of force in the field determined by KONST until the mirror point determined by the parameter INPUT is reached. "INPUT" is so designed to allow the tracing to stop when either the mirror colatitude, (TPR1) the mirror magnetic scalar potential, (XPR1) or the mirror magnetic field magnitude is attained. (BBPR1) The step size is given by the input parameter DEQ1. A "hunting" technique is employed to zero in on the given mirror point parameter (ie: DEQ1 is decreased during the last steps of the tracing of the line until the given mirror point is reached within a given percent error). The parameter LCTMAX limits the number of steps taken during the tracing of a line to guarantee that we stay away from very long lines near the polar regions.

(iv) ALPHAR:

Given an input point (RIN,TIN,PIN), a step-length DEQ and a magnetic field model KONST, the subroutine ALPHAR will trace the line of force passing through it until it reaches the magnetic equator where it again "hunts in" and computes the value of "α" corresponding to that line (defined as a function of the magnetic field at the equator). The output appears as the value of the magnetic equator's position, (RMIN, TMIN, PMIN) magnetic scalar potential, (VMIN) and field value (BMIN) at this minimum point where "α" is computed. The subroutine ALPHAR calls MAGNET, and RUNKT. ALPHA is the value of "α" for that input point.

(v) DLNVDA:

The subroutine DLVDA computes the value of $\frac{1}{\Delta V} \times \left.\frac{\partial \Delta V}{\partial \alpha}\right|_{\alpha_o}$ on the line passing through the given input point (REQ,TEQ,PEQ); where ΔV is the change in the magnetic scalar potential from the input point to its conjugate point in the other hemisphere, $\left.\frac{\partial \Delta V}{\partial \alpha}\right|_{\alpha_o}$ is the change in "ΔV" with "α", evaluated at the line passing through the input point. VPLU and VMNU are the values of the magnetic scalar potential at the input and its conjugate point, BMIRR is the corresponding magnetic field and QUOT is the output $\frac{1}{\Delta V} \left.\frac{\partial \Delta V}{\partial \alpha}\right|_{\alpha_o}$. Again, DEQ4 is the step-size and KONST the input field model selection.

(vi) GRADB:

GRADB, as its name implies evaluates the gradient of B at the input point (R,T,P). The increments determining the neighboring point from which the derivatives are obtained are given by (EPSR,EPST,EPSP), while the components of the gradient are given by (DELBR,DELBT,DELBP), and the magnitude denoted by ADELB. The definition of the derivative is used to compute this gradient in GRADB.

(vii) GRADAL:

In a similar way GRADAL computes the gradient of "α" at a given input point (R,T,P). The only difference is that, whereas in GRADB increments in MAGNET were taken to generate derivatives, now we must take increments in "α"; which calls the subroutine ALPHAR. Again, the increments in position are given by (EPSR,EPST,EPSP) and the step-size needed when computing "α" at neighboring points is given by DEQ2. The fundamental definition of the derivative is used as the computational technique in the subroutine yielding the output of (DELAR,DELAT, DELAP,ADELA) which are the components and magnitude of the gradient of "α".

(viii) CONSTA:

The subroutine CONSTA maps curves of constant "α" in the equitorial plane. Placing an input point into the subroutine (RINIT,TINIT,PINIT) and selecting a new value of azimuth "$d\phi$" away (DALTP) the code proceeds to "hunt" in the same input equitorial plane until it zeros in on the same value of "α" as the input point. The increment of "radial hunt" is given by DALTR. Again KONST selects the model, DEQ5 the step-length when computing "α" and LCTMAX the maximum number of steps.

(ix) NWALPR:

Given the input point (RR1,TT1,PP1) and the corresponding (ALPH1) and new value of "α_N" (ALPHN) at a new value of azimuth "d∅" from the first value, the subroutine NWALPR computes the position of the point in the magnetic equitorial plane where the value of "α" is ALPHN. A hunting technique similar to that used in CONSTA determines the new point, within a given percentage error in the true value of ALPHN compared to the calculated value from the newly found point. The new point has the coordinates (R2,T2,P2). DELTR is the radial "hunt" increment and KONST,LCTMAX,DEQ have the same meanings given previously.

(x) <u>LANDI, LINES, INTEG, CARMEL, START</u>:

The subroutine LANDI performs two functions, given the input point (R,T,P) and the field model KONST. It will compute the corresponding McIlwain "L" parameter for that point and the Integral Invariant "I" for that point. LANDI calls the remaining subroutines LINES, INTEG,CARMEL, and START. All subroutines call MAGNET. Except for some slight modifications in the input-output sequence, this group of subroutines was borrowed from McIlwain.[25] It has been treated essentially as a "Black Box".

B- Plots of Trapped Particle Shells

Invariant shells are mapped for the Hones and Mead models of the magnetosphere as follows.

Several "starting points" are selected in the equitorial plane and in the sub-solar direction. For each starting point a set of mirror points (in latitude) and rigidities is selected.

First, consider that set of particles that mirror at the magnetic equator. For each starting point an azimuthally neighboring point is located in the equatorial plane (call CONSTA) and the "$\nabla\alpha$" is calculated. From this azimuthally neighboring point the next point is located, in the same manner, and the calculation repeated until we cover 2π radians in azimuth. For each rigidity and azimuthal point equation IV-(82) is solved for the new (corrected) value of "$\alpha_c(\phi)$". The results are fed into the subroutine NWALPR which locates the equitorial crossing of the new (corrected) invariant shells. The output is then the corrected invariant shells, as a function of rigidity, for various radial distances from the earth.

Second, consider the set of particles that mirror at latitudes higher than the earth's magnetic equator. At the same starting points given above, the magnitude of the magnetic field and its associated magnetic scalar potential are computed for several mirror latitudes along the line of force passing through these points. At each mirror point $\frac{1}{\Delta V} \left.\frac{\partial \Delta V}{\partial \alpha}\right|_{\alpha_o}$ is computed (calling DLNVDA). Following the same procedure given on the previous page, other azimuthal points are located on constant "α" curves (in the equitorial plane) and the lines of force through these points are used (call LINESB) to compute the values B_{V+} which correspond to the same magnetic scalar potentials located on the starting lines. After this is repeated for all values of azimuth the results are placed in equation IV-(83) from which is computed the "new" value of "$\alpha_c(\emptyset)$". Again this is placed in NWALPR and the equitorial crossings of the new invariant shells are calculated. The new invariant shells through each starting point are split as a function of mirror point.

The results of the latter calculation are compared with the Integral Invariant and McIlwains "L" parameter as follows. Through each of the starting points and for each value of the mirror point, the value of "L" and "I" are computed (call LANDI). The equatorial crossing of the surfaces predicted by the constancy of these "invariants" are then computed and tabulated. A comparison of the previous computed results and these values is then tabulated.

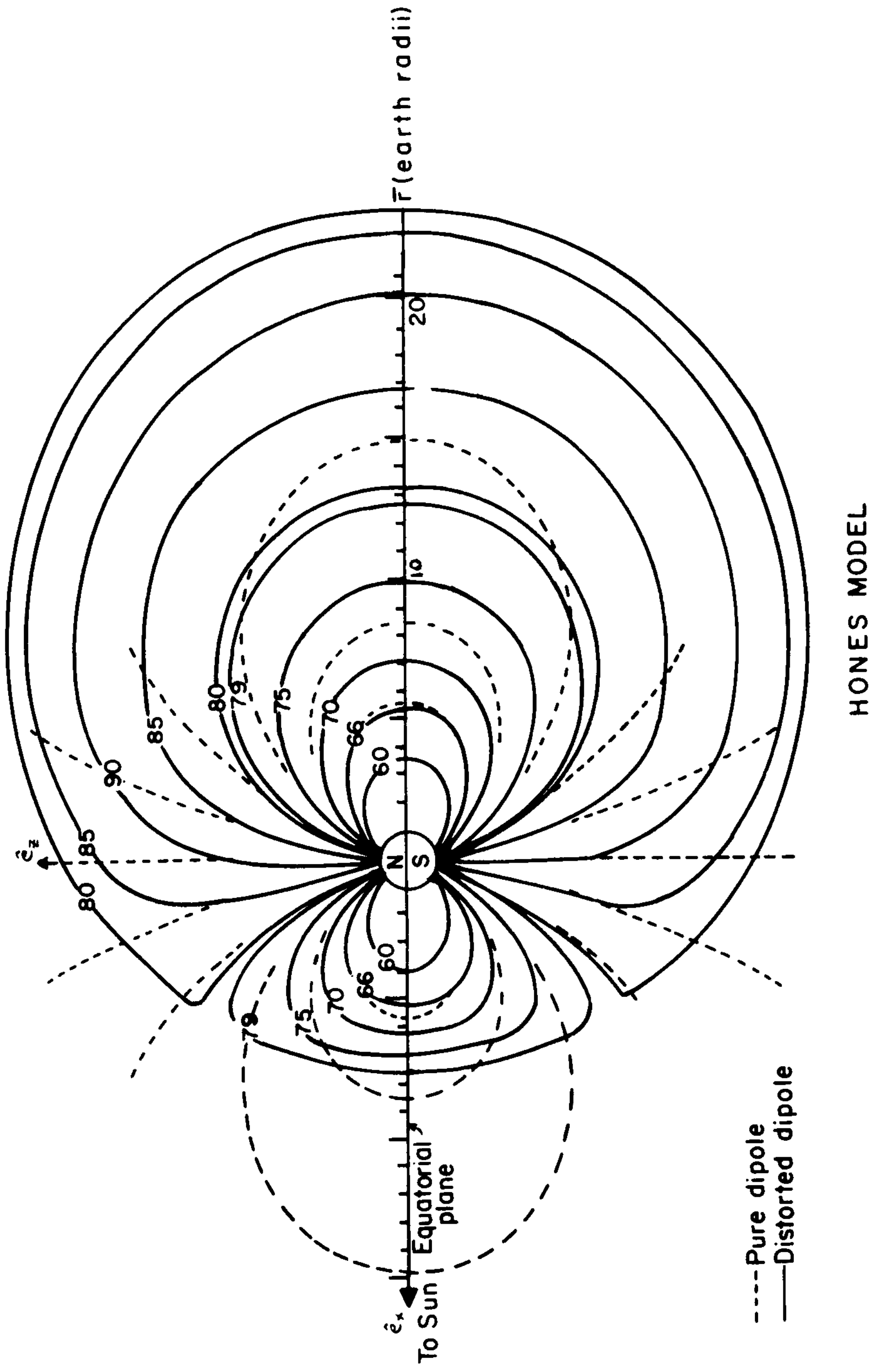
C- Corrections to Cosmic Ray Cut-Offs

Selecting several points on the surface of the earth, the vertical cut-off rigidity is computed with the aid of equation VI-(41). The subroutines which are called are ALPHAR, MAGNET, GRADAL and GRADB (and the respective subroutines which they call). The results of the computation of the above equation are then compared with the results of Shea, Dropkin, Ray. All rigidities are expressed in Bv (Billion-volts).

D- The Results

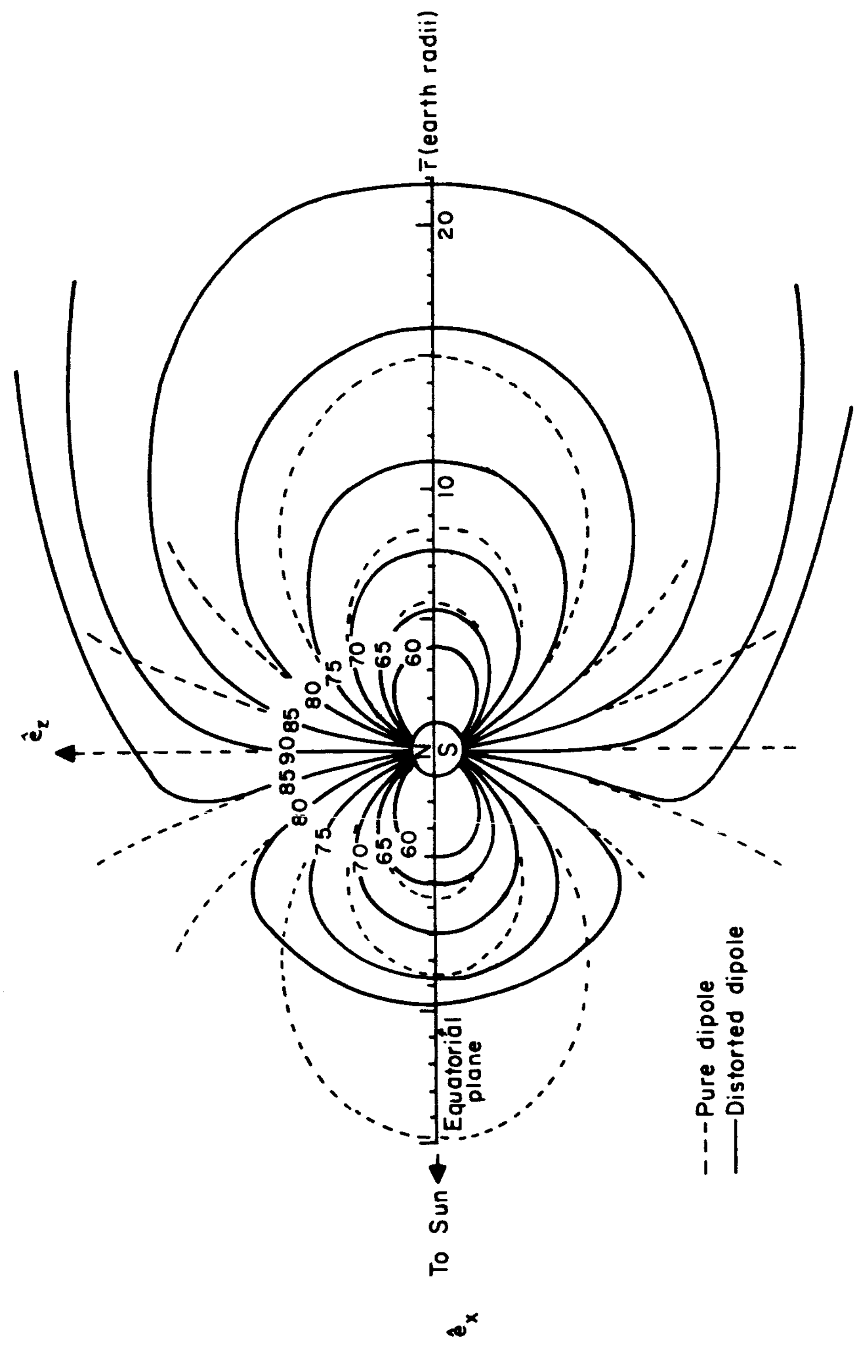
(i) Invariant Magnetic Shells

The two magnetosphere models of Hones[15] and Mead[26] are used as approximate models of the earth's field. The direction of the sun in the earth-sun line is chosen in the $\hat{e}_x$ direction while the direction of the north star is chosen as the $\hat{e}_z$ direction. The $\hat{e}_y$ direction is then out of the paper (see figures 13 and 14). In Figure 13 we plot the lines of force of the Hones field in the x-z plane, while in Figure 14 we plot the same for the Mead field. Superimposed on each figures are the lines of force of the dipole component of the geomagnetic field. Both figures are modifications of figures appearing in the Journal of Geophysical Research. Exact mathematical expressions for both field models are given in Appendix VIII. (The earth's dipole points south).



r̄ (earth radii)
20
10
HONES MODEL
To Sun
Equatorial plane
----Pure dipole
——Distorted dipole
90
85
80
79
75
70
66
60
N
S

figure 13



MEAD MODEL

figure 14

The results of computer calculations on the Cornell-Control Data 1604 computer are shown in figures 15 through 34.

For both field models, "initial" lines of force were selected in the sub-solar direction (ϕ=180°) which intersected the equator at $\overline{r}$ = 2, 8, 14, 20, 26 earth radii. (In these models the magnetic equator is "flat" and coincident with the geometrical equator.) For each initial line of force, mirror points were selected at λ_m = 80°, 50°, 25°, 0° except the lines passing through $\overline{r}$ = 2 where we choose λ_m = 50°, 30°, 15°, 0°. At each initial line of force and for each mirror point, equation III-(83) was used to compute $\overline{\gamma}$. The equation then was used to map out the invariant "h_c" shells which are now not constant-α shells. Since both field models are symmetrical about the x-z plane, the invariant shells are similarly symmetrical so that the figures 15 through 34 only show half of the magnetosphere. The intersections of constant - h_c surfaces with the equator is then plotted for each initial mirror point, line of force, and model. Superimposed on the constant - h_c plots are (a) the equatorial intersection of constant I (Integral invariant surfaces), and (b) the equatorial intersections of the α_c = constant surfaces which arise from the zeroth order first integral solution. Associated with each integral invariant surface, there

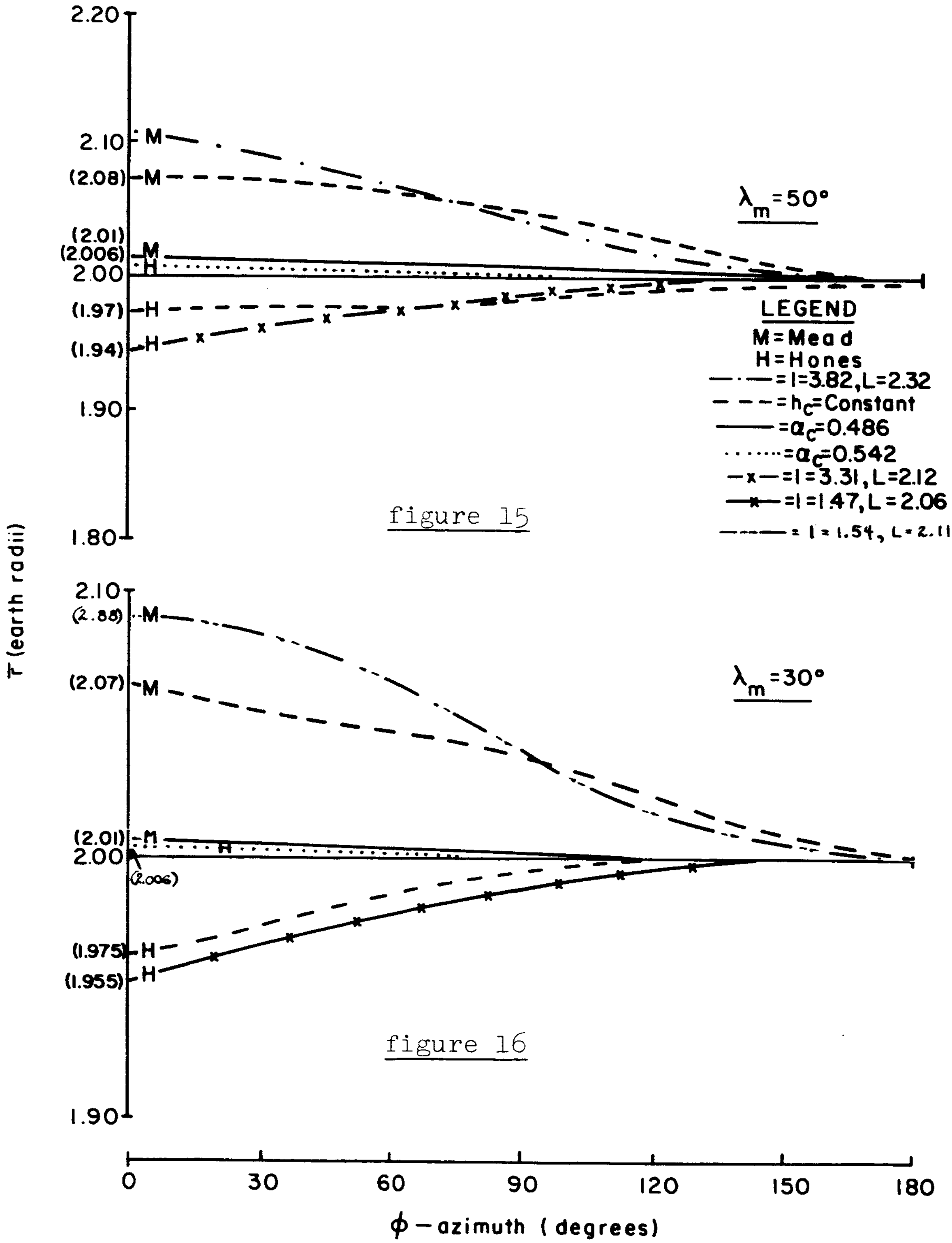


figure 15

figure 16

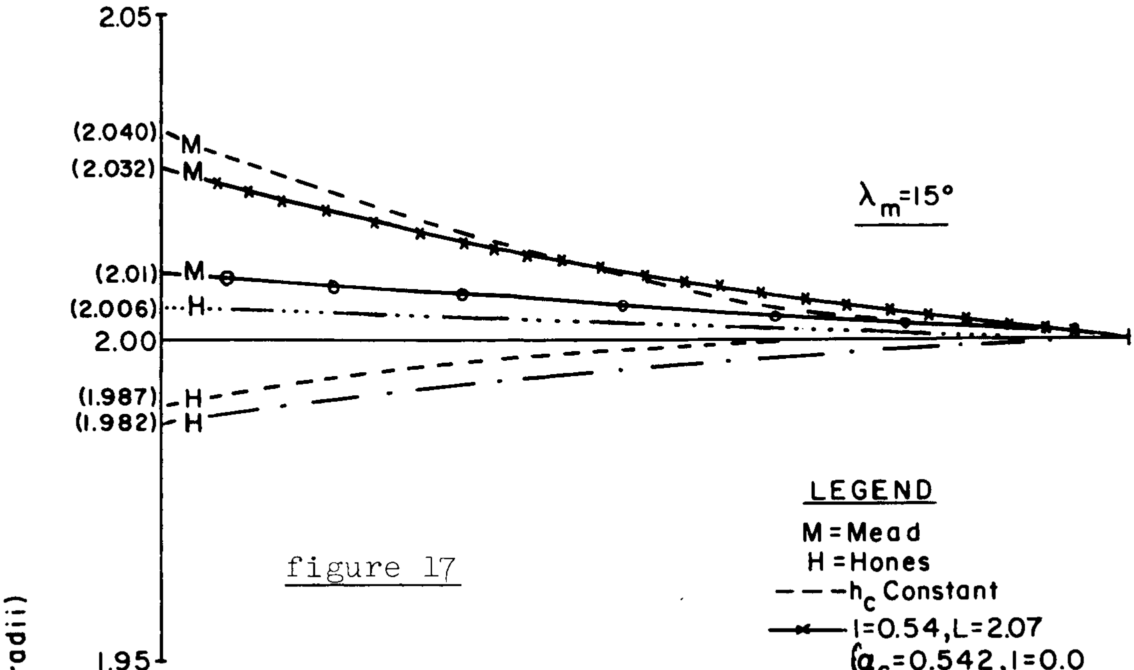


figure 17

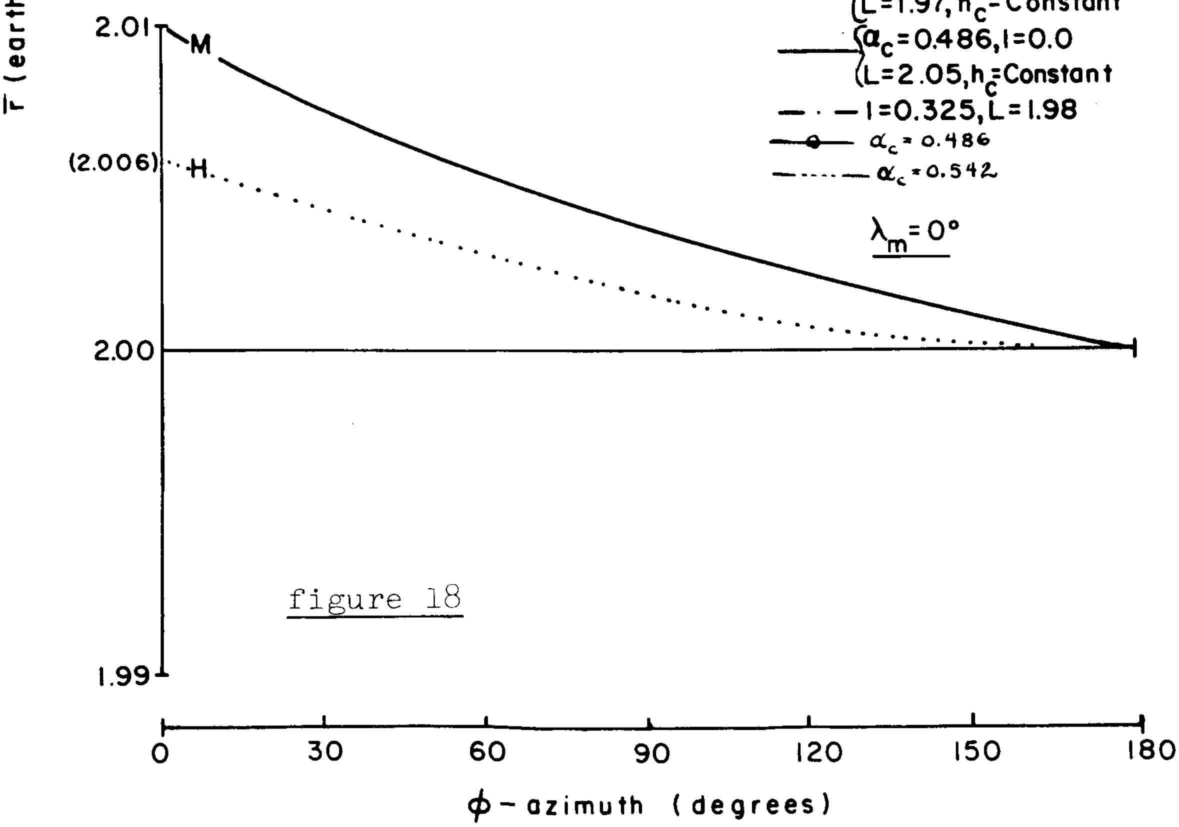


figure 18

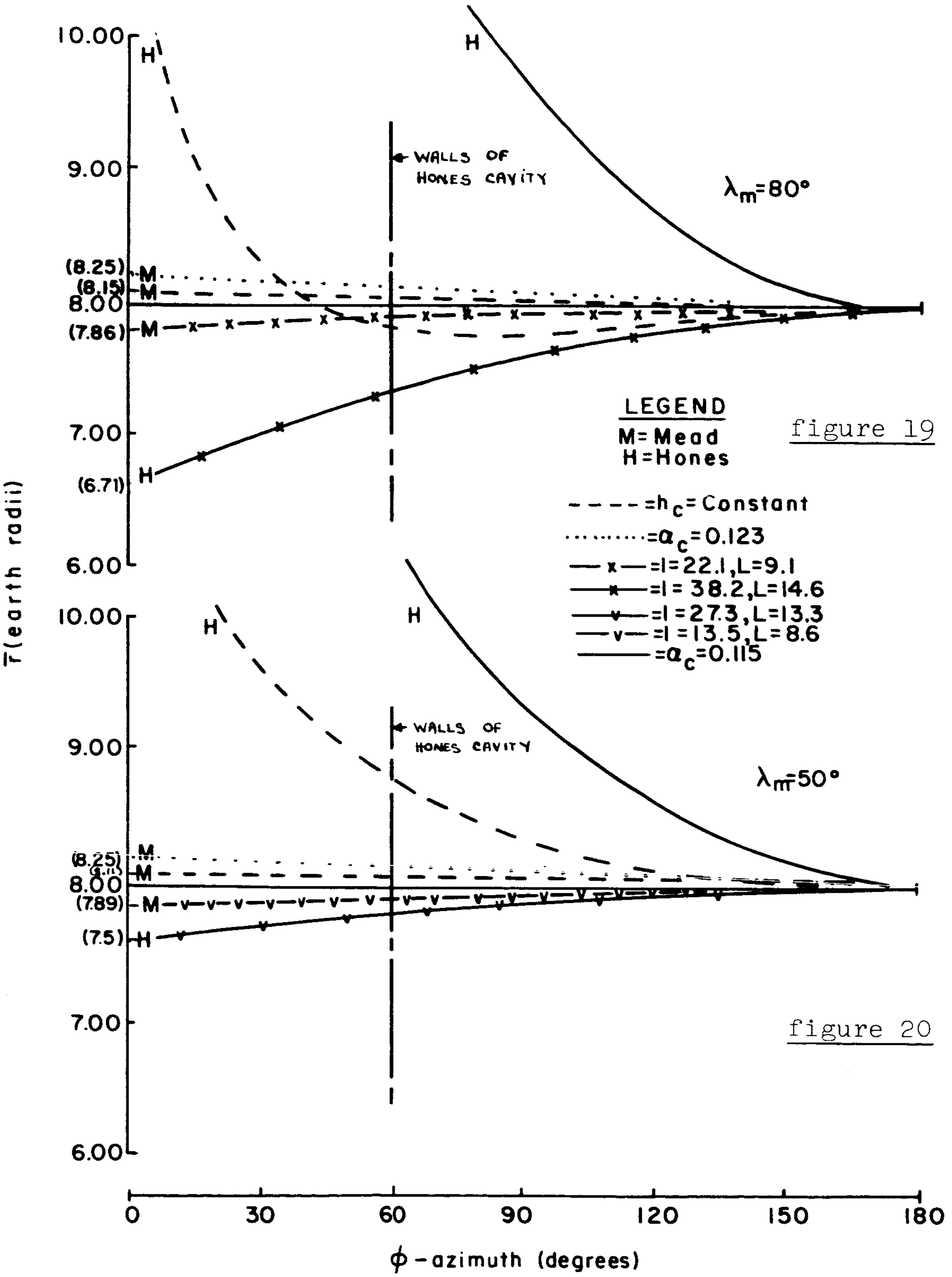


figure 19

figure 20

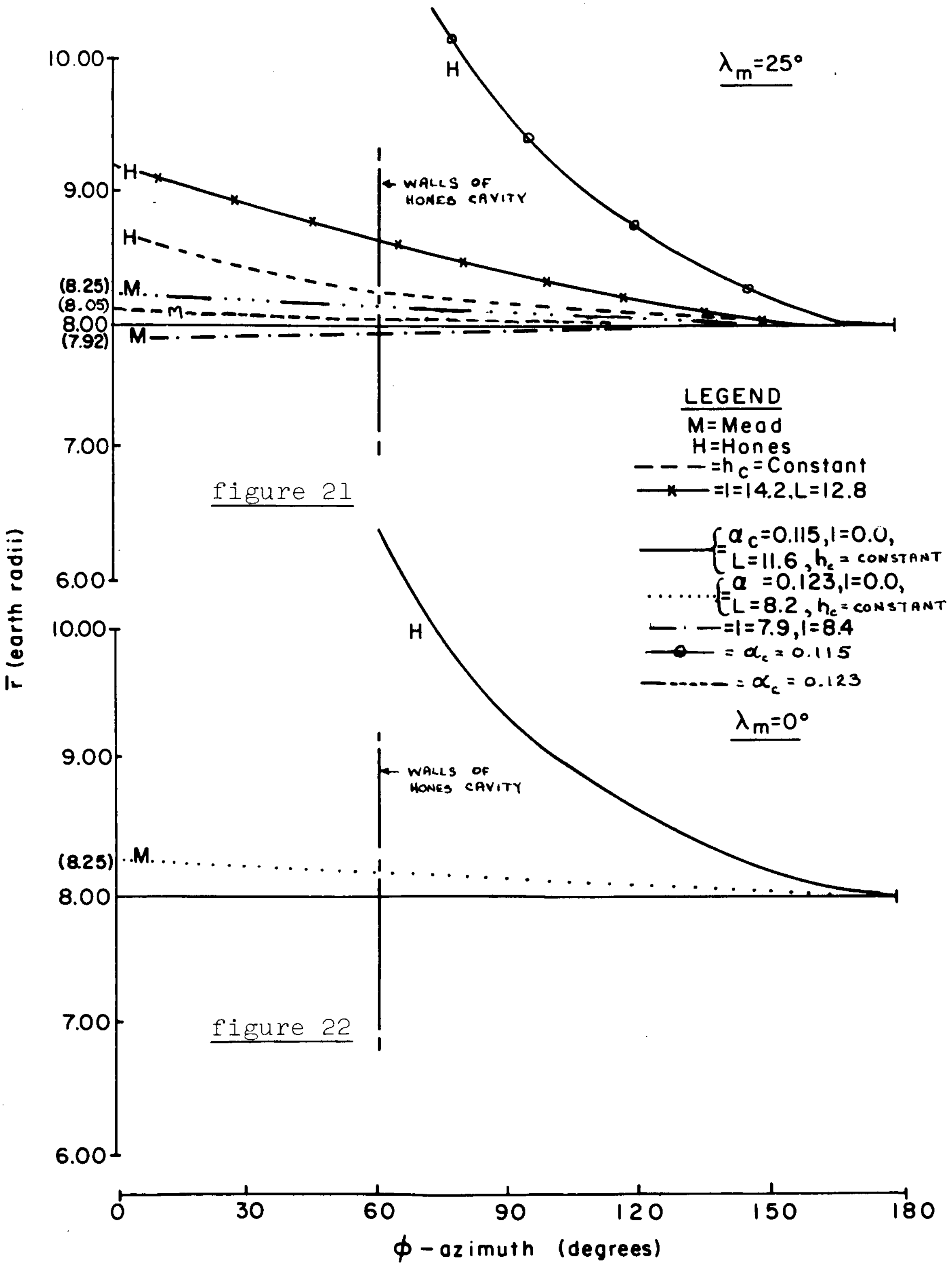


figure 21

figure 22

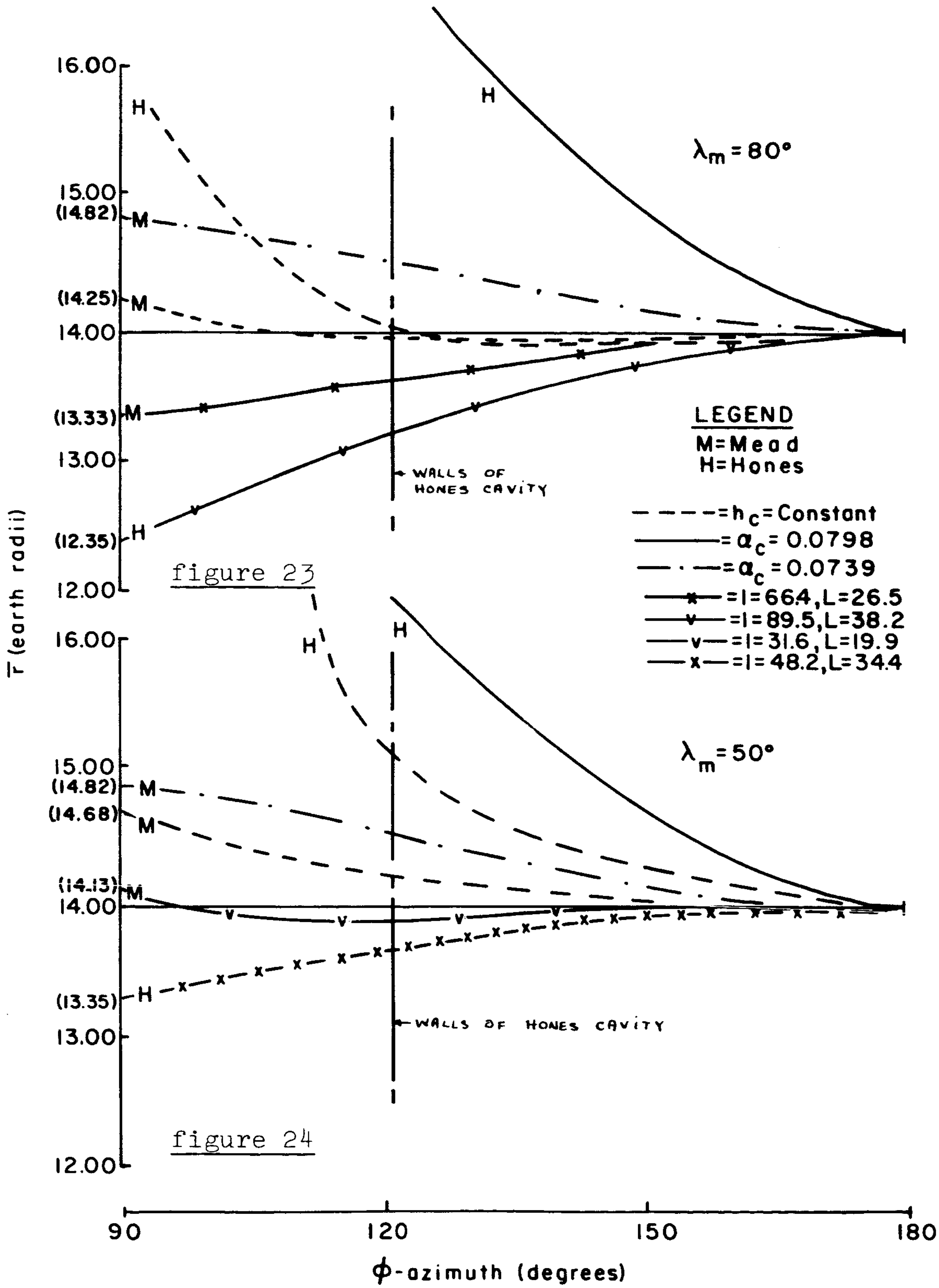
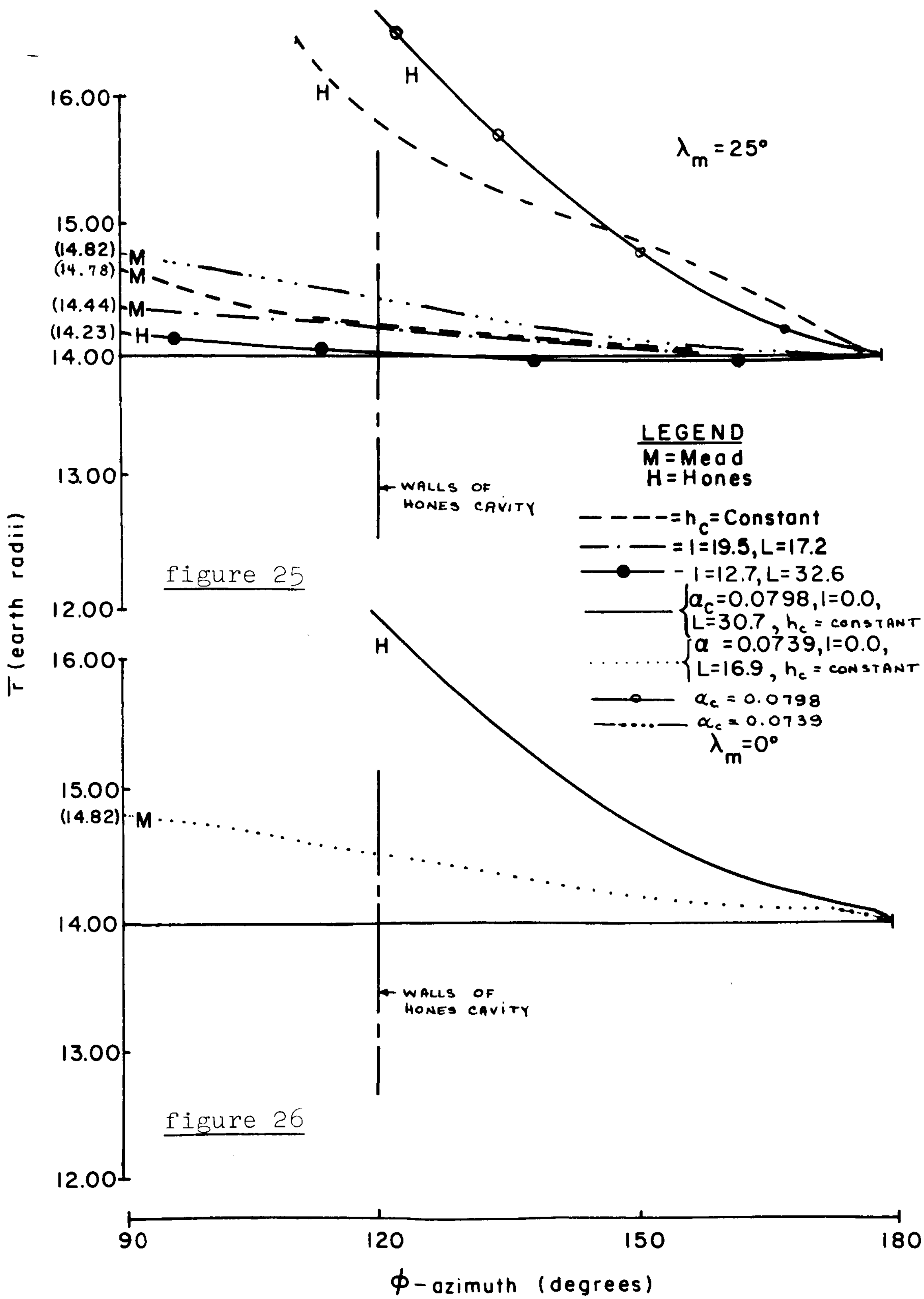


figure 23

figure 24



16.00
15.00
14.00
13.00
12.00
(14.82)
(14.78)
(14.44)
(14.23)
M
H
$\lambda_m = 25°$
WALLS OF HONES CAVITY
figure 25
LEGEND
M = Mead
H = Hones
= h_c = Constant
= I = 19.5, L = 17.2
I = 12.7, L = 32.6
α_c = 0.0798, I = 0.0, L = 30.7, h_c = CONSTANT
α = 0.0739, I = 0.0, L = 16.9, h_c = CONSTANT
α_c = 0.0798
α_c = 0.0739
$\lambda_m = 0°$
figure 26
$\overline{r}$ (earth radii)
90
120
150
180
ϕ - azimuth (degrees)

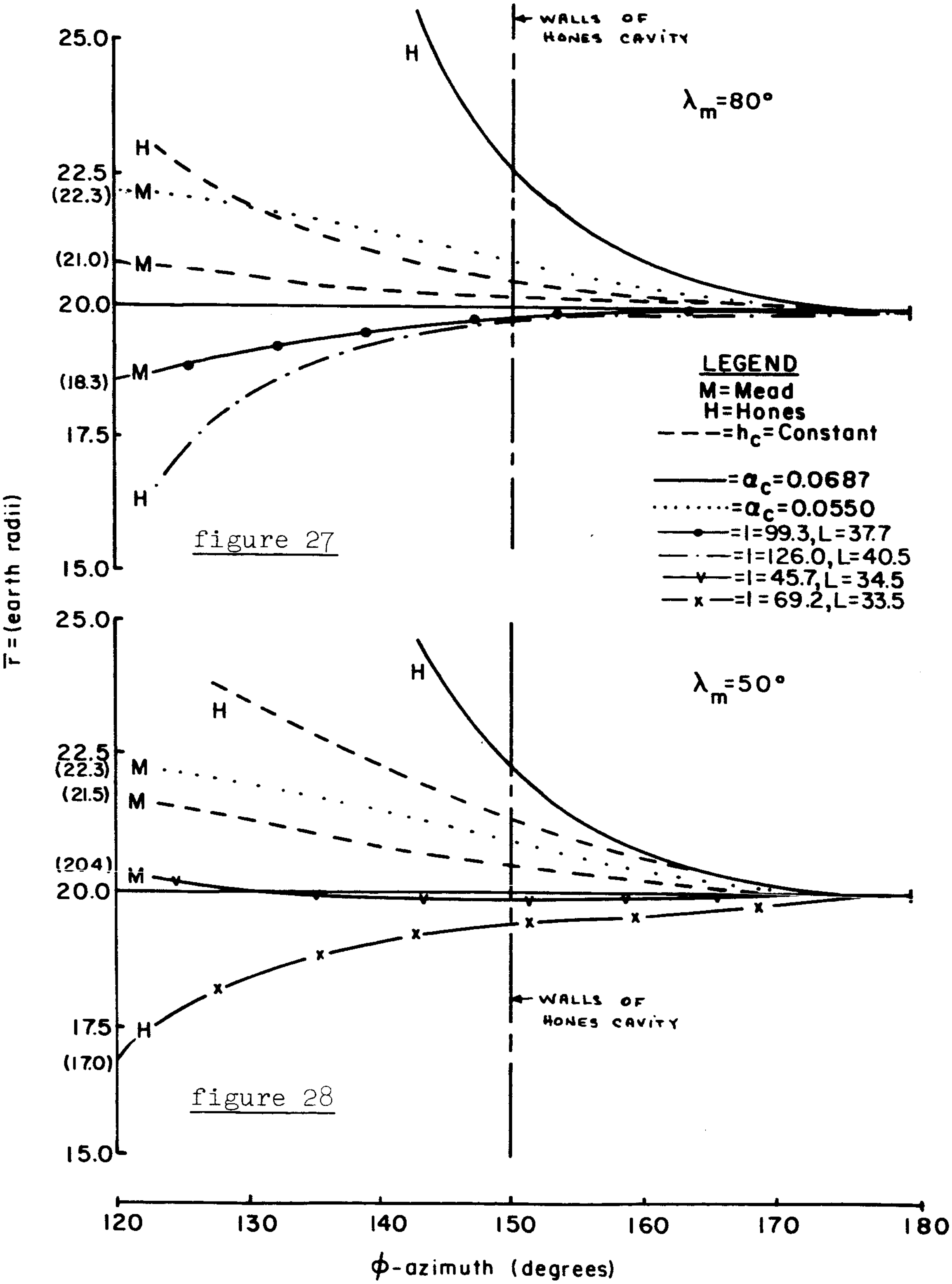


figure 27

figure 28

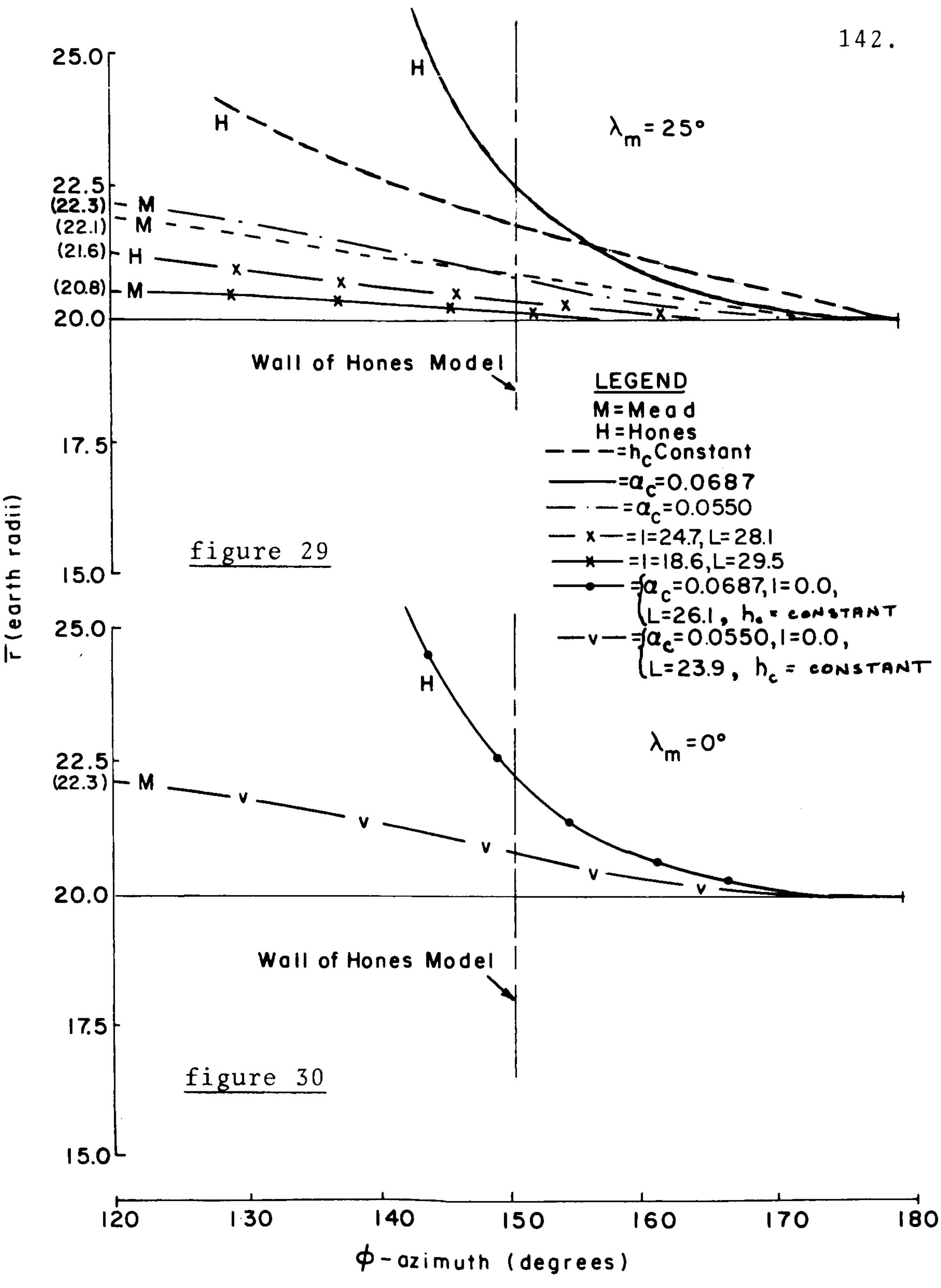


figure 29

figure 30

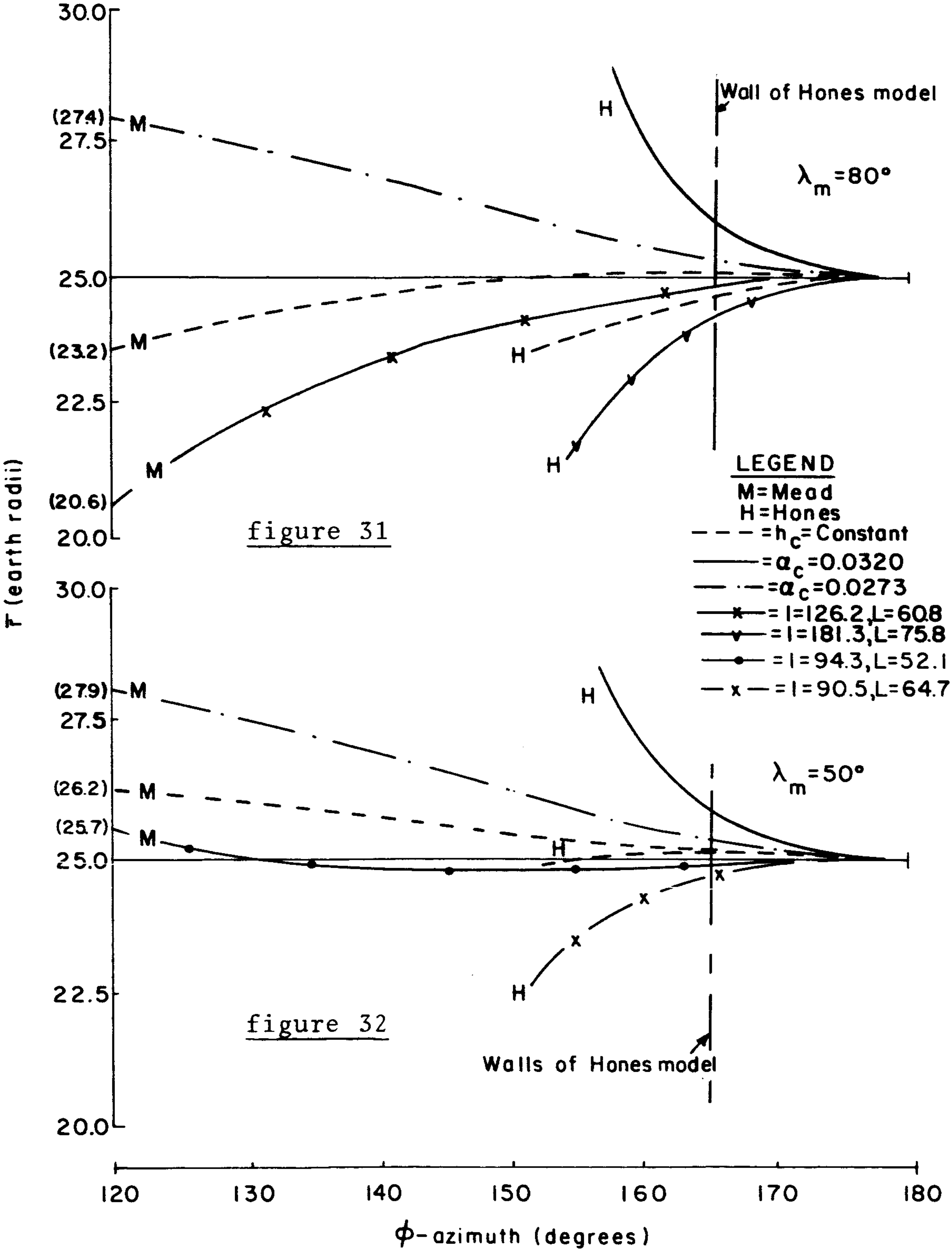


figure 31

figure 32

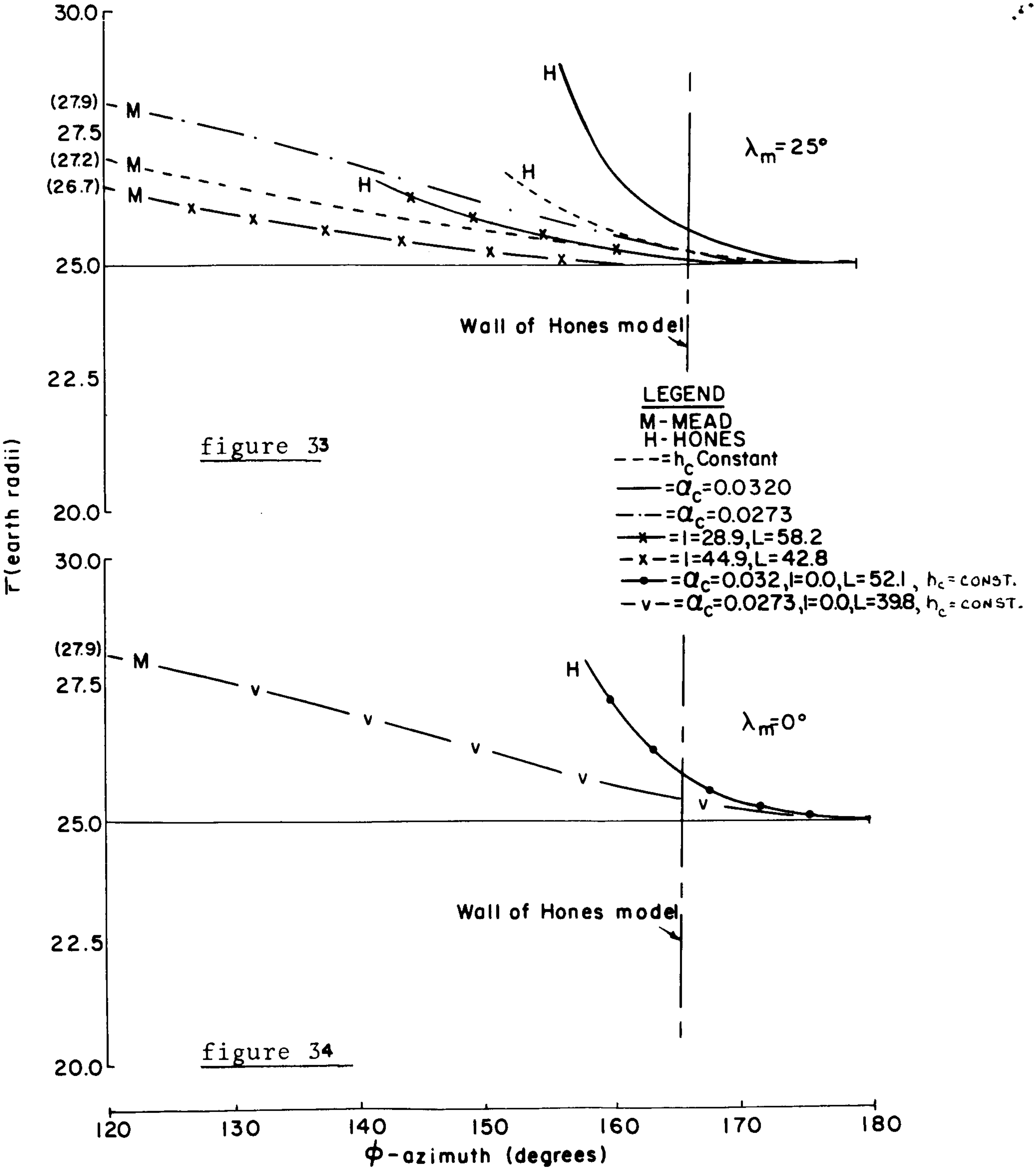


figure 33

figure 34

corresponds for the same mirror point, a constant L parameter. Therefore, the integral invariant surfaces are labled with the "L" parameters also. All abcissa's are in degrees, and $\overline{r}$, α_c, L, I and h_c are in earth radii.

The integral invariant shells are seen to diverge further and further from the α_c = constant shells as (a) the initial line of force is a greater distance from the earth in the subsolar direction and (b) the mirror point of the trapped particle is at a higher latitude for a fixed initial starting line at $\phi = 180^{\circ}$. When the first order correction is used, h_c=constant, it brings the new predicted invariant surface within 7% of the constant I surface.

For a given initial line, as the mirror point is increased it is seen that the McIlwain[25] "L" parameter remains constant to within 6% only closer than 7 earth radii to the earth. As distances are increased in the subsolar direction, L becomes a strong function of mirror point varying as much as 55% along the same line of force in the tail of the magnetosphere (about 26 earth radii out). That L should be approximately constant along a line of force is much poorer an approximation than the fact that α_c is an approximate constant as a function of particle mirror point. We see from the figures that h_c = constant (the first order correction to α_c =constant) surfaces only split to a maximum of 5% in the 26 earth radii tail of the magnetospheric models. Notice that this argument remains valid so long as the invariant surfaces remain within the magnetospheric boundary. Once the invariant surfaces pass beyond this point the shell concept loses meaning. That the McIlwain parameter does not remain even close to approximately constant ($\gtrsim 50\%$) as a function of mirror point in the tail of the magnetosphere is not very surprising. McIlwain showed that there exists a relationship between L and I which depends on B_m. Since his analysis defined L like R_{D_O} of the dipole field, there no reason to think that fields that severely diverge from the dipole have an equitorial parameter which remains constant as a function of the mirror point along a line of force.

The equatorial energy splitting is of the order 10^{-4} earth radii for up to 500 mev particles and is therefore not shown in these figures.

(ii) Cosmic Ray Cutoffs

The vertical cutoff rigidity of Cosmic Rays was calculated at various locations on the surface of the earth using equation V-(41) in a Finch and Leaton[7] field model. The calculations were performed on the Cornell Control-Data 1604 computer. The results are tabulated in Table II along with those results of Ray, Shea[46], Dropkin, Quenby and Wenk[37] and modified rigidities for the I.G.Y.

Shea's[46] results were found by simulating the trajectories of charged particles on a computer using the Finch and Leaton[7] field model. Ray obtained his results by, first, using

$$R_{\text{vert. cutoff}} = \frac{14.9}{L^2} \qquad (1)$$

where "L" is the McIlwain parameter calculated at the point, on the surface of the earth, of impact. Second, and more correctly, then by using

$$R_{\text{vert. cutoff}} = \frac{14.9}{L_c^2} \approx \frac{14.9}{L^2}\left(1 - \frac{2}{L}\,\vec{a}_c \cdot \nabla L\right) \qquad (2)$$

where now L_c is computed at the guiding center of the particle at the time of impact. Dropkin, again, simulated particle trajectories at the Goddard Space Flight Center and computed cutoffs from Størmer Theory.

In Table III the percent disagreement between the present results and those found by other authors has been tabulated. It will be noticed that the present cutoffs best agree with the results of Shea[46] and Quenby and Wenk[37], and least agree with the results by Dropkin. Particularly poor agreement is found at Brisbane.

TABLE II

STATION	LAT. (DEG.)	LONG. (DEG.)	VERTICAL CUTOFF RIGIDITIES IN BV						
			RAY $\frac{14.9}{L^2}$	RAY $\frac{14.9}{L_c^2}$	SHEA	DROP.	FRIED.	IGY	QUENBY and WENK
Ahmedabad	23.0	72.6	14.6	13.9	15.9	13.9	15.0	15.8	15.8
Brisbane	-27.5	153.0	6.6	6.5	5.6	6.2	6.1	----	7.0
Chacaltaya	-16.3	291.9	13.6	13.0	13.1	12.0	13.2	----	13.3
Cahmical	-30.3	293.8	11.7	11.6	11.3	10.9	11.9	----	12.0
Haleakala	20.7	203.7	11.8	11.7	13.3	10.4	12.9	----	13.3
Huancayo	-12.0	283.1	14.1	13.5	13.5	12.3	13.6	13.6	13.7
Kampala	0.3	32.6	14.8	15.8	15.0	13.3	14.9	----	15.3
Khartoum	15.6	32.6	15.6	15.1	15.6	14.8	15.9	----	15.6
Kodaikanal	10.2	77.5	17.5	18.2	17.5	17.0	17.8	17.6	17.5
Lae	-6.7	147.0	14.3	14.0	15.5	11.9	14.9	15.5	15.8
Mexico City	19.3	260.8	9.1	8.6	8.9	8.0	9.2	10.3	10.0
Mina Aquilar	-23.1	294.3	12.9	12.5	12.5	11.7	12.7	12.5	12.6
Posadas	-27.4	304.0	12.0	11.6	11.2	10.2	11.8	----	12.0
Rio de Janeiro	-22.9	316.8	12.1	11.6	11.2	10.6	11.9	11.5	12.1
Trivandrum	8.5	77.0	17.5	18.4	17.4	16.6	17.8	----	17.5
Tucuman	-26.9	294.6	12.3	12.0	11.8	11.6	12.1	----	12.3

TABLE III

STATION	LAT. (DEG.)	LONG. (DEG.)	% DISAGREEMENT RAY $\frac{14.9}{L^2}$	RAY $\frac{14.9}{L_c^2}$	SHEA	DROP.	IGY	QUENBY and WENK
Ahmedabad	23.0	72.6	2.7	7.4	6.0	7.4	5.3	5.3
Brisbane	-27.5	153.0	8.2	6.6	8.2	1.7	----	14.5
Chacaltaya	-16.3	291.9	3.1	1.5	0.7	9.3	----	0.7
Cahmical	-30.3	293.8	1.7	2.5	5.1	8.4	----	1.0
Haleakala	20.7	203.7	8.5	9.3	3.1	11.7	----	3.2
Huancayo	-12.0	283.1	3.7	0.8	0.8	9.6	0.0	0.7
Kampala	0.3	32.6	0.6	6.0	0.6	11.0	----	2.7
Khartoum	15.6	32.6	1.9	5.0	1.9	6.9	----	2.0
Kodaikanal	10.2	77.5	1.7	2.3	1.7	4.5	1.1	1.7
Lae	-6.7	147.0	4.0	6.1	4.0	20.0	4.1	6.1
Mexico City	19.3	260.8	1.1	6.5	3.3	13.0	12.0	8.7
Mina Aquilar	-23.1	294.3	1.6	1.6	1.6	7.9	1.6	0.8
Posadas	-27.4	304.0	1.7	1.7	5.2	13.0	----	1.7
Rio de Janeiro	-22.9	316.8	2.5	2.5	5.7	11.0	3.4	1.7
Trivandrum	8.5	77.0	1.7	3.4	2.3	6.8	----	1.7
Tucuman	-26.9	294.6	1.6	0.9	2.5	4.1	----	1.6
TOTAL % AVERAGE DISAGREEMENT			2.9	4.0	2.9	9.0	4.0	3.4

(VIII) NECESSARY CONDITIONS FOR THE EXISTENCE OF A STØRMER INTEGRAL

We will now derive the set of necessary conditions that a static magnetic field must satisfy in order that a Størmer Integral[38] exists. That is, we will derive the set of conditions that must be satisfied by the magnetic field such that there exists an $\bar{\alpha}$, $\bar{\beta}$ describing the field, that leads to the conservation equation (or first integral)

$$\frac{d(\bar{\alpha} + \vec{a}_c \cdot \nabla\bar{\alpha})}{dt} = 0 \tag{1}$$

where

$$\vec{a}_c = \frac{mc}{q} \vec{v} \times \vec{B} \quad . \tag{2}$$

A consequence of (1) will be that $\bar{\alpha}_c$ (at the guiding center) will be a constant of motion for sufficiently low rigidity particles.[1]

Let us proceed as follows. For a given magnetic field, $\vec{B}$, select any convenient function which is constant along lines of force and call it "α". From the partial differential equations

$$\vec{B} = \nabla\alpha \times \nabla\beta \tag{3}$$

we may then solve (in principle) for the corresponding "β". Furthermore from the partial differential equations

$$\vec{B} = \nabla V \tag{4}$$

we may solve for the corresponding "V". We now have a description of the field in terms of α, β, V. We further assume that this description has led to a non-vanishing $\frac{\partial \mathcal{L}}{\partial \beta}$, and hence the corresponding canonical momentum p_β is not a constant of the motion.

If there exists another description of the same magnetic field $\bar{\alpha}$, $\bar{\beta}$, $\bar{V}$ such that $p_{\bar{\beta}}$ is a constant of the motion it must be obtainable from the old description by a point transformation (canonical):

$$\bar{\alpha} = \bar{\alpha}(\alpha,\beta,V) \tag{5}$$

$$\bar{\beta} = \bar{\beta}(\alpha,\beta,V) \tag{6}$$

$$\bar{V} = \bar{V}(\alpha,\beta,V) \tag{7}$$

which is clear since all fields must be described in terms of coordinates only.

Assume that we have found such a description $\bar{\alpha}$, $\bar{\beta}$, $\bar{V}$ and $\frac{\partial \mathcal{L}}{\partial \bar{\beta}} = 0$. The necessary and sufficient conditions for this are

$$\frac{\partial |\nabla\bar{\alpha}|^2}{\partial \bar{\beta}} = \frac{\partial |\nabla\bar{\beta}|^2}{\partial \bar{\beta}} = \frac{\partial B}{\partial \bar{\beta}} = 0 \quad . \tag{8}$$

Intuitively $\bar{\alpha}$ and $\bar{\beta}$ in (5), (6) cannot depend on "V" because they must be constant functions along lines of force. Although α, β are constants along a line, V is not, and any dependence of our new $\bar{\alpha}$, $\bar{\beta}$ on it would destroy its invariance. We may prove this more rigorously as follows. From (5)-(7) and the chain rule we have

$$\nabla\bar{\alpha} = \frac{\partial\bar{\alpha}}{\partial\alpha}\nabla\alpha + \frac{\partial\bar{\alpha}}{\partial\beta}\nabla\beta + \frac{\partial\bar{\alpha}}{\partial V}\vec{B} \tag{9}$$

$$\nabla\bar{\beta} = \frac{\partial\bar{\beta}}{\partial\alpha}\nabla\alpha + \frac{\partial\bar{\beta}}{\partial\beta}\nabla\beta + \frac{\partial\bar{\beta}}{\partial V}\vec{B} \tag{10}$$

$$\vec{B} = \nabla\bar{V} = \frac{\partial\bar{V}}{\partial\alpha}\nabla\alpha + \frac{\partial\bar{V}}{\partial\beta}\nabla\beta + \frac{\partial\bar{V}}{\partial V}\vec{B} \ . \tag{11}$$

Since $\nabla\alpha$, $\nabla\beta$, $\nabla\bar{\alpha}$, and $\nabla\bar{\beta}$ are perpendicular to $\vec{B}$, we may dot $\vec{B}$ into both sides of (9) and (10) giving

$$\frac{\partial\bar{\alpha}}{\partial V} = \frac{\partial\bar{\beta}}{\partial V} = 0 \ . \tag{12}$$

Furthermore, we may satisfy (11) by simply making the identity transformation for V, ie:

$$\bar{V} = V \ . \tag{13}$$

However, $\bar{\alpha}$ and $\bar{\beta}$ must, in addition satisfy the same equation (3). Using (9) and (10) we then must have

$$\vec{B} = \nabla\bar{\alpha} \times \nabla\bar{\beta} = \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\frac{\partial\bar{\beta}}{\partial\beta} - \frac{\partial\bar{\beta}}{\partial\alpha}\frac{\partial\bar{\alpha}}{\partial\beta}\right)\nabla\alpha \times \nabla\beta \tag{14}$$

so that the Lagrange Bracket (Jacobian) of the transformation must be unity.

$$\left[\bar{\alpha},\ \bar{\beta}\right]_{\alpha,\beta} \equiv J\left\{\frac{\bar{\alpha},\bar{\beta}}{\alpha,\beta}\right\} = 1 \ . \tag{15}$$

We conclude that, if a transformation exists that takes α, β, $V, \frac{\partial\mathcal{L}}{\partial\beta} \neq 0$ to $\bar{\alpha},\bar{\beta},\bar{V}$, $\frac{\partial\mathcal{L}}{\partial\beta} = 0$ then it must necessarily satisfy

$$\overline{\alpha} = \overline{\alpha}(\alpha,\beta) \tag{16}$$

$$\overline{\beta} = \overline{\beta}(\alpha,\beta) \tag{17}$$

$$\overline{V} = V \qquad \text{(selected)} \tag{18}$$

$$\left[\overline{\alpha},\overline{\beta}\right]_{\alpha,\beta} = J\left\{\frac{\overline{\alpha},\overline{\beta}}{\alpha,\beta}\right\} = 1 \quad . \tag{19}$$

The necessary and sufficient conditions on the magnetic field can now be obtained from (8). First, consider the function $\psi(\alpha,\beta,V)$. From (16)-(18) we then have, by the chain rule

$$\frac{\partial\psi(\alpha,\beta,V)}{\partial\overline{\beta}} = \frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial\overline{\beta}} + \frac{\partial\psi}{\partial\beta}\frac{\partial\beta}{\partial\overline{\beta}} \quad . \tag{20}$$

We may now express $\frac{\partial\alpha}{\partial\overline{\beta}}$, $\frac{\partial\alpha}{\partial\overline{\alpha}}$, $\frac{\partial\beta}{\partial\overline{\alpha}}$, $\frac{\partial\beta}{\partial\overline{\beta}}$ in terms of $\frac{\partial\overline{\alpha}}{\partial\beta}$, $\frac{\partial\overline{\alpha}}{\partial\alpha}$, $\frac{\partial\overline{\beta}}{\partial\alpha}$, $\frac{\partial\overline{\beta}}{\partial\beta}$. To do this we simply take $\frac{\partial}{\partial\overline{\beta}}$ of both sides of (16), (17) yielding

$$\begin{aligned} 0 &= \frac{\partial\overline{\alpha}}{\partial\alpha}\frac{\partial\alpha}{\partial\overline{\beta}} + \frac{\partial\overline{\alpha}}{\partial\beta}\frac{\partial\beta}{\partial\overline{\beta}} \\ 1 &= \frac{\partial\overline{\beta}}{\partial\alpha}\frac{\partial\alpha}{\partial\overline{\beta}} + \frac{\partial\overline{\beta}}{\partial\beta}\frac{\partial\beta}{\partial\overline{\beta}} \quad . \end{aligned} \tag{21}$$

Using Cramer's rule and (19) then gives

$$\frac{\partial\alpha}{\partial\bar{\beta}} = -\frac{\partial\bar{\alpha}}{\partial\beta}$$

$$\frac{\partial\beta}{\partial\bar{\beta}} = \frac{\partial\bar{\alpha}}{\partial\alpha} \tag{22}$$

whereupon placing this result back in (20) expresses the partial derivative of $\psi(\alpha,\beta,V)$ with respect to $\bar{\beta}$, by its partial derivatives with respect to α and β. It follows

$$\frac{\partial\psi}{\partial\bar{\beta}} = \left[\bar{\alpha},\psi\right]_{\alpha,\beta} \tag{23}$$

From (9) and (10) we then have

$$|\nabla\bar{\alpha}|^2 = \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2|\nabla\alpha|^2 + \left(\frac{\partial\bar{\alpha}}{\partial\beta}\right)^2|\nabla\beta|^2 + 2\frac{\partial\bar{\alpha}}{\partial\alpha}\ \frac{\partial\bar{\alpha}}{\partial\beta}(\nabla\alpha\cdot\nabla\beta) \tag{24}$$

$$|\nabla\bar{\beta}|^2 = \left(\frac{\partial\bar{\beta}}{\partial\alpha}\right)^2|\nabla\alpha|^2 + \left(\frac{\partial\bar{\beta}}{\partial\beta}\right)^2|\nabla\beta|^2 + 2\frac{\partial\bar{\beta}}{\partial\alpha}\ \frac{\partial\bar{\beta}}{\partial\beta}(\nabla\alpha\cdot\nabla\beta)\quad . \tag{25}$$

Placing these in (8), using (23), we have the conditions

$$\left[\bar{\alpha},\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2|\nabla\alpha|^2 + \left(\frac{\partial\bar{\alpha}}{\partial\beta}\right)^2|\nabla\beta|^2 + 2(\nabla\alpha\cdot\nabla\beta)\frac{\partial\bar{\alpha}}{\partial\alpha}\ \frac{\partial\bar{\alpha}}{\partial\beta}\right]_{\alpha,\beta} = 0 \tag{26}$$

$$\left[\bar{\alpha},\left(\frac{\partial\bar{\beta}}{\partial\alpha}\right)^2|\nabla\alpha|^2 + \left(\frac{\partial\bar{\beta}}{\partial\beta}\right)^2|\nabla\beta|^2 + 2(\nabla\alpha\cdot\nabla\beta)\frac{\partial\bar{\beta}}{\partial\alpha}\ \frac{\partial\bar{\beta}}{\partial\beta}\right]_{\alpha,\beta} = 0 \tag{27}$$

$$\left[\bar{\alpha},B\right]_{\alpha,\beta} = 0 \tag{28}$$

$$\left[\bar{\alpha},\bar{\beta}\right]_{\alpha,\beta} = 1\quad . \tag{29}$$

We may rearrange (26)-(28) into another form by noting the Poisson[12] Bracket $[A,B]_{\alpha,\beta}=0$ and $[B,C]_{\alpha,\beta}=0$ imply $[A,C]_{\alpha,\beta}=0$.[36] We may then replace (26) to (28) by

$$\left[B,\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2|\nabla\alpha|^2+\left(\frac{\partial\bar{\alpha}}{\partial\beta}\right)^2|\nabla\beta|^2+2(\nabla\alpha\cdot\nabla\beta)\frac{\partial\bar{\alpha}}{\partial\alpha}\ \frac{\partial\bar{\alpha}}{\partial\beta}\right]_{\alpha,\beta}=0 \quad (30)$$

$$\left[B,\left(\frac{\partial\bar{\beta}}{\partial\alpha}\right)^2|\nabla\alpha|^2+\left(\frac{\partial\bar{\beta}}{\partial\beta}\right)^2|\nabla\beta|^2+2(\nabla\alpha\cdot\nabla\beta)\frac{\partial\bar{\beta}}{\partial\alpha}\ \frac{\partial\bar{\beta}}{\partial\beta}\right]_{\alpha,\beta}=0 \quad (31)$$

$$\left[B,\bar{\alpha}\right]_{\alpha,\beta}=0 \quad . \quad (32)$$

Our first condition on the magnetic field can be obtained by combining (30) and (32). Solving (32) for $\frac{\partial\bar{\alpha}}{\partial\beta}$ in terms of $\frac{\partial\bar{\alpha}}{\partial\alpha}$, $\frac{\partial B}{\partial\alpha}$, and $\frac{\partial B}{\partial\beta}$ and replacing the result in (30), we have, after some algebra

$$\left[B,\left(\frac{\left(\frac{\partial B}{\partial\alpha}\right)^2|\nabla\alpha|^2+\left(\frac{\partial B}{\partial\beta}\right)^2|\nabla\beta|^2+(\nabla\alpha\cdot\nabla\beta)\frac{\partial B}{\partial\alpha}\ \frac{\partial B}{\partial\beta}}{\left(\frac{\partial B}{\partial\alpha}\right)^2}\right)\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2\right]_{\alpha,\beta}=0 \quad . \quad (33)$$

If we define a unit vector $\hat{e}_B$ in the $\vec{B}$ direction and use the chain rule on $B(\alpha,\beta,V)$ we have the identity

$$\xi\equiv|\nabla B|^2-(\hat{e}_B\cdot\nabla B)^2\equiv\left(\frac{\partial B}{\partial\alpha}\right)^2|\nabla\alpha|^2+\left(\frac{\partial B}{\partial\beta}\right)^2|\nabla\beta|^2+(\nabla\alpha\cdot\nabla\beta)\frac{\partial B}{\partial\alpha}\ \frac{\partial B}{\partial\beta} \quad (34)$$

which we may replace in (33) to give

$$\left[B,\xi\,\frac{\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)^2}\right]_{\alpha,\beta}=0 \quad . \quad (35)$$

Now, expanding the Poisson Bracket (35), yields[12]

$$\left[B, \xi \frac{\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)^2}\right]_{\alpha,\beta} = \left[B, \xi\right]_{\alpha,\beta} \frac{\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)^2} + \xi \left[B, \frac{\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)^2}\right]_{\alpha,\beta} = 0 \; . \tag{36}$$

However, the second term on the right hand side of (36) is zero. This can be shown by, again, expanding it, ie:

$$\left[B, \frac{\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)^2}\right]_{\alpha,\beta} = \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2 \left[B, \left(\frac{\partial B}{\partial\alpha}\right)^{-2}\right]_{\alpha,\beta} + \frac{1}{\left(\frac{\partial B}{\partial\alpha}\right)^2} \left[B, \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2\right]_{\alpha,\beta} \; . \tag{37}$$

That the second term on the right of (37) just cancels the first is clear from the following

$$\left[B, \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2\right]_{\alpha,\beta} = 2\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right) \left[B, \frac{\partial\bar{\alpha}}{\partial\alpha}\right]_{\alpha,\beta} = 2\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right) \left[\frac{\partial B}{\partial\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\partial\bar{\alpha}}{\partial\beta}\right) - \frac{\partial B}{\partial\beta} \frac{\partial^2\bar{\alpha}}{\partial\alpha^2}\right] \; . \tag{38}$$

Now take $\frac{\partial\bar{\alpha}}{\partial\beta}$ from (32) and place it into (38) to give

$$\left[B, \left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2\right]_{\alpha,\beta} = 2\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right) \left\{\frac{\partial B}{\partial\alpha} \frac{\partial\bar{\alpha}}{\partial\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\frac{\partial B}{\partial\beta}}{\frac{\partial B}{\partial\alpha}}\right) + \frac{\partial B}{\partial\beta} \frac{\partial^2\bar{\alpha}}{\partial\alpha^2} - \frac{\partial B}{\partial\beta} \frac{\partial^2\bar{\alpha}}{\partial\alpha^2}\right\}$$

$$= \frac{2\left(\frac{\partial\bar{\alpha}}{\partial\alpha}\right)^2}{\left(\frac{\partial B}{\partial\alpha}\right)} \left\{B, \frac{\partial B}{\partial\alpha}\right\}_{\alpha,\beta} \; . \tag{39}$$

The right hand side of (39) is equal and opposite to the first term on the right hand side of (37). Hence our first condition takes the form

$$\left[B,\ |\nabla B|^2 - (\hat{e}_{\vec{B}} \cdot \vec{B})^2 \right]_{\alpha,\beta} = 0 \quad . \tag{40}$$

Recall the equivalences cited in Appendix II,

$$\frac{\partial}{\partial\alpha} \equiv \frac{\nabla\beta \times \vec{B}}{B^2} \cdot \nabla \tag{41}$$

$$\frac{\partial}{\partial\beta} \equiv \frac{\vec{B} \times \nabla\alpha}{B^2} \cdot \nabla \quad . \tag{42}$$

Our condition (40) then takes the form

$$\left(\frac{\nabla\beta \times \vec{B}\cdot\nabla B}{B^2}\right)\left(\frac{\vec{B} \times \nabla\alpha}{B^2}\cdot\nabla\left(|\nabla B|^2-(\hat{e}_{\vec{B}}\cdot\nabla B)^2\right)\right)$$

$$-\left(\frac{\vec{B}\times\nabla\alpha\cdot\nabla B}{B^2}\right)\left(\frac{\nabla\beta \times \vec{B}}{B^2}\cdot\nabla\left(|\nabla\beta|^2-(\hat{e}_{\vec{B}}\cdot\nabla B)^2\right)\right) = 0 \quad . \tag{43}$$

This condition may be reduced to a much simpler form after a great deal of vector algebra and use of (3). The result is

$$(\vec{B} \times \nabla B)\cdot\nabla\left\{|\nabla B|^2-(\hat{e}_{\vec{B}}\cdot\nabla B)^2\right\} = 0$$

or (44)

$$\hat{e}_B \times \nabla B \cdot \nabla \left|\hat{e}_B \times \nabla B\right| = 0 \quad .$$

This is the sufficient condition we have been seeking. The magnetic field which satisfies this condition has, as a consequence, a representation $\overline{\alpha}$, $\overline{\beta}$, $\overline{V}$ of such a form that $\frac{\partial \chi}{\partial \overline{\beta}} = 0$. The receipé for finding such a representation is as follows. Define <u>any</u> "α", say, as some function of the minimum value of magnetic field along a line of force of the field. Solve equation (3) which gives us the corresponding value of β. We may now invert these expressions to give, say, the cartesian coordinates as a function of α, β, V. Then all the field variables $\vec{B}$, $\nabla\alpha$, $\nabla\beta$, may be similarly expressed. From equation (32) we may now solve for $\overline{\alpha}(\alpha,\beta)$ which may then be transformed to $\overline{\alpha}(x,y,z)$, and from (31) we may now solve for $\overline{\beta}(\alpha,\beta)$ which similarly be transformed to $\overline{\beta}(x,y,z)$. The new $\overline{\alpha}$, $\overline{\beta}$ just found in addition to $\overline{V} = V$ are our new representation which lead to a Lagrangian that is independent of $\overline{\beta}$.

An example of a field that satisfies (44) is the dipole field I-(48) for the term $\vec{B} \times \nabla B$ is in the ϕ direction so that $\frac{\partial}{\partial \phi}$ of the curly bracket is zero, which of course is as it must be since all variables within the bracket are independent of ϕ and therefore have a zero derivative with respect to ϕ.

APPENDICES

APPENDIX - I

The Lagrangian for a relativistic charged particle in a static magnetic field is given by[12]

$$\mathcal{L}_r = -\frac{m_o}{\gamma} c^2 + \frac{q\vec{A}\cdot\vec{v}}{c} \qquad (1)$$

where

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}}$$

$\beta = v/c$

m_o = rest mass

$\vec{A}$ = magnetic vector potential

q = charge (cgs units)

c = speed of light

$\vec{v}$ = velocity

We will now proceed to show that the equations of motion arising from (1) are the same as those arising from the Lagrangian

$$\mathcal{L} = \frac{mv^2}{2} + \frac{q\,\vec{A}\cdot\vec{v}}{c} \qquad (2)$$

where

$m = m_o\gamma$ = mass = constant of the motion

provided we treat m as a constant (ie: $\dot{m} = 0$).

Since (1) and (2) differ only in the first term on the right hand side, which is strictly a function of speed, we may rewrite both equations as

$$\mathcal{L}_r = -m_o c^2 (1- \frac{v^2}{c^2})^{1/2} + \xi(\vec{x},\vec{V}) \tag{1'}$$

$$\mathcal{L} = \frac{m}{2} v^2 + \xi(\vec{x},\vec{V}) \quad . \tag{2'}$$

It is then clear from (1'), (2') that

$$\frac{\partial \mathcal{L}_r}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial \xi(\vec{x},\vec{V})}{\partial x_i} \qquad i = 1,2,3 \tag{3}$$

So we must proceed to show

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_r}{\partial \dot{x}_i}\right] = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right] \qquad i = 1,2,3 \tag{4}$$

in order that Lagranges equations be equivalent.

Since $\dot{x}_i$ is no more than v_i, we may now form, from (1'),(2')

$$\frac{\partial \mathcal{L}_r}{\partial v_i} = \frac{m_o v_i}{(1-v^2/c^2)^{1/2}} + \frac{\partial \xi(\vec{x},\vec{v})}{\partial v_i} \quad . \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial v_i} = mv_i + \frac{\partial \xi(\vec{x},\vec{v})}{\partial v_o} \tag{6}$$

The time derivatives of (5), (6) become

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_r}{\partial v_i}\right] = \frac{m_o \dot{v}_i}{(1- v^2/c^2)^{\frac{1}{2}}} + m_o v_i \frac{d(1- v^2/c^2)^{-\frac{1}{2}}}{dt} + \frac{d}{dt}\left[\frac{\partial \xi(\vec{x},\vec{V})}{\partial v_i}\right] \tag{10}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial v_i}\right] = \frac{m_o \dot{v}_i}{(1-v^2/c^2)^{1/2}} + \frac{d}{dt}\left[\frac{\partial \xi(\vec{x},\vec{v})}{\partial v_i}\right] \quad . \tag{11}$$

The only difference between (10) and (11) is in the term

$$m_o v_i \frac{d(1-v^2/c^2)^{-1/2}}{dt} \tag{12}$$

but this term is clearly zero since the speed of particle in a static magnetic field is constant, and thus the time derivative in (12) is zero. Hence the Lagranges equations coming from (1'), (2') are equivalent and we may replace (1) by (2) provided 'm' is treated as a constant (ie: $\dot{m} = 0$).

APPENDIX - II

We will now proceed to compute the Lagrangian, in the α, β, V coordinate system, for a charged particle in a static magnetic field. The derivation fills in the omitted steps in the paper by Ray.[38]

From a previous Appendix we have established the Lagrangian of such a particle as

$$\mathcal{L} = \frac{m}{2} v^2 + \frac{q}{c} \vec{v} \cdot \vec{A} \tag{1}$$

where $m = m_o \gamma$ and is a constant of the motion.

Let

$$\begin{array}{lll} \xi_1 = x & \xi_2 = y & \xi_3 = z \\ \eta_1 = \alpha & \eta_2 = \beta & \eta_3 = V \,. \end{array} \tag{2}$$

We have a point transformation relating the ξ_i's to the η_j's. Ie:

$$\xi_i = \xi_i(\eta_j) \qquad i,j = 1,2,3 \,. \tag{3}$$

The time derivative of (3) then becomes

$$\dot{\xi}_i = \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j \qquad i = 1,2,3 \,. \tag{4}$$

So that the kinetic energy term of (1) becomes

$$\frac{m}{2} v^2 = \frac{m}{2} \sum_{\ell=1}^{3} \dot{\xi}_\ell^2 = \frac{m}{2} \sum_{\ell=1}^{3} \sum_{j=1}^{3} \sum_{j'=1}^{3} \frac{\partial \xi_\ell}{\partial \eta_j} \frac{\partial \xi_\ell}{\partial \eta_{j'}} \dot{\eta}_j \dot{\eta}_{j'} \,. \tag{5}$$

Furthermore the remaining term takes the form

$$\frac{q}{c}\vec{A}\cdot\vec{v} = \frac{q}{c}\sum_{i=1}^{3} A_i\dot{\xi}_i = \frac{q}{c}\sum_{i=1}^{3}\sum_{j=1}^{3} A_i\frac{\partial\xi_i}{\partial\eta_j}\dot{\eta}_j \quad .(6)$$

Placing (5), (6) in (1) our Lagrangian becomes

$$\mathcal{L}(\eta,\dot{\eta}) = \frac{m}{2}\sum_{j=1}^{3}\sum_{j'=1}^{3}\left[\sum_{\ell=1}^{3}\frac{\partial\xi_\ell}{\partial\eta_j}\frac{\partial\xi_\ell}{\partial\eta_{j'}}\right]\dot{\eta}_j\dot{\eta}_{j'} + \frac{q}{c}\sum_{j=1}^{3}\left[\sum_{i=1}^{3}A_i\frac{\partial\xi_i}{\partial\eta_j}\right]\dot{\eta}_j \quad (7)$$

where now $\vec{A}$ is considered a function of η_j only. All that remains is to express the $\frac{\partial\xi_a}{\partial\eta_b}$'s in terms of the $\frac{\partial\eta_{a'}}{\partial\xi_{b'}}$'s. This can be done as follows. By the chain rule, we obtain from (3)

$$d\xi_i = \sum_{j=1}^{3}\frac{\partial\xi_i}{\partial\eta_j}d\eta_j \qquad i = 1,2,3 \qquad (8)$$

or, conversely

$$d\eta_{i'} = \sum_{j'=1}^{3}\frac{\partial\eta_{i'}}{\partial\xi_j}d\xi_{j'} \quad . \qquad i' = 1,2,3 \quad (9)$$

Placing (9) in (8) we obtain

$$d\xi_i = \sum_{j'=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial\xi_i}{\partial\eta_j}\frac{\partial\eta_j}{\partial\xi_{j'}}\right]d\xi_{j'} \qquad (10)$$

However, because the ξ_i are independent variables, in (10) we must have

$$\sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \eta_j} \frac{\partial \eta_j}{\partial \xi_{j'}} = \delta_{ij'}, \qquad i = 1,2,3 \quad . \tag{11}$$

The solution to these 9 equations (11), is given by Cramer's Rule as

$$\frac{\partial \xi_l}{\partial \eta_j} = \frac{\left[\frac{\partial \eta_k}{\partial \xi_m}\frac{\partial \eta_i}{\partial \xi_n} - \frac{\partial \eta_i}{\partial \xi_m}\frac{\partial \eta_k}{\partial \xi_n}\right]}{\sum_{p=1}^{3} \frac{\partial \eta_1}{\partial \xi_p}\left\{\frac{\partial \eta_2}{\partial \xi_q}\frac{\partial \eta_3}{\partial \xi_r} - \frac{\partial \eta_3}{\partial \xi_q}\frac{\partial \eta_2}{\partial \xi_r}\right\}} \tag{12}$$

where l,m,n and i,j,k, independently take on cyclic numbers. Also q and r are cyclic permutations of p.

Employing (12) we then find, omitting some algebra

$$\frac{\partial x}{\partial \alpha} = \frac{(\nabla\beta \times \vec{B})_x}{B^2} \qquad \frac{\partial y}{\partial \alpha} = \frac{(\nabla\beta \times \vec{B})_y}{B^2} \qquad \frac{\partial z}{\partial \alpha} = \frac{(\nabla\beta \times \vec{B})_z}{B^2} \tag{13a}$$

$$\frac{\partial x}{\partial \beta} = \frac{(\vec{B} \times \nabla\alpha)_x}{B^2} \qquad \frac{\partial y}{\partial \beta} = \frac{(\vec{B} \times \nabla\alpha)_y}{B^2} \qquad \frac{\partial z}{\partial \beta} = \frac{(\vec{B} \times \nabla\alpha)_z}{B^2} \tag{13b}$$

$$\frac{\partial x}{\partial V} = \frac{\bar{\mu} B_x}{B^2} \qquad \frac{\partial y}{\partial V} = \frac{\bar{\mu} B_y}{B^2} \qquad \frac{\partial z}{\partial V} = \frac{\bar{\mu} B_z}{B^2} \quad . \tag{13c}$$

Let us now define the matrix element $T_{jj'}$ as

$$T_{jj'} \equiv \sum_{\ell=1}^{3} \frac{\partial \xi_\ell}{\partial \eta_j}\frac{\partial \xi_\ell}{\partial \eta_{j'}} = \sum_{\ell=1}^{3} \frac{\left\{\frac{\partial \eta_k}{\partial \xi_m}\frac{\partial \eta_i}{\partial \xi_n} - \frac{\partial \eta_i}{\partial \xi_m}\frac{\partial \eta_k}{\partial \xi_n}\right\}\left\{\frac{\partial \eta_{k'}}{\partial \xi_m}\frac{\partial \eta_{i'}}{\partial \xi_n} - \frac{\partial \eta_{i'}}{\partial \xi_m}\frac{\partial \eta_{k'}}{\partial \xi_n}\right\}}{\left[\sum_{p=1}^{3}\frac{\partial \eta_1}{\partial \xi_p}\left\{\frac{\partial \eta_2}{\partial \xi_q}\frac{\partial \eta_3}{\partial \xi_r} - \frac{\partial \eta_3}{\partial \xi_q}\frac{\partial \eta_2}{\partial \xi_r}\right\}\right]^2} \tag{14}$$

where i,j,k; i',j',k'; l,m,n; p,q,r are to be taken in cyclid order. The elements of (14) may then be evaluated (we have used facts that $\nabla\alpha \times \nabla\beta = \vec{B}$) and tabulated as follows

$$\overline{\overline{T}} = \left|\begin{array}{ccc} \frac{|\nabla\beta|^2}{B^2} & -\frac{(\nabla\alpha\cdot\nabla\beta)}{B^2} & 0 \\ -\frac{(\nabla\alpha\cdot\nabla\beta)}{B^2} & \frac{|\nabla\alpha|^2}{B^2} & 0 \\ 0 & 0 & \frac{\overline{\mu}^2}{B^2} \end{array}\right| . \tag{15}$$

Furthermore, we may, in the same manner evaluate

$$\sum_{i=1}^{3} A_i \frac{\partial \xi_i}{\partial \alpha} = \frac{\vec{A}\cdot(\nabla\beta \times \vec{B})}{B^2} = 0; \quad \sum_{i=1}^{3} A_i \frac{\partial \xi_i}{\partial \beta} = \frac{\vec{A}\cdot(\nabla\alpha \times \vec{B})}{B^2} = \alpha;$$

$$\sum_{i=1}^{3} A_i \frac{\partial \xi_i}{\partial V} = \frac{\overline{\mu}\vec{A}\cdot\vec{B}}{B^2} = 0 . \tag{16}$$

Placing (15) and (16) back into equation (7) we thus finally obtain

$$\mathcal{L} = \frac{m}{2}\left[\frac{|\nabla\beta|^2}{B^2}\dot{\alpha}^2 + \frac{|\nabla\alpha|^2}{B^2}\dot{\beta}^2 - \frac{2(\nabla\alpha\cdot\nabla\beta)}{B^2}\dot{\alpha}\dot{\beta} + \left(\frac{\overline{\mu}^2}{B^2}\right)\dot{V}^2\right] + \frac{q}{c}\,\alpha\dot{\beta} \quad . \tag{17}$$

In equation (17) the mass 'm' is the relativistic mass, and all variables are assumed a function of α, β, V. The gradients of α, β, V may be performed in, say, a cartesian system of coordinates and then substitute equation (3) into the result.

The previous derivation may, in addition, be applied to derive the following relations between derivatives in the cartesian/spherical coordinate system to that of the α, β, V system.

Consider the function $f(\vec{r})$. Let us proceed to compute the partial derivatives $\frac{\partial f}{\partial\alpha}$, $\frac{\partial f}{\partial\beta}$, $\frac{\partial f}{\partial V}$. We have, by the chain rule,

$$\frac{\partial f}{\partial\eta_i} = \sum_{j=1}^{3} \frac{\partial f}{\partial\xi_j}\,\frac{\partial\xi_j}{\partial\eta_i} \quad . \tag{18}$$

Therefore we may now substitute (12) (or (13)) in (18), where upon we have, after some algebra, the result

$$\frac{\partial f}{\partial\alpha} = \frac{(\nabla\beta \times \vec{B})\cdot\nabla f}{B^2} \tag{19}$$

$$\frac{\partial f}{\partial\beta} = \frac{(\vec{B} \times \nabla\alpha)\cdot\nabla f}{B^2} \tag{20}$$

$$\frac{\partial f}{\partial V} = \frac{\overline{\mu}\;\vec{B}\cdot\nabla f}{B^2} \quad . \tag{21}$$

(19) thru (20) are very useful in simplifying rather large vector expressions.

APPENDIX - III

We will now show that it is always possible to find a vector potential to describe out static magnetic field that satisfies the gauge $\vec{A}\cdot\vec{B} = 0$.[38]

Consider any arbitrary vector potential $\vec{A}'$ satisfying

$$\vec{B} = \nabla \times \vec{A}' \qquad (1)$$

which is no more than a set of three partial differential equations from which $\vec{B}$ can be obtained given $\vec{A}'$. However, the vector function $\vec{A}'$ is not unique. Define an arbitrary function Y such that

$$\vec{A} = \vec{A}' + \nabla Y \quad . \qquad (2)$$

Obviously[37] $\vec{A}$ also satisfies (1), and we may further make the restriction that

$$\vec{A}\cdot\vec{B} = 0 \qquad (3)$$

whence from (1) we have

$$\vec{A}\cdot\vec{B} = 0 = \vec{A}'\cdot\vec{B} + \vec{B}\cdot\nabla Y \quad . \qquad (4)$$

Equation (4) is a partial differential equation for 'Y' which, once satisfied, can be used back in (2) to generate the vector potential $\vec{A}$ which satisfies the desired gauge (3).

APPENDIX - IV

The following relationship was noted by Hassit[13]. A slightly different proof is given below.

Consider the coordinate system set up at any arbitrary point in space shown in figure 35.

We have

$$\nabla\beta \cdot \vec{B} = 0 \qquad (1)$$

therefore $\nabla\beta$ must lie in the $\nabla\alpha$, $\vec{B} \times \nabla\alpha$ plane. However, the change in β in any arbitrary direction, $\overrightarrow{ds}$, is given by

$$d\beta = \nabla\beta \cdot \overrightarrow{ds} \quad . \qquad (2)$$

If we choose $\overrightarrow{ds}$ in the $\vec{B} \times \nabla\alpha$ direction ie.

$$\overrightarrow{ds} = \frac{\vec{B} \times \nabla\alpha}{B\,|\nabla\alpha|}\; ds \qquad (3)$$

then (2) becomes

$$d\beta = \frac{\nabla\beta \cdot (\vec{B} \times \nabla\alpha)}{B\,|\nabla\alpha|}\, ds = \frac{B}{|\nabla\alpha|}\; ds \; . \qquad (4)$$

We may therefore calculate the total change in β between two points in space by integrating the function $\frac{B}{|\nabla\alpha|}$ (expressed in, say, cartesian coordinates) along the curve traced out by the curve formed in the $\vec{B} \times \nabla\alpha$ direction. The obvious advantage of this, is that we now have a recipe' for computing changes in β, which is normally a very complicated solution to a set of partial differential equations.

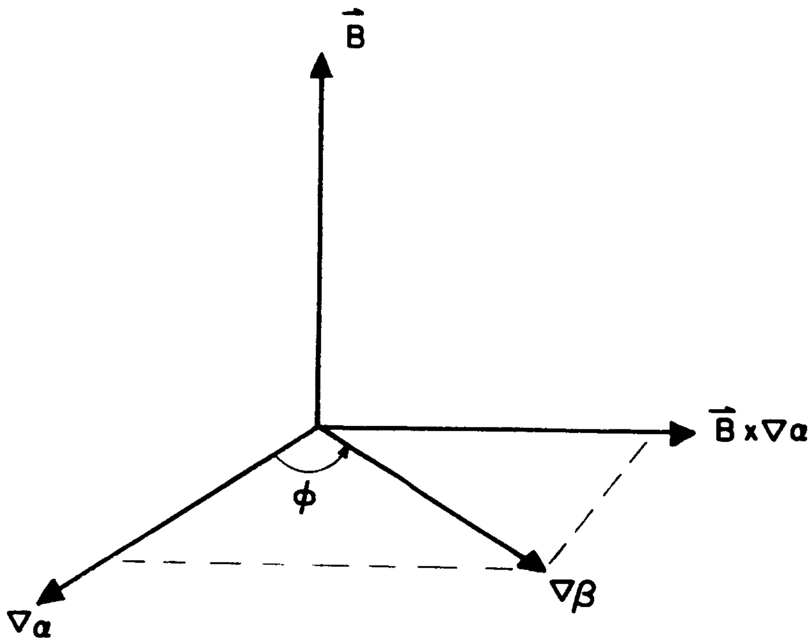


figure 35

APPENDIX - V

In this appendix we will prove the assertion[38] that a field with no variation of magnetic field along lines of force lends itself to choosing $\alpha = f(B)$, that is a function only of the magnitude of the magnetic field intensity. The proof is as follows:

Assume

$$\vec{B} = B(x,y)\hat{e}_z \quad . \tag{1}$$

α and β must then satisfy

$$\nabla\alpha \times \nabla\beta = B(x,y)\hat{e}_z \quad . \tag{2}$$

However, because of our above assumption α, in addition, must satisfy

$$\alpha = f(B) \quad . \tag{3}$$

Placing (3) in (2), gives us the set of partial differential equations that β must satisfy

$$B(x,y)\hat{e}_z = f'(B)(\nabla B \times \nabla\beta) \quad . \tag{4}$$

That is, we must have

$$\begin{aligned} \frac{\partial B}{\partial y}\frac{\partial \beta}{\partial z} &= \frac{\partial B}{\partial z}\frac{\partial \beta}{\partial y} \\ \frac{\partial B}{\partial z}\frac{\partial \beta}{\partial x} &= \frac{\partial B}{\partial x}\frac{\partial \beta}{\partial z} \\ B(x,y) &= f'(B)\left(\frac{\partial B}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial B}{\partial y}\frac{\partial \beta}{\partial x}\right) \quad . \end{aligned} \tag{5}$$

Excluding the uniform field ($\vec{B}$ = constant), the first two equations of (5) imply (using the general form (1) for $\vec{B}$)

$$\beta = \beta(x,y) \quad . \tag{6}$$

Placing this result back in the third equation of (5) we have

$$\frac{B(x,y)}{f'(B(x,y))} = \frac{\partial B(x,y)}{\partial x}\frac{\partial \beta(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial y}\frac{\partial \beta(x,y)}{\partial x} \quad . \tag{7}$$

Equation (7) is a partial differential equation for β, which can, in principle, be solved once the field $B(x,y)$, and the function $f(B)$ is prescribed. The conclusion to be drawn from this proof is that we may always describe any field $B(x,y)\hat{e}_z$ by the variables $\alpha(B)$, β, V. This configuration has particular merit for equatorial trapped particles in models of the magnetosphere (ie. Hones,[15] Mead[26]).

APPENDIX - VI

We will now prove that it is always possible to select an α', β', for a field of the form $\vec{B} = B(x,y)\hat{e}_z$ such that $\nabla\alpha'\cdot\nabla\beta' = 0$.

To prove this, consider a region of space where the currents are everywhere perpendicular to the static magnetic field. We assume

$$\vec{E} = 0 \tag{1}$$

$$\vec{B} = B(x,y)\hat{e}_z \tag{2}$$

$$\frac{\partial\vec{B}}{\partial t} = 0 \tag{3}$$

$$\vec{J}\cdot\vec{B} = 0 \quad . \tag{4}$$

It is always possible to express such a field by an α, β, V such that[1]

$$\vec{A} = \alpha\nabla\beta \tag{5}$$

$$\vec{B} = \nabla\alpha \times \nabla\beta \tag{6}$$

$$\vec{B} = \bar{\mu}\nabla V \quad . \tag{7}$$

Here we assume that

$$\nabla\alpha\cdot\nabla\beta \neq 0 \tag{8}$$

and will show that for a field of the form (2) we may always make the point transformation

$$\alpha' = \alpha'(\alpha,\beta) \tag{9}$$

$$\beta' = \beta'(\alpha,\beta) \tag{10}$$

$$V' = V \tag{11}$$

$$\bar{\mu}' = \bar{\mu} \tag{12}$$

so that the new variables satisfy (6) and (7) with the α', β', V' replacing α, β, V and further with

$$\nabla\alpha' \cdot \nabla\beta' = 0 \quad . \tag{13}$$

By the chain rule for differentiating an implicit function we have, using (9) and (10)

$$\nabla\alpha' = \frac{\partial\alpha'}{\partial\alpha}\nabla\alpha + \frac{\partial\alpha'}{\partial\beta}\nabla\beta \tag{14}$$

$$\nabla\beta' = \frac{\partial\beta'}{\partial\alpha}\nabla\alpha + \frac{\partial\beta'}{\partial\beta}\nabla\beta \quad . \tag{15}$$

Therefore we must find the transformation (9) through (12) satisfying

$$\nabla\alpha' \cdot \nabla\beta' = 0 = \frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\alpha}|\nabla\alpha|^2 + \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\beta}|\nabla\beta|^2 + \left(\frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta} + \frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\alpha}\right)(\nabla\alpha \cdot \nabla\beta) \tag{16}$$

and

$$\nabla\alpha' \times \nabla\beta' = \left(\frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta} - \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\alpha}\right)\nabla\alpha \times \nabla\beta = \nabla\alpha \times \nabla\beta = \vec{B} \ . \tag{17}$$

All we need find is a sufficient set of conditions such that (16) and (17) are satisfied and our task is completed.

From equation (17) it is clear that we must demand that the Jacobian of the transformation is unity. Ie:

$$J\left\{\frac{\alpha', \beta', V'}{\alpha, \beta, V}\right\} = \frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta} - \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\alpha} = 1 \quad . \tag{18}$$

Also, since $\nabla\alpha \cdot \nabla\beta \neq 0$ we may satisfy (16) <u>sufficiently</u> by demanding

$$\frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta}\;|\nabla\alpha|^2 + \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\beta}\;|\nabla\beta|^2 = 0 \tag{19}$$

and

$$\frac{\partial\alpha'}{\partial\alpha}\frac{\partial\beta'}{\partial\beta} + \frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta'}{\partial\alpha} = 0 \quad . \tag{20}$$

We may now treat equations (18) and (20) as two linear equations for the partial derivatives $\frac{\partial\beta'}{\partial\alpha}$, $\frac{\partial\beta'}{\partial\beta}$ in terms of $\frac{\partial\alpha'}{\partial\alpha}$ and $\frac{\partial\alpha'}{\partial\beta}$. Using Cramer's Rule and omitting algebra we obtain

$$\frac{\partial\beta'}{\partial\alpha} = -\frac{1}{2(\partial\alpha'/\partial\beta)} \tag{21}$$

$$\frac{\partial\beta'}{\partial\beta} = \frac{1}{2(\partial\alpha'/\partial\alpha)} \quad . \tag{22}$$

We may now replace (21) and (22) in (19) and therefore transformation (9), (10) must satisfy the partial differential equation

$$\frac{\partial\alpha'}{\partial\alpha}\;|\nabla\alpha| \pm \frac{\partial\alpha'}{\partial\beta}\;|\nabla\beta| = 0 \quad . \tag{23}$$

The solution of (23) for the $\alpha'(\alpha,\beta)$ may then be put back into (21) and (22) to find $\beta'(\alpha,\beta)$. Therefore we may always find an α',β',V' describing the field (2) that also satisfies $\nabla\alpha'\cdot\nabla\beta' = 0$.

A simple example will demonstrate the procedure. Consider the uniform field of unit magnitude.

$$\vec{B} = \hat{e}_z \quad . \tag{24}$$

One description of this field can be found by selecting

$$\alpha = x + y \tag{25}$$

$$\beta = y \tag{26}$$

$$V = z \tag{27}$$

$$\bar{\mu} = 1 \quad . \tag{28}$$

(25) through (28) will generate the proper field (24), satisfy the gauge $\vec{A}\cdot\vec{B} = 0$, but give $\nabla\alpha\cdot\nabla\beta \neq 0$. However, according to (23) the transformation we need to make $\nabla\alpha'\cdot\nabla\beta' = 0$ is the solution to

$$(2)^{1/2} \frac{\partial\alpha'}{\partial\alpha} \pm \frac{\partial\alpha'}{\partial\beta} = 0 \quad . \tag{29}$$

This is a linear, homogenious partial differential equation of the first order that may be solved by the method of characteristics to yield the general solution

$$\alpha' = \alpha'(\beta \pm \frac{\alpha}{\sqrt{2}}) \tag{30}$$

where α' is an arbitrary function of its argument. For simplicity, let us select the solution

$$\alpha' = \beta + \frac{\alpha}{\sqrt{2}} \quad . \tag{31}$$

Returning to (21) and (22) we then find, corresponding to (31) we have

$$\beta' = -\frac{\alpha}{2} + \frac{\beta}{\sqrt{2}} + c \quad . \tag{32}$$

Replacing α, β in terms of x,y from (25), (26) we finally obtain

$$\alpha' = \frac{x}{\sqrt{2}} + (1+\frac{\sqrt{2}}{2})y \tag{33}$$

$$\beta' = -\frac{x}{2} + \left(\frac{\sqrt{2} - 1}{2}\right)y \qquad . \tag{34}$$

A quick check will demonstrate that $\nabla\alpha' \cdot \nabla\beta' = 0$ and α', β' of (33) and (34) will generate the given B field, (24).

APPENDIX - VII

$$P_{m,n}(\cos\theta) = \begin{cases} \left[\dfrac{2(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\cos\theta) & m > o \\ \\ P_n^m(\cos\theta) & m = 0 \end{cases}$$

where $P_{m,n}(\cos\theta)$ are the Schmidt-normalized associated Legendre functions defined in terms of the conventional Legendre-functions given by

$$P_n^m(\cos\theta) = \frac{\sin^m\theta}{2^n\ n!}\ \frac{d^{n+m}(\cos^2\theta - 1)^n}{d(\cos\theta)^{n+m}}$$

TABLE IV

$P_{0,0}(\cos\theta) = 1$

$P_{1,0}(\cos\theta) = \cos\theta$

$P_{1,1}(\cos\theta) = \sin\theta$

$P_{2,0}(\cos\theta) = (1/2)(3\cos^2\theta-1)$

$P_{2,1}(\cos\theta) = \sqrt{3}\ \sin\theta\ \cos\theta$

$P_{2,2}(\cos\theta) = (\sqrt{3}/2)\sin^2\theta$

$P_{3,0}(\cos\theta) = (1/2)(5\cos^3\theta-3\cos\theta)$

$P_{3,1}(\cos\theta) = (\sqrt{6}/4)(5\sin\theta\ \mathrm{cso}^2\theta-\sin\theta)$

$P_{3,2}(\cos\theta) = (\sqrt{15}/2)\sin^2\theta\ \cos\theta$

$P_{3,3}(\cos\theta) = (\sqrt{10}/4)\sin^3\theta$

$P_{4,0}(\cos\theta) = (35/8)\cos^4\theta-(15/4)\cos^2\theta+(3/8)$

$P_{4,1}(\cos\theta) = (\sqrt{10}/4)(7\sin\theta\ \cos^3\theta-3\sin\theta\ \cos\theta)$

$P_{4,2}(\cos\theta) = (\sqrt{5}/4)(7\sin^2\theta\ \cos^2\theta-\sin^2\theta)$

$P_{4,3}(\cos\theta) = (\sqrt{70}/4)\sin^3\theta\ \cos\theta$

$P_{4,4}(\cos\theta) = (\sqrt{35}/8)\sin^4\theta$

$P_{5,0}(\cos\theta) = (63/8)\cos^5\theta-(35/4)\cos^3\theta+(15/8)\cos\theta$

$P_{5,1}(\cos\theta) = (\sqrt{15}/8)(21\sin\theta\ \cos^4\theta-7\sin\theta\ \cos^2\theta + \sin\theta)$

$P_{5,2}(\cos\theta) = (\sqrt{105}/4)(3\sin^2\theta\cos^3\theta-\sin^2\theta\ \cos\theta)$

$P_{5,3}(\cos\theta) = (\sqrt{70}/16)(9\sin^3\theta\ \cos^2\theta-\sin^3\theta)$

$$P_{5,4}(\cos\theta) = (3\sqrt{35}/8)\sin^4\theta\ \cos\theta$$

$$P_{5,5}(\cos\theta) = (3\sqrt{14}/16)\sin^5\theta$$

$$P_{6,0}(\cos\theta) = (231/16)\cos^6\theta-(315/16)\cos^4\theta+(105/16)\cos^2\theta-(1/16)$$

$$P_{6,1}(\cos\theta) = (\sqrt{21})((33/8)\sin\theta\ \cos^5\theta-(15/4)\sin\theta\ \cos^3\theta$$

$$+(5/8)\sin\theta\ \cos\theta)$$

$$P_{6,2}(\cos\theta) = (\sqrt{210})((33/32)\sin^2\theta\ \cos^4\theta-(9/16)\sin^2\theta\ \cos^2\theta$$

$$+(1/32)\sin^2\theta)$$

$$P_{6,3}(\cos\theta) = (\sqrt{210}/16)(11\sin^3\theta\ \cos^3\theta-3\sin^3\theta\ \cos\theta)$$

$$P_{6,4}(\cos\theta) = (\sqrt{7}/16)(33\sin^4\theta\ \cos^2\theta-3\sin^4\theta)$$

$$P_{6,5}(\cos\theta) = (3\sqrt{154}/16)\sin^5\theta\ \cos\theta$$

$$P_{6,6}(\cos\theta) = (\sqrt{462}/32)\sin^6\theta$$

TABLE V

$$Y_{m,n}(\theta) = \frac{dP_{m,n}(\cos\theta)}{d\theta}$$

$Y_{0,0}(\theta) = 0$

$Y_{1,0}(\theta) = -\sin\theta$

$Y_{1,1}(\theta) = \cos\theta$

$Y_{2,0}(\theta) = -3\sin\theta\ \cos\theta$

$Y_{2,1}(\theta) = \sqrt{3}(2\cos^2\theta-1)$

$Y_{2,2}(\theta) = \sqrt{3}\sin\theta\ \cos\theta$

$Y_{3,0}(\theta) = -(15/2)\sin\theta\ \cos^2\theta+(3/2)\sin\theta$

$Y_{3,1}(\theta) = (5\sqrt{6}/4)\cos^3\theta-(5\sqrt{6}/2)\sin^2\theta\ \cos\theta-(\sqrt{6}/4)\cos\theta$

$Y_{3,2}(\theta) = \sqrt{15}(\sin\theta\ \cos^2\theta-(1/2)\sin^3\theta)$

$Y_{3,3}(\theta) = (3\sqrt{10}/4)\sin^2\theta\ \cos\theta$

$Y_{4,0}(\theta) = (1/2)(-35\sin\theta\ \cos^3\theta+15\sin\theta\ \cos\theta)$

$Y_{4,1}(\theta) = (\sqrt{10}/4)(7\cos^4\theta-21\sin^2\theta\ \cos^2\theta-3\cos^2\theta+3\sin^2\theta)$

$Y_{4,2}(\theta) = (\sqrt{5}/2)(7\sin\theta\ \cos^3\theta-7\sin^3\theta\cos\theta-\sin\theta\ \cos\theta)$

$Y_{4,3}(\theta) = (\sqrt{70}/4)(3\sin^2\theta\ \cos^2\theta-\sin^4\theta)$

$Y_{4,4}(\theta) = (\sqrt{35}/2)(\sin^3\theta\ \cos\theta)$

$Y_{5,0}(\theta) = -(315/8)\sin\theta\ \cos^4\theta+(105/4)\sin\theta\ \cos^2\theta-(15/8)\sin\theta$

$Y_{5,1}(\theta) = (\sqrt{15})(-(21/2)\sin^2\theta\ \cos^3\theta+(21/8)\cos^5\theta+(7/2)\sin^2\theta\ \cos\theta$
$\quad -(7/4)\cos^3\theta+(1/8)\cos\theta)$

$Y_{5,2}(\theta) = (\sqrt{105})(-(9/4)\sin^3\theta\ \cos^2\theta+(3/2)\sin\theta\ \cos^4\theta\ +$
$\quad (1/4)\sin^3\theta-(1/2)\sin\theta\ \cos^2\theta)$

$Y_{5,3}(\theta) = (\sqrt{70})((27/16)\sin^2\theta\ \cos^3\theta-(3/16)\sin^2\theta\ \cos\theta$
$\quad -(\frac{9}{8})\sin^4\theta\ \cos\theta)$

$Y_{5,4}(\theta) = (\sqrt{35})((3/2)\sin^3\theta\ \cos^2\theta-(3/8)\sin^5\theta)$

$$Y_{5,5}(\theta) = (15\sqrt{14}/16)\sin^4\theta\ \cos\theta$$
$$Y_{6,0}(\theta) = -(693/8)\cos^5\theta\ \sin\theta+(315/4)\cos^3\theta\ \sin\theta-(105/8)\sin\theta\cos\theta$$
$$Y_{6,1}(\theta) = (\sqrt{21})((33/8)\cos^6\theta-(165/8)\sin^2\theta\ \cos^4\theta-(15/4)\cos^4\theta +(45/4)\sin^2\theta\ \cos^2\theta+(5/8)\cos^2\theta-(5/8)\sin^2\theta)$$
$$Y_{6,2}(\theta) = (\sqrt{210})((33/16)\sin\theta\ \cos^5\theta-(33/8)\sin^3\theta\cos^3\theta- (9/8)\sin\theta\ \cos^3\theta+(9/8)\sin^3\theta\ \cos\theta+(1/16)\sin\theta\cos\theta)$$
$$Y_{6,3}(\theta) = (\sqrt{210}/16)(33\sin^2\theta\ \cos^4\theta-33\sin^4\theta\ \cos^2\theta+3\sin^4\theta- 9\sin^2\theta\ \cos^2\theta)$$
$$Y_{6,4}(\theta) = (\sqrt{7})(-(33/8)\sin^5\theta\ \cos\theta+(33/4)\sin^3\theta\ \cos^3\theta- (3/4)\sin^3\theta\ \cos\theta)$$
$$Y_{6,5}(\theta) = (\sqrt{154}/16)(-3\sin^6\theta+15\sin^4\theta\ \cos^2\theta)$$
$$Y_{6,6}(\theta) = (3\sqrt{462}/16)(\sin^5\theta\ \cos\theta)$$

APPENDIX - VIII

GAUSSIAN COEFFICIENTS (GEOGRAPHIC)

Tabulation of the Gaussian coefficients for the geomagnetic field. Taken from Finch and Leaton.[7] A geographic coordinate system is assumed. All coefficients are in gauss.

Table VI

n \ m	0	1	2	3	4	5	6
1	0.3055	0.0227 -0.0590					g_n^m h_n^m
2	0.0152	-0.0303 0.0190	-0.0158 -0.0024				
3	-0.0118	0.0191 0.0045	-0.0126 -0.0029	-0.0091 0.0009			
4	-0.0095	-0.0080 -0.0015	-0.0058 0.0031	0.0038 0.0004	-0.0031 0.0017		
5	0.0027	-0.0032 -0.0002	-0.0020 -0.0010	0.0004 0.0005	0.0015 0.0014	0.0007 -0.0009	
6	-0.0010	-0.0005 0.0002	-0.0002 -0.0011	0.0024 0.0000	0.0003 0.0001	0.0000 0.0003	0.0011 0.0001

GAUSSIAN COEFFICIENTS (GEOMAGNETIC)

Tabulation of the Gaussian coefficients for the geomagnetic field. Coefficients obtained by a linear transformation relating the geographic coordinate system to the geomagnetic coordinate system. Calculations performed by the author on the Cornell-Control Data 1604 computer. The following coefficients apply in a geomagnetic coordinate system. All coefficients are in gauss.

Table VII

n \ m	0	1	2	3	4	5	6
1	0.3120	0.0000 0.0000					g'^m_n h'^m_n
2	0.0049	-0.0288 -0.0228	0.0193 -0.0043				
3	-0.0082	0.0115 0.0166	0.0107 -0.0111	0.0057 0.0055			
4	-0.0082	0.0051 -0.0100	0.0001 -0.0025	-0.0032 0.0006	0.0023 0.0034		
5	0.0017	-0.0014 -0.0030	0.0027 0.0006	-0.0006 -0.0012	0.0018 -0.0012	-0.0001 -0.0006	
6	-0.0002	0.0016 -0.0002	-0.0017 0.0006	-0.0007 -0.0020	0.0004 0.0018	-0.0008 -0.0013	-0.0026 -0.0030

MEAD MODEL OF MAGNETOSPHERE

Tabulation of the Gaussian coefficients for Mead model of the magnetosphere. South magnetic pole points at the north star, while azimuth is measured positive east of the sub-solar point. All coefficients are in gauss.

$$g_1^0 = 0.3120$$

Table VIII

n \ m	0	1	2	3	4	5	6
1	0.2511×10^{-3}						$\bar{g}_n^m$
2		-0.1242×10^{-4}					
3	0.0072×10^{-5}		0.0233×10^{-5}				
4		-0.0240×10^{-6}		-0.0016×10^{-6}			
5	-0.0057×10^{-7}		0.0108×10^{-7}		0.0010×10^{-7}		
6		0.0013×10^{-8}		-0.0019×10^{-8}		-0.0004×10^{-8}	

HONES MAGNETOSPHERIC MODEL

The Hones[15] model assumes two dipoles, one 28 times the strength of the other, placed 28 earth radii apart. The smaller dipole has the strength of the actual earth dipole.

The magnetic scalar potential for this field is given by

$$\frac{V(r,\Theta,\emptyset)}{r_e} = (0.3120\cos\Theta)\left[\frac{1}{\bar{r}^2} + \frac{28r}{(\bar{r}^2+784-56\bar{r}\ \sin\Theta\cos\emptyset)^{3/2}}\right]$$

where $\bar{r}$ is in earth radii, $\frac{V(r,\Theta,\emptyset)}{r_e}$ is in gauss. r_e = 6317.2 km. Θ = collatitude, $\emptyset$ = azimuth. Again the magnetic south pole points at the north star and azimuth is positive east of the sub-solar point.

APPENDIX - IX

Define the function $F_n^m(a^2,q^2)$ as

$$F_n^m(a^2,q^2) \equiv \int_0^1 \frac{(q^2-\xi^2)^{m/2}}{(a^2+\xi^2)^n}\, d\xi \quad . \qquad (1)$$

We will first evaluate $F_0^1(a^2,q^2)$ and $F_1^1(a^2,q^2)$, from them we will be able to generate all the functions that will be needed. So we have

$$F_0^1(a^2,q^2) = \int_0^1 (q^2-\xi^2)^{1/2} d\xi = \frac{(q^2-1)^{1/2}}{2} + \frac{q^2 \sin^{-1}(1/q)}{2} \quad . \qquad (2)$$

In addition we have

$$F_1^1(a^2,q^2) = \int_0^1 \frac{(q^2-\xi^2)^{1/2}}{a^2+\xi^2}\, d\xi \quad .$$

Make the substitution $\xi = q\cos\theta$ and this becomes

$$= \int_{\cos^{-1}(1/q)}^{\pi/2} \frac{\sin^2\theta \, d\theta}{(a^2/q^2)+\cos^2\theta}$$

or

$$F_1^1(a^2,q^2) = \cos^{-1}(1/q) + \frac{\pi}{2}\left[\frac{(a^2+q^2)^{1/2}}{a} - 1\right] - \frac{(a^2+q^2)^{1/2}}{a} \tan^{-1}\frac{a(q^2-1)^{\frac{1}{2}}}{(a^2+q^2)^{\frac{1}{2}}} \, . \qquad (3)$$

We may then generate the remaining functions

$$F_0^1(a^2,q^2)=(1/2)\left[(q^2-1)^{1/2}+q^2\sin^{-1}(1/q)\right] \tag{4}$$

$$F_0^1(a^2,q^2) = \pi/4 \tag{5}$$

$$F_1^1(a^2,q^2)= \frac{(a^2+q^2)^{1/2}}{a}\left[(\pi/2)-\tan^{-1}\frac{a(q^2-1)^{1/2}}{(a^2+q^2)^{1/2}}\right]+\cos^{-1}(1/q)$$

$$- \pi/2 \tag{6}$$

$$F_1^1(a^2,1)= \frac{\pi}{2}\left[\frac{(1+a^2)^{1/2}}{a} -1\right]$$

$$F_2^1(a^2,q^2)= -\frac{\partial}{\partial a^2}\left[F_1^1(a^2,q^2)\right] \tag{7}$$

$$F_2^1(a^2,q^2)= \frac{q^2\left[(\pi/2)-\tan^{-1}\left(\frac{a(q^2-1)^{1/2}}{(a^2+q^2)^{1/2}}\right]\right.}{2a^3(a^2+q^2)^{1/2}} + \frac{(q^2-1)^{1/2}}{2a^2(1+a^2)} \tag{8}$$

$$F_2^1(a^2,1) = \frac{\pi}{4a^3(1+a^2)^{1/2}} \tag{9}$$

$$F_0^{-1}(a^2,q^2)=2\frac{\partial}{\partial q^2}(F_0^1(a^2,q^2))=\sin^{-1}(1/q) \tag{10}$$

$$F_0^{-1}(a^2,1) = \pi/2 \tag{11}$$

$$F_1^{-1}(a^2,q^2)= 2\frac{\partial}{\partial q^2}(F_1^1(a^2,q^2))=\frac{\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{a(q^2-1)^{1/2}}{(a^2+q^2)^{1/2}}\right)\right]}{a(a^2+q^2)^{1/2}} \tag{12}$$

$$F_1^{-1}(a^2,q^2) = \frac{\pi}{2a(1+a^2)^{1/2}} \tag{13}$$

APPENDIX - X

In this appendix we will establish that "β" is a non-decreasing function along any "α" surface, and therefore an appropriate variable for measuring distance along the curves which are formed by the intersections of constant "α" and constant "V" surfaces.

We assume (true for all real magnetic field models of the earth) that the magnetic field is continuous and single-valued in all of space, and may be expressed in the form of II-(16). With the choice of α being some function of the minimum value of the magnetic field along a line of force, the surfaces of constant α will be shells made up of lines of force, concentrically arranged, about the origin (the earth). The surfaces of constant V will intersect the surfaces of constant α along space curves. Fields which display axial symmetry will have their magnetic fields constant in magnitude along these curves (see figure 36). Since β is constant along lines of force, it is somewhat intuitively obvious that its changes in azimuth should be some measure of the azimuthal angle (see figure 36).

Consider a space curve made up of the intersection of an arbitrary constant α and constant V surface as shown in figure 37, the curve need not be planar.

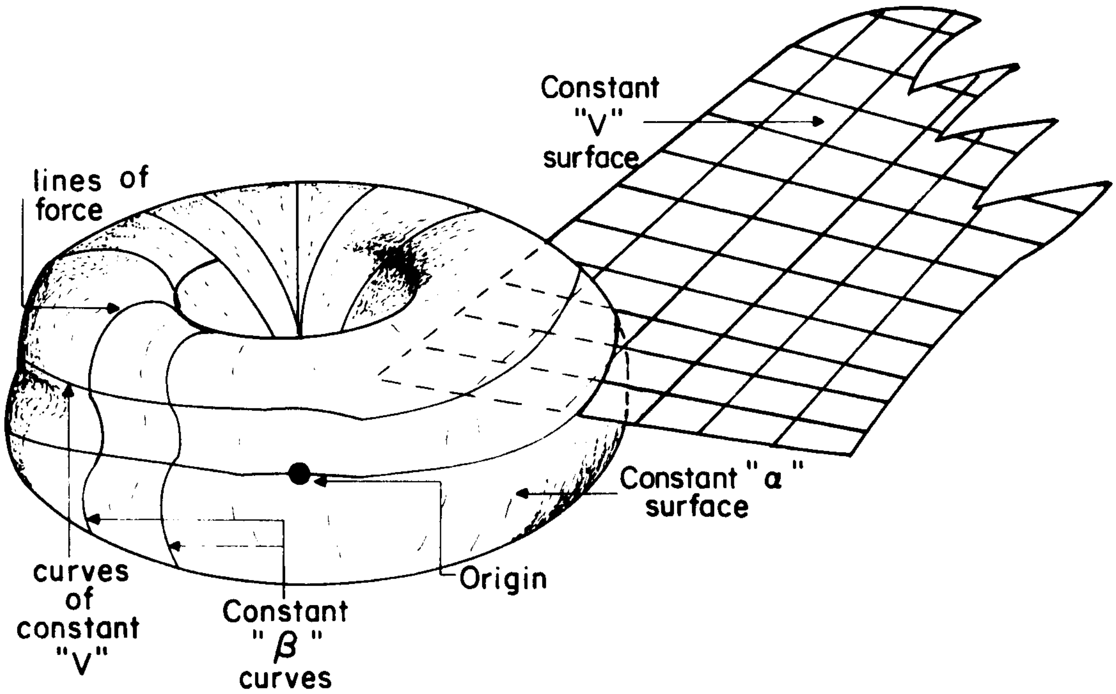

Constant
"V"
surface
lines of
force
Constant "α"
surface
Origin
curves
of
constant
"V"
Constant
"β"
curves

figure 36.

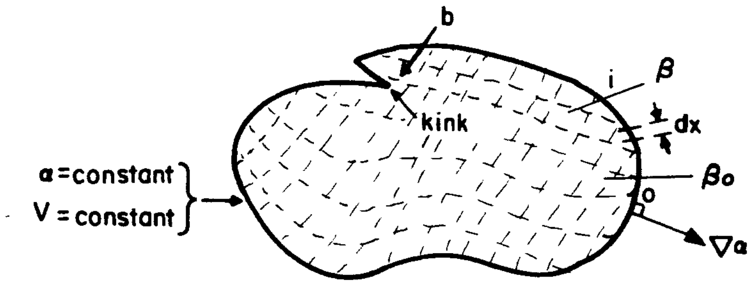


figure 37

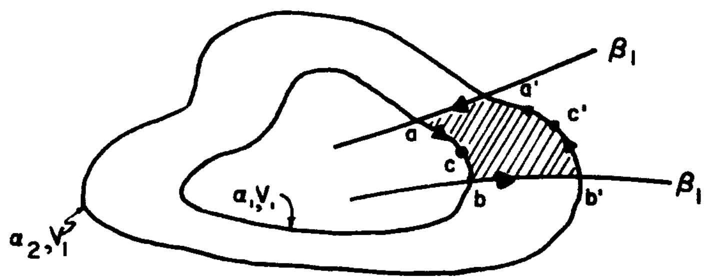


figure 38

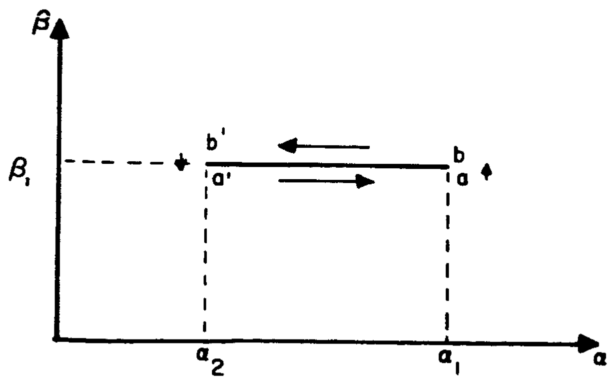


figure 39

Two lines of force pass through "i" and "o", respectively, in the above figure. The values of β on these two lines of force are β_o and β respectively. Starting with β_o we may generate all other values of β along this curve by[13]

$$\beta - \beta_o = \int_o^i \frac{B}{|\nabla\alpha|}\, dx \qquad (1)$$

where $\overrightarrow{dx} = \frac{\vec{B} \times \nabla\alpha}{B}\, dx$ is the unit length along the curve formed by constant "V", constant "α". From our assumption given on previous page, B is positive semi-definite and continuous, however, "$|\nabla\alpha|$" need not be continuous although it must be positive semi-definite. This is demonstrated by the "kink" in the above contour, "b". At this point the $\nabla\alpha$ changes discontinuously. The distance "dx" is non-decreasing and it therefore follows, that "β" must be a monatonically increasing function of "x" even over discontinuities in the integrand (ie. at b). This is clear from the fact that the integrand, although perhaps discontinuous, is always positive semi-definite, ergo, the integral is non-decreasing. This, however, is not enough if "β" is to qualify as a measure of length. In fact, it must be the case that β be a "strictly increasing" function of azimuth. That this is indeed the case may be shown as follows.

Consider the shaded area between two contours of constant - V, constant - α (shown in figure 38).

Assume that the two surfaces bounding the shaded area have the same value β_1. (The case if β is simply non-decreasing). β must be constant and equal to β_1 along the entire length a-b and a'-b'. This follows from the above where we showed that β is a non-decreasing function of "x". We may now map b'-a'-a-b-b' onto the α-β plane (see figure 17). The flux linking the hatched area of figure 16 is given by[33]

$$\Phi = \int_S \vec{B}\cdot\vec{dS} = \int_S (\nabla x\vec{A})\cdot\vec{dS} = \oint_C \vec{A}\cdot\vec{dl} \quad (2)$$

where $\vec{dS}$ is normal to the hatched area and $\vec{dl}$ is along the contour b'-a'-a-b-b'. However, from I-(3) $\vec{A} = \alpha\nabla\beta$ so that (2) becomes

$$\Phi = \oint \alpha\nabla\beta\cdot\vec{dl} = \oint \alpha d\beta \quad (3)$$

where (3) is no more than the area enclosed by b'-a'-a-b-b' in the α-β plane of figure 39. Noting from the figure that this is zero, we conclude that the arbitrarily selected hatched area of figure 38 encloses no flux. Since the real geomagnetic fields we will be treating have no flux free regions in space, we conclude that the case arising in figure 38 never occurs and therefore β is a strictly increasing function indeed. We may therefore choose β as azimuthal measure of "distance".

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