ON AN APPROXIMATION THEOREM OF KUPKA AND SMALE

by

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Introduction.

We present here a somewhat simpler version of the proof of an important approximation theorem of Kupka and Smale [1,2], concerning differential equations defined on a compact manifold M^n. We also say something about the non-compact case.

As is the case with their proofs, by which the present one is much inspired, the whole matter is essentially a transversality affaire a la Thom [3], and the present treatment makes this point even more clear. The simplification and streamlining introduced here stem from the use of a theorem of P. Hartman (1.8) (which takes the place of many computations) and from the argument in (3.2) using the existence of a minimum $\tau > 0$ for the period of the closed orbits. This last fact allows us to avoid a delicate argument involving the iterates of the Poincare transformation and instead use a simple transversality argument on the transformation itself.

The results of Kupka and Smale are equivalent, Smale working first with diffeomorphisms and then extending the result to vector fields and Kupka, as we do here, working directly with vector fields; the corresponding result for diffeomorphisms then follows immediately. As for Kupka's work [2] an equivalent but more palatable version of it can be found in his thesis at IMPA, written in Portuguese. A weaker version of the theorem considered here has been announced, without proof, by Markus [5]. For $n = 2$ the theorem is contained in a previous result of the author [6]. Both Kupka and Smale consider only the case where M^n is compact. At the end of the present paper we extend their result to the case of an open manifold.

But in this extension the behavior at infinity is not taken into account at all and in this respect the problem may be considered to be wide open. Most of what follows was the object of a series of three lectures given at the University of California at Berkeley in the Summer of 1965, and thanks are due to S. Smale, I. Kupka, C. Pugh, M. Shub and R. Abraham for lively discussions. The author is also thankful to R. Thom for comments on a previous draft of this paper.

1. Preliminaries.

Let $M = M^n$ be a compact C^∞-differentiable manifold and let $\mathfrak{X}$ be the space of all C^r-vector fields X on M with the C^r-topology, $r \geqq 1$. We suppose that a metric has been fixed in $\mathfrak{X}$, say by covering M with a finite number of coordinate neighborhoods; $\mathfrak{X}$ then becomes a Banach space. We assume also that M is endowed with a Riemannian metric.

We now fix some terminology and recall some definitions and known results. See for instance [4], and for results not explicitly there, the forthcoming lecture notes by the author.

(1.1) Call $\varphi_t(X)$ or simply $\varphi_t: M \to M$ the 1-parameter group of diffeomorphisms generated by a vector field $X \in \mathfrak{X}$.

(1.2) A singularity of X is a point $p \in M$ such that $X(p) = 0$; it is said to be generic if no eigenvalue of the Jacobian matrix of X at p, $dX(p)$, has zero real part.

(1.3) a) The stable (W^+) and unstable (W^-) manifolds associated to a generic singularity p are defined as follows. Let k, $0 \leqq k \leqq n$, be the number of eigenvalues of $dX(p)$ with negative real part. The set W^+ of all points of M such that the trajectory of X through it tend to p as $t \to \infty$ is an immersed k-dimensional submanifold of M passing through p i.e. there is an 1-1 immersion $\psi: R^k \to M$ such that $\psi(0) = p$, $\psi(R^k) = W^+$. In general, even if $n = 2$, W^+ is not a submanifold of M. In a similar way, changing $t \to \infty$ by $t \to -\infty$ in the above definition one gets the $(n-k)$-dimensional unstable manifold W^- associated to p.

b) There exists a $(k-1)$-dimensional sphere $S^+ \subset W^+$, transversal to X, dividing W^+ into two connected components, the one containing p being a k-dimensional ball B^+. Considering all sufficiently small arcs of geodesic starting at S^+ and normal there to W^+ (and so along (n-k) independent directions) one gets a (n-1)-dimensional manifold Σ^+, with boundary, transversal to X. We say that Σ^+ is a "fence" associated to S^+ and clearly $S^+ = \partial B^+ = B^+ \cap \Sigma^+$. Now once S^+ and Σ^+ are fixed there is a neighborhood $\mathscr{U}$ of X in $\mathscr{G}_1$ such that whenever $Y \in \mathscr{U}$ then Y is transversal to Σ^+ and has exactly one critical point p(Y) such that the corresponding k-dimensional stable manifold $W^+(Y)$ intersects Σ^+ at a (k-1)-dimensional sphere $S^+(Y)$ which is the boundary of a k-dimensional ball $B^+(Y)$ containing p(Y) and contained in $W^+(Y)$. Besides $S^+(Y) = \partial B^+(Y) = B^+(Y) \cap \Sigma^+$ can be isotopically deformed onto S^+, the isotopy taking place in Σ^+; and $B^+(Y)$ can be made arbitrarily C^r-close to B^+ be taking $\mathscr{U}$ small enough. If K=n we put $\Sigma^+ = S^+$. In exactly the same manner we define Σ^-, S^-, B^-,... for the unstable manifold W^-.

(1.4) The Poincaré transformation Φ associated to a closed orbit γ of X is defined as follows. Let $p \in \gamma$ and call Σ a <u>cross-section</u> at p i.e. a small piece of an (n-1)-dimensional submanifold of M containing p and transversal to X. There is no loss of generality if we identify Σ with a neighborhood of the origin in R^{n-1} and put p = 0. There exists a neighborhood of p, $\Sigma_o \subset \Sigma$, so small that whenever $q \in \Sigma_o$ the trajectory of X through q meets Σ at a point which we call $\Phi(q)$. If X is perturbed to $X + \delta X$ with δX C^r-small then the corresponding Poincaré transformation is changed to $\Phi + \delta\Phi$ with $\delta\Phi$ C^r-small, and conversely.

(1.5) <u>A tubular neighborhood</u> $T(\gamma)$ of a closed orbit γ is a neighborhood of γ having γ as basis and B^{n-1} as fiber and they consitute a fundamental system of neighborhoods of γ; they are always either diffeomorphic to the product $S^1 \times B^{n-1}$ (solid torus) or to the corresponding twisted product (Klein bottle K^n).

(1.6) a) A generic closed orbit γ of X is one such that the Jacobian matrix of Φ at 0, $d\Phi(0)$, has no eigenvalue of modulo 1. If k, $0 \leqq k \leqq n-1$, is the number of such eigenvalues with modulo < 1 then the set of all trajectories of X tending to γ as $t \to \infty$ (i.e. whose ω-limit set is γ) is an immersed $(k+1)$-dimensional submanifold W^+ of M called the stable manifold associated to γ; similarly one has the unstable manifold W^-, of dimension $(n-k)$, associated to γ. b) A result similar to (1.3b) holds here with the difference that now B^+ is not a k-dimensional ball but either a product $S^1 \times B^{k-1}$ (solid torus) or the corresponding twisted product and $S^+ = \partial B^+ = B^+ \cap \Sigma^+$ is no more a sphere.

(1.7) The stable and unstable manifolds W^+ and W^- of γ intersect transversally along γ. The connected component of $W^+ \cap \Sigma_0$ containing $p = 0$, V^+ is the stable manifold of Φ at the fixed point $p = 0 \in \Sigma_0$ (see 1.4). Similarly we have the unstable manifold V^- of Φ at 0. Now it is easy to verify that γ being generic the map $\psi : \Sigma_0 \to \Sigma \times \Sigma$ defined by $\psi(q) = (q, \Phi(q))$ is transversal to the diagonal Δ of $\Sigma \times \Sigma$ at (p,p). Also, if ψ is transversal to Δ at $(p,p) \in \Delta$ (so that (p,p) is an isolated point of $\Delta \cap \Phi(\Sigma_0)$) then it is possible to make a C^r-small change $\delta\psi$ in ψ is that $\delta\psi(0) = 0$ and $d(\psi + \delta\psi)(q)$ has no eigenvalue with modulo 1; besides $(\psi + \delta\psi)(q) = (q, (\Phi + \delta\Phi)(q))$.

(1.8) Let

$$dx/dt = Ax + f(x) \quad , \quad f(0) = 0 \; , \; df(0) = 0 \; ,$$

have $0 \in R^n$ as a generic singularity and consider the associated linear system $dy/dt = Ay$. Let φ_t and λ_t respectively be the corresponding 1-parameter group of diffeomorphisms. Then there is a homeomorphism $T: x \to y$ defined in the neighborhood of the origin and such that $\lambda_t = T\varphi_t T^{-1}$. This means that φ_t is topologically equivalent to λ_t. So the homeomorphism T maps trajectories of the non-linear equation onto trajectories of the linear equation preserving the parametrization i.e. the t-interval between two points on one integral curve is the same as the t interval between the corresponding points on the image. There is also a corresponding theorem for transformations $\Phi : R^n \to R^n$ having a fixed point 0 with $d\Phi(0)$ having no eigenvalue with modulus 1. These results are due to Hartman [4].

2. <u>The theorem.</u>

Let $\mathcal{G}_i$, $i = 1,2,3$, be the set of all $X \in \mathfrak{X}$ satisfying conditions G_i;

G_1: the singularities of X are generic (and so finite in number)

G_2: the closed orbits of X are generic

G_3: the stable and unstable manifolds associated to the generic singularities and closed orbits are transversal.

In dimension $n = 2$ condition G_3 says that there is no trajectory connecting saddle points.

Let $\mathcal{G}_{12} = \mathcal{G}_1 \cap \mathcal{G}_2$, $\mathcal{G} = \mathcal{G}_{123} = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$. Recall that a subset of $\mathfrak{X}$ is called <u>residual</u> if it contains the countable intersection of sets open and dense in $\mathfrak{X}$; from Baire's theorem it is necessarily dense in $\mathfrak{X}$, since $\mathfrak{X}$ is a complete metric space. We now state the theorem of Kupka and Smale.

Theorem. $\mathcal{G}$ is residual in $\mathfrak{X}$.

The fact that $\mathcal{G}_1$ is open and dense in $\mathfrak{X}$ is an easy consequence of Thom's tranversality lemma, see for instance [6]. A natural way to prove this theorem would then be to prove that $\mathcal{G}_2$ is residual in $\mathcal{G}_1$ and that $\mathcal{G}_3$ is residual in $\mathcal{G}_2$. For technical reasons we proceed as follows.

Let $T > 0$ be an integer and call:

$\mathfrak{X}(T)$: the subset of $\mathcal{G}_1$ such that $X \in \mathfrak{X}(T)$ implies that all closed orbits of X of period $\leq T$ are generic;

$\tilde{\mathfrak{X}}(T)$: the subset of $\mathfrak{X}(T)$ such that when $X \in \tilde{\mathfrak{X}}(T)$ then the stable and unstable manifolds of all singularities and of all closed orbits of X with period $\leq T$ are transversal.

Since

$$\mathcal{G} = \mathcal{G}_{123} = \bigcap_{T=1}^{\infty} \tilde{\mathfrak{X}}(T)$$

the theorem will be proved once we prove the following propositions.

Proposition 1. $\mathfrak{X}(T)$ is open and dense in $\mathfrak{X}$

Proposition 2. $\tilde{\mathfrak{X}}(T)$ is residual in $\mathfrak{X}(T)$.

We remark that since

$$\mathcal{G}_{12} = \bigcap_{T=1}^{\infty} \mathfrak{X}(T) ,$$

from Proposition 1 it follows that $\mathcal{G}_{12}$ is residual in $\mathcal{G}_1$ and so also in $\mathfrak{X}$. We now proceed to the proof of these propositions after which the theorem is proved.

3. Proof of Proposition 1.

To that end we need three Lemmas.

Lemma 1. If p is a singular point of $X \in \mathcal{G}_1$ and $T > 0$ then there is a neighborhood U of p in M and a neighborhood $\mathcal{U}$ of X in $\mathcal{G}_1$ such that whenever $Y \in \mathcal{U}$ then Y has in U exactly one singular point $p(Y)$ which depends continuously on Y and every closed orbit of Y meeting U has period $> T$.

Proof. The part concerning U, and $p(Y)$ follows from the fact that $\mathcal{G}_1$ is open and dense in $\mathfrak{X}$. From (1.8) one gets that U may be taken so small that every trajectory of X meeting U spends there a time $> 2T$. Now a simple semi-continuity argument shows that there exists a neighborhood $\mathcal{U}$ of X in $\mathcal{G}_1$ such that every trajectory of $Y \in \mathcal{U}$ meeting U spends there a time $> T$, proving Lemma 1.

An analogous holds for **closed** orbits.

Lemma 2. Let $T > 0$ and γ be a generic closed orbit of $X \in \mathcal{G}_1$ with period $\leq T$. Then there is a tubular neighborhood V of γ and a neighborhood $\mathcal{V}$ of X in $\mathcal{G}_1$ every $Y \in \mathcal{V}$ has a generic closed orbit $\gamma(Y) \subset V$ and besides, with the eventual exception of $\gamma(Y)$ every closed orbit of Y meeting V has period $> T$; $\gamma(Y)$ varies continuously with Y.

Proof. Let γ have period $\tau \leq T$ and put $N = 1 + [T/\tau]$ where the bracket stands for the greatest integer contained in T/τ. Referring to (1.4)-(1.6), choose the cross-section Σ so small that $\phi\colon \Sigma_0 \to \Sigma$ and all its iterates ϕ^k, $k = 1,\dots,N$ have $p = \gamma \cap \Sigma$ as the only fixed point. To ϕ^k there is associated the map $\psi^k\colon \Sigma_0 \to \Sigma \times \Sigma$ and the graph of ϕ^k intersects generically the diagonal of $\Sigma \times \Sigma$ at (p,p) and so this intersection is isolated. Now a C^r-small change from X to Y gives raise to a C^r-small change in ψ^k with the result that the corresponding graph again intersects generically the diagonal. From this it results that one may find neighborhoods V of γ and $\mathcal{V}$ of X

such that for $Y \in \mathscr{V}$ the corresponding Poincare transformation $\phi(Y)$ and all its iterates up to order N have only one fixed point, $p(Y)$. To $p(Y)$ there corresponds for Y a generic closed orbit $\gamma(Y) \subset V$ and any other closed orbit of Y meeting V corresponds to a fixed point of $\phi^{\ell}(Y)$ with $\ell > N$. Therefore its period T' is close to ℓ times the period of $\gamma(Y)$ and since this one is very close to τ, if $\mathscr{V}$ is small enough, we get that $T' > \tau N > T$. Lemma 2 is proved. From Lemma 2 we get immediately the following.

Corollary. If $X \in \mathfrak{X}(T)$ then X has only a finite number of closed orbits of period $\leq T$.

Lemma 3. Let K be a compact subset of M and assume that no point of K is a singularity or belongs to a closed orbit of a vector field X, of period $\leq T$. Then there exists a neighborhood $\mathscr{U}$ of X in $\mathfrak{X}$ such that every closed orbit of $Y \in \mathscr{U}$ meeting K has period $> T$.

Proof. To every point in K we associate a neighborhood U of it in M and a neighborhood $\mathscr{V}$ of X in $\mathfrak{X}$ such that whenever $Y \in \mathscr{V}$ then every closed orbit of Y intersecting U has period $> T$. We then extract a finite covering of K by sets U and call $\mathscr{U}$ the intersection of the corresponding $\mathscr{V}$. Lemma 3 is proved.

We now prove

(3.1) $\mathfrak{X}(T)$ is open in $\mathscr{G}_1$.

Proof. Let $p \in M$ be such that it is neither a singularity of $X \in \mathfrak{X}(T)$ or is situated on a closed orbit of period $\leq T$. Let $\mathscr{W}$ be a neighborhood of X in $\mathscr{G}_1$ such that whenever $Y \in \mathscr{W}$ then every closed orbit of Y meeting W has period $> T$. If p is a singularity of X or is on one of its closed orbits of period $\leq T$ we apply Lemma 1 or Lemma 2 to get neighborhoods U and $\mathscr{U}$ or

V and $\mathscr{V}$. Since M is compact we find a finite number of U, V and W covering M. The intersection of the corresponding $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ contains an open set made up of points of $\mathscr{X}(T)$, proving (3.1). To end the proof of Proposition 1 we need only to prove that

(3.2) $\mathscr{X}(T)$ is dense in $\mathscr{G}_1$.

Proof. Let $X \in \mathscr{G}_1$. We want to find $Y \in \mathscr{X}(T)$ arbitrarily close to X and we do so by means of a finite number of successive modifications on X.

Let $\tau > 0$ be such that no closed orbit of X has period less than τ. Such τ exists for otherwise we would find a sequence of closed orbits, whose periods tend to zero, converging to a singular point of X; from (1.8) this is clearly impossible.

Now let $\Gamma = \Gamma(\tau, 3\tau/2)$ be the set of all closed orbits of X whose periods are within the closed interval $[\tau, 3\tau/2]$. Clearly Γ is a closed set in M. We now cover every trajectory in Γ by two concentric neighborhoods V and U, $V \subset \overline{V} \subset U$, and call Σ and Σ', $\Sigma \subset \Sigma'$ the corresponding cross-sections; Σ' is chosen so small that it meets every trajectory of Γ just once. Since Γ is compact we extract from the covering a finite set V_i, U_i, $V_i \subset \overline{V}_i \subset U_i$, $i = 1,\dots,k$, such that the V_i's cover Γ. Then, from (1.4), by making a C^r-small change in the Poincaré transformation, vanishing outside Σ_1 (cross-section corresponding to V_1) we get, by transversality, a vector field Y_1', C^r-close to X, agreeing eith X outside of U_1, and such that in V_1 every closed orbit of Y_1' of period $\leq 3\tau/2$ is generic. An important point here is that by taking Y_1' close enough to X every closed orbit of Y_1' in V_1 which corresponds to m turns, $m > 1$, (i.e. to a fixed point of the m-th iterate of the Poincaré transformation) has period close to $m\overline{\tau}$, $\tau \leq \overline{\tau} \leq 3\tau/2$ and so greater than $3\tau/2$; every trajectory corresponding to $m = 1$ is generic. From the Corollary to Lemma 2 it follows that, in V_1, Y_1' has only a finite number of closed orbits of period $\leq 3\tau/2$.

We now proceed as above and perturb Y_1' slightly inside U_2 getting a system Y_2' such that inside V_2 all trajectories of Y_2' of period $\leq 3\tau/2$ are generic; from Lemma 2 one gets that for Y_2' close enough to Y_1' then we do not disturb the situation that we had before in V_1 i.e. inside $V_1 \cup V_2$ all trajectories of Y_2' of period $\leq 3\tau/2$ are generic. Repeating this argument up to V_k we obtain $Y_1 = Y_k'$ such that it is arbitrarily close to X and besides all of its trajectories of period $\leq 3\tau/2$ contained in $V' = \bigcup_{i=1} V_i$ are generic. Outside V', Y_1 might have nongeneric closed orbits of period $\leq 3\tau/2$. But applying Lemma 3 with $K = M\text{-}V'$ we see that for Y_1 close enough to X all periodic orbits of Y_1 meeting K have period $> 3\tau/2$. So all periodic orbits of Y_1 of period $\leq 3\tau/2$ are generic.

We now essentially repeat the procedure and work with the set $\Gamma(3\tau/2, 2\tau)$ of all closed orbits of Y_1 with period within the interval $[3\tau/2, 2\tau]$. Compared with the previous case there is a little difference here namely that Y_1 may have periodic orbits of period $< 3\tau/2$ (whereas X had none of period $< \tau$). But these are generic and finite in number and we can find a neighborhood W of their union disjoint from the U's employed in covering $\Gamma(3\tau/2, 2\tau)$. We then do as before taking $K = M\text{-}V'\text{-}W$ and get Y_2 such that all of its closed orbits of period $\leq 2\tau$ are generic.

Proceeding that way we have that ℓ being an integer such that $\ell\tau/2 > T$, $Y_\ell = Y$ can be made arbitrarily close to X and such that all of its closed orbits of period $\leq T$ are generic i.e. $Y \in \mathcal{X}(T)$. This proves (3.2) and also Proposition 1.

4. Proof of Proposition 2.

To show that $\tilde{\mathcal{X}}(T)$ is residual in $\mathcal{X}(T)$ we need only to show that

(4.1) given any $X \in \mathcal{X}(T)$ there exists a neighborhood $\mathcal{U}$ of X in $\mathcal{X}(T)$ such that $\tilde{\mathcal{X}}(T) \cap \mathcal{U}$ contains a residual set in $\mathcal{U}$.

To prove (4.1) let p_i, $i = 1,\ldots,N$ be the critical elements of X

i.e. its singularities and closed orbits of period $\leq T$; these are generic. Now let U_i and $\mathscr{U}_i$ be neighborhoods of p_i in M and X in $\mathscr{X}(T)$, respectively, such that whenever $Y \in \mathscr{U}_i$ then, inside U_i, Y has one and only one critical element $p_i(Y)$, as in Lemmas 1 and 2. Assume these U_i to be disjoint and that one chooses the fences $\Sigma_i^{\pm} \subset U_i$ and $\mathscr{U}_i$ so small that $B_i^{\pm}(Y) \subset U_i$ and meets $\Sigma_i^{\pm}$ transversally when $Y \in \mathscr{U}_i$; see (1.3b) and 1.6b).

Define the "t-expansion", $t > 0$, of the stable or unstable ball $B_i^{\pm}(Y)$ as

$$B_i^{\pm}(Y; \mp t) = \varphi_t(Y)B_i^{\pm}(Y) \qquad \text{(see (1.1))} .$$

For $t < 0$ we have a "t-contraction". To each pair of indices i, j, $1 \leq i, j \leq N$ and integer $\tau > 0$ define $\mathscr{F}_1(i,j;\tau)$ as the set of all $Y \in \mathscr{U}_i \cap \mathscr{U}_j = \mathscr{U}_{ij}$ such that $B_i^{\pm}(Y; \mp \tau)$ is transversal to $B_j^{\mp}(Y; \pm \tau)$. We now claim that to prove (4.1) have only to prove that for every $\tau > 0$

(4.2) $\mathscr{F}_1(i,j; \tau)$ is open and dense in $\mathscr{U}_{ij}$.

In fact, from (4.2) it follows that

$$\bigcap_{i,j=1}^{N} \mathscr{F}_1(i,j; \tau) = \mathscr{F}(\tau)$$

is open and dense in $\bigcap_{i=1}^{N} \mathscr{U}_i = \mathscr{U}$.

Now (4.1) follows from the fact that

$$\tilde{\mathscr{X}}(T) = \bigcap_{\tau=1}^{\infty} \mathscr{F}(\tau) .$$

To prove (4.2) it is enough to prove that

(4.3) the set $\mathscr{F}(i,j ; \tau)$ of all fields $Y \in \mathscr{U}_{ij}$ such that $B_i^{-}(Y;\tau)$ is transversal to $B_j^{+}(Y; -\tau)$ is open and dense in $\mathscr{U}_{ij}$

Since the case of $B_i^{+}(Y;-\tau)$ and $B_j^{-}(Y; \tau)$ is treated the same way.

We now proceed to prove (4.3).

That $\mathcal{F}(i,j\ ;\ \tau)$ is open in $\mathcal{U}_{ij}$ is quite obvious from (1.3b) and from the standard fact that the tranversality condition is preserved under a C^r-small perturbation of the manifold invloved, $r \geq 1$.

To prove density we show that $\mathcal{V}$ being any neighborhood of $Y \in \mathcal{U}_{ij}$ then we can find $\bar{Y} \in \mathcal{V} \cap \mathcal{F}(i,j;\ \tau)$ and this is done as follows. If $B_i^-(Y;\ \tau)$ is transversal to $B_j^+(Y;\ -\tau)$ we take $\bar{Y} = Y$. If not, we make a small C^r-change in these manifolds so as to put them transversally; after which it remains to show that this change can be obtained by changing Y into $\bar{Y}$. It is a situation analogous to (1.4) but more complex. We proceed now to do this and we consider first the case $i \neq j$.

From (1.3b) we can find $\nu > 0$ so big and choose a neighborhood of Y, $\mathcal{V}_1 \subset \mathcal{V}$ and the fence $\Sigma_i^- = \Sigma$ so small that $B_i^-(Z;-\nu) \cap B_j^+(Z;-\tau) = \emptyset$, $\varphi_{-\nu}(Z)\Sigma \cap B_j^+(Z;\ -\tau) = \emptyset$ for $Z \in \mathcal{V}_1$. From well known facts about the local behavior of trajectories of differential equations we can choose the fence Σ so small that the application

$$\lambda\colon \Sigma \times [-\nu,\ \tau] \to M$$

defined by

$$\lambda(x,\ t) = \varphi_t(Y)x \quad , \quad x \in \Sigma, \quad t \in [-\nu,\ \tau]$$

be a diffeomorphism. The image

$$\lambda(\Sigma \times [-\nu,\ \tau]) = L \subset M$$

is a manifold with boundary. Clearly

$$B_i^-(Z,\ \tau) \cap B_j^+(Z;\ -\tau) \subset L \quad , \quad Z \in \mathcal{V}_1 \ ,$$

if $\mathcal{V}_1$ is small enough, so that we need only to care for what happens inside L.

Let $L_1 = \varphi_\tau(Y)Z$ be a face of L and

$$C(\tau) = B_j^+(Y; -\tau) \cap L_1$$

Since $B_i^-(Y; \tau)$ and $B_j^+(Y; -\tau)$ can only intersect along pieces of trajectories of Y the fact that they are transversal at a given point implies that they are transversal all along the common piece of trajectory passing through that point. From this it follows that $B_i^-(Y; \tau)$ and $B_j^+(Y; -\tau)$ are transversal if and only if $\varphi_\tau(Y)S_i^-(Y)$ is transversal to $C(\tau)$. We proceed to construct our field $\bar{Y} \in {}_1$ in such a way that $\varphi_\tau(\bar{Y})S_i^-(\bar{Y})$ is transversal to $C(\bar{Y}; \tau) =$ $L_i \cap B_j^+(\bar{Y}; -\tau)$ which implies that $B_i^-(\bar{Y}; \tau)$ is transversal to $\bar{B}_j(\bar{Y}; -\tau)$, proving (4.3) . Let $C = \varphi_{-\tau}(Y)C(\tau)$ be the compact manifold obtained by pulling back $C(\tau)$ to Σ through the trajectories of Y. In Σ we have then the two compact sub-manifolds $S_i^-(Y)$ and C which may or may not be transversal. If they are, we are through for then $C(\tau)$ and $\varphi_\tau(Y)S_i^-(Y)$ are also transversal. If not, we apply to Σ , $S_i^-(Y)$ and C a classical isotopy lemma of Thom [3, p.26].

It follows then that given a neighborhood Ω of $\partial\Sigma$, disjoint from C and $S_i^-(Y)$, and $\varepsilon > 0$ we can find a 1-parameter family of diffeomorphisms

$$\psi_s : \Sigma \to \Sigma, \quad 0 \leq s \leq 1$$

each C^r-close to the identity by less than ε and such that

$$\psi_0 = \text{id} ,$$

$$\psi_1(S_i^-(Y)) \text{ is transversal to } C ,$$

$$\psi_s = \text{id in } \Omega ,$$

$$\partial^n\psi_s(x)/\partial s^n = 0 \text{ for } x \in \Sigma, \text{ at } s = 0,1 \text{ and } n = 1,2,\ldots$$

(reparameterizing if necessary).

From the isotopy ψ_s we pass to the field $\overline{Y}$ as follows. Let

$$\psi: \Sigma \times [-\nu, \tau] \to \Sigma \times [-\nu, \tau]$$

be a diffeomorphism defined by

$$\psi(x, \tau) = (x, t) \quad \text{if} \quad -\nu \leq t \leq 0$$
$$\psi(x, t) = (\psi_{t/\tau}(x), t) \quad \text{if} \quad 0 \leq t \leq \tau \ .$$

We are then spreading the isotopy ψ_s between time $t = 0$ and $t = \tau$. This induces in a natural way a deformation of the "horizontal" lines in $\Sigma \times [-\nu, \tau]$, corresponding to the trajectories of Y in L, and the new lines so obtained are the integral curves of $\overline{Y}$. Precisely $\overline{Y}$ is defined as follows. Let $E = (0,1)$ be the horizontal unit field in $\Sigma \times [-\nu, \tau]$, image through λ^{-1} of Y in L and define

$$\overline{Y}(x) = Y(x) \quad \text{if} \quad x \in M\text{-}L$$
$$Y(x) = \lambda_* \circ \psi_* E(\psi^{-1} \circ \lambda^{-1}(x)) \quad \text{if} \quad x \in L$$

where the star stands for differential. From the conditions satisfied by the isotopy ψ_s it follows that $\overline{Y}$ is in fact a vector field on M and it is also clear that by choosing ε small enough one has that $Y \in \mathscr{V}_1$. Clearly

$$B_j^+(Y; -\tau) \cap L_1 = B_j^+(Y; -\tau) \cap L_1 = C(\tau) = C(\overline{Y}, \tau)$$

As remarked before we need only to show that $C(\tau)$ is transversal to $\varphi_\tau(\overline{Y})S_i^-(\overline{Y})$. Now from the way $\overline{Y}$ and λ were defined we have $S_i^-(Y) = S_i^-(\overline{Y})$ and

$$\varphi_\tau(\overline{Y})S_i^-(\overline{Y}) = \lambda(\psi_1 S_i^-(\overline{Y}), \tau) = \lambda(\psi_1 S_i^-(\overline{Y}), \tau) \ .$$

On the other hand $C(\tau) = \lambda(C, \tau)$. Since $\psi_1(\overline{Y})S_i^-(\overline{Y})$ is transversal to C, $\lambda(\psi_1(\overline{Y})S_i^-(Y),\tau)$ is transversal to $\lambda(C, \tau)$, completing the proof of (4.3) in case $i \neq j$.

To the case $i = j$ we can apply the above argument with little modification. In this case $B_i^-(Y; \tau)$ and $B_i^-(Y; -\tau)$ meet outside L along the critical element $p_i(Y)$. But there they are transversal so that again only matters what happens inside L, and this goes unchanged. The proof of Proposition 2 is now complete. The Theorem is proved.

5. The non compact case.

If $M = M^n$ is non compact the C^r-topology, $r \geqq 1$, in the space $\mathfrak{X}$ of all vector fields in M is defined as follows. Let

$$(5.1) \qquad K_1 \subset K_2 \subset \ldots \subset K_i \subset K_{i+1} \ldots \subset M$$

be a decomposition of M into an expanding sequence of compact sets each K_i having non empty interior $\dot{K}_i \subset K_{i+1}$. If $X \in \mathfrak{X}$ and $\delta(x) > 0$ is a continuous function defined on M let

$$(5.2) \qquad \begin{aligned} &\delta_i = \inf \delta(x) \quad , \quad x \in K_i - \dot{K}_{i-1} \quad , \quad K_o = \emptyset \\ &A(X,\delta(x)) = \bigcap_{i=1}^{\infty} \{Y \mid d(X,Y;\ K_i - K_{i-1}) < \delta_i\} \end{aligned}$$

where $d(X,Y;\ K_i-K_{i-1})$ stands for the usual C^r-distance between X and Y in K_i-K_{i-1}. When $\delta(x)$ varies in the set of positive continuous functions on M the sets $A(X,\delta(x))$ form a basis in X for a system of neighborhoods of a topology in $\mathfrak{X}$, which does not depend on the decomposition (5.1) or on the metric chosen on each K_i.

Vector fields in M are maps of M inot its tangent bundle $T(M)$, $X\colon M \to T(M)$, and Whitney [7] has introduced a C^r-topology in the set of mappings of one manifold into another. The above topology is exactly the topology that would arise that way and we call it the C^r-topology of Whitney in $\mathfrak{X}$ If M is compact we get the metrizable topology we had before. For a non-compact M the Whitney topology in $\mathfrak{X}$ is nonmetrizable but it has the Baire property [8]. One might be tempted to consider instead of the Whitney topology, say the compact open topology or the topology of uniform convergence, both of which would make $\mathfrak{X}$ metrizable. But the consideration of a vector field X in R^2 with infinitely many generic singularities which as they go to infinity have their eigenvalues tending to zero shows that in both of these topologies we may have a field Y very close to X which no longer exhibits these singularities.Therefore these topologies are no good for dealing with "generic" properties as we are doing here.

The statement of the theorem of Kupa-Smale, in the compact or the non-compact case is exactly the same. We now indicate the straightforward modifications needed to cover the non-compact case. We assume that the decomposition (5.1) has been chosen and use the notations of paragraph 2.

(5.3) $\mathcal{G}_1$ is open and dense in $\mathfrak{X}$.

Proof. Follows from a transversality argument as in the compact case.

Call

$\mathfrak{X}(T; K_i)$: the subset of elements $X \in \mathcal{G}_1$ such that every closed orbit of X which meets K_i and is of period $\leq T$ is generic.

$\tilde{\mathfrak{X}}(T; K_i)$: the subset of $\mathfrak{X}(T; K_i)$ such that when $X \in \tilde{\mathfrak{X}}(T; K_i)$ then the stable and unstable manifolds of the singularities of X in K_i or of the closed orbits of period $\leq T$ meeting K_i are transversal.

We have, T and i being fixed,

(5. 4) $\mathfrak{X}(T; K_i)$ is open and dense in $\mathcal{G}_1$

Proof. Let $X \in \mathfrak{X}(T; K_i)$ then the argument of (3.1) shows that there is a certain $\delta_i > 0$ such that whenever $Y \in A(X;\delta_i)$ then $Y \in \mathfrak{X}(T; K_i)$ proving openess.

To prove density let $X \in \mathcal{G}_1$ and let $A(X; \delta(x))$, $\delta(x) > 0$ on M, be a neighborhood of X in $\mathcal{G}_1$.

Now repeating the argument of (3.2), for the case where we use the compact set K_i in place of the compact manifold, it is easily seen that one can get a vector field Y on M, $Y \in \mathfrak{X}(T;K_i)$, obtained by perturbing X on a small neighborhood of K_i contained in K_{i+1} and such that

$$d(X,Y;\ K_j-\dot{K}_{j-1}) < \delta_j \quad , \quad j \leq i,$$

the δ's being as in (5.2). Since $Y = X$ outside K_{i+1} it is quite clear that $Y \in A(X, \delta(x))$ and so

$$Y \in \mathfrak{X}(T;K_i) \cap A(X, \delta(x)) \ ,$$

proving density and completing the proof of (5.4).

(5.5) $\tilde{\mathfrak{X}}(T;\ K_i)$ is residual in $\mathfrak{X}(T;\ K_i)$

Proof. The argument used in the proof of Proposition 2 can be carried through here with no essential difference.

We then have the theorem of Kupka-Smale:

(5.6) $\mathcal{G}$ is residual in $\mathfrak{X}$

Proof. Follows immediately from (5.3), (5.4), (5.5) and from the fact that

$$\mathcal{G} = \bigcap_{T=1}^{\infty} \bigcap_{i=1}^{\infty} \tilde{\mathfrak{X}}(T;\ K_i) \ .$$

From (5.4) it follows immediately that $\mathfrak{X}(T)$ is residual in $\mathfrak{X}$ but actually we can show that, as in the compact case, (5.7) $\mathfrak{X}(T)$ is open and dense in $\mathfrak{X}$.

Proof. Let $X \in \mathfrak{X}(T)$ and let η_i be such that whenever $d(X,Y; K_i) < \eta_i$ then $Y \in \mathfrak{X}(T; K_i)$, see (5.4). Now any continuous $\eta(x) > 0$ on M such that $\eta(x) < \eta_i$ for $x \in K_i$ is such that $Y \in A(X; \eta(x))$ implies $Y \in \mathfrak{X}(T)$ and this proves $\mathfrak{X}(T)$ open density of $\mathfrak{X}(T)$ follows from (5.4).

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