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by

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INTRODUCTION

"Squeeze-film" action describes the viscous flow process by which the pressure in a liquid bearing film increases above ambient when the bearing gap is closing down and decreases below ambient when the bearing gap is opening up. In the case of a gaseous film in an oscillating gap, aside from the corresponding oscillation in the film pressure, its temporal average also becomes elevated above the ambient by an amount which increases with the frequency and magnitude of oscillation. This latter phenomenon can be used to support a bearing load and is the subject of the present paper.

Several studies on gaseous squeeze-film bearings have been made, (Refs. 1, 2 and 3); they concern simple bearing shapes with certain prescribed mode of gap oscillation. More complicated bearing shapes, e.g., conical and spherical bearings, are also of interest. The mode of gap oscillation likely varies from one situation to another. It is useful to have a method of analysis which is applicable to arbitrary bearing shape and arbitrary mode of gap oscillation.

The time dependent gas lubrication equation is of the diffusion type and can be solved by standard techniques applicable to initial value problems. In the case of a gaseous squeeze-film bearings, the "steady-state" solution is of primary interest. Because the oscillation frequency is very high, the transient period would encompass many cycles of oscillation. Thus, if one treats the analysis with the initial value problem approach, the required computation would be rather lengthy due to the transient period. Alternately, in seeking the "steady-state" solution, one can use an asymptotic analysis to take advantage of the condition that the frequency is very high. Effectiveness of the latter approach has already been demonstrated in References 2 and 3 for special cases; this paper will deal with the general asymptotic treatment of gaseous squeeze-film bearings.

REVIEW OF BACKGROUND

According to the theory of hydrodynamic lubrication the fluid film pressure satisfies Reynolds' equation, which can be written in the vector form as follows (Ref. 4).

$\vec{V}$ is the vector sum of the absolute sliding velocities of the bearing surfaces.

$$\mathrm{div}\left\{-\frac{\rho h^3}{12\mu}\,\mathrm{grad}\,p+\frac{\rho h\vec{V}}{2}\right\}+\frac{\partial}{\partial t}(\rho h)=0 \qquad (1)$$

Typically, the boundary condition requires the pressure to become ambient at the peripheries. With gaseous lubricant and metallic bearing materials, the fluid film is essentially isothermal; thus, for most gases, viscosity can be regarded as a constant and density can be replaced by pressure in Reynolds' equation.

This equation can be rendered dimensionless by normalizing various variables with appropriate reference quantities, which are:

Variable		Reference Quantity		Normalized Variable
Fluid Film Pressure	p	Ambient Pressure	p_a	$P = p/p_a$
Fluid Film Thickness	h	Mean Bearing Gap	C	$H = h/C$
Time	t	Time Constant	$1/\nu$	$\tau = \nu t$
Surface Differential Operators	div, grad	Reciprocal of Typical Dimension	$1/R$	Div = R div Grad = R grad
Velocity	$\vec{V}$		ωR	$U = \frac{V}{\omega R}$

The normalized equation is

$$\text{Div}\left\{-PH^3\ \text{Grad}P + \vec{\Lambda U}PH\right\} + \sigma\frac{\partial}{\partial\tau}(PH) = 0 \qquad (2)$$

where,

Λ = compressibility number

$$= \frac{6\mu\omega}{P_a}\left(\frac{R}{C}\right)^2$$

σ = squeeze number

$$= \frac{12\mu\nu}{P_a}\left(\frac{R}{C}\right)^2$$

Consider now, that the bearing surface can be described in terms of orthogonal curvilinear coordinates α and β as shown in Fig. 1. The peripheries of the bearing film are at α_1 and α_2. Let g_α and g_β be respectively the linear measures of the two coordinates, then Eq. (2) can be rewritten as (Ref. 5):

$$\frac{1}{g_\alpha g_\beta}\left\{\frac{\partial}{\partial\alpha}\left(-\frac{g_\beta}{g_\alpha}PH^3\frac{\partial P}{\partial\alpha} + g_\beta\,\Lambda PHU_\alpha\right) + \frac{\partial}{\partial\beta}\left(-\frac{g_\alpha}{g_\beta}PH^3\frac{\partial P}{\partial\beta} + g_\alpha\,\Lambda PHU_\beta\right)\right\} + \sigma\frac{\partial}{\partial\tau}(PH) = 0 \qquad (3)$$

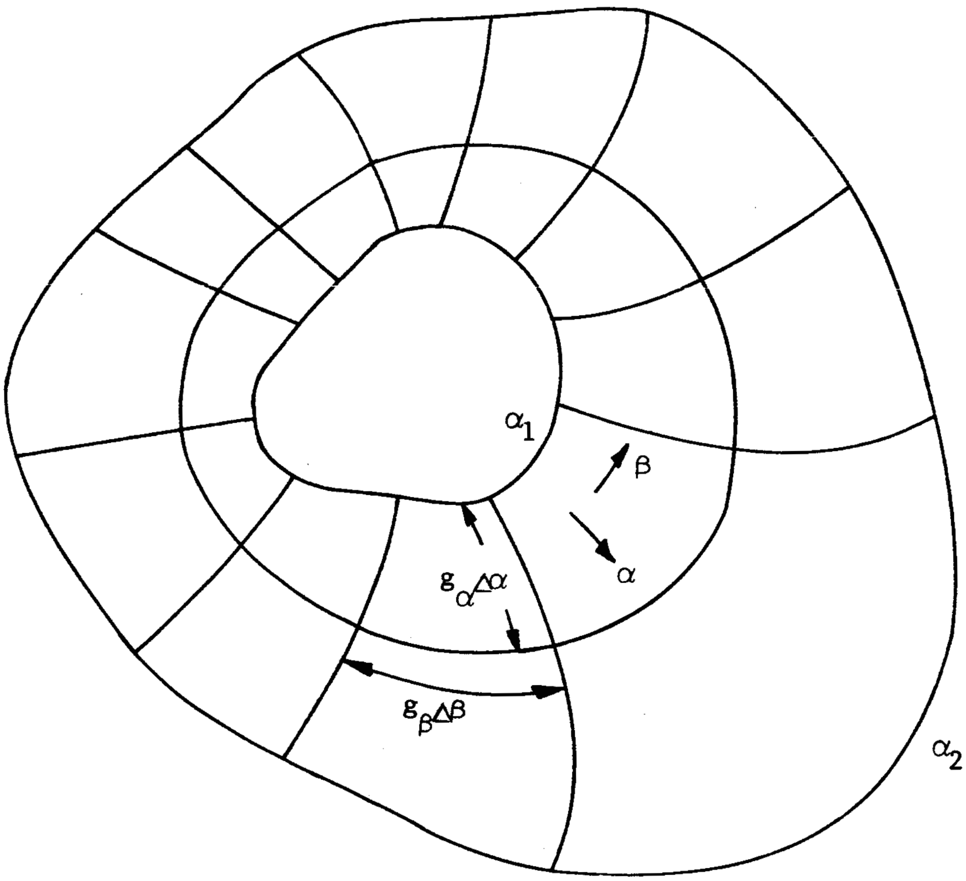

Fig. 1 Schematic of Arbitrary Bearing Surface Described in Curvilinear Coordinates

For typical squeeze film bearings, one of the bearing surfaces undergoes a high frequency oscillation, so that the squeeze number is very large (often $> 10^3$). Thus, provided that Λ is finite, g_α and g_β are well behaved functions, and that $\frac{\partial P}{\partial \alpha}$ and $\frac{\partial P}{\partial \beta}$ are bounded, then it is of interest to consider the asymptotic solution

$$\lim_{\sigma \to \infty} PH = \psi_\infty (\alpha,\beta). \tag{4}$$

This asymptotic solution, however, is not uniformly valid because it is unable to satisfy the boundary conditions:

$$\left. \begin{aligned} PH\Big|_{\alpha_1,\beta,\tau} &= H(\alpha_1,\beta,\tau) \\ PH\Big|_{\alpha_2,\beta,\tau} &= H(\alpha_2,\beta,\tau) \end{aligned} \right\} \tag{5}$$

Aside from the fact that edge corrections are needed, which would link the asymptotic solution and the true boundary conditions; formulation of the asymptotic analysis further requires the generation of the governing equation for the asymptotic solution and its own boundary conditions.

GOVERNING EQUATION FOR THE ASYMPTOTIC SOLUTION

Consider the periodic problem, such that

$$H(\alpha,\beta,\tau + 2\pi) = H(\alpha,\beta,\tau) \tag{6}$$

$$P(\alpha,\beta,\tau + 2\pi) = P(\alpha,\beta,\tau) \tag{7}$$

Defining,

$$PH = \psi, \tag{8}$$

then,

$$\psi(\alpha,\beta,\tau + 2\pi) = \psi(\alpha,\beta,\tau) \tag{9}$$

In terms of ψ, Eq. (3) can be rewritten as

$$\frac{1}{g_\alpha g_\beta}\left\{\frac{\partial}{\partial\alpha}\left[-\frac{g_\beta}{g_\alpha}\left(H\psi\frac{\partial\psi}{\partial\alpha} - \psi^2\frac{\partial H}{\partial\alpha}\right) + g_\beta U_\alpha \Lambda\psi\right]\right.$$
$$\left. + \frac{\partial}{\partial\beta}\left[-\frac{g_\alpha}{g_\beta}\left(H\psi\frac{\partial\psi}{\partial\beta} - \psi^2\frac{\partial H}{\partial\beta}\right) + g_\alpha U_\beta \Lambda\psi\right]\right\}$$
$$+ \sigma\frac{\partial\psi}{\partial\tau} = 0 \tag{10}$$

Integrate Eq. (10) with respect to τ for one full period and make use of Eq. (9):

$$\int_0^{2\pi} d\tau\left\{\frac{\partial}{\partial\alpha}\left[-\frac{g_\beta}{g_\alpha}\left(H\psi\frac{\partial\psi}{\partial\alpha} - \psi^2\frac{\partial H}{\partial\alpha}\right) + g_\beta U_\alpha \Lambda\psi\right]\right.$$
$$\left. + \frac{\partial}{\partial\beta}\left[-\frac{g_\alpha}{g_\beta}\left(H\psi\frac{\partial\psi}{\partial\beta} - \psi^2\frac{\partial H}{\partial\beta}\right) + g_\alpha U_\beta \Lambda\psi\right]\right\} = 0 \tag{11}$$

Using the asymptotic approximation for ψ, then one gets

$$\frac{\partial}{\partial\alpha}\left[-\frac{g_\beta}{g_\alpha}\left(\bar{H}\,\psi_\infty\frac{\partial\psi_\infty}{\partial\alpha}-\psi_\infty^2\frac{\partial\bar{H}}{\partial\alpha}\right)+g_\beta U_\alpha \Lambda\psi_\infty\right]$$

$$+\frac{\partial}{\partial\beta}\left[-\frac{g_\alpha}{g_\beta}\left(\bar{H}\,\psi_\infty\frac{\partial\psi_\infty}{\partial\beta}-\psi_\infty^2\frac{\partial\bar{H}}{\partial\beta}\right)+g_\alpha U_\beta \Lambda\psi_\infty\right]=0 \qquad (12)$$

where

$$\bar{H}=\frac{1}{2\pi}\int_0^{2\pi} H\,d\tau \qquad (13)$$

Equation (12) is the governing equation for ψ_∞ . Assuming that the problem is the Dirichlet type, then the appropriate boundary conditions ought to be

$$\left.\begin{aligned}\psi_\infty(\alpha_1,\beta)&=F_1(\beta)\\ \psi_\infty(\alpha_2,\beta)&=F_2(\beta)\end{aligned}\right\} \qquad (14)$$

The exact forms for F_1 and F_2 remain to be determined.

EDGE CORRECTIONS

Because the exact boundary conditions, Eq. (5), can not be satisfied by ψ_∞, an edge correction is required to construct a uniformly valid solution.

Let the uniformly valid solution be

$$\psi = \psi_\infty(\alpha,\beta) + \psi_e(\alpha,\beta,\tau) \tag{15}$$

when σ is very large, ψ_e must be negligibly small everywhere except in the edge regions near the boundaries α_1 and α_2. The role of ψ_e is to permit a rapid transition between ψ_∞ and the boundary conditions of ψ. Therefore, α-wise variation of ψ_e has a stretched scale in each of the two edge regions.

For instance, for $\alpha \overset{>}{\approx} \alpha_1$, we have the stretched coordinate

$$\zeta_1 = \sigma^n(\alpha-\alpha_1) \tag{16}$$

n is a positive number yet to be determined.

We have

$$\left.\begin{aligned} \frac{\partial\psi}{\partial\alpha} &= \frac{\partial\psi_\infty}{\partial\alpha} + \sigma^n \frac{\partial\psi_e}{\partial\zeta_1} \\ \frac{\partial^2\psi}{\partial\alpha^2} &= \frac{\partial^2\psi_\infty}{\partial\alpha^2} + \sigma^{2n} \frac{\partial^2\psi_e}{\partial\zeta_1^2} \end{aligned}\right\} \tag{17}$$

Substitute into Eq. (10):

$$-\frac{\sigma^{2n}H}{2(g_\alpha)^2}\frac{\partial^2}{\partial\zeta_1^2}(\psi_e)^2 + \sigma\frac{\partial\psi_e}{\partial\tau} = 0\left\{\text{Lower powers of } \sigma\right\} \tag{18}$$

In the limit, the righthand side is neglected and for both terms on the lefthand side to be of similar order, it is required that

$$n = \frac{1}{2} \tag{19}$$

Furthermore, since H and g_α do not vary with the stretched scale, they can be replaced by their edge values

$$\left.\begin{aligned} H(\alpha_1,\beta,\tau) &= H_1(\beta,\tau) \\ g_\alpha(\alpha_1,\beta) &= G_1(\beta) \end{aligned}\right\} \tag{20}$$

Thus, the governing differential equation for ψ_e near α_1 is

$$\frac{H_1}{2(G_1)^2}\frac{\partial^2(\psi_e)^2}{\partial\zeta_1^2} = \frac{\partial\psi_e}{\partial\tau} \tag{21}$$

Its boundary conditions are

$$\left.\begin{aligned} \psi_e(\zeta_1 = 0,\beta,\tau) &= H_1 - F_1 \\ \psi_e(\zeta_1 \rightarrow \infty,\beta,\tau) &= 0 \end{aligned}\right\} \tag{22}$$

Similarly, near α_2

$$\frac{H_2}{2(G_2)^2}\frac{\partial^2(\psi_e)^2}{\partial\zeta_2^{\ 2}} = \frac{\partial\psi_e}{\partial\tau} \tag{23}$$

where

$$\left.\begin{aligned} H_2 &= H(\alpha_2,\beta,\tau) \\ G_2 &= g_\alpha(\alpha_2,\beta) \end{aligned}\right\} \tag{24}$$

$$\zeta_2 = \sqrt{\sigma}\,(\alpha_2-\alpha) \tag{25}$$

with the boundary conditions

$$\left.\begin{aligned} \psi_e(\zeta_2 = 0,\beta,\tau) &= H_2-F_2 \\ \psi_e(\zeta_2 \rightarrow \infty,\beta,\tau) &= 0 \end{aligned}\right\} \tag{26}$$

Note that although Eqs. (21) and (23) do not have β-derivatives, β-dependence of ψ_e is caused by the β-dependence of the coefficients and of the boundary conditions.

ψ_e obeys the diffusion equation; the nature of its solution is well known. It is exponentially small except in a narrow band bordering each ambient edge. The width of each ambient edge is of the order $1/\sqrt{\sigma}$.

BOUNDARY CONDITIONS FOR THE ASYMPTOTIC SOLUTION

Integrate Eq. (11) with respect to α once:

$$\int_0^{2\pi} d\tau \left[- \frac{g_\beta}{g_\alpha} \left(H\psi \frac{\partial\psi}{\partial\alpha} - \psi^2 \frac{\partial H}{\partial\alpha}\right) + g_\beta U_\alpha \Lambda\psi \right]$$

$$+ \int d\alpha \int_0^{2\pi} d\tau \ \frac{\partial}{\partial\beta} \left[- \frac{g_\beta}{g_\alpha} \left(H\psi \frac{\partial\psi}{\partial\beta} - \psi^2 \frac{\partial H}{\partial\beta}\right) + g_\alpha U_\beta \Lambda\psi \right]$$

$$= A(\beta) \tag{27}$$

The first term can be slightly rewritten:

$$H\psi \frac{\partial\psi}{\partial\alpha} = \frac{1}{2} \frac{\partial}{\partial\alpha} (H\psi^2) - \frac{\psi^2}{2} \frac{\partial H}{\partial\alpha} \tag{28}$$

Substitute Eq. (28) into Eq. (27), multiply by g_α/g_β, integrate with respect to α once more and rearrange:

$$\int_0^{2\pi} d\tau \ H\psi^2$$

$$= \int d\alpha \left\{ 3\psi^2 \frac{\partial H}{\partial\alpha} + 2 \ g_\alpha \ U_\alpha \Lambda\psi - 2 \frac{g_\alpha}{g_\beta} A(\beta) \right\}$$

$$+ \int 2 \frac{g_\alpha}{g_\beta} d\alpha \int d\alpha \int_0^{2\pi} d\tau \ \frac{\partial}{\partial\beta} \left[- \frac{g_\beta}{g_\alpha} \left(H\psi \frac{\partial\psi}{\partial\beta} - \psi^2 \frac{\partial H}{\partial\beta}\right) + g_\alpha U_\beta \Lambda\psi \right]$$

$$+ B(\beta) \tag{29}$$

Because all integrands on the righthand side of Eq. (29) are bounded, the entire righthand side of Eq. (29) must be some well behaved function of α and β; that is

$$\int_0^{2\pi} d\tau \; H\psi^2 = I\,(\alpha,\beta) \tag{30}$$

More specifically, $\frac{\partial I}{\partial \alpha}$ is bounded. Therefore,

$$\begin{aligned} I(\alpha_1^{+},\beta) &= I(\alpha_1,\beta) + 0\left\{1/\sqrt{\sigma}\right\} \\ &= \int_0^{2} d\tau \; H^3(\alpha_1,\beta,\tau) + 0\left\{1/\sqrt{\sigma}\right\} \end{aligned} \tag{31}$$

where α_1^{+} denotes a location just internal of the edge region near α_1. But, because of the nature of the edge correction, we also have

$$\begin{aligned} I(\alpha_1^{+},\beta) &= \int_0^{2\pi} d\tau \; H(\alpha_1^{+},\beta,\tau)\, \psi_\infty^2(\alpha_1^{+},\beta) \\ &\quad + 0\left\{\exp(-\sqrt{\sigma})\right\} \\ &= 2\pi\, \bar{H}\,(\alpha_1^{+},\beta)\, \psi_\infty^2(\alpha_1^{+},\beta) + 0\left\{\exp\,(-\sqrt{\sigma})\right\} \end{aligned} \tag{32}$$

Therefore,

$$\psi_\infty^2(\alpha_1^{+},\beta) = \frac{\int_0^{2\pi} d\tau \;\; H^3(\alpha_1,\beta,\tau)}{2\pi\; \bar{H}(\alpha_1^{+},\beta)} + 0\left\{\frac{1}{\sqrt{\sigma}}\right\} + 0\left\{\exp(-\sqrt{\sigma})\right\}$$

Neglecting $0\left\{\frac{1}{\sqrt{\sigma}}\right\}$ and $0\left\{\exp(-\sqrt{\sigma})\right\}$, and noting that it is no longer necessary to distinguish α_1^+ from α_1; we have

$$\psi_\infty^2(\alpha_1,\beta) = \frac{\int_0^{2\pi} d\tau \;\; H^3(\alpha_1,\beta,\tau)}{2\pi \;\bar{H}(\alpha_1,\beta)} \tag{33}$$

Similarly, it can be derived,

$$\psi_\infty^2(\alpha_2,\beta) = \frac{\int_0^{2} d\tau \; H^3(\alpha_2,\beta,\tau)}{2\pi \;\bar{H}(\alpha_2,\beta)} \tag{34}$$

Equations (33) and (34) are the desired boundary conditions for the asymptotic solution. Now, formulation of the asymptotic analysis is complete. Comparing Eqs. (10) and (12), it is seen that the governing differential equation for ψ_∞ is quasi-linear and is formally identical to that for a non-time-dependent self-acting gas bearing. For given bearing surface geometry, g_α and g_β; sliding velocity, U_α and U_β; and prescribed squeeze motion, $H(\alpha,\beta,\tau)$; the solution can be found by methods already developed for self-acting gas bearings.

COMPUTATION OF BEARING FORCE

Suppose the bearing gap can be expressed as

$$H = \bar{H}(\alpha,\beta) + H_1(\alpha,\beta)\cos\tau \tag{35}$$

and it is of interest to calculate the temporal average of the component of the squeeze-film bearing reaction in a given direction. Let the angle of inclination of a normal vector of the bearing surface from the given direction be $\theta(\alpha,\beta)$, then the desired average force is

$$F = \frac{p_a R^2}{2\pi} \left\{ \iint g_\alpha g_\beta \cos\theta \, d\alpha d\beta \int_0^{2\pi} \left(\frac{\psi}{H} - 1\right) d\tau \right\} \tag{36}$$

Because the solution of ψ has been divided into three regions, the force can be accordingly so divided. That is

$$\frac{F}{p_a R^2} = f_\infty + f_1 + f_2 \tag{37}$$

$$f_\infty = \frac{1}{2\pi} \iint g_\alpha g_\beta \cos\theta \, d\alpha d\beta \int_0^{2\pi} \left(\frac{\psi_\infty}{H} - 1\right) d\tau$$

Now,

$$\int_0^{2\pi} \left(\frac{\psi_\infty}{H} - 1\right) d\tau$$

$$= \int_0^{2\pi} \frac{\psi_\infty \, d\tau}{\bar{H} + H_1 \cos\tau} - 2\pi$$

$$= \frac{2\psi_\infty}{\sqrt{(\bar{H})^2 - (H_1)^2}} \tan^{-1} \sqrt{\frac{\bar{H} - H_1}{\bar{H} + H_1}} \tan\frac{\tau}{2} \Bigg|_{\tau = 0}^{\tau = 2\pi} - 2\pi$$

$$= 2\pi \left\{ \frac{\psi_\infty}{\sqrt{(\bar{H})^2 - (H_1)^2}} - 1 \right\}$$

Therefore,

$$f_\infty = \iint \left\{ \frac{\psi_\infty}{\sqrt{(\bar{H})^2 - (H_1 0^2}} - 1 \right\} g_\alpha g_\beta \cos\theta \, d\alpha d\beta \tag{38}$$

Due to the edge correction near α_1, we also have

$$f_1$$

$$= \frac{1}{2\pi} \iint_{\alpha \approx \alpha_1} g_\alpha g_\beta \cos\theta \, d\alpha d\beta \int_0^{2\pi} \frac{\psi_e}{H} \, d\tau$$

$$\approx \frac{1}{2\pi\sqrt{\sigma}} \left\{ \int G_1 \, g_\beta(\alpha_1, \beta) \cos\theta_1 \, d\beta \int_0^\infty d\zeta_1 \int_0^{2\pi} \frac{\psi_e}{H_1} \, d\tau \right\} \tag{39}$$

where $\theta_1 = \theta(\alpha_1, \beta)$.

Similarly, due to the edge correction near α_2, we have

$$f_2 = \frac{1}{2\pi\sqrt{\sigma}} \left\{ \iint_{G_2} g_\beta(\alpha_2,\beta)\cos\theta_2 \, d\beta \int_0^\infty d\zeta_2 \int_0^{2\pi} \frac{\psi_e}{H_2} \, d\tau \right\} \tag{40}$$

where $\theta_2 = \theta(\alpha_2,\beta)$

SUMMARY AND CONCLUSIONS

The high frequency gaseous squeeze-film bearing can be analyzed by considering the internal region and edge regions separately.

The internal region obeys the asymptotic equation which is formally similar to the equation for the time-independent self-acting gas bearing; consequently, its solution can be obtained by any of many methods already developed for the latter problem.

In an edge region, the governing equation is a form of the diffusion equation with relatively simple coefficients.

So far as the temporal average of the bearing force is concerned, the contribution of the edge region relative to that of the internal region is $O(1/\sqrt{\sigma})$, where σ is a dimensionless frequency. In typical situations, σ is often in excess of 10^3, so that the effects in the edge region(s) can be neglected.

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