A NEW APPROACH TO THE KINETIC THEORY OF PLASMAS

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ABSTRACT

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A new approach to the kinetic theory of plasmas is presented. This approach is based upon the hierarchy of equations obtained from the Liouville equation by integration over position coordinates. A generalization of the Mayer cluster expansion is used to rewrite the distribution function in terms of a set of generalized correlation functions. The resulting hierarchy of equations share with the Liouville equation the characteristic that they are linear, and each may be solved by a straightforward operator method. The final reduction in description is accomplished by integrating the solution over the unwanted velocity coordinates. Three problems are studied. First, the initial reaction of the plasma to the presence of a small amplitude disturbance is shown to agree with the predictions of Landau. Second, a general kinetic (master) equation is derived for a spatially-homogeneous, stable plasma in the limit that the plasma parameter is small but not zero. This general equation, first derived by McCune, is shown explicitly to reduce, after integration over N-1 velocities, to the kinetic equation of Balescu and Lenard. Finally, the behavior of an unstable collisionless plasma is considered. The hierarchy of equations may be solved, in the limit that the plasma parameter approaches zero, to obtain an explicit solution for the distribution function which includes all wave-coupling effects. The solution is shown to reduce to a form which agrees with the results of quasi-linear theory in the limit that the initial amplitude and growth rate of the disturbance are small. The application of the method to the problem of a strongly-unstable plasma is briefly discussed.

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NOMENCLATURE

Several symbols defined in the text and used but briefly are not included in this list.

$C(1\|t)$	eq. (7.2-7)
$C_{\vec{k}}(1)$	eq. (7.2-11)
$\vec{E}_{\vec{k}}$	electric field
e	electronic charge
$F^{N,N}$	ensemble distribution function
$f^{N,\nu}$	generalized correlation function
f_0	spatially-homogeneous part of single-particle function
$f_{\vec{k}}$	single-particle function
$\vec{k}$	wave vector
$L(\vec{k},p)$	eq. (3.2-21)
N	number of particles in the system
n	number density
p	Laplace transform variable
$\vec{p}_i$	momentum of ith particle
$p_{\vec{k}}$	position of unstable root in p-plane (see eq. (7.2-8))
$R_{\vec{k}}(1\|p)$	eq. (7.2-9)
$R_{\vec{k}}(1\|-i\vec{k}\cdot\vec{v}_1)$	eq. (7.2-1)
r_D	Debye length
t	time

U_{ij}	intermolecular potential
u_i	component of velocity of i^{th} particle parallel to the wave vector
V	volume of system
$\vec{V}$	phase velocity of wave
$\vec{v}_i$	velocity of i^{th} particle
$\vec{x}_i$	position of i^{th} particle
γ	growth rate of unstable mode
ε	plasma parameter
$\varepsilon(\vec{k},p)$	eq. (3.2-24)
σ	parameter used to order amplitude of initial disturbance
$\varphi(\vec{v})$	spatially homogeneous part of distribution function at t = 0
$\Phi(u)$	eq. (3.2-25)
$\psi(k)\ (4\pi)$	Fourier transform of intermolecular potential
$\omega_{\vec{k}}$	frequency of $\vec{k}$ th mode
ω_p	plasma frequency

Operators

$\mathcal{D}_{\vec{k}}(i)$	eq. (3.2-4)
$\mathcal{D}_{\vec{k}}(ij)$	eq. (3.2-4)
$\mathcal{H}^{\nu}$	eq. (2-17)
$\mathcal{H}_{\vec{k}}(1)$	eq. (3.3-3)
$\underset{\sim}{H}$	eq. (3.2-13)
$\mathcal{J}^{\nu}$	eq. (2-3)

$\mathcal{J}_{\vec{K}}(12)$	eq. (4.2-9)
$\mathcal{I}(ij)$	eq. (2-3)
$\mathcal{K}^{\nu}$	eq. (2-3)
$\mathcal{L}^{\nu}$	eq. (2-17)
$L(ij)$	eq. (4.2-6)
$L_{\vec{K}}(1)$	eq. (7.2-17)
$\mathcal{M}^{\nu}$	eq. (2-17)
$M(ij)$	eq. (5.3-1)
$M_{\vec{K}-\vec{K}',\vec{K}'}(i,j)$	eq. (7.4.2-5)
$\mathcal{P}_{\vec{K}}(1\|t)$	eq. (3.2-1)
$\mathscr{P}_{\vec{K}}(1\|t)$	Table I
$\mathcal{R}^{\nu}$	eq. (2-17)
$S^{(n)}_{\vec{K}}(1,i\|j\cdots n\|t)$	Table I
$\mathcal{T}_{\vec{K}}(1)$	eq. (8-9)
$\underset{\sim}{V}$	eq. (3.2-8)
$\underset{\sim}{V}'$	eq. (3.2-26)

<u>Superscripts</u>

$\sim$	denotes integration over velocity
$\rightharpoonup$	denotes a vector quantity
$*$	denotes complex conjugate

CHAPTER 1

INTRODUCTION

A charged particle in a fully-ionized plasma interacts simultaneously with a large number of other particles. While most of these interactions are weak, the combined effect can be significant, giving rise to the so-called "collective" phenomena in plasmas. In addition to the large number of weak interactions, there are a small number of strong interactions that produce large deviations in the trajectory of a particle. The latter interactions give rise to such phenomena as microscopic density fluctuations and bremstrahlung emission and absorption in the plasma. A kinetic theory description of a plasma must include in a systematic way the effects due to these two extreme types of interactions between particles. One statistical mechanical treatment of a system of N-particles has been formulated by Gibbs.[1] The particles are viewed classically as charge and mass localized at a (moving) point in space, and each is assumed to interact with the N-1 particles through the laws of Newtonian mechanics. An ensemble of similar, non-interacting systems is introduced, and each system of the ensemble is represented by a point in a 6N-dimensional phase space. The number of systems is assumed to be so large that the cloud of points in phase space may be represented by a distribution function which is continuous in the 6N variables. The time rate of change of the ensemble function is governed by the Liouville equation, which prescribes that the systems of the ensemble can neither be created nor destroyed.

Much effort has been directed towards finding a solution to the Liouville equation. Due to the complexity of this equation one seeks, at the present time, an approximation to the solution. The approximations that are in common use are based upon the possibility of sorting out the simpler from the more complex (but less likely) interactions.

Practically, one has used either a hierarchy of equations, the BBGKY[2,3,4,5,6] hierarchy, obtained from the Liouville equations by an integration over position and velocity coordinates, or a diagram method of one kind or another[7,8,9] to represent a direct solution of the Liouville equation. The first approach takes advantage of the observation that all quantities of interest can be determined from the reduced functions f_S which are obtained from the ensemble distribution function by an integration over N-S velocity and position coordinates. The level of description of the problem is therefore reduced immediately by integrating the Liouville equation over N-S velocity and position coordinates. The result is a system of coupled equations (the BBGKY hierarchy) in which the equation for the function f_s has a term which contains f_{s+1} and so on. No simplification has been achieved at this point except through the explicit use of the symmetry properties of the distribution functions. The problem has simply been transformed from that of solving the Liouville equation to that of solving a system of N coupled equations. In order to obtain a simplification of the equations it is necessary to introduce some assumptions which enable one to compute the lower order functions f_1, f_2, ... without knowing the higher order functions f_3, f_4, ... These assumptions usually take the form of a Mayer cluster expansion[10], as well as an ordering of the successively higher correlation functions in terms of some small parameter appropriate to the system. The problem is reduced in this way to that of solving the first two or three equations of the hierarchy. For a plasma the appropriate small parameter is the inverse of the number of particles in a Debye sphere -- often called the plasma parameter. In the limit that the plasma parameter approaches zero the correlations are assumed to vanish altogether and the entire BBGKY system of equations is reduced to its lowest member, the non-linear Vlasov equation. The neglect of correlations in this limit is equivalent to neglecting interparticle collisions in an ordinary gas.

An alternative approach (the diagram method) is taken by Prigogine, Balescu and Brout.[7,8,9] A direct solution of the Liouville is obtained first and the reduction of the level of description comes only after the solution has been obtained. The advantage of formulating the problem in this manner is that the Liouville equation is linear, and the investigator has at his disposal the well-established techniques for solving linear partial differential equations. The solution is expressed in the form of an infinite series of increasingly complicated terms. Diagrammatic methods are required to simplify the notational problem. Finally, the dominant contribution to each term of the solution is determined on the basis of certain criteria which serve to define the problem.

A somewhat different approach to the kinetic theory of a plasma is presented in the following Chapters. The starting point is the hierarchy of equations obtained from the Liouville equation by an integration over the position coordinates only. This system of equations, derived by Higgins[11] and McCune[12] has been used by McCune to derive the master equation for plasmas. The distribution functions in the hierarchy depend upon all N momentum and ν position coordinates. The functions are written in terms of generalized correlation functions, a procedure which can be viewed as a generalization of the Mayer cluster expansion.

A new system of equations is obtained for the correlation functions. Each equation of this hierarchy is coupled to four other equations. However, all are linear. Even wave coupling effects appear in a linear way in this formulation. While the equations are to be solved at this level of description, considerable simplification of the solution is achieved subsequently by integrating the solutions over the "unwanted" velocity coordinates.

The above approach has two characteristic features. First, as mentioned above, the equations of the hierarchy are coupled. For the general case where the plasma parameter ε is small (but not zero) it is necessary to include arguments similar to those used with the BBGKY hierarchy in order to truncate the system of equations and to reduce the problem to the solution of the two or three lowest members. Second, the basic equations are linear and may be solved by simple operator methods. To illustrate this advantage we shall consider in Chapters 5 - 8 a plasma in the collisionless limit ($\varepsilon \to 0$) where the BBGKY hierarchy, as already mentioned, reduces to the non-linear Vlasov equation (under the assumed Mayer cluster expansion).

While the approach described above has much of the generality of the approach of Prigogine, et. al., it has two distinct advantages over the latter. First, the number of terms is not so large as to require the introduction of diagrams. Second, the choice of the dominant terms in the solution can here be made on clear physical grounds whereas the basis for the choice of the dominant diagrams of Prigogine, et.al., is often somewhat obscure.

Before proceeding, let us briefly discuss an alternative view of plasma statistical mechanics which could be adopted, based upon the "exact distribution function" for a given system of N point particles.* The evolution in time of the exact distribution function is governed by the Vlasov equation.[13] In general, the distribution must be treated as a random function of time[14,15], and the Vlasov equation must be averaged over an ensemble of systems. The result is a hierarchy of equations in which each equation is coupled to the one which follows it. This hierarchy is equivalent to the BBGKY hierarchy. Dupree[16,17] has shown that a consistent ordering of the terms in the system of equations is possible, and that the (small) plasma parameter ε can be used to prescribe a systematic perturbation procedure for obtaining a solution valid to any order.

---

*The "exact distribution function" is for the classical plasma discussed here a sum of N delta functions

$$\mathcal{N}(\vec{X},t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\vec{X} - \vec{X}_i(t))$$

where $\vec{X}_i(t)$ is the trajectory of the i th particle in velocity and position space.

While the approach based upon the Liouville equation differs somewhat in point of view and methodology from the above approach based upon the Vlasov equation, the results of both are in complete agreement. We adopt henceforward the first point of view and seek approximate solutions to the Liouville equation by an extension of the methods of McCune.

Three problems are to be studied. First, the initial value problem of a small amplitude disturbance in a collisionless plasma is considered. The equations of the hierarchy are found to reduce to particularly simple forms, and each may be solved independently of the others. The solution for the single-particle distribution function agrees, to within terms that are small in the limit that the number of particles becomes large, with that obtained by Landau to the linearized Vlasov equation.[18] The solution to the first problem assumes a central position in the theory. The methods developed are illustrative of the methods employed in later chapters to solve more complex problems. Also the first problem serves as a simple introduction to the more complicated general operators which appear at later stages.

The second problem is that of studying for a stable, spatially homogeneous plasma the general kinetic (or master) equation derived by McCune. We show in Chapter 4 that when the level of description is reduced by an integration over N-1 velocity coordinates McCune's kinetic equation becomes identical with that obtained by Balescu[19] and Lenard[20]. Finally, the "bump-in-tail" instability[21] in a collisionless plasma is discussed. The equations are again simplified in the limit that the plasma parameter approaches zero. Each member is coupled only to the two which come directly after it in the hierarchy. Since the equations are linear, each can be solved by a generalization of the operator method introduced by Dupree.[22] The solutions are then combined to write the complete solution for the single-particle distribution function in terms of the initial conditions on the problem. This approach eliminates the need for the adiabatic hypothesis or multiple time scales usually required for this type of problem. The solution is simplified by assuming the initial amplitude and growth rate of the unstable disturbance to be small and the time not too large. The results are found to agree for an "intermediate" period of time with those obtained from quasi-linear theory.[23,24]

A simple method is proposed to calculate the growth and self-limiting of a weakly-unstable disturbance. The results of the approximate calculation are found to compare favorably with the numerical calculation of Drummond and Pines.[23]

CHAPTER 2

HIERARCHY OF EQUATIONS FOR THE GENERALIZED CORRELATION FUNCTIONS

We consider an ensemble of systems of charged particles. In the interests of simplicity we limit ourselves to the case of a single species of charged particles in a neutralizing background of immobile particles. There are to be N such particles, each of which has associated with it a position coordinate $\vec{x}_i$ and a momentum coordinate $\vec{p}_i$. The gas is assumed to be "classical", that is quantum mechanical and relativistic effects may be neglected. The theory is thus not applicable to the very high temperature plasmas, such as thermonuclear plasmas, in which radiation from relativistic particles constitutes an important part of the energy of the system. It can, however, be applied to the relatively cool plasmas, such as the solar wind where $T \sim 10^5\,{}^\circ K$.

The distribution of systems within the ensemble is described by $F^{N,N}(\{N\}|t)$ which is a function of N (vector) position coordinates and N (vector) momentum coordinates. The particles are assumed to be indistinguishable so that

$F^{N,N}(\{N\}|t)$ is symmetric to the interchange of any pair of indices. All quantities are non-dimensionalized by introducing a length which characterizes the range of interaction between particles and a frequency which characterizes the time scale on which microscopic changes take place. These plasma quantities are assumed to be the Debye length r_D and the plasma frequency ω_p.

The special case of no external force fields is considered. Furthermore, since we are interested in those properties of the plasma which are independent of the size of the container, the walls are removed to infinity in such a way that the mean number density of particles $n = \frac{N}{V}$ is a constant. If $\frac{r_D^3}{V} << 1$, edge effects may be neglected.

The reduced function $F^{N,\nu}(\{\nu\}|t)$ is obtained from the ensemble distribution function by an integration over N-spatial variables.

$$F^{N,\nu}(\{\nu\}|t) = \int \left(\frac{r_D^3 d\vec{x}}{V} \right)^{N-\nu} F^{N,N}(\{N\}|t) \qquad (2\text{-}1)$$

We adopt the notation that $F^{A,B}(\{B\}|t)$ is a function of A velocity and B position coordinates. The indices in the set $\{B\}$ are associated with both position and velocity coordinates; those in the set $\{A-B\}$ with velocity coordinates only. While the ensemble function $F^{N,N}(\{N\}|t)$ is symmetric to an interchange of any two indices, the reduced functions $F^{N,\nu}(\{\nu\}|t)$ are symmetric only to the interchange of indices **within** the sets $\{\nu\}$ and $\{N-\nu\}$. It is not symmetric to an interchange of indices **between** these two sets.

The following equation for the function $F^{N,\nu}(\{\nu\}|t)$ has been derived by Higgins[11] and McCune.[12] It is obtained from the Liouville equation by an integration over N-ν spatial coordinates (see eq. (10) of ref. (12)).

$$\frac{\partial F^{N,\nu}(\{\nu\}|t)}{\partial t} + \mathcal{K}^{\nu} F^{N,\nu}(\{\nu\}|t) - \mathcal{E}\,\mathcal{I}^{\nu} F^{N,\nu}(\{\nu\}|t) =$$

$$= \mathcal{E}\left\{ \sum_{i}^{\{\nu\}} \sum_{j}^{\{N-\nu\}} \int \frac{r_D^3 d\vec{x}_j}{V} I(ij) F^{N,\nu+1}(\{\nu\}j|t) \right. \tag{2-2}$$

$$\left. + \sum_{i<j}^{\{N-\nu\}}\sum \int \frac{r_D^3 d\vec{x}_i}{V} \frac{r_D^3 d\vec{x}_j}{V} I(ij) F^{N,\nu+2}(\{\nu\}ij|t) \right\}$$

where

$$\mathcal{K}^{\nu} \equiv \sum_{i=1}^{\nu} \vec{v}_i \cdot \frac{\partial}{\partial \vec{x}_i}$$

$$\mathcal{I}^{\nu} \equiv \sum_{i<j}^{\{\nu\}}\sum \frac{\partial u_{ij}}{\partial \vec{x}_i} \cdot \left(\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j} \right) \equiv \sum_{i<j}^{\{\nu\}}\sum I(ij) \tag{2-3}$$

$$\mathcal{E} \equiv (4\pi n r_D^3)^{-1}$$

The non-dimensional potential u_{ij}, assumed to be a spherically-symmetric function of the distance between the points $\vec{x}_i$ and $\vec{x}_j$, has been measured in units of e^2/r_D.

The symmetry of the reduced functions $F^{N,\nu}(\{\nu\}|t)$ and the condition that $F^{N,\nu}(\{\nu\}|t)$ vanish at the boundaries of the phase space have been used in the derivation of (2-2). Equation (2-2) is the $\nu^{\underline{th}}$ member of a hierarchy of equations in which each is coupled to two higher equations (in contrast to the BBGKY hierarchy where each member is coupled to only one higher equation).

The generalized correlation functions $f^{N,\nu}(\{\nu\}|t)$ are defined in the following way.

$$
\begin{aligned}
F^{N,0}(t) &= F^{N,0}(t) \\
F^{N,1}(1|t) &= F^{N,0}(t) + f^{N,1}(1|t) \\
F^{N,2}(12|t) &= F^{N,0}(t) + f^{N,1}(1|t) + f^{N,1}(2|t) + f^{N,2}(12|t) \qquad (2\text{-}4) \\
F^{N,3}(123|t) &= F^{N,0}(t) + f^{N,1}(1|t) + f^{N,1}(2|t) + f^{N,1}(3|t) + f^{N,2}(12|t) \\
&\quad + f^{N,2}(13|t) + f^{N,2}(23|t) + f^{N,3}(123|t) \\
&\vdots
\end{aligned}
$$

The relations (2-4) may be considered a generalization of the Mayer cluster expansion.[10] To show the relation to the Mayer cluster expansion, the expression (2-4) for $F^{N,2}(12|t)$ is integrated over the N-2 "spare" velocity coordinates.

$$
F^{2,2}(12|t) = F^{2,0}(t) + f^{2,1}(1|t) + f^{2,1}(2|t) + f^{2,2}(12|t) \qquad (2\text{-}5)
$$

The Mayer cluster expansion for the two-particle distribution function is

$$
F^{2,2}(12|t) = F^{1,1}(1|t)\,F^{1,1}(2|t) + g(12|t) \qquad (2\text{-}6)
$$

where $\mathcal{F}^{1,1}(1|t)$ is the single-particle function and $g(12|t)$ the correlation function. We write $\mathcal{F}^{1,1}(1|t)$ as the sum of a spatially-homogeneous part $f_0(1|t)$ and a spatially-inhomogeneous part $f_1(1|t)$ and substitute into (2-6)

$$\mathcal{F}^{2,2}(12|t) = f_0(1|t)f_0(2|t) + f_1(1|t)f_0(2|t) + f_0(1|t)f_1(2|t) + f_1(1|t)f_1(2|t) + g(12|t) \tag{2-7}$$

When the result (2-7) is compared with (2-5) we find

$$\begin{aligned} f^{2,1}(1|t) &= f_1(1|t)f_0(2|t) \\ f^{2,1}(2|t) &= f_0(1|t)f_1(2|t) \\ f^{2,2}(12|t) &= f_1(1|t)f_1(2|t) + g(12|t) \end{aligned} \tag{2-8}$$

The function $f^{2,2}(12|t)$ thus contains the effects of correlations between the particles 1 and 2. If at any time $f^{2,2}(12|t) = f^{1,1}(1|t)\,f^{1,1}(2|t)$ then the particles 1 and 2 are statistically independent.

A hierarchy of equations for the generalized correlation functions is obtained by substituting the expressions (2-4) into the hierarchy (2-2). The equation for $\mathcal{F}^{N,0}(t)$ is simplified by noting that the interaction potential U_{ij} is spherically symmetric. Therefore, the only part of $\mathcal{F}^{N,2}(ij|t)$ that remains after the integrations over $\vec{x}_i$ and $\vec{x}_j$ is $f^{N,2}(ij|t)$ and we find

$$\frac{\partial F^{N,0}(t)}{\partial t} = \mathcal{E} \sum_{i<j}^{\{N\}}\!\!\sum \int \frac{r_D^3 d\vec{x}_i}{V} \frac{r_D^3 d\vec{x}_j}{V} I(ij) f^{N,2}(ij|t) \tag{2-9}$$

The second equation of the hierarchy (2-2), rewritten in terms of the generalized correlations (2-4), may be simplified by using (2-9) and the spherical symmetry of $\mathcal{U}_{ij}$ with the following result.

$$\begin{aligned} \frac{\partial f^{N,1}(1|t)}{\partial t} &+ \mathcal{K}^1 f^{N,1}(1|t) - \mathcal{E}\sum_{j=2}^{N} \int \frac{r_D^3 d\vec{x}_j}{V} I(1j) f^{N,1}(j|t) = \\ &= \mathcal{E}\Big\{ \sum_{j=2}^{N} \int \frac{r_D^3 d\vec{x}_j}{V} I(1j) f^{N,2}(1j|t) \\ &\qquad - \sum_{j=2}^{N} \int \frac{r_D^3 d\vec{x}_1}{V} \frac{r_D^3 d\vec{x}_j}{V} I(1j) f^{N,2}(1j|t) \\ &\qquad + \sum_{i<j}^{\{N-1\}}\!\!\sum \int \frac{r_D^3 d\vec{x}_i}{V} \frac{r_D^3 d\vec{x}_j}{V} I(ij) f^{N,3}(1ij|t) \Big\} \end{aligned} \tag{2-10}$$

The generalized single-particle function $f^{N,1}(1|t)$ is related by (2-10) to the <u>two</u> higher functions $f^{N,2}(1i|t)$ and $f^{N,3}(1ij|t)$.

If the relations (2-9) and (2-10) are used to eliminate some terms from the third equation of the hierarchy, we find the following equation for $f^{N,2}(12|t)$.

$$\begin{aligned} \frac{\partial f^{N,2}(12|t)}{\partial t} &+ \mathcal{K}^2 f^{N,2}(12|t) - \mathcal{E}\sum_{j=3}^{N} \int \frac{r_D^3 d\vec{x}_j}{V}\Big(I(1j) f^{N,2}(2j|t) + I(2j) f^{N,2}(1j|t)\Big) - \mathcal{E}\,\mathcal{J}^2 f^{N,2}(12|t) = \\ &= \mathcal{E}\Big\{ \mathcal{J}^2 F^{N,0}(t) + \mathcal{J}^2 f^{N,1}(1|t) - \int \frac{r_D^3 d\vec{x}_1}{V} I(12) f^{N,1}(1|t) + \mathcal{J}^2 f^{N,1}(2|t) \\ &\quad - \int \frac{r_D^3 d\vec{x}_2}{V} I(12) f^{N,1}(2|t) - \int \frac{r_D^3 d\vec{x}_1}{V} I(12) f^{N,2}(12|t) + \int \frac{r_D^3 d\vec{x}_1}{V} \frac{r_D^3 d\vec{x}_2}{V} I(12) f^{N,2}(12|t) \\ &\quad - \int \frac{r_D^3 d\vec{x}_2}{V} I(12) f^{N,2}(12|t) + \sum_{j=3}^{N} \int \frac{r_D^3 d\vec{x}_j}{V}\big(I(1j) + I(2j)\big) f^{N,3}(12j|t) \end{aligned} \tag{2-11}$$

$$-\sum_{j=3}^{N}\int \frac{r_D^3 d\vec{x}_1}{V}\frac{r_D^3 d\vec{x}_j}{V} I(1j) \overset{N,3}{f}(12j|t) - \sum_{j=3}^{N}\int \frac{r_D^3 d\vec{x}_2}{V}\frac{r_D^3 d\vec{x}_j}{V} I(2j) \overset{N,3}{f}(12j|t)$$
$$+\sum_{i<j}^{\{N-2\}}\sum\int \frac{r_D^3 d\vec{x}_i}{V}\frac{r_D^3 d\vec{x}_j}{V} I(ij) \overset{N,4}{f}(12ij|t)\Big\}$$

This result may be simplified somewhat if we consider the order of magnitude of the term

$$\mathcal{E}\int \frac{r_D^3 d\vec{x}_1}{V} I(12) \overset{N,1}{f}(1|t) \tag{2-12}$$

The range of the interaction potential U_{ij} is assumed to be of the order of the Debye length so that U_{ij} is very small for $|\vec{x}_i - \vec{x}_j| > 1$ (the position coordinate has been non-dimensionalized with respect to the Debye length). Thus, we may characterize the order of magnitude of the term (2-12) in the following way

$$\mathcal{E}\frac{r_D^3}{V}\left(I(12)\overset{N,1}{f}(1|t)\right)_{AV} \tag{2-13}$$

where $\left(I(12)\overset{N,1}{f}(1|t)\right)_{AV}$ is the average value of the integrand of (2-12) inside a sphere with a radius of one Debye length centered at the point $\vec{x}_2$. We can write, from the definition (2-3) of $\mathcal{E}$, $\mathcal{E}\,{r_D^3}/{V} = 1/4\pi N$. We see from (2-13) that the term (2-12) becomes vanishingly small in the limit $N \rightarrow \infty$ and may therefore be dropped from the equation (2-11). There are other terms of the form

$$\varepsilon \sum_{j=3}^{N} \int \frac{\mathcal{D}^3 d\vec{x}_j}{V} I(1j) f^{N,3}(12j) \tag{2-14}$$

on the right-hand side of (2-11). Each term of the sum (2-14) is, by the arguments presented above, of order $(1/N)$. However, since there are N such terms their combined effect is of order $\left(I(1j)f^{N,3}(12j)\right)_{AV}$, and they cannot be discarded from equation (2-11). The equation for $f^{N,2}(12)$ becomes

$$\frac{\partial f^{N,2}(12|t)}{\partial t} + \kappa^2 f^{N,2}(12|t) - \varepsilon \sum_{j=3}^{N} \int \frac{\mathcal{D}^3 d\vec{x}_j}{V}\left(I(1j) f^{N,2}(2j|t) + I(2j) f^{N,2}(1j|t)\right) - \varepsilon \mathcal{J}^2 f^{N,2}(12|t) =$$

$$= \varepsilon \Bigg\{ \mathcal{J}^2 \left(F^{N,0}(t) + f^{N,1}(1|t) + f^{N,1}(2|t)\right) + \sum_{j=3}^{N} \int \frac{\mathcal{D}^3 d\vec{x}_j}{V} \left(I(1j) + I(2j)\right) f^{N,3}(12j|t)$$

$$- \sum_{j=3}^{N} \int \frac{\mathcal{D}^3 d\vec{x}_1}{V} \frac{\mathcal{D}^3 d\vec{x}_j}{V} I(1j) f^{N,3}(12j|t) - \sum_{j=3}^{N} \int \frac{\mathcal{D}^3 d\vec{x}_2}{V} \frac{\mathcal{D}^3 d\vec{x}_j}{V} I(2j) f^{N,3}(12j|t) \tag{2-15}$$

$$+ \sum_{i<j}^{\{N-2\}} \sum \int \frac{\mathcal{D}^3 d\vec{x}_i}{V} \frac{\mathcal{D}^3 d\vec{x}_j}{V} I(ij) f^{N,4}(12ij|t) \Bigg\}$$

where terms of $\mathcal{O}(\frac{2}{N})$ have been dropped. We find from (2-15) that the two-particle function is coupled to four other functions, the two ($F^{N,0}(t), f^{N,1}(i|t)$) that come just before it in the hierarchy and the two ($f^{N,3}(ij\ell|t), f^{N,4}(ij\ell m|t)$) that come just after.

If these same arguments are used to obtain an equation for $f^{N,3}(123|t)$ we find (the error is now of $\mathcal{O}(\frac{3}{N})$)

$$\frac{\partial f^{N,3}_{(123|t)}}{\partial t} + \mathcal{K}^3 f^{N,3}_{(123|t)} - \varepsilon \sum_{j=3}^{N} \int \frac{r_D^3 d\vec{x}_j}{V} \Big(I(1j) f^{N,3}_{(23j|t)} + I(2j) f^{N,3}_{(13j|t)}$$

$$+ I(3j) f^{N,3}_{(12j|t)} \Big) - \varepsilon\, \mathcal{J}^3 f^{N,3}_{(123|t)} = \varepsilon \Big\{ I(12) f^{N,1}_{(3|t)} + I(13) f^{N,1}_{(2|t)}$$

$$+ I(23) f^{N,1}_{(1|t)} + \big(I(13) + I(23)\big) f^{N,2}_{(12|t)} + \big(I(12) + I(23)\big) f^{N,2}_{(13|t)}$$

$$+ \sum_{j}^{\{N-3\}} \int \frac{r_D^3 d\vec{x}_j}{V} \big(I(1j) + I(2j) + I(3j)\big) f^{N,4}_{(123j|t)} + \big(I(13) + I(12)\big) f^{N,2}_{(23|t)} \qquad (2\text{-}16)$$

$$- \sum_{j}^{\{N-3\}} \int \frac{r_D^3 d\vec{x}_1}{V} \frac{r_D^3 d\vec{x}_j}{V} I(1j) f^{N,4}_{(123j|t)} - \sum_{j}^{\{N-3\}} \int \frac{r_D^3 d\vec{x}_2}{V} \frac{r_D^3 d\vec{x}_j}{V} I(2j) f^{N,4}_{(123j|t)}$$

$$- \sum_{j}^{\{N-3\}} \int \frac{r_D^3 d\vec{x}_3}{V} \frac{r_D^3 dx_j}{V} I(3j) f^{N,4}_{(123j|t)} + \sum_{i<j}\sum^{\{N-3\}} \int \frac{r_D^3 d\vec{x}_i}{V} \frac{r_D^3 d\vec{x}_j}{V} I(ij) f^{N,5}_{(123ij|t)} \Big\}$$

We see that the three-particle function is coupled to four other functions. As with the generalized two-particle function, these functions are the two ($f^{N,1}_{(i|t)}$, $f^{N,2}_{(ij|t)}$) which come just before and the two ($f^{N,4}_{(ij\ell m|t)}$, $f^{N,5}_{(ij\ell mq|t)}$) which come just after $f^{N,3}_{(123|t)}$ in the hierarchy. We find the same result in the general case. The ν^{th} equation may be written in the following formal way (terms of order (ν/N) have been discarded)

$$\left(\frac{\partial}{\partial t} + \mathcal{H}^{\nu} - \varepsilon \mathcal{J}^{\nu}\right) f^{N,\nu} = \varepsilon \mathcal{R}^{\nu}\left(f^{N,\nu-2}, f^{N,\nu-1}\right) + \mathcal{M}^{\nu}\left(f^{N,\nu+1}\right) + \mathcal{T}^{\nu}\left(f^{N,\nu+2}\right) \qquad (2\text{-}17)$$

The hierarchy of equations, of which (2-17) is the ν^{th} member, has several important properties. First, the equations are quite general in that no special assumptions have been made about correlations in the plasma. Second, the equations are linear. The introduction of the generalized correlations (2-4) was chosen in such a way that the resulting hierarchy would retain the linear nature of the Liouville equation. Third, the differential and integral operators in (2-17) are independent of time, a property which, as we will see, enables one to solve the problem of a weakly unstable plasma without the use of an adiabatic hypothesis or multiple time scales. Finally, we shall see in the next Chapter that the operator $V \equiv -i\mathcal{H}'_{\vec{k}}$ in the equation for $f^{N,1}(1|t)$ ($\mathcal{H}'_{\vec{k}}$ and $\mathcal{H}'$ are related by a Fourier transform in the spatial variable $\vec{x}_1$) is self-adjoint.

CHAPTER 3

SHORT-TIME BEHAVIOR OF A SMALL-AMPLITUDE DISTURBANCE

3.1 Introduction

We consider the time-response of a small-amplitude electrostatic disturbance in a spatially-uniform collisionless plasma (there is no magnetic field). With the assumption that the amplitude of the disturbance is small, the equations of the hierarchy can be decoupled (in the limit $\varepsilon \to 0$) and each equation solved independently of the others. The equation for the generalized single-particle function may be written in matrix notation. We show that the operator $V = -i\mathcal{H}_{\vec{K}}$ in the equation for $f^{N,1}_{(1|t)}$ is self-adjoint, which implies that the eigenvalues of the operator are real and the solution remains bounded in time. Thus, the theory as formulated has a mathematically stable nature even when the initial conditions are such that the amplitude of the disturbance initially begins to grow.

However, as we shall see, the self-adjoint properties of the operator matrix represent information which is inaccessible to the theory when the level of description is reduced by an integration over the N-1 extra velocity coordinates. We show that if terms of $O(\frac{1}{N})$ are neglected the solution for the single-particle function $f_k(1|t)$ agrees for "short" times with that obtained by Landau, and it can therefore represent "stable" or "unstable" behavior in the sense of the usual linearized theory of the Vlasov equation. As is well known, the Landau solution of the initial value problem exhibits unstable behavior for certain plasma equilibria. The amplitude of the disturbance, according to the linearized theory, continues to grow indefinitely. In order to reconcile the possibility of the continued growth of $f_k(1|t)$ with the established[12] self-adjoint properties of the operator matrix $\underset{\sim}{V}$ we show that the discarded terms (of $O(\frac{1}{N})$), which add little to the behavior of the plasma for times observable in the laboratory, contribute in an important way to the mathematical properties of the matrix.

The solution for the generalized single-particle function may be rewritten by introducing the operator $\mathcal{P}_k(1|t)$ which relates the function $f_k^{N,1}(1|t)$ to its initial value. The usefulness of the $\mathcal{P}_k(1|t)$ operators becomes evident when we show that the solution for the ν -particle function may be written as a product of ν $\mathcal{P}_k(1|t)$ operators acting on an appropriate initial value function. The integration over the N-ν extra velocity coordinates reduces the solution to a product of the single-particle propagators $\mathcal{T}_k(1|t)$, first discussed by Dupree.[22] The methods used to obtain the forms of the operators $\mathcal{P}_{k}(1|t)$ and $\mathcal{T}_{k}(1|t)$ illustrate the general procedure employed in later chapters to solve more complex forms of the equations of the hierarchy, thus yielding solutions valid for times longer than those treated in the present chapter.

3.2 Short-Time Behavior of the Single-Particle Function, $f^{N,1}(1|t)$

The hierarchy of equations (2-17) for the generalized correlation functions may be simplified if the amplitude of the disturbance is assumed to be small. A parameter σ associated with the amplitude of the initial disturbance is used to order each term of the hierarchy. $F^{N,0}(t)$ is considered to be of $O(1)$. For a small initial disturbance of the homogeneous state the function $f^{N,1}(1|t)$ is taken of order σ since it contains the spatial coordinates of a single particle, $f^{N,2}(12|t)$ is of $O(\sigma^2)$ and so on. With the above ordering procedure the terms on the left-hand side of equation (2-10) for $f^{N,1}(1|t)$ are of $O(\sigma)$, and the terms on the right, which involve the functions $f^{N,2}(1i|t)$ and $f^{N,3}(1ij|t)$, are of $O(\sigma^2)$ and $O(\sigma^3)$, respectively. The assumption that σ is very small is used to justify neglecting the terms from the right-hand side of (2-10) and to reduce the equation for $f^{N,1}(1|t)$ to the following homogeneous form.

$$\frac{\partial f^{N,1}(1|t)}{\partial t} + \mathcal{H}'_1 f^{N,1}(1|t) - \varepsilon \sum_{j=2}^{N} \int \frac{\xi_D^3 d\vec{x}_j}{V} I(1j) f^{N,1}(j|t) = 0 \qquad (3.2-1)$$

The Fourier transform of the function $f^{N,1}(1|t)$

$$f^{N,1}_{\vec{K}}(1|t) = \int d\vec{x}_1 \, e^{-i\vec{K}\cdot\vec{x}_1} f^{N,1}(1|t) \tag{3.2-2}$$

is introduced to further simplify the equation for the single particle function. We find, from (3.2-1)

$$\frac{\partial f^{N,1}_{\vec{K}}(1|t)}{\partial t} + i\vec{K}\cdot\vec{v}_1 f^{N,1}_{\vec{K}}(1|t) - \frac{i}{N}\vec{K}\,\Psi(K)\cdot\sum_{j=2}^{N}\left(\frac{\partial}{\partial\vec{v}_1} - \frac{\partial}{\partial\vec{v}_j}\right) f^{N,1}_{\vec{K}}(j|t) = 0 \tag{3.2-3}$$

where the relation $\mathcal{E}\, r_D^3/V = 1/4\pi N$ has been used to eliminate the parameter $\mathcal{E}$. The function $4\pi\,\Psi(K)$ is the Fourier transform of the intermolecular potential. For the Coulomb potential $\Psi(K) = \frac{1}{K^2}$. To simplify the writing we introduce the operators

$$\begin{aligned} \mathcal{D}_{\vec{K}}(i) &\equiv \vec{K}\,\Psi(K)\cdot\frac{\partial}{\partial\vec{v}_i} \\ \mathcal{D}_{\vec{K}}(ij) &\equiv \vec{K}\,\Psi(K)\cdot\left(\frac{\partial}{\partial\vec{v}_i} - \frac{\partial}{\partial\vec{v}_j}\right) = \mathcal{D}_{\vec{K}}(i) - \mathcal{D}_{\vec{K}}(j) \end{aligned} \tag{3.2-4}$$

Equation (3.2-3), rewritten in terms of the operators (3.2-4), becomes

$$\frac{\partial f^{N,1}_{\vec{K}}(1|t)}{\partial t} + i\vec{K}\cdot\vec{v}_1 f_{\vec{K}}(1|t) - \frac{i}{N}\sum_{j=2}^{N}\mathcal{D}_{\vec{K}}(1j) f^{N,1}_{\vec{K}}(j|t) = 0 \tag{3.2-5}$$

The mathematical properties of equation (3.2-5) are most easily investigated by rewriting it in matrix notation. We introduce the N-dimensional column vector $\underset{\sim}{f}^{N,1}(t)$

$$\underset{\sim}{f}^{N,1}(t) \equiv \begin{pmatrix} f_{\vec{K}}^{N,1}(1|t) \\ f_{\vec{K}}^{N,1}(2|t) \\ \cdot \\ \cdot \\ \cdot \\ f_{\vec{K}}^{N,1}(N|t) \end{pmatrix} \qquad (3.2\text{-}6)$$

and write the equation for the time rate of change of $\underset{\sim}{f}^{N,1}(t)$ as

$$\frac{\partial}{\partial t}\underset{\sim}{f}^{N,1}(t) + i\underset{\sim}{V}\,\underset{\sim}{f}^{N,1}(t) = 0 \qquad (3.2\text{-}7)$$

The N x N-dimensional operator matrix $\underset{\sim}{V}$ has been defined to be

$$\underset{\sim}{V} \equiv \begin{bmatrix} \vec{K}\cdot\vec{v}_1 & -\frac{1}{N}\mathcal{D}_{\vec{K}}(12) & \cdot\;\cdot\;\cdot & -\frac{1}{N}\mathcal{D}_{\vec{K}}(1N) \\ -\frac{1}{N}\mathcal{D}_{\vec{K}}(21) & \vec{K}\cdot\vec{v}_2 & \cdot\;\cdot\;\cdot & -\frac{1}{N}\mathcal{D}_{\vec{K}}(2N) \\ -\frac{1}{N}\mathcal{D}_{\vec{K}}(31) & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\frac{1}{N}\mathcal{D}_{\vec{K}}(N1) & -\frac{1}{N}\mathcal{D}_{\vec{K}}(N2) & \cdot\;\cdot\;\cdot & \vec{K}\cdot\vec{v}_N \end{bmatrix} \qquad (3.2\text{-}8)$$

where the elements of the operator $\underset{\sim}{V}$ are real. The matrix is antisymmetric ($\mathcal{D}_{\vec{K}}(12) = -\mathcal{D}_{\vec{K}}(21)$).

The adjoint $\hat{\underset{\sim}{V}}$ of the operator $\underset{\sim}{V}$ is defined by the relation[25]

$$\int (d\vec{v})^N \left\{ \underset{\sim}{f}^{N,1}, \underset{\sim}{V}\, \underset{\sim}{f}^{N,1} \right\} = \int (d\vec{v})^N \left\{ \hat{\underset{\sim}{V}}\, \underset{\sim}{f}^{N,1}, \underset{\sim}{f}^{N,1} \right\} \qquad (3.2\text{-}9)$$

where $\{\underset{\sim}{A}, \underset{\sim}{B}\}$ is the N-dimensional inner product of the vectors $\underset{\sim}{A}^*$ and $\underset{\sim}{B}$ (the superscript * denotes the complex conjugate). The properties of the matrix (3.2-8) can be used to establish that the operator $\underset{\sim}{V}$ is self-adjoint ($\underset{\sim}{V} = \hat{\underset{\sim}{V}}$). It follows that the eigenvalues λ of $\underset{\sim}{V}$ are real[25], and the corresponding eigenvectors $\underset{\sim}{h}_\lambda(t)$ are neutrally stable functions of time (substitute any eigenvector for $\underset{\sim}{f}^{N,1}(t)$ in (3.2-7) and use the relation $\underset{\sim}{V}\,\underset{\sim}{h}_\lambda = \lambda\, \underset{\sim}{h}_\lambda$). Finally, solutions of the equation (3.2-7) remain bounded in time.[25]

For illustrative purposes the eigenvalue problem for the special case N=2 is investigated in some detail in Appendix A. The eigenvectors, which form a complete set, are used to write a solution to equation (3.2-7) (for N=2) which satisfies arbitrary initial conditions.

However, for large N the stable behavior of $f^{N,1}(1|t)$ at the N-momentum level is essentially lost to the theory when the level of description is reduced to a single velocity and spatial coordinate.

The solution for $f_{\vec{k}}^{N,1}(1|t)$ may be found in terms of its initial value by taking a Laplace transform of (3.2-7) in time. We introduce the transformed function ($\mathrm{Re}\, p > 0$)*

$$f_{\vec{K}}^{N,1}(1|p) = \int_0^\infty dt\, e^{-pt} f_{\vec{K}}^{N,1}(1|t) \tag{3.2-10}$$

and rewrite the equation for $f_{\vec{K}}^{N,1}(1|p)$ as

$$(p + i\vec{K}\cdot\vec{v}_1) f_{\vec{K}}^{N,1}(1|p) - \frac{i}{N}\sum_{j=2}^{N} D_{\vec{K}}(1j) f_{\vec{K}}^{N,1}(j|p) = f_{\vec{K}}^{N,1}(1|t=0) \tag{3.2-11}$$

Both sides of (3.2-11) are divided by the quantity $(p+i\vec{K}\cdot\vec{v}_1)$, and the column vector $\underset{\sim}{f}^{N,1}(p)$, which is the Laplace transform of the vector $\underset{\sim}{f}^{N,1}(t)$, is introduced to obtain the new matrix equation.

$$(\underset{\sim}{1} - \underset{\sim}{H})\underset{\sim}{f}^{N,1}(p) = \underset{\sim}{S}(p) \tag{3.2-12}$$

$\underset{\sim}{1}$ is the unit matrix. The operator matrix $\underset{\sim}{H}$ and the "source" matrix $\underset{\sim}{S}(p)$ are defined below.

$$\underset{\sim}{H} \equiv \begin{bmatrix} 0 & \frac{i}{N}\frac{D_{\vec{K}}(12)}{p+i\vec{K}\cdot\vec{v}_1} & \cdots & \frac{i}{N}\frac{D_{\vec{K}}(1N)}{p+i\vec{K}\cdot\vec{v}_1} \\ \frac{i}{N}\frac{D_{\vec{K}}(21)}{p+i\vec{K}\cdot\vec{v}_2} & 0 & \cdots & \frac{i}{N}\frac{D_{\vec{K}}(2N)}{p+i\vec{K}\cdot\vec{v}_2} \\ \vdots & \vdots & & \vdots \\ \frac{i}{N}\frac{D_{\vec{K}}(N1)}{p+i\vec{K}\cdot\vec{v}_N} & \frac{i}{N}\frac{D_{\vec{K}}(N2)}{p+i\vec{K}\cdot\vec{v}_N} & \cdots & 0 \end{bmatrix} \tag{3.2-13}$$

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*The scalar Laplace transform variable p should not be confused with the vector momentum $\vec{p}$.

$$\underset{\sim}{S}(p) = \begin{pmatrix} \dfrac{f_{\vec{K}}^{N,1}(1|t=0)}{p+i\vec{K}\cdot\vec{v}_1} \\ \dfrac{f_{\vec{K}}^{N,1}(2|t=0)}{p+i\vec{K}\cdot\vec{v}_2} \\ \vdots \\ \dfrac{f_{\vec{K}}^{N,1}(N|t=0)}{p+i\vec{K}\cdot\vec{v}_N} \end{pmatrix} \tag{3.2-14}$$

The formal solution for $\underset{\sim}{f}^{N,1}(p)$ may be written in the following way

$$\begin{aligned} \underset{\sim}{f}^{N,1}(p) &= (\underset{\sim}{1} - \underset{\sim}{H})^{-1}\underset{\sim}{S}(p) \\ &= (1 + \underset{\sim}{H} + \underset{\sim}{H}\underset{\sim}{H} + \underset{\sim}{H}\underset{\sim}{H}\underset{\sim}{H} + \cdots)\underset{\sim}{S}(p) \end{aligned} \tag{3.2-15}$$

where the right-hand side of (3.2-15) contains an infinite number of terms.

The solution (3.2-15) relates the Laplace-transformed function to its initial value. However, the function $f^{N,1}(1|p)$, which involves N different velocity coordinates, contains more information than would be required for a particular problem. Indeed, one could not specify an initial condition in such detail. A more useful quantity is the single-particle distribution function $f_{\vec{K}}(1|p)$ obtained from $f_{\vec{K}}^{N,1}(1|p)$ by an integration over all velocities except $\vec{v}_1$. We isolate from (3.2-15) the first element in the matrix $\underset{\sim}{f}^{N,1}(p)$ and integrate over all velocities except $\vec{v}_1$, to obtain the following result.

$$f_{\vec{k}}(1|p) = \frac{f_{\vec{k}}^{1,1}(1|t=0)}{p+i\vec{k}\cdot\vec{v}_1} + \frac{iD_{\vec{k}}(1)}{p+i\vec{k}\cdot\vec{v}_1}\int d\vec{v}_\ell \frac{f_{\vec{k}}^{2,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell}$$

$$+\frac{iD_{\vec{k}}(1)}{p+i\vec{k}\cdot\vec{v}_1}\int d\vec{v}_\ell d\vec{v}_m \frac{iD_{\vec{k}}(m)}{p+i\vec{k}\cdot\vec{v}_m}\frac{f_{\vec{k}}^{3,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell} \tag{3.2-16}$$

$$+\frac{iD_{\vec{k}}(1)}{p+i\vec{k}\cdot\vec{v}_1}\int d\vec{v}_\ell d\vec{v}_m d\vec{v}_n \frac{iD_{\vec{k}}(n)}{p+i\vec{k}\cdot\vec{v}_n}\frac{iD_{\vec{k}}(m)}{p+i\vec{k}\cdot\vec{v}_m}\frac{f_{\vec{k}}^{4,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell} + \cdots$$

Terms of $\Theta\left(\frac{1}{N}\right)$ have been disregarded in writing (3.2-16). To be exact, the coefficients of the second, third and fourth terms on the right-hand side of (3.2-16) should be

$$\frac{N-1}{N}, \quad \frac{(N-1)(N-2)}{N^2}, \quad \frac{(N-1)(N-2)(N-3)}{N^3} \tag{3.2-17}$$

respectively. The coefficients (3.2-17) all approach unity in the limit of large N. Furthermore, we have dropped from the third term of (3.2-16) N-1 terms of $\Theta\left(\frac{1}{N^2}\right)$ and from the fourth term (3N-5)(N-1) terms of $\Theta\left(\frac{1}{N^3}\right)$. An increasing number of terms must be discarded from each higher term in the series solution for $f_{\vec{k}}(1|p)$. The omission of these terms is justified in two ways. First, our interest lies in the determination of the bulk behavior of the plasma, not in those properties which depend upon the total number of particles present within the system. The equations of the hierarchy were derived on the basis that the walls of the container could be moved to infinity, and the limit of very large N has been used throughout. The coefficients of the terms which are dropped, being inversely proportional to N, becomes vanishingly small in the limit of very large N. Second, it can be shown that the correction terms (of $O(\frac{1}{N})$) significantly influence the solution only after a time T which is much longer than the time required by the system to come to a macroscopic equilibrium. In order to estimate T we note that each term of the series (3.2-15) can be divided into two parts, one which contributes to the solution (3.2-16) and one which represents corrections to the solution. While the first part dominates the lower terms, the correction part becomes increasingly important as one goes to higher terms in the series solution. Far out in the series there is a term which contains equal contributions from both parts. The time at which this term in the series becomes of $O(1)$ then provides an estimate of T. We show in Appendix B that T is of the order of $\sqrt{N}$ plasma periods, a time considered to be much longer than the time required for the plasma to come to an equilibrium. Thus, while the solution (3.2-16) is to be considered an approximation to the exact solution of (3.2-12) the error introduced in using (3.2-16) is negligible for all times of interest.

The function $f^{2,1}(\ell 1 t=0)$ which appears (after taking a Fourier transform in the spatial variable $\vec{x}_\ell$) in the second term of the solution (3.2-16) for $f_k(1|\rho)$ contains one velocity variable ($\vec{v}_1$ in this case) which does not have an associated spatial coordinate. Balescu[9] has shown that the function $f^{2,1}(\ell|t=0)$ may be written as a product of two functions; $f(\ell|t=0)$ which depends upon the velocity $\vec{v}_\ell$ and the position $\vec{x}_\ell$, and $\varphi(\vec{v}_1)$ which depends only upon the velocity $\vec{v}_1$. To summarize the argument (for more details see sections 2 and 3 of ref. 9) we note that two particles become statistically independent if they are separated sufficiently far from one another (shielding effects within the plasma are assumed to limit the interaction range to a distance on the order of the Debye length). At large separation distances $f^{2,1}(\ell|t=0)$ may therefore be written as a product of two distribution functions.

$$f^{2,1}(\ell|t=0) = f^{1,1}(\ell|t=0)\ \varphi(\vec{v}_1) \qquad (3.2\text{-}18)$$

However, since $f^{2,1}(\ell|t=0)$ is independent of the position coordinate $\vec{x}_1$ and therefore of the distance between the particles, the product form (3.2-18) must be valid for all interparticle distances. The result (3.2-18) is easily generalized to include a larger number of particles. The function $f^{\nu+1,\nu}(\{\nu\}|t)$ may be written as the product of the functions $f^{\nu,\nu}(\{\nu\}|t)$ and $f^{1,0}(i|t)$ whenever it is possible to separate the particle i sufficiently far (more than one Debye length) from <u>all</u> the particles in the set $\{\nu\}$. It is readily seen that the necessary separation can be achieved only if $\nu << N$.

The above arguments are used to write $f_{\vec{K}}^{\nu,1}(\ell|t=0)$ as a product of $f_{\vec{K}}(\ell|t=0)$ and ν-1 functions $\varphi(\vec{v})$.

$$f_{\vec{K}}^{\nu,1}(\ell|t=0) = f_{\vec{K}}(\ell|t=0) \prod_{i\neq\ell}^{\{\nu\}} \varphi(i) \tag{3.2-19}$$

The function $\varphi(\vec{v})$ is the spatially-homogeneous part of $f_{\vec{K}}^{2,1}(\ell|t=0)$ at the initial instant of time, and $f_{\vec{K}}(\ell|t=0) \equiv f_{\vec{K}}^{1,1}(\ell|t=0)$. The form (3.2-19) for the functions $f_{\vec{K}}^{\nu,1}(\ell|t=0)$ is introduced into the solution (3.2-16) for $f_{\vec{K}}(1|p)$ to obtain

$$f_{\vec{K}}(1|p) = \frac{f_{\vec{K}}(1|t=0)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{i D_{\vec{K}}(1)\varphi(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_\ell \frac{f_{\vec{K}}(\ell|t=0)}{p+i\vec{K}\cdot\vec{v}_\ell}\left[1+L(\vec{K},p)+L^2(\vec{K},p)+\cdots\right] \tag{3.2-20}$$

where we have defined the quantity

$$L(\vec{K},p) \equiv \int \frac{i\vec{K}\,\psi(K)\cdot\frac{\partial\varphi}{\partial\vec{v}_1}}{p+i\vec{K}\cdot\vec{v}_1}\,d\vec{v}_1 \tag{3.2-21}$$

Balescu[9] has found the same result by using diagram methods. The infinite series of terms on the right-hand side of (3.2-20) converges only in those regions of the p-plane where $L(\vec{K},p) < 1$. However, using the relation

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \tag{3.2-22}$$

we note that

$$f_{\vec{k}}(1|p) = \frac{f_{\vec{k}}(1|t=0)}{p+i\vec{k}\cdot\vec{v}_1} + \frac{i\vec{k}\,\psi(k)\cdot\frac{\partial\varphi}{\partial\vec{v}_1}\int d\vec{v}_2\,\frac{f_{\vec{k}}(2|t=0)}{p+i\vec{k}\cdot\vec{v}_2}}{(p+i\vec{k}\cdot\vec{v}_1)\ \varepsilon(\vec{k},p)} \qquad (3.2\text{-}23)$$

represents the analytic continuation of (3.2-20) with

$$\varepsilon(\vec{k},p) \equiv 1 - L(\vec{k},p) \qquad (3.2\text{-}24)$$

since it is valid for all values of $L(\vec{k},p)$ (except, of course, the (isolated) zeros of $\varepsilon(\vec{k},p)$) and agrees with the solution (3.2-20) in the regions where $L(\vec{k},p) < 1$. The result (3.2-23) is precisely that obtained by Landau[18] from the linearized Vlasov equation.

If the theory of residues is used to determine the inverse Laplace transform of (3.2-23), then the time behavior of $f_{\vec{k}}(1|t)$ is related to the poles of the function $f_{\vec{k}}(1|p)$. Two different types of physical behavior are included. Balescu[9] has noted that the contribution of the pole $p = -i\vec{k}\cdot\vec{v}_1$ represents the individual particle behavior of the plasma; the tendency for a local density excess to be spread over larger regions of space by the free-streaming motion of the particles. On the other hand, the poles of $(\varepsilon(\vec{k},p))^{-1}$ represent the collective behavior of the plasma. While there may be, in general, an infinite number of collective poles, most are heavily damped[26] and contribute little to the long-time behavior of $f_{\vec{K}}(1|t)$. Landau has shown that in the limit of small $\vec{K}$ (long wavelength) the dominant pole (the one furthest to the right in the p-plane) implies plasma oscillations close to the plasma frequency which grow or decay at an exponential rate which depends upon $\partial\Phi(u)/\partial u$, where u is the component of velocity parallel to $\vec{K}$ and

$$\Phi(u_1) \equiv \int d\vec{v}_1 \; \delta\left(u_1 - \frac{\vec{K}\cdot\vec{v}_1}{K}\right) \varphi(1) \tag{3.2-25}$$

The amplitude of the disturbance grows if $\left(\partial\Phi(u)/\partial u\right)_{u=\frac{\omega_K}{K}}$ is positive, decays if it is negative (for $\omega_K/K > 0$).

The growth or decay rate of the above (Landau) pole is independent of time. If the plasma is unstable then the disturbance, as predicted by the solution (3.2-23), grows indefinitely. However, we have noted earlier that the self-adjoint properties of the operator matrix $\underset{\sim}{V}$ imply that the solution for $f_{\vec{K}}^{N,1}(1|t)$ is bounded in time. In order to reconcile the bounded properties of $f_{\vec{K}}^{N,1}(1|t)$ with the possibility of an infinite growth of $f_{\vec{K}}(1|t)$ we consider the operator matrix $\underset{\sim}{V}'$, defined in the following way

$$\underset{\sim}{V}' \equiv \begin{bmatrix} K\cdot\vec{v}_1 & \frac{1}{N}D_{\vec{K}}(1) & & \frac{1}{N}D_{\vec{K}}(1) \\ \frac{1}{N}D_{\vec{K}}(2) & \vec{K}\cdot\vec{v}_2 & & \frac{1}{N}D_{\vec{K}}(2) \\ & & & \\ \frac{1}{N}D_{\vec{K}}(N) & \frac{1}{N}D_{\vec{K}}(N) & & \vec{K}\cdot\vec{v}_N \end{bmatrix} \tag{3.2-26}$$

The matrix (3.2-26) is obtained from $\underset{\sim}{V}$ (equ. (3.2-8)) by dropping a term from each of the off-diagonal elements (replace $\mathcal{D}_k(ij)$ by $\mathcal{D}_k(i)$). We observe that the operator $\underset{\sim}{V}'$, unlike $\underset{\sim}{V}$, is not self-adjoint.

$$\int (d\vec{v})^N \left\{ \underset{\sim}{f}^{N,1}, \underset{\sim}{V}' \underset{\sim}{f}^{N,1} \right\} \neq \int (d\vec{v})^N \left\{ \underset{\sim}{V}' \underset{\sim}{f}^{N,1}, \underset{\sim}{f}^{N,1} \right\} \tag{3.2-27}$$

The deletion of the second velocity derivative from the operator $\mathcal{D}_k(ij)$ has destroyed the self-adjointness of the operator matrix.

Despite the mathematical difference between the operators $\underset{\sim}{V}$ and $\underset{\sim}{V}'$, both predict (in the limit $N \to \infty$) the same behavior for the single particle function after a reduction in the level of description. We consider the matrix equation

$$\frac{\partial}{\partial t} \underset{\sim}{f} + i \underset{\sim}{V}' \underset{\sim}{f} = 0 \tag{3.2-28}$$

The initial value problem may be solved by taking a Laplace transform in time, exactly as was done above. The solution for the first element in the vector $\underset{\sim}{f}$ becomes, after integrating over all velocities except $\vec{v}_1$, identical to the solution (3.2-16) for $f_k(1|p)$. The only difference between the two solutions is that some terms of $O(\frac{1}{N})$ which were discarded in the writing of (3.2-16) do not appear in the solution (3.2-28). For instance, the term

$$-\frac{1}{N}\,\frac{iD_{\vec{k}}(1)}{p+i\vec{k}\cdot\vec{v}_1}\int d\vec{v}_\ell\, d\vec{v}_m\,\frac{iD_{\vec{k}}(1)\,f_{\vec{k}}^{3,1}(1|t=0)}{(p+i\vec{k}\cdot\vec{v}_m)(p+i\vec{k}\cdot\vec{v}_1)} \qquad (3.2\text{-}29)$$

has been discarded from the third term of the solution (3.2-16) for $f_{\vec{k}}(1|p)$. The third term of the solution of equation (3.2-28) has no similar contribution. The elements of the matrix $\underset{\sim}{V}$ which were dropped to obtain $\underset{\sim}{V}'$ can therefore be associated with

(a) properties of the plasma which depend upon N, the total number of particles present within the system.

(b) events which occur on a time scale much longer than the time required for the system to come to a macroscopic equilibrium.

The self-adjoint properties of the operator $\underset{\sim}{V}$ represent information inaccessible to a theory which attempts to describe the bulk behavior of a system.

Two physical interpretations of the behavior represented by the dominant collective (Landau) pole of (3.2-23) are possible. The first is based upon the motion of a particle in the force field of a wave. Jackson[27] has discussed the energy transferred to and from the wave by the "trapping" of particles in a potential well of the disturbance. However, Dawson[28] has noted that trapping, being a non-linear process, cannot be the source of Landau damping which comes from the solution of a linear equation. Dawson has interpreted Landau damping in terms of the energy transferred between the wave and the particles which have a velocity near the wave velocity. Particles which travel slightly faster than the wave are slowed down as they transfer energy to the wave, and particles which travel slightly slower than the wave are accelerated as the wave transfers energy to the particles. The net transfer of energy to the wave depends upon the relative number of particles travelling slower and faster than the wave and therefore to the slope of the distribution function at the wave velocity. An alternative interpretation of the interaction between a wave and particles discussed by Pearson[29], is based upon the absorption and emission of plasma waves by a particle. A particle in the field of a plasma wave can, in this view, either absorb or emit wave energy. A wave is damped if the absorption of all particles exceeds the emission and is amplified if the emission exceeds the absorption. Finally, we note that Landau damping may not be visible at all. Both Backus[30] and Dawson[28] have shown that non-linear processes may effect the behavior of the disturbance before Landau damping can be observed. If the trapping time[44] is less than the Landau damping time then the terms which were discarded from the right-hand side of the equations of the hierarchy (2-17) become important before the Landau part of the homogeneous solution has an opportunity to dominate the behavior.

The essential elements of the physical picture of Landau damping have been verified in ref. 31 where the spatial damping of electrostatic waves in an effectively collisionless plasma was measured experimentally. The damping rates were found to agree, to within experimental error, with the results of Landau. Furthermore, it was found that if the plasma contained no particles with velocities near that of the wave (the high velocity tail of the distribution function was cut off) then the electrostatic waves propagated undamped in the plasma.

3.3 The Generalized Operator $\mathcal{P}_k(1|t)$

The homogeneous equation (3.2-3) for the generalized single-particle function is used with an appropriate undisturbed state to define the operator $\mathcal{P}_k(1|t)$. The operator, which is a function of N velocities, the wave vector $\vec{k}$ and time, is not symmetric to the interchange of $\vec{v}_1$ with any other velocity (the N-1 other velocities may be exchanged symmetrically). Although $\mathcal{P}_k(1|t)$ may be used to rewrite the solution for $f_k^{N,1}(1|t)$ in a simple, formal manner, its real importance becomes evident when we consider other equations of the hierarchy. We find below and in Chapter 5 that the operators $\mathcal{P}_k(t)$ may be used to write the solution to each equation. The solution contains two types of terms. First, there is a product of $\mathcal{P}_k(t)$ operators and an initial value function; second, there is a convolution integral in time of $\mathcal{P}_{\vec{K}}(t)$ operators with a (time-dependent) source term. The latter term does not enter the present discussion as it does not significantly influence the short-time behavior of the distribution functions (it is discussed in detail in Chapters 5-8).

The operator $\mathcal{P}_{\vec{K}}(1|t)$ is defined by the differential equation

$$\frac{\partial}{\partial t}\mathcal{P}_{\vec{K}}(1|t) + \mathcal{H}_{\vec{K}}(1)\,\mathcal{P}_{\vec{K}}(1|t) = 0 \qquad (3.3\text{-}1)$$

with the initial condition

$$\mathcal{P}_{\vec{K}}(1|t=0) = 1 \qquad (3.3\text{-}2)$$

We have introduced the notation

$$\mathcal{H}_{\vec{K}}(1) \equiv i\vec{K}\cdot\vec{v}_1 - \frac{i}{N}\vec{K}\,\psi(K)\cdot\sum_{j=2}^{N}\left(\frac{\partial}{\partial\vec{v}_1} - \frac{\partial}{\partial\vec{v}_j}\right)(1\leftrightarrow j) \qquad (3.3\text{-}3)$$

where the parenthesis ($1 \leftrightarrow j$) indicates that the velocities $\vec{v}_1$ and $\vec{v}_j$ are to be interchanged. The operator $\mathcal{P}_{\vec{K}}(1|t)$ may be used to write the solution for $f_{\vec{K}}^{N,1}(1|t)$ in the following way

$$f_{\vec{K}}^{N,1}(1|t) = \mathcal{P}_{\vec{K}}(1|t)\,f_{\vec{K}}^{N,1}(1|t=0) \qquad (3.3\text{-}4)$$

An explicit expression for the operator is found by taking a Laplace transform of (3.3-4) and equating the result with the solution for $f_{\vec{K}}^{N,1}(1|p)$ found above in Section 3.2 (equ. 3.2-15). The operator $\mathcal{P}_{\vec{K}}(1|p)$ written in terms of the Laplace transform variable p, has the following form

$$\mathcal{P}_{\vec{K}}(1|p) = \frac{1}{p+i\vec{K}\cdot\vec{v}_1} + \sum_j{}' \frac{\frac{i}{N}\mathcal{D}_{\vec{K}}(1j)}{p+i\vec{K}\cdot\vec{v}_1}\,\frac{(1\leftrightarrow j)}{p+i\vec{K}\cdot\vec{v}_j}$$
$$+ \sum_j{}' \frac{\frac{i}{N}\mathcal{D}_{\vec{K}}(1j)}{p+i\vec{K}\cdot\vec{v}_1}\sum_l{}' \frac{\frac{i}{N}\mathcal{D}_{\vec{K}}(jl)}{p+i\vec{K}\cdot\vec{v}_j}\,\frac{(1\leftrightarrow l)}{p+i\vec{K}\cdot\vec{v}_l} + \cdots \tag{3.3-5}$$

The single-particle distribution function $f_{\vec{K}}(1|t)$ is obtained from $f_{\vec{K}}^{N,1}(1|t)$ by integrating over all velocities except $\vec{v}_1$. We write, from (3.3-4)

$$f_{\vec{K}}(1|t) = \int (d\vec{v})^{N-1}\mathcal{P}_{\vec{K}}(1|t) f_{\vec{K}}^{N,1}(1|t) \equiv \mathcal{T}_{\vec{K}}(1|t) f_{\vec{K}}(1|t=0) \tag{3.3-6}$$

where the reduced operator $\mathcal{T}_{\vec{K}}(1|t)$ depends only upon the velocity $\vec{v}_1$ (as well as the wave number $\vec{K}$ and time). When the expression (3.3-6) is compared (after taking a Laplace transform in time) with the solution (3.2-23) for $f_{\vec{K}}(1|p)$ we find

$$\mathcal{T}_{\vec{K}}(1|p) = \frac{1}{p+i\vec{K}\cdot\vec{v}_1} + \frac{i\vec{K}\,\psi(\kappa)\cdot\frac{\partial\varphi(1)}{\partial\vec{v}_1}\int\frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\epsilon(\vec{K},p)} \tag{3.3-7}$$

in agreement with the result obtained by Dupree.[22] The operator $\mathcal{P}_{\vec{k}}(1|t)$, obtained by taking an inverse Laplace transform of (3.3-7) propagates the function $f_{\vec{k}}(1|t=0)$ in time according to the linearized Vlasov equation.

The form to which the operator $\mathcal{R}_{\vec{k}}(1|\rho)$ reduces after an integration over N-1 velocities depends upon the function on which it is operating $\left(f^{N,1}(1|t=0)\right.$ in the case of $\left.(3.3\text{-}4)\right)$. We show in Chapter 6 that a hierarchy of "reduced" operators may be constructed from a sequence of functions $\gamma_n^{N,1}(1)$ which have the same overall properties as $f^{N,1}(1|t=0)$ but differ as to how the appropriate symmetry is obtained.

We have shown in Chapter 2 that the equation for $f^{N,\nu}(\{\nu\}|t)$ (where $\nu \geq 2$) may be written in the following way

$$\left(\frac{\partial}{\partial t} + \mathcal{H}^{\nu}\right) f^{N,\nu} - \varepsilon\, \mathcal{J}^{\nu} f^{N,\nu} = \\ = \varepsilon\, \mathcal{R}^{\nu}\left(f^{N,\nu-1}, f^{N,\nu-2}\right) + \mathcal{M}^{\nu}\left(f^{N,\nu+1}\right) + \mathcal{L}^{\nu}\left(f^{N,\nu+2}\right) \qquad (3.3\text{-}8)$$

If a small parameter σ is used to order the distribution functions (as in Section 3.2, $f^{N,\nu} \sim \mathcal{O}(\sigma^{\nu})$) and the parameter ε is of order σ^3 or smaller, then the entire right-hand side of (3.3-8) is at most of order $\sigma^{\nu+1}$. Furthermore, the last term on the left-hand side of (3.3-8) is of order $\sigma^{\nu+3}$. For small values of σ , terms of $\mathcal{O}(\sigma^{\nu+1})$ or higher may be dropped from (3.3-8), and the ν^{th} equation of the hierarchy reduced to the following homogeneous form

$$\left(\frac{\partial}{\partial t} + \mathcal{H}^{\nu}\right) f^{N,\nu}(\{\nu\}|t) = 0 \qquad (3.3\text{-}9)$$

The operator $\mathcal{H}^{\nu}_{\vec{K}_1\vec{K}_2\cdots\vec{K}_\nu}$ for $\nu = 2$ may be written (see (2.13) , after taking a double Fourier transform in the spatial variables $\vec{x}_1$ and $\vec{x}_2$),

$$\begin{aligned}\mathcal{H}^{2}_{\vec{K}_1\vec{K}_2} = {} & i\vec{K}_1\cdot\vec{v}_1 - \frac{i}{N}\vec{K}_1\,\psi(K_1)\cdot\sum_{j=3}^{N}\left(\frac{\partial}{\partial\vec{v}_1} - \frac{\partial}{\partial\vec{v}_j}\right)(1\longleftrightarrow j) \\ & + i\vec{K}_2\cdot\vec{v}_2 - \frac{i}{N}\vec{K}_2\,\psi(K_2)\cdot\sum_{j=3}^{N}\left(\frac{\partial}{\partial\vec{v}_2} - \frac{\partial}{\partial\vec{v}_j}\right)(2\longleftrightarrow j)\end{aligned} \qquad (3.3\text{-}10)$$

If we add to (3.3-10) two terms of $O(\frac{1}{N})$ then we may write

$$\begin{aligned}\mathcal{H}^{2}_{\vec{K}_1\vec{K}_2} &= \left(i\vec{K}_1\cdot\vec{v}_1 - \frac{i}{N}\vec{K}_1\,\psi(\vec{K}_1)\cdot\sum_{j}{}'\left(\frac{\partial}{\partial\vec{v}_1} - \frac{\partial}{\partial\vec{v}_j}\right)(1\longleftrightarrow j)\right) \\ &\quad + \left(i\vec{K}_2\cdot\vec{v}_2 - \frac{i}{N}\vec{K}_2\,\psi(K_2)\cdot\sum_{j}{}'\left(\frac{\partial}{\partial\vec{v}_2} - \frac{\partial}{\partial\vec{v}_j}\right)(2\longleftrightarrow j)\right) \qquad (3.3\text{-}11) \\ &= \mathcal{H}_{\vec{K}_1}(1) + \mathcal{H}_{\vec{K}_2}(2)\end{aligned}$$

and the equation for $f^{N,2}_{\vec{K}_1\vec{K}_2}(12|t)$ becomes

$$\left(\frac{\partial}{\partial t} + \mathcal{H}_{\vec{K}_1}(1) + \mathcal{H}_{\vec{K}_2}(2)\right) f^{N,2}_{\vec{K}_1\vec{K}_2}(12|t) = 0 \qquad (3.3\text{-}12)$$

The operators $\mathcal{H}_{\vec{K}_1}(1)$ and $\mathcal{H}_{\vec{K}_2}(2)$ commute to $\mathcal{O}(\frac{1}{N})$.[12] The only terms which do not commute are those which have been added to (3.3-10) in order to write (3.3-11). We may now use the $\mathcal{P}_{\vec{K}}(t)$ operators to write the solution of (3.3-12) as

$$f^{N,2}_{\vec{K}_1\vec{K}_2}(12|t) = \mathcal{P}_{\vec{K}_1}(1|t)\,\mathcal{P}_{\vec{K}_2}(2|t)\,f^{N,2}_{\vec{K}_1\vec{K}_2}(12|t=0) \tag{3.3-13}$$

In general, (to within an error of $O(\frac{\nu}{N})$) the $\nu^{\underline{th}}$ equation of the hierarchy may be written

$$\left(\frac{\partial}{\partial t} + \sum_{i=1}^{\nu} \mathcal{H}_{K_i}(i)\right) f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t) = 0 \tag{3.3-14}$$

with the solution

$$f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t) = \prod_{i=1}^{\nu} \mathcal{P}_{\vec{K}_i}(i|t)\, f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t=0) \tag{3.3-15}$$

If both sides of (3.3-15) are integrated over the velocities in the set $\{N-\nu\}$ we find

$$f^{\nu,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t) = \prod_{i=1}^{\nu} \mathcal{P}_{\vec{K}_i}(i|t)\, f^{\nu,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t=0) \tag{3.3-16}$$

The case ν = 2 is discussed in detail in section 2 of Appendix C. We note that if the initial conditions are such that $f^{\nu,\nu}_{\vec{K}_1 \cdots \vec{K}_\nu}(\{\nu\}|t=0)$ may be written as a product of single particle functions (the particles in the set are statistically uncorrelated) then, for short times at least, the ν-particle function remains factorized.

$$f^{\nu,\nu}_{\vec{K}_1 \cdots \vec{K}_\nu}(\{\nu\}|t) = \prod_{i=1}^{\nu}\left(\mathcal{P}_{\vec{K}_i}(i|t) f_{\vec{K}}(i|t=0)\right) \qquad (3.3\text{-}17)$$

3.4 Discussion

The initial response of a plasma to a small disturbance has been studied above. The assumptions that the amplitude of the disturbance and the plasma parameter ε were small quantities were used to decouple each equation of the hierarchy from the others. Each equation was solved in terms of the generalized propagator $\mathcal{P}_{\vec{K}}(t)$ and the initial value of the correlation function, $f^{N,\nu}(\{\nu\}|t=0)$. The result for the single-particle distribution function $f_{\vec{K}}(1|t)$ was found to agree with that obtained by Landau when the level of description was reduced by an integration over the N-1 extra velocity coordinates. Furthermore, if the ν-particle function could be factored initially (which means physically that initially the particles were statistically uncorrelated) it was found to remain factorized, for short times at least. The solution for $f^{\nu,\nu}(\{\nu\}|t)$ became a product of ν time-dependent, single-particle functions.

If a plasma is stable then the solutions to the homogeneous equations (3.3-9) have a part which oscillates as $e^{-i\vec{k}\cdot\vec{v}t}$, and a part which decays exponentially with time (Landau damping). In the limit of large times it is the "source" terms on the right-hand side of each equation that determine the long-time behavior of a stable plasma. We show in Chapter 4 that if the plasma is spatially homogeneous ($f^{N,1}(1|t=0) = 0$) and if the parameter ε is small, but not zero, it is the first term on the right-hand side of (2-15) for $f^{N,2}(12|t)$ which determines the dominant behavior at long times. The inclusion of this term, which involves $F^{N,0}(t)$, links the third equation of the hierarchy to the first. We show that the simultaneous solution of the first and third equations leads, upon a reduction in the level of description, to the Balescu-Lenard collision term.

On the other hand, if the plasma is unstable there is at least one mode of oscillation of the homogeneous solution (3.3-17) which grows exponentially with time and dominates the solution of (3.3-9) in the limit of large times. If $f^{1,1}(1|t)$ grows as $e^{\gamma t}$, the solution for $f^{2,2}(12|t)$, since it involves a product of two $P_{\vec{k}}(t)$ operators, grows as $e^{2\gamma t}$, $f^{3,3}(123|t)$ as $e^{3\gamma t}$, and so on. It is a feature of the theory that the growth rate $\gamma_{\vec{k}}$ is independent of time, and the solution of the homogeneous part of each equation of the hierarchy grows indefinitely in time. If the amplitude of each function $f^{\nu,\nu}(\{\nu\}|t)$ is not to become infinite it must be prevented from so doing by the source terms on the right-hand side of each equation. These terms, which could be discarded initially because they were small, must be included in the limit of large times. If a disturbance, which is initially of $\mathcal{O}(\sigma)$, grows exponentially we find after time t

$$f_{\vec{k}}(t) \sim \mathcal{O}(\sigma e^{\gamma t}) \tag{3.4-1}$$

The time at which $f_{\vec{k}}(t)$ becomes of $\mathcal{O}(1)$ is then

$$t \sim -\frac{\ln \sigma}{\gamma} \tag{3.4-2}$$

However, not only does $f_{\vec{k}}(t)$ become of $\mathcal{O}(1)$, but $f^{2,2}(t)$, $f^{3,3}(t)$, ... also become of $\mathcal{O}(1)$ at time t (3.4-2). The terms on the right-hand side of the equation (3.3-8) are no longer small compared with the ones on the left, and each equation of the hierarchy becomes coupled with at least one other equation. The complete hierarchy of equations must be solved for an unstable plasma, a problem considered in detail in Chapters 5 through 8.

CHAPTER 4

KINETIC EQUATION FOR A STABLE, HOMOGENEOUS PLASMA

4.1 Introduction

A kinetic equation for a spatially-homogeneous, stable plasma is derived from the hierarchy of equations for the N-momentum functions $f^{N,\nu}(\{v\}|t)$. The discussion is limited to a plasma in which there is no magnetic field present. Quantum and relativistic effects are neglected. To proceed, we derive from the hierarchy (2-17) a general kinetic (master) equation of the form (4.1-1) for the distribution function $\mathcal{F}^{N,0}(t)$.

$$\frac{\partial \mathcal{F}^{N,0}(t)}{\partial t} = C\left[\mathcal{F}^{N,0}(t)\right] \tag{4.1-1}$$

The "collision" operator on the right-hand side of (4.1-1) contains only the N-momentum function $\mathcal{F}^{N,0}(t)$. Once the general kinetic equation has been obtained the level of description may be reduced by an integration over all but one of the velocities. As discussed by McCune[12] the above represents an alternative approach to that usually taken in the derivation of a kinetic equation for the single particle function.

The plasma parameter ε , assumed small, is used to order the terms of the hierarchy (2-17). We show in Section 4.2 that, with the adopted ordering procedure, the hierarchy can be truncated at the equation for the two-particle function $f^{N,2}(12|t)$. The problem is reduced (to within an error of $\mathcal{O}(\varepsilon^2)$) to the simultaneous solution of two equations. McCune[12] has solved these equations by the method of multiple time scales to obtain a generalized kinetic equation of the form (4.1-1). The methods of Appendix C are used in Section 4.3 to reduce the general kinetic equation to the kinetic equation of Balescu and Lenard. Finally, we discuss in Section 4.4 an alternative derivation of (4.1-1) which stresses the statistical nature of an assumption which must be made to obtain a kinetic equation for a plasma.

4.2 Generalized Kinetic Equation

The single-particle function vanishes identically in a spatially-homogeneous plasma. The terms of the equations are ordered by means of the plasma parameter ε , assumed to be a small quantity. The N-momenta function $F^{N,0}(t)$ is assumed to be of $\mathcal{O}(1)$. The equation (2-17) for $f^{N,2}(12|t)$ contains the term $\varepsilon \mathcal{R}^2(F^{N,0})$ which is of $\mathcal{O}(\varepsilon)$. The function $f^{N,2}(12|t)$ is thus of $\mathcal{O}(\varepsilon)$. The fourth equation of the hierarchy (for $f^{N,3}(\{3\}|t)$) contains the term $\varepsilon \mathcal{R}^3(f^{N,2}(\{2\}|t))$ (remember $f^{N,1}(1|t) \equiv 0$), and we conclude that $f^{N,3}(\{3\}|t)$ is of $\mathcal{O}(\varepsilon^2)$. The fifth equation has the term $\varepsilon \mathcal{R}^4(f^{N,2}, f^{N,3})$ so $f^{N,4}(\{4\}|t)$ is also of $\mathcal{O}(\varepsilon^2)$. The higher functions, $f^{N,\nu}(\{\nu\}|t)$ for $\nu > 4$, are of $\mathcal{O}(\varepsilon^3)$ or smaller. If terms which are of second order in ε or smaller are discarded from the hierarchy, the formulation is reduced to two coupled equations.

$$\frac{\partial F^{N,0}(t)}{\partial t} = \varepsilon \sum_{i<j}^{\{N\}}\sum \int \frac{\lambda_D^3 d\vec{x}_i}{V} \frac{\lambda_D^3 d\vec{x}_j}{V} I(ij) f^{N,2}(ij|t) \qquad (4.2\text{-}1)$$

$$\frac{\partial f^{N,2}(12|t)}{\partial t} + \left(\mathcal{H}(1) + \mathcal{H}(2)\right) f^{N,2}(12|t) = \varepsilon\, \mathcal{G}^2 F^{N,0}(t) \qquad (4.2\text{-}2)$$

We have assumed throughout that the initial conditions on the problem are consistent with the above ordering procedure.

The two-particle function for a spatially-homogeneous plasma depends only upon the distance between the points 1 and 2.

$$f^{N,2}(\vec{x}_1, \vec{x}_2|t) = \varepsilon\, g^{N,2}(|\vec{x}_1 - \vec{x}_2||t) \qquad (4.2\text{-}3)$$

where the function $g^{N,2}(12|t)$ is now considered to be of $\mathcal{O}(1)$. It is convenient to rewrite the right-hand side of (4.2-1) in terms of a Fourier transform in the variable

$\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. The convolution theorem for Fourier transforms is used to write

$$\varepsilon \frac{\lambda_D^3}{V}\int d\vec{x}_i \frac{\lambda_D^3 dx_j}{V} I(ij) f^{N,2}(ij|t) = \frac{\varepsilon}{4\pi N}\int d\vec{x}_{ij}\, I(ij) g^{N,2}(ij|t)$$

$$= \frac{\varepsilon}{N} L(ij) g_{\vec{K}}^{N,2}(ij|t) \tag{4.2-4}$$

where we have defined the Fourier transform of $g^{N,2}(ij|t)$ as

$$g_{\vec{K}}^{N,2}(ij|t) = \int d\vec{x}_{ij}\, e^{-i\vec{K}\cdot(\vec{x}_i - \vec{x}_j)} g^{N,2}(ij|t) \tag{4.2-5}$$

and introduced the new operator

$$L(ij) = -\frac{i}{(2\pi)^3}\int d\vec{K}\, \vec{K}\, \psi(K)\cdot\left(\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j}\right)\begin{pmatrix}\vec{K}_i \to \vec{K}\\ \vec{K}_j \to -\vec{K}\end{pmatrix} \tag{4.2-6}$$

The notation $\begin{pmatrix}\vec{K}_i \to \vec{K}\\ \vec{K}_j \to -\vec{K}\end{pmatrix}$ is used to denote that, when the operator $L(ij)$ acts on a function of $\vec{K}_i$ and $\vec{K}_j$, $\vec{K}_i$ is to be replaced by $\vec{K}$ and $\vec{K}_j$ is to be replaced by $-\vec{K}$. The equations (4.2-1) and (4.2-2) become

$$\frac{\partial F^{N,0}(t)}{\partial t} = \frac{\varepsilon}{N}\sum_{i<j}^{\{N\}}\sum L(ij) g^{N,2}(ij|t) \tag{4.2-7}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{H}_{\vec{K}}(1) + \mathcal{H}_{-\vec{K}}(2)\right) g_{\vec{K}}^{N,2}(12|t) = \mathcal{J}_{\vec{K}}(12)\mathcal{F}^{N,0}(t) \tag{4.2-8}$$

where

$$\mathcal{J}_{\vec{K}}(12) \equiv i\vec{K}\Psi(K)\cdot\left(\frac{\partial}{\partial\vec{v}_1} - \frac{\partial}{\partial\vec{v}_2}\right) \tag{4.2-9}$$

We note that the right-hand side of (4.2-7) is of $\mathcal{O}(\varepsilon)$ compared to the left which suggests that there are two time scales on which events can be expected to occur. McCune[12] has solved equations (4.2-7) and (4.2-8) by the method of multiple time scales.[32, 33, 34, 35] If the function is expanded in powers of ε

$$\mathcal{F}^{N,0}(t) = \mathcal{F}_0^{N,0}(t) + \varepsilon\mathcal{F}_1^{N,0}(t) \tag{4.2-10}$$

and the multiple time scales τ and $\varepsilon\tau$ introduced, then equations (4.2-7) and (4.2-8) may be solved to obtain a general kinetic equation (equation (51) of ref. (12)) for the evolution of $\mathcal{F}_0^{N,0}(t)$ on the slow ($\varepsilon\tau$) time scale. The condition that the initial correlations $g^{N,2}(12|t=0)$ disappear from the plasma in the limit of large times by the process of phase mixing is used in the derivation. We do not repeat the calculations but merely quote the result (rewritten in the present notation)

$$\frac{\partial \mathcal{F}_0^{N,0}(\epsilon\tau)}{\partial(\epsilon\tau)} = \frac{1}{N}\sum_{i<j}^{\{N\}}\sum \int_0^\infty d\tau' L(ij) \mathcal{P}_{\vec{k}}(i|\tau') \mathcal{P}_{-\vec{k}}(j|\tau') \mathcal{J}_{\vec{k}}(12) \mathcal{F}_0^{N,0}(\epsilon\tau) \tag{4.2-11}$$

Equation (4.2-11) predicts the rate of change of the function $\mathcal{F}_0^{N,0}(\epsilon\tau)$ on the slow time scale. We show in the next section that the linear equation (4.2-11), when integrated over (N-1) velocities, reduces to the non-linear kinetic equation of Balescu and Lenard.

4.3 Reduction of the Level of Description

Equation (4.2±11) becomes, after an integration over the N-1 velocities $\vec{v}_2 \ldots\ldots \vec{v}_N$

$$\frac{\partial f(\vec{v}_1)}{\partial(\epsilon\tau)} = \int (d\vec{v})^{N-1} L(12) \int_0^\infty \mathcal{P}_{\vec{k}}(1|\tau') \mathcal{P}_{-\vec{k}}(2|\tau') \mathcal{J}_{\vec{k}}(12) \mathcal{F}_0^{N,0}(\epsilon\tau) \tag{4.3-1}$$

where we have defined the single particle function

$$f(\vec{v}_1) = \int (d\vec{v})^{N-1} \mathcal{F}^{N,0}(\epsilon\tau) \tag{4.3-2}$$

and $(d\vec{v})^{N-1}$ denotes the velocity element $d\vec{v}_2 \ldots\ldots d\vec{v}_N$. We noted in Chapter 3 that the spatially-homogeneous function $\mathcal{F}^{\nu,0}(t)$ could be written as a product of ν functions $f(\vec{v})$ for values of ν much less than N. This observation is used with the result of Section 2, Appendix C, to write

$$\int (d\vec{v})^{N-1} \mathcal{P}_{\vec{K}}(1|\tau') \mathcal{P}_{-\vec{K}}(2|\tau') \mathcal{Q}_{\vec{K}}(12) F_0^{N,0}(\epsilon\tau) =$$

$$= \int (d\vec{v}_2) \mathcal{P}_{\vec{K}}(1|\tau') \mathcal{P}_{-\vec{K}}(2|\tau') \mathcal{Q}_{\vec{K}}(12) f(\vec{v}_1) f(\vec{v}_2) \qquad (4.3\text{-}3)$$

The propagator $\mathcal{P}_{\vec{K}}(1|\tau')$ is the same as that obtained from a solution to the linearized Vlasov equation except that here the Laplace transform of $\mathcal{P}_{\vec{K}}(1|\tau')$ (in the variable τ') depends upon the slow time scale ($\epsilon\tau$) through the presence of the function $f(\vec{v}_1)$ in the operator.

$$\mathcal{P}_{\vec{K}}(1|p) = \frac{1}{p + i\vec{K}\cdot\vec{v}_1} + \frac{i\vec{K}\,\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}\int \frac{d\vec{v}_1}{p + i\vec{K}\cdot\vec{v}_1}}{(p + i\vec{K}\cdot\vec{v}_1)\;\mathcal{E}(\vec{K},p)} \qquad (4.3\text{-}4)$$

where

$$\mathcal{E}(\vec{K},p) = 1 - \int d\vec{v}_1 \; \frac{i\vec{K}\,\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}}{p + i\vec{K}\cdot\vec{v}_1}$$

Equation (4.3-1) becomes with the result (4.3-3)

$$\frac{\partial f(\vec{v}_1)}{\partial(\epsilon\tau)} = \int d\vec{v}_2\, L(12) \int_0^\infty d\tau'\, \mathcal{P}_{\vec{K}}(1|\tau') \mathcal{P}_{-\vec{K}}(2|\tau') \mathcal{Q}_{\vec{K}}(12) f(\vec{v}_1) f(\vec{v}_2) \qquad (4.3\text{-}5)$$

The convolution theorem for Laplace transforms is used to rewrite (4.3-5) as

$$\frac{\partial f(\vec{v}_1)}{\partial(\epsilon\tau)} = \frac{1}{2\pi i} \int d\vec{v}_2\, L(12) \int_C dp\, \mathcal{P}_{\vec{K}}(1|p) \mathcal{P}_{-\vec{K}}(2|-p) \mathcal{Q}_{\vec{K}}(12) f(\vec{v}_1) f(\vec{v}_2) \qquad (4.3\text{-}6)$$

where the contour C passes to the right of the poles of $\mathcal{P}_{\vec{K}}(1|p)$ and to the left of the poles of $\mathcal{P}_{-\vec{K}}(2|-p)$.

The definitions (4.3-4) of the operator $\mathcal{P}_{\vec{K}}(1|p)$ and the function $\varepsilon(\vec{K},p)$ are used to write

$$\mathcal{P}_{\vec{K}}(1|p)\, i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial\vec{v}_1} = \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)} \tag{4.3-7}$$

and

$$\int d\vec{v}_2\, \mathcal{P}_{-\vec{K}}(2|-p)\, i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_2)}{\partial\vec{v}_2} = 1 - \frac{1}{\varepsilon(-\vec{K},-p)} \tag{4.3-8}$$

The right-hand side of (4.3-6) may be written in the following convenient way (we leave out for the moment the operator $L(12)$)

$$\frac{1}{2\pi i}\int_C dp\, \mathcal{P}_{\vec{K}}(1|p)\int d\vec{v}_2\, \mathcal{P}_{-\vec{K}}(2|p)\, \mathcal{D}_{\vec{K}}(12) f(\vec{v}_1) f(\vec{v}_2) =$$

$$\begin{aligned} = \frac{1}{2\pi i}\int_C dp \Bigg\{ & \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)}\;\frac{\int d\vec{v}_2\,\frac{f(\vec{v}_2)}{-p-i\vec{K}\cdot\vec{v}_2}}{\varepsilon(-\vec{K},-p)} \\ & + \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial\vec{v}_1}\int d\vec{v}_1\,\frac{f(\vec{v}_1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)\,\varepsilon(-\vec{K},-p)} - \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial\vec{v}_1}\int d\vec{v}_1\,\frac{f(\vec{v}_1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)} \\ & - \frac{f(\vec{v}_1)}{p+i\vec{K}\cdot\vec{v}_1}\;\frac{\int d\vec{v}_2\,\frac{i\vec{K}\psi(K)\cdot\partial f(\vec{v}_2)/\partial\vec{v}_2}{-p-i\vec{K}\cdot\vec{v}_2}}{\varepsilon(-\vec{K},-p)} \Bigg\} \end{aligned} \tag{4.3-9}$$

The contour C passes to the right of the pole $p = -i\vec{K}\cdot\vec{v}_1$ in the first term of (4.3-9). If the contour is moved from the right to the left of this pole, the first term may be rewritten as a contour integral plus 2πi times the residue at the pole. We find

$$\frac{1}{2\pi i}\int_C dp \, \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}\int d\vec{v}_2 \frac{f(\vec{v}_2)}{-p-i\vec{K}\cdot\vec{v}_2}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)\,\varepsilon(-\vec{K},-p)} =$$

$$= \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}}{\left|\varepsilon(\vec{K},-i\vec{K}\cdot\vec{v}_1)\right|^2}\int d\vec{v}_2 \frac{f(\vec{v}_2)}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma}$$

$$-\frac{1}{2\pi i}\int_{C'} dp' \, \frac{-i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}\int d\vec{v}_2 \frac{f(\vec{v}_2)}{p'-i\vec{K}\cdot\vec{v}_2}}{(p'-i\vec{K}\cdot\vec{v}_2)\,\varepsilon(-\vec{K},p')\,\varepsilon(\vec{K},-p')} \qquad (4.3\text{-}10)$$

where

$$\varepsilon(\vec{K},-i\vec{K}\cdot\vec{v}_1) = 1 - \int d\vec{v}_2 \, \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_2)}{\partial \vec{v}_2}}{i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma}$$

The small positive parameter σ has been introduced to specify the way in which the path of integration is to go around the pole. The new variable $p' = -p$ has been used to write the integral of (4.3-10). The contour C' passes to the left of the poles of $\left(\varepsilon(\vec{K},-p')\right)^{-1}$ and to the right of the rest of the poles of the integrand. We note that the integrand in the second term of (4.3-10) is the complex conjugate of the integrand in the second term of (4.3-9). The difference between these two integrands, divided by $2\pi i$, is then a purely real quantity. We note from equation (4.3-6) that since $\partial f(\vec{v}_1)/\partial(\varepsilon\tau)$ is real and the operator $L(12)$ is purely imaginary, we require only the imaginary part of (4.3-9). The second terms of (4.3-9) and (4.3-10) do not contribute to this imaginary part.

The third term of (4.3-9) has no poles to the right of the contour C and vanishes when the contour is closed to the right. The last term has a single pole $p = -i\vec{K}\cdot\vec{v}_1$ to the left of the contour, and the integral may be evaluated by closing the contour to the left. We find the following result

$$\frac{1}{2\pi i}\int_C P_{\vec K}(1|p)\int d\vec{v}_2\, P_{-\vec K}(2|-p)\,\vartheta_{\vec K}(12)\, f(\vec{v}_1) f(\vec{v}_2)\,dp =$$

$$= \frac{i\vec{K}\Psi(K)\cdot\dfrac{\partial f(\vec{v}_1)}{\partial \vec{v}_1}}{|\mathcal{E}(\vec{K},-i\vec{K}\cdot\vec{v}_1)|^2}\int d\vec{v}_2\,\frac{f(\vec{v}_2)}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma}$$

$$-\frac{f(\vec{v}_1)}{\mathcal{E}(-\vec{K}, i\vec{K}\cdot\vec{v}_1)}\int d\vec{v}_2\,\frac{i\vec{K}\Psi(K)\cdot\dfrac{\partial f(\vec{v}_2)}{\partial \vec{v}_2}}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma} + \text{REAL TERMS} \qquad (4.3\text{-}11)$$

The numerator and denominator of the second term of (4.3-11) are multiplied by $\mathcal{E}(\vec{K},-i\vec{K}\cdot\vec{v}_1)$ to obtain

$$\frac{f(\vec{v}_1)}{\varepsilon(-\vec{K}, i\vec{K}\cdot\vec{v}_1)} \int \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_2)}{\partial \vec{v}_2}}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma} d\vec{v}_2 = f(\vec{v}_1) \frac{\int d\vec{v}_2 \frac{i\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_2)}{\partial \vec{v}_2}}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma}}{|\varepsilon(\vec{K}, -i\vec{K}\cdot\vec{v}_1)|^2}$$

$$- f(\vec{v}_1) \frac{\left| \int d\vec{v}_2 \frac{i\vec{K}\psi(K)\cdot\partial f(\vec{v}_2)/\partial \vec{v}_2}{-i\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+\sigma} \right|^2}{|\varepsilon(\vec{K}, -i\vec{K}\cdot\vec{v}_1)|^2} \tag{4.3-12}$$

where the second term of (4.3-12) is a purely real quantity.

The above integrals may be simplified by an integration over the components of velocity perpendicular to the wave vector $\vec{K}$. If the component of $\vec{v}_1$, parallel to $\vec{K}$ is denoted by u_1, then we find

$$\int d\vec{v}_2 \frac{\vec{K}\psi(K)\cdot\frac{\partial f(\vec{v}_2)}{\partial \vec{v}_2}}{\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+i\sigma} = \int du_2 \frac{K\psi(K)\frac{\partial F(u_2)}{\partial u_2}}{K(u_2-u_1)+i\sigma}$$

$$\int d\vec{v}_2 \frac{f(\vec{v}_2)}{\vec{K}\cdot(\vec{v}_2-\vec{v}_1)+i\sigma} = \int du_2 \frac{F(u_2)}{K(u_2-u_1)+i\sigma} \tag{4.3-13}$$

where we have defined the new function of velocity

$$F(u_2) \equiv \int d\vec{v}_2 \, \delta\left(u_2 - \frac{\vec{K}\cdot\vec{v}_2}{K}\right) f(\vec{v}_2) \tag{4.3-14}$$

The Plemelj formula[26]

$$\lim_{\sigma\to 0} \int du \frac{F(u)}{u-v+i\sigma} = \oint du \frac{F(u)}{u-v} - \pi i \int du \, F(u)\, \delta(u-v) \tag{4.3-15}$$

($⨍$ denotes the Cauchy principal value of the integral) is used with the relations (4.3-12) and (4.3-13) to rewrite (4.3-11) as

$$\frac{1}{2\pi i}\int_C dp\, P_{\vec{K}}(1|p)\int d\vec{v}_2\, P_{-\vec{K}}(2|-p)\,\vartheta_{\vec{K}}(12)\, f(\vec{v}_1) f(\vec{v}_2) =$$

$$= \frac{i}{(2\pi)^3}\frac{\pi\,\Psi(K)}{|\varepsilon(\vec{K}, -i\vec{K}\cdot\vec{v}_1)|^2}\left(\frac{\partial f(\vec{v}_1)}{\partial u_1}F(u_1) - f(\vec{v}_1)\frac{\partial F(u_1)}{\partial u_1}\right) \qquad (4.3\text{-}16)$$

$$+ \text{REAL TERMS}$$

where

$$\frac{\partial}{\partial u_1} = \frac{\vec{K}}{K}\cdot\frac{\partial}{\partial \vec{v}_1}$$

The relation (4.3-16) is substituted into equation (4.3-6) for $\partial f(\vec{v}_1)/\partial(\varepsilon\tau)$ to obtain the final result

$$\frac{\partial f(\vec{v}_1)}{\partial(\varepsilon\tau)} = \frac{\partial}{\partial\vec{v}_1}\cdot\int\frac{d\vec{K}}{(2\pi)^3}\frac{\pi\vec{K}\,\Psi^2(K)}{|\varepsilon(\vec{K}, -i\vec{K}\cdot\vec{v}_1)|^2}\left(\frac{\partial f(\vec{v}_1)}{\partial u_1}F(u_1) - f(\vec{v}_1)\frac{\partial F(u_1)}{\partial u_1}\right) \qquad (4.3\text{-}17)$$

in agreement with the kinetic equation of Balescu[19] and Lenard.[20] We note that this is a non-linear equation for the evolution in time of the single-particle function. Whereas the behavior at the N-momentum level of description is linear ((4.2-11) is a linear equation for $F^{N,0}(t)$), the behavior becomes non-linear when the level of description is reduced.

4.4 Discussion

The small parameter ε has been used to order the generalized correlation functions and to determine the long time behavior of a stable plasma. The hierarchy of equations (2-27), truncated at the equation for the two-particle function, was then solved in Section 4.2 by the method of multiple time scales to obtain a linear equation for the time rate of change of the function $\mathcal{F}^{N,0}(t)$ of the N momentum of the system. McCune[12] has pointed out that any function of the total energy of the system is a stationary solution of (4.2-11). Further, he has noted that a large class of these functions of the total energy (subject to the condition that they be normalizable) reduce upon integration over N-1 velocities to a Maxwellian distribution in the limit of large N. We have shown in Section 4.3 that the general kinetic equation (4.2-11) upon integration over N-1 velocities reduces (in the limit $N \rightarrow \infty$) to the Balescu-Lenard equation. The only stationary solution to this reduced equation is the Maxwellian distribution[19], in agreement with the findings of McCune.

Both the Balescu-Lenard equation and the master equation thus predict the irreversible approach of the single-particle function towards a Maxwellian distribution. It should be noted that the set of coupled equations (4.2-7) and (4.2-8) from which this solution was obtained are time reversible. The reversibility was lost when we introduced the assumption that the weakness of the correlations in the plasma ($f^{N,2}(12|t) \sim O(\epsilon)$) led to two widely separated time scales on which events could occur.[12,36] Events which occur on the fast time scale happen so quickly that their asymptotic behavior may be used as "initial" conditions for events which occur on the slow time scale, and the latter appear to the observer to evolve irreversibly in time.

Another derivation of the results of Section 4.2 is possible which brings this approach more into line with the ideas of Prigogine, Resibois and Balescu.[37,9] The derivation begins with an exact solution of the equations (4.2-7) and (4.2-8). The solution of equation (4.2-8) for $g_{\vec{K}}^{N,2}(12|t)$ may be written in terms of the operator $\mathcal{P}_{\vec{K}}(t)$ in the following way.

$$g_{\vec{K}}^{N,2}(12|t) = \mathcal{P}_{\vec{K}}(1|t)\,\mathcal{P}_{-\vec{K}}(2|t)\,g_{\vec{K}}^{N,2}(12|t=0)$$

$$+\int_0^t d\tau\, \mathcal{P}_{\vec{K}}(1|t-\tau)\,\mathcal{P}_{-\vec{K}}(2|t-\tau)\,\vartheta_{\vec{K}}(12)\,F^{N,0}(\tau) \qquad (4.4\text{-}1)$$

The equation (4.2-7) for $F^{N,0}(t)$ now becomes with the substitution of (4.4-1)

$$\frac{\partial F^{N,0}(t)}{\partial t} = \frac{\epsilon}{N}\sum_{i<j}^{\{N\}}\sum L(ij)\,\mathcal{P}_{\vec{K}}(i|t)\,\mathcal{P}_{-\vec{K}}(j|t)\,g_{\vec{K}}^{N,2}(ij|t=0)$$

$$+\frac{\epsilon}{N}\sum_{i<j}^{\{N\}}\sum L(ij)\int_0^t d\tau\, \mathcal{P}_{\vec{K}}(i|t-\tau)\,\mathcal{P}_{-\vec{K}}(j|t-\tau)\,\vartheta_{\vec{K}}(ij)\,F^{N,0}(\tau) \qquad (4.4\text{-}2)$$

The equation (4.4-2) for the time rate of change of the N-momentum function has two characteristics which are typical of a general kinetic equation which is valid for all times (compare with equation (3.2-11) of ref. (37)). First, there is a term which represents the effect at time t of the initial correlations in the plasma. This term is important during the initial stages of the evolution of the plasma. Second, there is a term which is non-Markovian. We see from (4.4-2) that the rate of change of $\mathcal{F}^{N,0}(t)$ at any time t depends upon the values of $\mathcal{F}^{N,0}(t)$ at all earlier times. The non-Markovian behavior arises from the convolution that is a consequence of the finite time of collision between particles and the finite correlation length in the plasma. In order to bring equation (4.4-2) into agreement with equation (4.2-11) of Section 4.2 we need to change (4.4-2) to a Markovian form. This is accomplished by introducing the assumption that the collision time (memory) is sufficiently short that the change of $\mathcal{F}^{N,0}(t)$ during this period may be neglected (compare with the discussion of Grad[38]). With this assumption $\mathcal{F}^{N,0}(\tau)$ is replaced by $\mathcal{F}^{N,0}(t)$ in the more general integral of (4.4-2) and the kinetic equation becomes Markovian. In the limit of large times the initial conditions die out and we may rewrite (4.4-2) as

$$\frac{\partial \mathcal{F}^{N,0}(t)}{\partial t} = \frac{\varepsilon}{N}\int_0^\infty d\tau \sum_{i<j}^{\{N\}}\sum L(ij)\,\mathcal{P}_{\vec{k}}(i|\tau)\,\mathcal{P}_{-\vec{k}}(j|\tau)\,\mathcal{Q}_{\vec{k}}(ij)\,\mathcal{F}^{N,0}(t) \qquad (4.4\text{-}3)$$

in agreement with the general kinetic equation of Section 4.2.

We have discussed in Section 4.3 the reduction of equation (4.2-11) to the Balescu-Lenard equation for the single particle function. The reduction was accomplished by rewriting the integral on the right-hand side of (4.2-11) as a convolution integral in the Laplace variable p and making use of the argument that the function $F^{\nu 0}(t)$ in this integral could be written as a product of ν single particle functions $f(t)$. However, note that the integral on the right-hand side of (4.4-2), when rewritten as a convolution integral, contains the function $F^{N,0}(p)$. The reduced function $F^{\nu,0}(p)$ of the Laplace variable p cannot be written as a product of ν single particle functions $f(p)$. However, it is still possible to reduce the level of description of the collision term to obtain an equation for the single particle function. The result is expressed in terms of a convolution integral in time. Thus, the exact solution of the equations (4.2-7) and (4.2-8) does not require that the N momenta be specified as the evolution of the system can still be described in terms of the single-particle function. However, a very detailed description is required as the behavior of the system at any given time is dependent upon its entire past history. To further simplify the description an adiabatic assumption must be introduced to reduce the equation to its familiar Markovian form.

CHAPTER 5

FORMAL SOLUTION OF THE HIERARCHY FOR AN UNSTABLE PLASMA IN THE LIMIT $\varepsilon \to 0$

5.1 Introduction

We consider in the remaining Chapters the problem of a weak instability in a low-density, high-temperature plasma. In such a plasma collisions between particles occur so infrequently that their effect upon the behavior of the gas may often be neglected. For example, the plasma in interplanetary space has a mean free path on the order of 1 astronomical unit and a Debye length (the scale on which collective effects are important) on the order of 10 meters. If such a plasma is unstable the mechanism which limits the instability and brings about its final decay cannot depend upon direct collisions between particles. We propose to use the hierarchy of equations (2-17) for the distribution functions $f^{N,\nu}(\{v\}/t)$ in N-momentum space to study collective interactions in an unstable plasma.

There are two advantages to be gained from a consideration of the problem from the point of view of the equations (2-17). First, the equations are linear. Even the effects of mode coupling are formulated in a linear way, a feature which is in sharp contrast to the non-linear character of the Vlasov equation. Second, the operators $\mathcal{H}^{\nu}$, $\mathcal{M}^{\nu}$ and $\mathcal{L}^{\nu}$ which appear in the equations are independent of time, and the problem of an unstable collisionless plasma may be solved without the introduction of an adiabatic hypothesis or multiple time scales. The disadvantage is that the entire hierarchy of equations must be retained in the analysis. It is not possible to truncate the hierarchy at the ν^{th} member and to consider only the ν lowest equations.

Collective interactions in a plasma have been the object of much research in recent years. Analytic investigations of self-limiting of linearly-unstable plasmas have been confined to the so-called "bump-in-tail" instability in the weakly-unstable limit.[21] In such an unstable plasma the interactions between particles and waves dominate over wave-wave interactions, and the latter may be neglected in the lowest order approximation. The simplified equations of "quasi-linear" theory represent a first correction to the linearized theory and describe the self-limiting of the instability.[23,24] The non-linear mode-coupling terms, which remain small, are treated as a perturbation to the quasi-linear solution and lead to a redistribution of the energy throughout the wave spectrum.[39,40,41] The results of quasi-linear theory are reviewed briefly below. The remainder of this Chapter is devoted to a discussion of the basic assumptions, the simplifications of the hierarchy (2-17) that result from these assumptions and the formal solution of the simplified equations.

We consider in Chapter 6 the problem of reducing the level of description of the solution for $f_{\vec{k}}^{N,1}(1|t)$ from N velocities to that of a single velocity. A hierarchy of operators is uncovered in the course of the reduction. The "reduced" operators are found to be simply related to one another. The form of the solution in the limit of large times is determined in Chapter 7. Further simplifications of the solution are found in Chapter 8 by taking the limit that the initial growth rate of the disturbance is very small. The equation for the spatially-homogeneous function $F^{1,0}(t)$ is found, after these simplifications, to be a diffusion-type of equation in agreement with the results of quasi-linear theory. A method is proposed in Chapter 9 for obtaining a simple expression to approximate the growth with time of the energy in the initially most unstable mode in the plasma. The approximate solution is found to agree qualitatively with results obtained by Drummond and Pines[23] from a numerical calculation.

5.2 Review of Quasi-Linear Theory

The quasi-linear theory of unstable plasmas is based upon two assumptions. The first is that the number density is so low and the plasma temperature so high that collisions between particles may be neglected. The second is that the energy in the disturbance is much less than the mean kinetic energy of the particles.

In the limit that the plasma is "collisionless" it is possible to find a "solution" to the BBGKY hierarchy of equations.[26] This special "solution", the non-linear Vlasov equation, is itself an equation for the evolution in time of the single-particle distribution function ($\nu = 1$). It is sometimes referred to as a correlationless kinetic equation because to obtain it correlations between the velocities of different particles must be neglected for all times.

Only one exact solution to the Vlasov equation has been found, that of a steady-state, one-dimensional electrostatic wave in a plasma of ions and electrons.[42] No time-dependent problem has yet been solved exactly. To attack the problem of an unstable disturbance it is customary to separate the single-particle distribution function into a spatially-homogeneous part $g(\vec{v}|t)$ and a spatially inhomogeneous part $f_{\vec{K}}(\vec{v}|t)$. The equations for these functions are[23]

$$\frac{\partial f_{\vec{K}}}{\partial t} + i\vec{K}\cdot\vec{v} f_{\vec{K}} - i\vec{K}\,\psi(K)\cdot\frac{\partial g}{\partial \vec{v}}\int d\vec{v}\, f_{\vec{K}} = \tag{5.2-1}$$

$$= \sum_{\vec{q}\neq 0} i(\vec{K}-\vec{q})\,\psi(|\vec{K}-\vec{q}|)\cdot\frac{\partial f_{\vec{q}}}{\partial \vec{v}}\int d\vec{v}\, f_{\vec{K}-\vec{q}}$$

$$\frac{\partial g}{\partial t} = -\sum_{\vec{q}} i\vec{q}\,\psi(q)\cdot\frac{\partial f_{\vec{q}}}{\partial \vec{v}}\int d\vec{v}\, f_{-\vec{q}} \tag{5.2-2}$$

where we have written the disturbance as a sum of discrete modes. We have further assumed that there is no applied electric field.

There has been as yet no simplification as equation (5.2-1) for $f_{\vec{K}}$ is still non-linear. To simplify the analysis it is necessary to introduce the second of the above two assumptions and to consider a plasma in which the mean kinetic energy of the particles is much greater than the energy of the disturbance. A parameter σ is used to characterize the order of magnitude of the disturbance $\left(f_{\vec{K}} \sim O(\sigma)\right)$. If σ is small then the non-linear term on the right-hand side of (5.2-1) is small compared with the terms on the left. We may then neglect (for times which are not too long) the non-linear terms in (5.2-1) and reduce the formulation to the pair of coupled linear equations (5.2-2) and (5.2-3).

$$\frac{\partial f_{\vec{K}}}{\partial t} + i\vec{K}\cdot\vec{v} f_{\vec{K}} - i\vec{K}\,\psi(K)\cdot\frac{\partial g}{\partial \vec{v}}\int d\vec{v}\, f_{\vec{K}} = 0 \qquad (5.2\text{-}3)$$

However, even the linear "zero order" equations have not been solved exactly because equation (5.2-3) for $f_{\vec{K}}$ contains the operator $i\vec{K}\psi(K)\cdot\frac{\partial g}{\partial \vec{v}}\int d\vec{v}$ which is dependent upon time. The approximate methods which are employed to solve (5.2-3) require either the introduction of an adiabatic hypothesis[21,23,39] or the use of multiple time scales.[43,41] The basic assumption is that for a weak disturbance $\left(f_{\vec{K}} \sim O(\sigma) << 1\right)$ the function g changes very slowly in time so that one may solve equation (5.2-3) for $f_{\vec{K}}$ holding g

constant. Equation (5.2-3), solved to $\mathcal{O}(\sigma^2)$ in the limit of large times (so that the free-streaming $e^{i\vec{k}\cdot\vec{v}t}$ terms damp out), is combined with (5.2-2) to obtain a diffusion equation for g. The diffusion of g in velocity space eventually limits the disturbance to some maximum amplitude.

However, in order for the adopted ordering procedure to be valid for all times, the disturbance must be limited to a small equilibrium amplitude. The requirement that $f_{\vec{k}}$ remain small is used to show that the growth rate γ of the disturbance must be very small ($\gamma \sim \mathcal{O}(\sigma^2)$). If $f_{\vec{k}} \sim \mathcal{O}(\sigma)$ for all times then the non-linear terms of (5.2-1) remain small and may be treated by perturbation methods. The non-linear terms have been found to lead to a redistribution of the energy throughout the wave spectrum and a gradual damping of the disturbance.

The present approach has the advantage over "quasi-linear" theory that the equations for a collisionless plasma can be solved directly without introducing an adiabatic hypothesis or multiple time scales. Further, the equations are linear so that no perturbation methods are required to deal with the mode-coupling terms. We find in Chapters 6 through 8 that the problems of this theory are not involved with the solution of the basic equations but with the reduction of the solution to some simple form. We show that if the initial amplitude of the disturbance is small then many terms may be approximated by their asymptotic forms with a resulting simplification of the solution. Further, we find that the order in which events occur depends upon the growth rate γ. If γ is less than unity then the "mode coupling" terms in the solution do not become important until some long time characterized by $t \sim T$ whereas the "quasi-linear" terms become significant at a shorter time $\tau < T$. We show that if γ is sufficiently small then the quasi-linear readjustment of the plasma takes place completely before the mode-coupling terms enter the solution. In this case the amplitude of the disturbance remains small for all times. The results are consistent with those of "quasi-linear" theory.

5.3 Basic Assumptions

We assume that collisional effects may be neglected and take the limit that the parameter $\varepsilon \to 0$. Terms like $\varepsilon \mathcal{D}^{\nu} f^{N,\nu}$ in the hierarchy equation (2-17) become vanishingly small in this limit. However, the order of magnitude of terms like $\varepsilon \sum_{j}^{\{N-\nu\}} \int \frac{r_D^3 \, dx_j}{V} \mathcal{I}(ij) f^{N,\nu}_{(ij)}$ is (by the arguments of Chapter 2)

$$(N-\nu)\varepsilon \frac{r_D^3}{V} O(f^{N,\nu}) = \frac{N-\nu}{4\pi N} O(f^{N,\nu})$$

and must be retained. The hierarchy becomes for a collisionless plasma

$$\left(\frac{\partial}{\partial t} + \mathcal{H}^{\nu}\right) f^{N,\nu}_{(\{\nu\}|t)} = \mathcal{M}^{\nu}\left(f^{N,\nu+1}_{(\{\nu+1\}|t)}\right) + \mathcal{I}^{\nu}\left(f^{N,\nu+2}_{(\{\nu+2\}|t)}\right) \qquad (5.3\text{-}1)$$

In order to interpret the terms on the right-hand side of (5.3-1) we take a Fourier transform in the spatial variable $\vec{x}_1$ of the equation for $f^{N,1}(1|t)$. We find

$$\left(\frac{\partial}{\partial t}+i\vec{k}\cdot\vec{v}\right)f_{\vec{k}}^{N,1}(1|t)-\frac{i}{N}\vec{k}\,\psi(k)\cdot\sum_j{}'\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_j}\right)f_{\vec{k}}^{N,1}(j|t)=$$

$$=-\frac{i}{(2\pi)^3N}\sum_j{}'\int d\vec{q}\,(\vec{k}-\vec{q})\,\psi(|\vec{k}-\vec{q}|)\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_j}\right)f_{\vec{k}-\vec{q},\vec{q}}^{N,2}(1j|t)$$

$$+\frac{i}{(2\pi)^3N}\,\delta(\vec{k})\sum_j{}'\int d\vec{q}\,(-\vec{q})\,\psi(q)\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_j}\right)f_{-\vec{q},\vec{q}}^{N,2}(1j|t) \tag{5.3-2}$$

$$-\frac{i}{(2\pi)^3N}\sum_{j<\ell}^{\{N-1\}}{}'\int d\vec{q}\,\vec{q}\,\psi(q)\cdot\left(\frac{\partial}{\partial\vec{v}_j}-\frac{\partial}{\partial\vec{v}_\ell}\right)f_{\vec{k},\vec{q},-\vec{q}}^{N,3}(1j\ell|t)$$

The second term on the right-hand side of (5.3-2) (with the $\delta(\vec{k})$) removes the spatially-homogeneous part of the first term from the equation. Since we consider equation (5.3-2) only for non-zero values of $\vec{k}$ $\left(f_{\vec{k}=0}^{N,1}(1|t)=0\right.$ by the condition $\left.\int d\vec{x}_1\,F^{N,1}(1|t)=F^{N,0}(t)\right)$ the second term does not enter the discussion. The first term on the right-hand side of (5.3-2) has an analogous form to the non-linear term on the right-hand side of the Vlasov equation (5.2-1). Similarly, the last term on the right-hand side of (5.3-2) has an analogous form to the term on the right-hand side of equation (5.2-2) for $g(\vec{v}|t)$. We expect that the first term leads to a redistribution of the wave energy and the last term to a diffusion of the distribution function in the (N-1) dimensional velocity space $\vec{v}_2$, $\vec{v}_3$, ..., $\vec{v}_N$. This is found to be the case.

We assume that the plasma is unstable to disturbances whose wave numbers $\vec{K}$ lie within a certain range $\Delta\vec{K}$. The discussion is limited to the "bump-in-tail" instability. Each disturbance is assumed to have a unique initial growth rate $\gamma_{\vec{K}}$ and an initial amplitude characterized by a parameter σ . We have argued in Chapter 3 that if σ is small the terms on the left-hand side of the ν th equation (5.3-1) of the hierarchy are initially an order of magnitude larger than those on the right so that for short times we need solve only the homogeneous equation for each function $f^{N,\nu}(\{\nu\}|t)$. However, we find that $f^{N,\nu}(\{\nu\}|t)$ grows as $e^{\nu\gamma t}$ where γ is a <u>constant</u> and that after a time t the terms of (5.3-1) have the orders of magnitude

$$
\begin{aligned}
\mathcal{H}^{\nu} f^{N,\nu} &\sim O(\sigma e^{\gamma t})^{\nu} \\
\mathcal{M}^{\nu} f^{N,\nu+1} &\sim O(\sigma e^{\gamma t})^{\nu+1} \\
\mathcal{L}^{\nu} f^{N,\nu+2} &\sim O(\sigma e^{\gamma t})^{\nu+2}
\end{aligned}
\qquad (5.3\text{-}3)
$$

When $t \sim -\frac{\ln\sigma}{\gamma}$ the terms on the right-hand side of (5.3-1) are of the same order of magnitude as those on the left and cannot be discarded from each equation. The time $t \sim -\frac{\ln\sigma}{\gamma}$ characterizes the time at which each equation becomes coupled to other equations of the hierarchy.

The smaller the initial amplitude of the disturbance and the smaller its growth rate the longer is the time before the terms which contain $f^{N,\nu+1}$ and $f^{N,\nu+2}$ begin to influence significantly the solution. Since these terms represent the processes of diffusion in velocity space and redistribution of wave energy we expect them to influence the behavior of our system slowly if σ and γ are small.

For long times, therefore, it is not possible to truncate the hierarchy at the ν th equation by discarding the terms which involve $f^{N,\nu+1}$ and $f^{N,\nu+2}$ because $f^{N,\nu}$, which would then grow at the rate $e^{\nu\gamma t}$ indefinitely, can be expected to lead to anomolies in the solution. The problem of an unstable plasma involves the solution of the complete hierarchy of equations (5.3-1) The solution for the single-particle distribution function involves an infinite number of terms, reminiscent of the solution (3.2-29) for the short-time behavior of $f_R(1|t)$.

For the moment we neglect in each equation the terms $\mathcal{M}^{\nu}(f^{N,\nu+1})$ which lead to a spreading of the wave energy. We emphasize that this step is not necessary to obtain a solution of (5.3-1) since the terms which are discarded are linear. Indeed, the contribution of the terms $\mathcal{M}^{\nu}(f^{N,\nu+1})$ is discussed in Section 7.4. However, their inclusion increases the complexity of the solution, and it is desirable to anticipate the quasi-linear result that in a weakly-unstable plasma the diffusion in velocity space takes place quickly compared with the redistribution of wave energy. Thus our attention is confined to some "intermediate" times. We calculate in Chapter 7 the contributions of the terms that are discarded and determine under what conditions they are small and may be neglected.

The ν th equation of our hierarchy now becomes

$$\frac{\partial f^{N,\nu}(\{\nu\}|t)}{\partial t} + K^{\nu} f^{N,\nu}(\{\nu\}|t) - \frac{1}{4\pi N}\sum_{i}^{\{\nu\}}\sum_{j}^{\{N-\nu\}}\int d\vec{x}_j\, I(ij)\, f^{N,\nu}(\{\nu-1\}j|t) =$$

$$= \frac{1}{4\pi N}\sum_{i<j}^{\{N-\nu\}}\sum\int d\vec{x}_i\, \frac{r_D^3\, dx_j}{V}\, I(ij)\, f^{N,\nu+2}(\{\nu\}ij|t) \qquad (5.3\text{-}4)$$

Both sides of (5.3-4) may be integrated over the velocities in the set $\{N-\nu\}$. We may demonstrate the equivalence of the resulting equations with the quasi-linear theory. If, for example, the distribution functions are factored as

$$f^{\nu,\nu}(\{\nu\}|t) = f(1)f(2)\cdots f(\nu)$$

$$f^{\nu+1,\nu}(\{\nu\}|t) = f(1)f(2)\cdots f(\nu)f_0(\nu+1) \qquad (5.3\text{-}5)$$

where

$$f(1) = f(\vec{x}_1, \vec{v}_1, t) \quad , \quad f_0(1) = f_0(\vec{v}_1, t)$$

and substituted into (5.3-4) the equation reduces to a sum of ν equations of the form

$$\frac{\partial f(1)}{\partial t} + \vec{v}_1 \frac{\partial f(1)}{\partial \vec{x}_1} - \frac{N-\nu}{4\pi N}\frac{\partial f_0(1)}{\partial \vec{v}_1}\cdot\int dx_j\, d\vec{v}_j\, \frac{\partial u_{1j}}{\partial \vec{x}_1} f(j) = 0 \qquad (5.3\text{-}6)$$

Similarly, the lowest equation of the hierarchy, when integrated over all velocities except $\vec{v}_1$, becomes

$$\frac{\partial f_0(1)}{\partial t} = \frac{1}{4\pi} \int d\vec{x}_1 \, \frac{r_D^3 dx_j}{V} \, I(1j) f(1) \int d\vec{v}_j \, f(j) \tag{5.3-7}$$

The two equations (5.3-6) and (5.3-7) are identical in the limit $\frac{N-\nu}{N} \to 1$ to the "zero order" equations for quasi-linear theory. Thus the hierarchy (5.3-4) of equations is equivalent to the equations of quasi-linear theory (without the non-linear terms).

However, we do not choose to reduce the level of description at this point to that of a single-particle distribution function. As was pointed out in Section 5.2 the equations of the quasi-linear theory are linear, but the time-dependent operator in the linearized equation (5.3-6) for $f(1)$ requires that some adiabatic hypothesis or multiple time scales be introduced. The only solutions that have been obtained to the equations (5.3-6) and (5.3-7) have been approximate ones. However, we can, by considering the hierarchy (5.3-4) obtain an exact solution to these equations.

5.4 Formal Solution

We present below the formal solution of the equations (5.3-4). Note that in order to solve for the function $f^{N,\nu}(\{\nu\}|t)$ we need to know a function which comes after it in the hierarchy. We take a ν-dimensional Fourier transform in the variables $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_\nu$, and write (we add ν terms of $O(\frac{1}{N})$ as we did in Chapter 3 to write (3.3-14))

$$\left(\frac{\partial}{\partial t} + \mathcal{H}_{\vec{K}_1}(1) + \mathcal{H}_{\vec{K}_2}(2) + \cdots + \mathcal{H}_{\vec{K}_\nu}(\nu)\right) f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t) = \frac{1}{N}\sum_{i<j}^{\{N-\nu\}}\sum L(ij)\, f^{N,\nu+2}_{\vec{K}_1\cdots\vec{K}_\nu\vec{K}_i\vec{K}_j}(\{\nu\}ij|t) \tag{5.4-1}$$

$L(ij)$ operates on a function of $\vec{K}_i$ and $\vec{K}_j$ (see (4.2-6))

$$L(ij) = -\frac{i}{(2\pi)^3}\int d\vec{K}'\,\vec{K}'\,\psi(K)\cdot\left(\frac{\partial}{\partial\vec{\nu}_i} - \frac{\partial}{\partial\vec{\nu}_j}\right)\begin{pmatrix}\vec{K}_i \to \vec{K}' \\ \vec{K}_j \to -\vec{K}'\end{pmatrix} \tag{5.4-2}$$

As was mentioned in Chapter 3 the operators $\mathcal{H}_{\vec{K}_i}$ and $\mathcal{H}_{\vec{K}_j}$ commute to $\mathcal{O}(\frac{1}{N})$ so that we may write the formal solution of (5.4-1) in terms of the operators $\mathcal{P}_{\vec{K}}(t)$.

$$\begin{aligned} f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t) &= \mathcal{P}_{\vec{K}_1}(1|t)\,\mathcal{P}_{\vec{K}_2}(2|t)\cdots\mathcal{P}_{\vec{K}_\nu}(\nu|t)\, f^{N,\nu}_{\vec{K}_1\cdots\vec{K}_\nu}(\{\nu\}|t=0) \\ &+ \frac{1}{N}\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\cdots\mathcal{P}_{\vec{K}_\nu}(\nu|t-\tau)\sum_{i<j}^{\{N-\nu\}}\sum L(ij)\, f^{N,\nu+2}_{\vec{K}_1\cdots\vec{K}_\nu\vec{K}_i\vec{K}_j}(\{\nu\}ij|\tau) \end{aligned} \tag{5.4-3}$$

The first term in the solution for $f^{N,1}(1|t)$ contains the initial value of $f^{N,1}(1|t=0)$ and the second the time dependent function $f^{N,3}(t)$. The latter function may be eliminated from the solution for $f^{N,1}(t)$ by substituting into (5.4-3) the solution for $f^{N,3}(t)$. However, the solution for $f^{N,3}(t)$ has two terms, one containing $f^{N,3}(t=0)$ and the other $f^{N,5}(t)$, and it is necessary to further substitute for the function $f^{N,5}(t)$ and so on. Each substitution for a function $f^{N,\nu}(t)$ introduces a term which contains $f^{N,\nu+2}(t)$.

The solution for $f^{N,1}_{\vec{K}_1}(1|t)$ has an infinite number of terms.

$$f^{N,1}_{\vec{K}_1}(1|t) = \mathcal{P}_{\vec{K}_1}(1|t) f^{N,1}_{\vec{K}_1}(1|t=0) + \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)$$

$$+ \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum L(ij)\int_0^\tau d\tau'\, \mathcal{P}^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau-\tau') \times \qquad (5.4\text{-}4)$$

$$+ \cdots \qquad \times \frac{1}{N}\sum_{\ell<m}^{\{N-3\}}\sum L(\ell m)\,\mathcal{P}^5_{\vec{K}_1\vec{K}_i\vec{K}_j\vec{K}_\ell\vec{K}_m}(1ij\ell m|\tau') f^{N,5}_{\vec{K}_1\vec{K}_i\vec{K}_j\vec{K}_\ell\vec{K}_m}(1ij\ell m|t=0)$$

where we have written $\mathcal{P}^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) = \mathcal{P}_{\vec{K}_1}(1|\tau)\mathcal{P}_{\vec{K}_i}(i|\tau)\mathcal{P}_{\vec{K}_j}(j|\tau)$, etc.. For t very small the convolution integrals approach zero (from (3.3-2), $\mathcal{P}_{\vec{K}_1}(1|t=0) = 1$) and the solution becomes

$$\text{for} \quad t \to 0 \quad f^{N,1}_{\vec{K}_1}(1|t) \simeq \mathcal{P}_{\vec{K}_1}(1|t) f_{\vec{K}_1}(1|t=0) \qquad (5.4\text{-}5)$$

We have shown in Chapter 3 that, when both sides of (5.4-5) are integrated over all velocities except $\vec{v}_1$, the solution reduces to that obtained by Landau. Thus, the leading term in (5.4-4) is a generalization of the Landau result. The rest of the terms, which become important at large times contribute to the quasi-linear behavior. Note further that the solution (5.4-4) contains only the initial values of the generalized correlation functions. Thus, in principle, one can determine from (5.4-4) the evolution in time of the generalized single-particle function from a knowledge of the initial values of the generalized ν-particle. However, the solution for $f^{N,1}_{\vec{K}}(1|t)$ represents too much information as it contains all N velocities. Having the general solution it is useful to reduce our description to the quantity

$f_{\vec{R}}(1|t)$, obtained from $f_{\vec{R}}^{N,1}(1|t)$ by an integration over all velocities except $\vec{v}_1$. The reduction of the level of description of the solution (5.4-4) is the subject of the next chapter.

CHAPTER 6

REDUCTION OF THE LEVEL OF DESCRIPTION OF THE SOLUTION

6.1 Introduction

The solution for the generalized single-particle function as written in the form (5.4-4) at the end of Chapter 5 is unwieldy because of the large number of velocity coordinates which are present. Indeed, as discussed in Chapter 3 the specification of the initial values of the functions present on the right-hand side of equation (5.4-4) is out of the question. However, we can integrate both sides of (5.4-4) over all velocities except $\vec{v}_1$, to obtain an expression for the single-particle function $f_{\vec{K}}(1|t)$. The reduction of the level of description of (5.4-4) is the main concern of this Chapter.

6.2 Operators

We discussed in Chapter 3 integration of the quantity $\mathcal{P}_{\vec{K}}(1|t) f_{\vec{K}}^{N,1}(1|t=0)$ over all velocities except $\vec{v}_1$. The result is $\mathcal{T}_{\vec{K}}(1|t) f_{\vec{K}}(1|t=0)$ where $\mathcal{T}_{\vec{K}}(1|t)$ propagates the function $f_{\vec{K}}(1|t=0)$ in time in agreement (to $O(\frac{1}{N})$) with the solution of the linearized Vlasov equation obtained by Landau. The result that $\mathcal{P}_{\vec{K}}(1|t)$ reduces to the operator $\mathcal{T}_{\vec{K}}(1|t)$ upon an integration over N-1 velocity coordinates depends crucially upon the properties of the function $f_{\vec{K}}^{N,1}(1|t=0)$. This function, when integrated over the velocities in the set $\{N-S\}$, becomes

$$\int (d\vec{v})^{N-S} f_{\vec{K}}^{N,1}(1|t=0) = f_{\vec{K}}^{S,1}(1|t=0) \tag{6.2-1}$$

where the function $f_{\vec{K}}^{S,1}(1|t=0)$ is a function of velocity which is symmetric to the interchange of any two velocities in the set $\{S-1\}$. However, the operator $\mathcal{P}_{\vec{K}}(1|t)$ may not necessarily be followed by a function of this particular form. For instance, in the second term of the solution for $f_{\vec{K}}^{N,1}(1|t)$ we find the quantity

$$\frac{1}{N} \sum_{i<j}^{\{N\}}\sum L(ij) \mathcal{P}_{\vec{K}_1}(1|T) \mathcal{P}_{\vec{K}_i}(i|T) \mathcal{P}_{\vec{K}_j}(j|T) f_{\vec{K}_1\vec{K}_i\vec{K}_j}^{N,3}(1ij|t=0) \tag{6.2-2}$$

It is important to remember that the operator $L(ij)$ contains derivatives with respect to $\vec{v}_i$ and $\vec{v}_j$. With the condition that the functions $f^{N,\nu}(\{v\}|t)$ vanish as $\vec{v}_i \rightarrow \infty$ we find by integration by parts

$$\int d\vec{v}_i\, d\vec{v}_j\, L(ij) \mathcal{P}_{\vec{K}_1}(1|T) \mathcal{P}_{\vec{K}_i}(i|T) \mathcal{P}_{\vec{K}_j}(j|T) f_{\vec{K}_1\vec{K}_i\vec{K}_j}^{N,3}(1ij|t=0) = 0 \tag{6.2-3}$$

The expression (6.2-2) may be integrated over the velocities in the set $\{N-S\}$ to obtain

$$\int (d\vec{v})^{N-S} \frac{1}{N} \sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_j}(j|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) =$$

$$= \frac{1}{N} \sum_{i<j}^{\{S-1\}}\sum L(ij) \int (d\vec{v})^{N-S}\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_j}(j|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)$$

$$+ \frac{N-S}{N} \sum_{i}^{\{S-1\}} \int (d\vec{v})^{N-S} L(i\,s{+}1)\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_{S+1}}(s{+}1|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1i\,s{+}1|t=0) \tag{6.2-4}$$

The first term on the right-hand side of (6.2-4) contains S^2 terms of $\mathcal{O}(\frac{1}{N})$, the second $S(N-S)$ terms of $\mathcal{O}(\frac{1}{N})$. If $S << N$ the second terms dominate the first and we find

$$\int (d\vec{v})^{N-S} \frac{1}{N} \sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_j}(j|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) =$$

$$= \sum_{i}^{\{S-1\}} \int (d\vec{v})^{N-S} L(i\,s{+}1)\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_{S+1}}(s{+}1|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1i\,s{+}1|t=0) + \mathcal{O}(\tfrac{S}{N}) \tag{6.2-5}$$

The right-hand side of (6.2-5) contains (S -1) different terms. If each term is examined individually we find that it is <u>not</u> symmetric to the interchange of the velocity $\vec{v}_i$ with any other velocity in the set $\{S-1\}$. Nevertheless, the right-hand side of (6.2-5), taken as a whole, is symmetric to the interchange of any two velocities in the set $\{S-1\}$, by virtue of the sum. Finally, we note from (6.2-3) that the i th term vanishes upon an integration over the velocity $\vec{v}_i$.
Thus if we write

$$\int (d\vec{v})^{N-S} \frac{1}{N} \sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}_{\vec{K}_1}(1|T)\,\mathcal{P}_{\vec{K}_i}(i|T)\,\mathcal{P}_{\vec{K}_j}(j|T)\, f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \equiv \equiv \sum_{i}^{\{S-1\}} h_1^{S,1}(1|i) \tag{6.2-6}$$

the function $h_1^{S,1}(1|i)$ has the properties that (a) it is not symmetric to the interchange of the velocity $\vec{v}_i$ with any other velocity in the set $\{S-1\}$ and (b) the integral of $h_1^{S,1}(1|i)$ over the velocity $\vec{v}_i$ vanishes.

We show that the operator $\mathcal{P}_{\vec{K}}(1|t)$, when followed by a function $\gamma_1^{N,1}(1)$ which has the property

$$\int (d\vec{v})^{N-S} \gamma_1^{N,1}(1) = \gamma_1^{S,1}(1) = \sum_i^{\{S-1\}} h_1^{S,1}(1|i) \tag{6.2-7}$$

does not reduce to the Landau operator $P_{\vec{K}}(1|t)$ upon an integration over the velocities in the set $\{S-1\}$. The expression (3.3-5) for $\mathcal{P}_{\vec{K}}(1|p)$ is used to write

$$\int (d\vec{v})^{N-1} \mathcal{P}_{\vec{K}}(1|p)\, \gamma_1^{N,1}(1) =$$

$$= \int (d\vec{v})^{N-1} \Bigg\{ \frac{\gamma_1^{N,1}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \sum_i \frac{\frac{i}{N} D_{\vec{K}}(1i)}{p+i\vec{K}\cdot\vec{v}_1} \frac{\gamma_1^{N,1}(i)}{p+i\vec{K}\cdot\vec{v}_i}$$
$$+ \sum_i \frac{\frac{i}{N} D_{\vec{K}}(1i)}{p+i\vec{K}\cdot\vec{v}_1} \sum_j \frac{\frac{i}{N} D_{\vec{K}}(ij)}{p+i\vec{K}\cdot\vec{v}_i} \frac{\gamma_1^{N,1}(j)}{p+i\vec{K}\cdot\vec{v}_j} + \cdots \Bigg\} \tag{6.2-8}$$

where the integration $(d\vec{v})^{N-1}$ is over all velocities except $\vec{v}_1$. The first term vanishes by the condition

$$\int (d\vec{v})^{N-1} \gamma_1^{N,1}(1) = 0 \tag{6.2-9}$$

which follows from (6.2-7) and property (b) of $h_1^{S,1}(1|i)$. The second term does not vanish; we find

$$\int (d\vec{v})^{N-1} \sum_i \frac{\frac{i}{N} D_{\vec{K}}(1i)}{p+i\vec{K}\cdot\vec{v}_i} \frac{\gamma_1^{N,1}(1)}{p+i\vec{K}\cdot\vec{v}_1} = \frac{i D_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} \int d\vec{v}_i \frac{h_1^{2,1}(1|i)}{p+i\vec{K}\cdot\vec{v}_i} \tag{6.2-10}$$

where the condition that the function $\gamma_1^{N,1}(1)$ vanishes at the boundaries of velocity space has been used. The third term when integrated over all velocities except $\vec{v}_1$ reduces to

$$\int (d\vec{v})^{N-1} \sum_i \frac{\frac{i}{N} D_{\vec{K}}(1i)}{p+i\vec{K}\cdot\vec{v}_1} \sum_j \frac{\frac{i}{N} D_{\vec{K}}(ij)}{p+i\vec{K}\cdot\vec{v}_i} \frac{\gamma_1^{N,1}(j)}{p+i\vec{K}\cdot\vec{v}_j} =$$

$$= \frac{i D_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} \int d\vec{v}_i\, d\vec{v}_j \frac{i D_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i} \frac{\gamma_1^{3,1}(j)}{p+i\vec{K}\cdot\vec{v}_j} \tag{6.2-11}$$

However, $\gamma_1^{3,1}(j)$ is composed of two terms, and we find from $\gamma_1^{3,1}(j) = h_1^{3,1}(j|1) + h_1^{3,1}(j|i)$ that (6.2-11) becomes

$$\frac{i D_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} \int d\vec{v}_i\, d\vec{v}_j \frac{i D_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i} \frac{h_1^{3,1}(j|1)}{p+i\vec{K}\cdot\vec{v}_j}$$

$$+ \frac{i D_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} \int d\vec{v}_i\, d\vec{v}_j \frac{i D_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i} \frac{h_1^{3,1}(j|i)}{p+i\vec{K}\cdot\vec{v}_j} \tag{6.2-12}$$

The fourth term of (6.2-8), since it involves the function $\gamma_1^{4,1}(\ell)$, consists of three terms, two of which are identical.

$$\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell \frac{iD_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\,\frac{iD_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}\,\frac{\gamma_1^{4,1}(\ell)}{p+i\vec{K}\cdot\vec{v}_\ell} =$$

$$= \frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell \frac{iD_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\,\frac{iD_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}\,\frac{h_1^{4,1}(\ell|1)}{p+i\vec{K}\cdot\vec{v}_\ell} \tag{6.2-13}$$

$$+2\,\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell \frac{iD_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\,\frac{iD_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}\,\frac{h_1^{4,1}(\ell|i)}{p+i\vec{K}\cdot\vec{v}_\ell}$$

We will show (see also Section 3 of Appendix C) that it is possible to write $h_1^{3,1}(j|1)$ as the product of the functions $h_1^{2,1}(j|1)$ and $\varphi(i)$. Similarly, we can factor $h_1^{4,1}(\ell|1)$ as

$$h_1^{4,1}(\ell|1) = h_1^{2,1}(\ell|1)\,\varphi(i)\,\varphi(j) \tag{6.2-14}$$

The results (6.2-14) are substituted above to obtain

$$\int (d\vec{v})^{N-1}\, P_{\vec{K}}(1|p)\,\gamma_1^{N,1}(1) =$$

$$= \frac{iD_{\vec{K}}(1)}{p+i\vec{K}_1\cdot\vec{v}_1}\int d\vec{v}_i \frac{h_1^{2,1}(i|1)}{p+i\vec{K}\cdot\vec{v}_i}\left(1+L(\vec{K},p)+L^2(\vec{K},p)+\cdots\right) \tag{6.2-15}$$

$$+\frac{iD_{\vec{K}}(1)\,\varphi(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_1\, d\vec{v}_i \frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\,\frac{h_1^{2,1}(i|1)}{p+i\vec{K}\cdot\vec{v}_i}\left(1+2L(\vec{K},p)+3L^2(\vec{K},p)+\cdots\right)$$

where we have once again used

$$L(\vec{K},p) = \int d\vec{v}_1 \frac{iD_{\vec{K}}(1)\,\varphi(1)}{p+i\vec{K}\cdot\vec{v}_1} \tag{6.2-16}$$

The two series on the right-hand side of (6.2-14) can be summed and the result written in terms of $\mathcal{E}(\vec{K},p) = 1 - L(\vec{K},p)$ as

$$\int (d\vec{v})^{N-1} \mathcal{P}_{\vec{K}}(1|p)\, \gamma_1^{N,1}(1) = S_{\vec{K}}(1,i|p)\, h_1^{2,1}(i|1) \tag{6.2-17}$$

where we have introduced the new operator

$$S_{\vec{K}}(1,i|p) = \left(\frac{i\mathcal{D}_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{i\mathcal{D}_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1 \frac{i\mathcal{D}_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\mathcal{E}(\vec{K},p)} \right) \frac{\int d\vec{v}_i \frac{1}{p+i\vec{K}\cdot\vec{v}_i}}{\mathcal{E}(\vec{K},p)} \tag{6.2-18}$$

We note that $S_{\vec{K}}(1,i|p)$ may be written as the product of three operators.

$$S_{\vec{K}}(1,i|p) = \mathcal{P}_{\vec{K}}(1|p) \int d\vec{v}_i\, \mathcal{P}_{\vec{K}}(i|p)\, i\mathcal{D}_{\vec{K}}(1) \tag{6.2-19}$$

The time behavior of the operator $S_{\vec{K}}(1,i|t)$ is closely allied with that of $\mathcal{P}_{\vec{K}}(1|t)$. The primary difference is that in $S_{\vec{K}}(1,i|p)$ the poles $(\mathcal{E}(\vec{K},p))^{-1}$ are double poles. If the growth rate for an unstable mode is $\gamma_{\vec{K}}$, then in the limit as $t \to \infty$ the first term of the operator $S_{\vec{K}}(1,i|t)$ grows as $e^{\gamma_{\vec{K}} t}$ and the second term as $t e^{\gamma_{\vec{K}} t}$.

The operator $S_{\vec{K}}(1,i|p)$ was derived on the basis that it was possible to write $h_1^{S,1}(1|i)$ as the product of $h_1^{2,1}(1|i)$ and $S-1$ functions $\varphi(j)$. We show that this factorization follows from the factorization of the initial conditions.

We have from (6.2-5) and (6.2-6)

$$h_1^{S,1}(1|i) =$$

$$= \int d\vec{v}_{S+1} L(i\,S+1) \int (d\vec{v})^{N-S-1} \mathcal{P}_{\vec{K}_1}(1|t) \mathcal{P}_{\vec{K}_i}(i|t) \mathcal{P}_{\vec{K}_{S+1}}(S+1|t) f^{N,3}_{\vec{K}_1 \vec{K}_i \vec{K}_{S+1}}(1\,i\,S+1|t=0) \qquad (6.2\text{-}20)$$

We discuss in Appendix C the product of a single operator $\mathcal{P}_{\vec{K}}(1|t)$ and a function $f^{N,1}_{\vec{K}}(1)$ integrated over all but S velocities. The extension to the case of three operators is straightforward. The methods of Section 2.4 of Appendix C are used to write

$$\int (d\vec{v})^{N-S-1} \mathcal{P}_{\vec{K}_1}(1|t) \mathcal{P}_{\vec{K}_i}(i|t) \mathcal{P}_{\vec{K}_{S+1}}(S+1|t) f^{N,3}_{\vec{K}_1 \vec{K}_i \vec{K}_{S+1}}(1\,i\,S+1|t=0) =$$

$$= \mathcal{P}_{\vec{K}_1}(1|t) \mathcal{P}_{\vec{K}_i}(i|t) \mathcal{P}_{\vec{K}_{S+1}}(S+1|t) f^{S+1,3}_{\vec{K}_1 \vec{K}_i \vec{K}_{S+1}}(1\,i\,S+1|t=0) \qquad (6.2\text{-}21)$$

If the function $f^{S+1,3}_{\vec{K}_1 \vec{K}_i \vec{K}_{S+1}}(1\,i\,S+1|t=0)$ is written as the product of $f^{3,3}_{\vec{K}_1 \vec{K}_i \vec{K}_{S+1}}(1\,i\,S+1|t=0)$ and $S-2$ functions of velocity $\varphi(j)$ and the result (6.2-21) substituted into (6.2-20) we find

$$h_1^{S,1}(1|2) =$$

$$= \left(\int d\vec{v}_j L(2j) \mathcal{P}_{\vec{K}_1}(1|t) \mathcal{P}_{\vec{K}_2}(2|t) \mathcal{P}_{\vec{K}_j}(j|t) f^{3,3}_{\vec{K}_1 \vec{K}_2 \vec{K}_j}(12j|t=0) \right) \varphi(3)\, \varphi(4) \cdots \varphi(S) \qquad (6.2\text{-}22)$$

thus justifying the statements made earlier that $h_1^{3,1}(1|i)$ and $h_1^{4,1}(1|i)$ could be written in the form (6.2-14).

In addition to the function $\gamma_1^{S,1}(1)$ which comes from the second term in the solution for $f^{N,1}_{\vec{K}}(1|t)$ we find in the third term a function of the form

$$\int (d\vec{v})^{N-S} \gamma_2^{N,1}(1) = \gamma_2^{S,1}(1) = \sum_{i \neq j}^{\{S-1\}}\sum h_2^{S,1}(1|i,j) \qquad (6.2\text{-}23)$$

where the function $h_2^{S,1}(1|i,j)$ has two unsymmetric indices in the set $\{S-1\}$. This function is not, in general, symmetric to an interchange of the velocities $\vec{v}_i$ and $\vec{v}_j$. The appropriate symmetry for $h_2^{S,1}(1|ij)$ is obtained through the double summation in the indices i and j over the set $\{S-1\}$. The function $h_2^{S,1}(1|i,j)$ has the property that

$$\int d\vec{v}_i \, h_2^{S,1}(1|i,j) = \int d\vec{v}_j \, h_2^{S,1}(1|i,j) = 0 \qquad (6.2\text{-}24)$$

The operator $\mathcal{P}_{\vec{K}}(1|p)$ when followed by a function of the form of $\gamma_2^{N,1}(1)$ reduces to yet another operator after an integration over all velocities except $\vec{v}_1$. To demonstrate this result we note that

$$\int (d\vec{v})^{N-1} \frac{\gamma_2^{N,1}(1)}{p+i\vec{K}\cdot\vec{v}_1} = \frac{1}{p+i\vec{K}\cdot\vec{v}_1} \int (d\vec{v})^{N-1} \gamma_2^{N,1}(1) = 0$$

$$\int (d\vec{v})^{N-1} \sum_i \frac{\frac{i}{N} D_{\vec{K}}(1i)}{p+i\vec{K}\cdot\vec{v}_1} \frac{\gamma_2^{N,1}(i)}{p+i\vec{K}\cdot\vec{v}_i} = \frac{i D_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} \int (d\vec{v})^{N-1} \frac{\gamma_2^{N,1}(i)}{p+i\vec{K}\cdot\vec{v}_i} = 0$$

(6.2-25)

by the property (6.2-24). Thus, the first two terms in the expansion $\int (d\vec{v})^{N-1} \mathcal{P}_{\vec{K}}(1|p)\gamma_2^{N,1}(1)$ vanish. The third term does not vanish. We find

$$\frac{i\mathcal{D}_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, \frac{i\mathcal{D}_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\;\frac{h_2^{3,1}(j|1,i)+h_2^{3,1}(j|i,1)}{p+i\vec{K}\cdot\vec{v}_j} \tag{6.2-26}$$

The fourth term is

$$\frac{i\mathcal{D}_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell\, \frac{i\mathcal{D}_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\;\frac{i\mathcal{D}_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}\times$$

$$\times\left\{\frac{\begin{array}{l} h_2^{4,1}(\ell|1,i)+h_2^{4,1}(\ell|1,j)+h_2^{4,1}(\ell|i,j) \\ +h_2^{4,1}(\ell|i,1)+h_2^{4,1}(\ell|j,1)+h_2^{4,1}(\ell|j,i)\end{array}}{p+i\vec{K}\cdot\vec{v}_\ell}\right\} \tag{6.2-27}$$

It is in all cases possible to write the function $h_2^{4,1}(1|i,j)$ as the product of $h_2^{3,1}(1|i,j)$ and the function $\varphi(\ell)$. We continue to higher terms in the expansion of $\mathcal{P}_{\vec{K}}(1|p)$ to obtain the following result

$$\begin{aligned}
\int (d\vec{v})^{N-1}\mathcal{P}_{\vec{K}}(1|p)\gamma_2^{N,1}(1) &= \frac{i\mathcal{D}_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_\ell\, \frac{i\mathcal{D}_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\;\frac{h_2^{3,1}(\ell|1,i)+h_2^{3,1}(\ell|i,1)}{p+i\vec{K}\cdot\vec{v}_\ell}\times \\
&\qquad \times\left(1+2L(\vec{K},p)+3L^2(\vec{K},p)+\cdots\right) \\
&+\frac{i\mathcal{D}_{\vec{K}}(1)\varphi(1)}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell\, \frac{i\mathcal{D}_{\vec{K}}(i)}{p+i\vec{K}\cdot\vec{v}_i}\;\frac{i\mathcal{D}_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}\;\frac{h_2^{3,1}(\ell|i,j)+h_2^{3,1}(\ell|j,i)}{p+i\vec{K}\cdot\vec{v}_\ell}\times \\
&\qquad \times\left(1+3L(\vec{K},p)+4\cdot 3L^2(\vec{K},p)+\cdots\right) \\
&= S_{\vec{K}}^{(1)}(1,\ell|j|p)\left(h_2^{3,1}(\ell|1,j)+h_2^{3,1}(\ell|j,1)\right)
\end{aligned} \tag{6.2-28}$$

where we have defined the new operator

$$S_{\vec{K}}^{(1)}(1,\ell|j|p) = \left(\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{iD_{\vec{K}}(1)\,\varphi(1)\int d\vec{v}_1 \frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\mathcal{E}(\vec{K},p)}\right) \times \frac{\int d\vec{v}_j \frac{iD_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}}{\mathcal{E}(\vec{K},p)} \; \frac{\int \frac{d\vec{v}_\ell}{p+i\vec{K}\cdot\vec{v}_\ell}}{\mathcal{E}(\vec{K},p)} \tag{6.2-29}$$

Note that we can write

$$S_{\vec{K}}^{(1)}(1,\ell|j|p) = P_{\vec{K}}(1|p)\int d\vec{v}_j\, P_{\vec{K}}(j|p)\int d\vec{v}_\ell\, P_{\vec{K}}(\ell|p)\, iD_{\vec{K}}(1)\, iD_{\vec{K}}(j) \tag{6.2-30}$$

We can continue and consider the function $\gamma_3^{N,1}(1)$ which has the property

$$\int (d\vec{v})^{N-S}\, \gamma_3^{N,1}(1) = \gamma_3^{S,1}(1) = \sum_{i\neq j\neq \ell}^{\{S-1\}}\sum\sum h_3^{S,1}(1|i,j,\ell) \tag{6.2-31}$$

where $h_3^{S,1}(1|i,j,\ell)$ is not symmetric to an interchange of the indices i , j or ℓ or to an interchange of any of these indices with any other index in the set $\{S-1\}$. The appropriate symmetry is obtained only after the triple summation over i , j and ℓ . The above methods may be used to write

$$\int (d\vec{v})^{N-1} P_{\vec{K}}(1|p)\, \gamma_3^{N,1}(1) = 2\, S_{\vec{K}}^{(2)}(1,i|j,\ell|p) \begin{pmatrix} h_3^{4,1}(i|1,j,\ell) + h_3^{4,1}(i|j,1,\ell) \\ + h_3^{4,1}(i|j,\ell,1) \end{pmatrix} \tag{6.2-32}$$

where

$$S_{\vec{K}}^{(2)}(1,i|j,\ell|p) = \left(\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{iD_{\vec{K}}(1)\,\varphi(1)\int d\vec{v}_1 \frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\mathcal{E}(\vec{K},p)}\right) \times \frac{\int d\vec{v}_j \frac{iD_{\vec{K}}(j)}{p+i\vec{K}\cdot\vec{v}_j}}{\mathcal{E}(\vec{K},p)} \; \frac{\int d\vec{v}_\ell \frac{iD_{\vec{K}}(\ell)}{p+i\vec{K}\cdot\vec{v}_\ell}}{\mathcal{E}(\vec{K},p)} \; \frac{\int \frac{d\vec{v}_i}{p+i\vec{K}\cdot\vec{v}_i}}{\mathcal{E}(\vec{K},p)} \tag{6.2-33}$$

The factor 2 in equation (6.2-32) comes from the property that the operator $S_K^{(2)}(1,i|j,\ell|p)$ is symmetric in the indices j and ℓ so that

$$S_{\vec{K}}^{(2)}(1,i|j,\ell|p)h_3^{4,1}(i|1,j,\ell) = S_{\vec{K}}^{(2)}(1,i|j,\ell|p)h_3^{4,1}(i|1,\ell,j) \qquad (6.2\text{-}34)$$

even though $h_3^{4,1}(i|1,j,\ell)$ is not symmetric to an interchange of these indices. We can also write $S_{\vec{K}}^{(2)}(1,i|j,\ell|p)$ as a product of $P_{\vec{K}}(p)$ and $D_{\vec{K}}$ operators.

$$S_{\vec{K}}^{(2)}(1,i|j,\ell|p)=(i)^3 P_{\vec{K}}(1|p)\int d\vec{v}_j\, P_{\vec{K}}(j|p)\int d\vec{v}_\ell\, P_{\vec{K}}(\ell|p)\int d\vec{v}_i\, P_{\vec{K}}(i|p) D_{\vec{K}}(1) D_{\vec{K}}(j) D_{\vec{K}}(\ell) \qquad (6.2\text{-}35)$$

We can continue to generate even higher operators $S_{\vec{K}}^{(n)}(1,i|j,\ell,\cdots,n|p)$. The operators may be arranged in a hierarchy with $P_{\vec{K}}(1|p)$ as the lowest member, $S_{\vec{K}}(1,i|p)$ as the second member, $S_{\vec{K}}^{(1)}(1,i|j|p)$ the third, and so on. These operators have been tabulated for reference in Table I.

6.3 Equations for Operators

Each operator of the hierarchy is related to another operator by a differential equation. We start with the operator $P_{\vec{K}}(1|p)$ and multiply it by $p+i\vec{K}\cdot\vec{v}_1$ to obtain

$$(p+i\vec{K}\cdot\vec{v}_1)P_{\vec{K}}(1|p) = 1+iD_{\vec{K}}(1)\varphi(1)\,\frac{\int\frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1}}{\varepsilon(\vec{K},p)} \qquad (6.3\text{-}1)$$

If the operator $P_{\vec{K}}(1|p)$ is integrated over the velocity $\vec{v}_1$ we find

$$\int d\vec{v}_1 \mathcal{P}_{\vec{K}}(1|p) = \int \frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1} + \left(\int d\vec{v}_1 \frac{iD_{\vec{K}}(1)\varphi(1)}{p+i\vec{K}\cdot\vec{v}_1}\right) \frac{\int \frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1}}{\mathcal{E}(\vec{K},p)}$$

$$= \frac{\int \frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1}}{\mathcal{E}(\vec{K},p)} \tag{6.3-2}$$

where we have used the definition (3.2-24) of the function $\mathcal{E}(\vec{K},p)$. We see that the second term of (6.3-1) may be written $iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1 \mathcal{P}_{\vec{K}}(1|p)$ and equation (6.3-1) becomes

$$(p+i\vec{K}\cdot\vec{v}_1)\mathcal{P}_{\vec{K}}(1|p) - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1 \mathcal{P}_{\vec{K}}(1|p) = 1 \tag{6.3-3}$$

The right-hand side of (6.3-3) comes simply from the initial condition for $\mathcal{P}_{\vec{K}}(1|t)$ which is found from the relation for Laplace transforms

$$\lim_{t\to 0} \mathcal{P}_{\vec{K}}(1|t) = \lim_{p\to\infty} p\,\mathcal{P}_{\vec{K}}(1|p) \tag{6.3-4}$$

We find after an inverse Laplace transform

$$\left(\frac{\partial}{\partial t} + H_{\vec{K}}(1)\right)\mathcal{P}_{\vec{K}}(1|t) = 0$$

$$\mathcal{P}_{\vec{K}}(1|t=0) = 1 \tag{6.3-5}$$

where

$$H_{\vec{K}}(1) = i\vec{K}\cdot\vec{v}_1 - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1$$

The operator $\mathcal{P}_{\vec{K}}(1|t)$ satisfies the linearized Vlasov equation (see Chapter 3) where we have linearized about the distribution function at $t = 0$.

The same procedure may be used to obtain equations for other operators of the hierarchy. Thus, if we multiply $S_{\vec{K}}(1,i|p)$ by $p + i\vec{K}\cdot\vec{v}_1$ and rearrange terms we find

$$\left(p + H_{\vec{K}}(1)\right) S_{\vec{K}}(1,i|p) = i\widetilde{\mathcal{P}}_{\vec{K}}(i|p)\mathcal{D}_{\vec{K}}(1) \tag{6.3-6}$$

where we have introduced the notation

$$\widetilde{\mathcal{P}}_{\vec{K}}(1|p) = \int d\vec{v}_1\, \mathcal{P}_{\vec{K}}(1|p) \tag{6.3-7}$$

The right-hand side of equation (6.3-6) for $S_{\vec{K}}(1,i|p)$ contains a source term which is the product of the differential operator $i\mathcal{D}_{\vec{K}}(1)$ and the density operator $\widetilde{\mathcal{P}}_{\vec{K}}(i|p)$. The initial value of $S_{\vec{K}}(1,i|t)$ is found from a relation similar to (6.3-4) to be

$$S_{\vec{K}}(1,i|t=0) = 0 \tag{6.3-8}$$

We find upon taking the inverse Laplace transform of (6.3-6) and using (6.3-8)

$$\left(\frac{\partial}{\partial t} + H_{\vec{K}}(1)\right) S_{\vec{K}}(1,i|t) = \widetilde{\mathcal{P}}_{\vec{K}}(i|t)\, i\mathcal{D}_{\vec{K}}(1) \tag{6.3-9}$$

The solution of the homogeneous part of the equation (6.3-9) (which is $\mathcal{P}_{\vec{K}}(1|t)S_{\vec{K}}(1,i|t=0)$) vanishes by the initial condition (6.3-8). The particular solution of (6.3-9) is the only non-zero contribution.

The same type of result is found for other operators of the hierarchy. We may write in general:

$$\left(\frac{\partial}{\partial t}+H_{\vec{K}}(1)\right)S^{(n)}_{\vec{K}}(1,i|j,\ell,\cdots,n|t)=\tilde{S}^{(n-1)}_{\vec{K}}(n,i|j,\ell,\cdots,n-1|t)i\mathcal{D}_{\vec{K}}(1)$$

$$S^{(n)}_{\vec{K}}(1,i|j,\ell,\cdots,n|t=0) = 0 \qquad (6.3\text{-}10)$$

The source term in the equation for $S^{(n)}_{\vec{K}}(1,i|j,\ell,\cdots,n|t)$ is the product of the differential operator $i\mathcal{D}_{\vec{K}}(1)$ and the next lowest operator of the hierarchy. ($S^{(n-1)}_{\vec{K}}(n,i|j,\ell,\cdots,n-1|t)$) integrated over the velocity $\vec{v}_n$. Each operator of our hierarchy is coupled to the one which comes just before it, so that the hierarchy is built up from the bottom. The lowest operator $\mathcal{P}_{\vec{K}}(1|t)$ is uniquely determined by the differential equation and initial condition (6.3-5). This operator is then used to determine the higher operator $S_{\vec{K}}(1,i|t)$ from the differential equation (6.3-9) and the initial condition (6.3-8). The operator $S_{\vec{K}}(1,i|t)$ is then used to determine the next operator $S^{(1)}_{\vec{K}}(1,i|j|t)$ and so on.

The reduction of many terms in the solution is not quite as straightforward as we have indicated above. The reason for the difficulty is that the solution (5.4-4) contains not only a term where the single operator $\mathcal{P}_{\vec{K}}(1|t)$ operates on a function $\gamma_1^{N,1}(1)$ but also terms in which a product of these operators $\mathcal{P}_{\vec{K}_1}(1|t)\mathcal{P}_{\vec{K}_2}(2|t)\cdots\mathcal{P}_{\vec{K}_\nu}(\nu|t)$ is followed by a function of ν wave vectors $\gamma_1^{N,\nu}(12\cdots\nu)$. The reduced function $\gamma_1^{S,\nu}(\{\nu\})$ has the same properties as does $\gamma_1^{N,1}(1)$, i.e.

$$\gamma_1^{S,\nu}(\{\nu\}) = \sum_i^{\{S-\nu\}} h_1^{S,\nu}(\{\nu\}|i) \qquad (6.3\text{-}11)$$

so that the term when reduced contains both the operators $\mathcal{P}_{\vec{K}}(1|t)$ and $S_{\vec{K}}(1,i|t)$. However, now there will be ν such terms instead of one as we obtained earlier. The techniques that are used to handle these more complicated terms are discussed in Appendix C.

6.4 Factorization of Initial Conditions

We assume in all that is written below and in the following Chapters that the initial conditions may be factored. We have discussed in Chapter 3 the reasons for writing $f^{S,\nu}(\{\nu\}|t=0)$ as the product of $S-\nu$ functions of velocity $\phi(j)$ and a function $f^{\nu,\nu}(\{\nu\}|t=0)$ of ν spatial and velocity coordinates. We now assume further that the initial correlations between particles vanish, and that we may write $f^{\nu,\nu}(\{\nu\}|t=0)$ as a product of ν single-particle functions. This assumption represents a limitation upon the analysis analogous to that imposed by the use of the Mayer cluster expansion (see ref. (26) and Chapter 1) in the BBGKY hierarchy. However, here we impose the restriction only as an <u>initial</u> condition. We show in Chapter 7 directly from the solution that in a collisionless plasma two particles which are statistically independent initially, remain so in the limit of large times.

6.5 Reduced Solutions

The first three terms of the solutions for the single particle function $f_{\vec{K}_1}(1|t)$ and the two-particle function $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$ are written below.

$$
\begin{aligned}
f_{\vec{K}_1}(1|t) &= P_{\vec{K}_1}(1|t) f_{\vec{K}_1}(1|t=0) + \int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau) \int d\vec{v}_j\, L(1j)\, P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \\
&+ \int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau) \int d\vec{v}_j\, L(1j)\, \times \\
&\times \left\{ \begin{aligned}
& P^2_{\vec{K}_i\vec{K}_j}(ij|\tau) f^2_{\vec{K}_i\vec{K}_j}(ij|t=0) \int_0^\tau d\tau'\, S_{\vec{K}_1}(1,\ell|\tau-\tau') \int d\vec{v}_m\, L(1m)\, P^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|\tau') f^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|t=0) \\
&+ P^2_{\vec{K}_1\vec{K}_j}(1j|\tau) f^2_{\vec{K}_1\vec{K}_j}(1j|t=0) \int_0^\tau d\tau'\, S_{\vec{K}_i}(i,\ell|\tau-\tau') \int d\vec{v}_m\, L(im)\, P^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|\tau') f^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|t=0) \\
&+ P^2_{\vec{K}_1\vec{K}_i}(1i|\tau) f^2_{\vec{K}_1\vec{K}_i}(1i|t=0) \int_0^\tau d\tau'\, S_{\vec{K}_j}(j,\ell|\tau-\tau') \int d\vec{v}_m\, L(jm)\, P^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|\tau') f^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|t=0)
\end{aligned} \right\} \\
&+ \int_0^t d\tau\, S^{(1)}_{\vec{K}_1}(1,i|\ell|t-\tau) \int d\vec{v}_j\, L(1j)\, \times \qquad\qquad (6.5\text{-}1) \\
&\times \left\{ \begin{aligned}
& P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \int_0^\tau d\tau' \int d\vec{v}_m\, L(\ell m)\, P^2_{\vec{K}_\ell\vec{K}_m}(\ell m|\tau') f^2_{\vec{K}_\ell\vec{K}_m}(\ell m|t=0) \\
&+ \int d\vec{v}_m\, L(\ell m)\, P^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|\tau) f^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|t=0) \int_0^{\tau'} d\tau'\, P^2_{\vec{K}_1\vec{K}_j}(1j|\tau') f^2_{\vec{K}_1\vec{K}_j}(1j|t=0)
\end{aligned} \right\} \\
&+ \cdots
\end{aligned}
$$

$$
\begin{aligned}
f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t) &= \mathcal{P}^{2}_{\vec{K}_1\vec{K}_2}(12|t)\, f^{2}_{\vec{K}_1\vec{K}_2}(12|t=0) \\
&+\int_0^t d\tau \begin{bmatrix} \mathcal{P}_{\vec{K}_2}(2|t-\tau) S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij) \\ +\mathcal{P}_{\vec{K}_1}(1|t-\tau) S_{\vec{K}_2}(2,i|t-\tau)\int d\vec{v}_j\, L(2j) \end{bmatrix} \mathcal{P}^{4}_{\vec{K}_1\cdots\vec{K}_j}(12ij|\tau)\, f^{4}_{\vec{K}_1\cdots\vec{K}_j}(12ij|t=0) \\
&+\Bigg\{ \int_0^t d\tau\, \mathcal{P}_{\vec{K}_2}(2|t-\tau)\, S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij)\,\times \\
&\quad\times \begin{bmatrix}
\mathcal{P}^{3}_{\vec{K}_2\vec{K}_i\vec{K}_j}(2ij|\tau) f^{3}_{\vec{K}_2\vec{K}_i\vec{K}_j}(2ij|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_1}(1,\ell|\tau-\tau')\int d\vec{v}_m\, L(im)\, \mathcal{P}^{3}_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|\tau') f^{3}_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|t=0) \\
+\mathcal{P}^{3}_{\vec{K}_1\vec{K}_2\vec{K}_j}(12j|\tau) f^{3}_{\vec{K}_1\vec{K}_2\vec{K}_j}(12j|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_i}(i,\ell|\tau-\tau')\int d\vec{v}_m\, L(im)\, \mathcal{P}^{3}_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|\tau') f^{3}_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|t=0) \\
+\mathcal{P}^{3}_{\vec{K}_1\vec{K}_2\vec{K}_i}(12i|\tau) f^{3}_{\vec{K}_1\vec{K}_2\vec{K}_i}(12i|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_j}(j,\ell|\tau-\tau')\int d\vec{v}_m\, L(jm)\, \mathcal{P}^{3}_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|\tau') f^{3}_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|t=0) \\
+\mathcal{P}^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_2}(2,\ell|\tau-\tau')\int d\vec{v}_m\, L(2m)\, \mathcal{P}^{3}_{\vec{K}_2\vec{K}_\ell\vec{K}_m}(2\ell m|\tau') f^{3}_{\vec{K}_2\vec{K}_\ell\vec{K}_m}(2\ell m|t=0)
\end{bmatrix} \\
&+\int_0^t d\tau\, \mathcal{P}_{\vec{K}_2}(2|t-\tau)\, S^{(1)}_{\vec{K}_1}(1,i|\ell|t-\tau)\int d\vec{v}_j\, L(ij)\,\times \qquad\qquad (6.5\text{-}2) \\
&\quad\times \begin{bmatrix}
\mathcal{P}^{4}_{\vec{K}_1\cdots\vec{K}_j}(12ij|\tau) f^{4}_{\vec{K}_1\cdots\vec{K}_j}(12ij|t=0)\int_0^\tau d\tau'\int d\vec{v}_m\, L(\ell m)\, \mathcal{P}^{2}_{\vec{K}_\ell\vec{K}_m}(\ell m|\tau')\, f^{2}_{\vec{K}_\ell\vec{K}_m}(\ell m|t=0) \\
+\int d\vec{v}_m\, L(\ell m)\, \mathcal{P}^{4}_{\vec{K}_2\cdots\vec{K}_m}(2i\ell m|\tau)\, f^{4}_{\vec{K}_2\cdots\vec{K}_m}(2i\ell m|t=0)\int_0^\tau d\tau'\, \mathcal{P}^{2}_{\vec{K}_1\vec{K}_j}(1j|\tau')\, f^{2}_{\vec{K}_1\vec{K}_j}(1j|t=0)
\end{bmatrix} \Bigg\} \\
&+\Big\{ 1 \longleftrightarrow 2 \Big\} \\
&+\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij)\, S_{\vec{K}_2}(2,\ell|t-\tau)\int d\vec{v}_m\, L(2m)\,\times \\
&\quad\times \begin{bmatrix}
\mathcal{P}^{4}_{\vec{K}_1\cdots\vec{K}_\ell}(1ij\ell|\tau) f^{4}_{\vec{K}_1\cdots\vec{K}_\ell}(1ij\ell|t=0)\int_0^\tau d\tau'\, \mathcal{P}^{2}_{\vec{K}_2\vec{K}_m}(2m|\tau')\, f^{2}_{\vec{K}_2\vec{K}_m}(2m|t=0) \\
+\mathcal{P}^{4}_{\vec{K}_2\cdots\vec{K}_m}(2i\ell m|\tau) f^{4}_{\vec{K}_2\cdots\vec{K}_m}(2i\ell m|t=0)\int_0^\tau d\tau'\, \mathcal{P}^{2}_{\vec{K}_1\vec{K}_j}(1j|\tau')\, f^{2}_{\vec{K}_1\vec{K}_j}(1j|t=0)
\end{bmatrix} \\
&+ \cdots
\end{aligned}
$$

The term $\{1 \leftrightarrow 2\}$ in the solution for $f^{2,2}_{\vec{K}_1 \vec{K}_2}(12|t)$ is identical to the quantity within the brackets $\{\ \}$ which precedes it except that the indices 1 and 2 are to be interchanged.

The second term in the solution (6.5-1) for $f_{\vec{K}_1}(1|t)$ is characterized by a single integration over time. The third has five different parts, each of which contains two integrations over time. Three different operators are present in this term. The fourth term (not shown) has 46 parts, each of which has three integrations over time. The operator $S^{(2)}_{\vec{K}_1}(1,i|j,\ell|t-\tau)$ appears for the first time in this term. While the formal solutions (6.5-1) and (6.5-2) are valid for all times, they are unwieldy. We show in the next Chapter that many simplifications may be made in the limit as time becomes large.

CHAPTER 7

TIME - ASYMPTOTIC BEHAVIOR OF THE SOLUTION

7.1 Introduction

The solution (6.5-1) for the single-particle distribution function is a general solution of the hierarchy of equations (5.3-4). We study below the long time behavior of (6.5-1). If the plasma is unstable each term of the solution has a part which grows exponentially with time and dominates the remaining parts in the limit as time becomes large. The form of the solution can be greatly simplified if only the dominant part of each term is included.

We determine in Section 7.2 the asymptotic form of the solutions for the single-particle function $f_{\vec{k}_1}(1|t)$ and the two-particle function $f^{2,2}_{\vec{k}_1\vec{k}_2}(12|t)$. We then show in Section 7.3 that (time-asymptotically) the solution for $f_{\vec{k}}(1|t)$ satisfies a linearized form of the Vlasov equation and that the two-particle function is equal to a product of two single-particle functions (two particles which were initially statistically uncorrelated remain so). We note in Section 7.4 that the terms of (6.5-1) which represent corrections to the first (Landau) term remain small for some short time (to be defined) if the amplitude σ of the initial disturbance is small. Finally, we observe that the assumptions (a) the plasma is "collisionless", and (b) the redistribution of energy throughout the wave spectrum may be neglected have been used to derive the hierarchy (5.3-4). Assumption (b) is not a necessary one. Without it a solution for $f_{\vec{K}}(1|t)$ could still be written in a form similar to (6.5-1). However, the solution would be even more involved in that the number of terms would be greatly increased. In order to avoid the complications that arise from the increased number of terms we have made assumption (b) and have argued in Chapter 5 that the solution should be valid for some "intermediate" times. We show that assumption (b) requires that the initial growth rate of the disturbance be sufficiently small.

7.2 Time-Asymptotic Form of the Solution

The first term in the solution (6.5-1) is identical to the Landau result. We assume that for each wave number $\vec{K}$ in a certain range $\Delta\vec{K}$ there is a single mode which grows exponentially with time. This mode is represented by one of the zeros of $\mathcal{E}(\vec{K},p)$. The remaining zeros of $\mathcal{E}(\vec{K},p)$ (for the wave number $\vec{K}$) are assumed to lie in the left-half p -plane and so to represent damped modes. In addition to the collective modes, there is the "free-streaming" mode which arises from the pole at $p=-i\vec{K}_1\cdot\vec{v}_1$. We denote by $R_{\vec{K}_1}(1)$ and $R_{\vec{K}_1}(1|-i\vec{K}_1\cdot\vec{v}_1)$ the residues of $P_{\vec{K}_1}(1|p)f_{\vec{K}_1}(1|t=0)$ at the poles $p_{\vec{K}_1}$ and $-i\vec{K}_1\cdot\vec{v}_1$ respectively, and write the time-asymptotic behavior of the first term as:

$$\lim_{t\to\infty} P_{\vec{K}_1}(1|t) f_{\vec{K}_1}(1|t=0) \simeq R_{\vec{K}_1}(1)\, e^{\rho_{\vec{K}_1} t} + R_{\vec{K}_1}(1|-i\vec{K}_1\cdot\vec{v}_1)\, e^{-i\vec{K}_1\cdot\vec{v}_1 t} \tag{7.2-1}$$

We need below both the result (7.2-1) and the integral of (7.2-1) over the velocity $\vec{v}_1$. If $R_{\vec{K}_1}(1|-i\vec{K}_1\cdot\vec{v}_1)$ is an absolutely integrable function of velocity then the second of the above two terms decays to zero in the limit of large times.[45] The asymptotic behavior (of the integral over $\vec{v}_1$) is determined by the first term, and we find

$$\lim_{t\to\infty} \int d\vec{v}_1\, P_{\vec{K}_1}(1|t) f_{\vec{K}_1}(1|t=0) \simeq \tilde{R}_{\vec{K}_1}\, e^{\rho_{\vec{K}_1} t} \tag{7.2-2}$$

where

$$\tilde{R}_{\vec{K}_1} = \int d\vec{v}_1\, R_{\vec{K}_1}(\vec{v}_1|\rho_{\vec{K}_1}) \tag{7.2-3}$$

The second term of the solution for $f_{\vec{K}_1}(1|t)$ contains the factor

$$\int d\vec{v}_j\, L(1j)\, P_{\vec{K}_1}(1|\tau) f_{\vec{K}_1}(1|t=0)\, P_{\vec{K}_j}(j|\tau) f_{\vec{K}_j}(j|t=0) \tag{7.2-4}$$

which we write in detail as

$$\frac{-i}{(2\pi)^3}\int d\vec{K}'\, \vec{K}'\psi(K')\cdot\frac{\partial}{\partial\vec{v}_1} P_{\vec{K}'}(\vec{v}_1|\tau) f_{\vec{K}'}(1|t=0) \int d\vec{v}_j\, P_{-\vec{K}'}(v_j|\tau) f_{-\vec{K}'}(j|t=0) \tag{7.2-5}$$

The time-asymptotic forms (7.2-1) and (7.2-2) for the operators are used to rewrite (7.2-5) as

$$-\frac{i}{(2\pi)^3}\int d\vec{K}'\,\vec{K}'\,\Psi(K)\cdot\frac{\partial}{\partial\vec{v}_1}\left(\mathcal{R}_{\vec{K}'}(1)\,e^{p_{\vec{K}'}t}+\mathcal{R}_{\vec{K}'}(1|-i\vec{K}'\cdot\vec{v}_1)\,e^{-i\vec{K}'\cdot\vec{v}_1 t}\right)\widetilde{\mathcal{R}}_{-\vec{K}'}\,e^{p_{-\vec{K}'}t} \tag{7.2-6}$$

The factor (7.2-4), when it appears in the solution, is always preceded by the operator $i\mathcal{D}_{\vec{K}_1}(1)$ (see the form of the operator $\mathcal{S}_{\vec{K}_1}(1,i|p)$ in Table 1). Thus, there are some terms which involve the second velocity derivative of the exponential $e^{-i\vec{K}'\cdot\vec{v}_1 t}$, a quantity which grows in time as t^2. However, these terms are integrated over $\vec{K}'$. We require that the residue $\mathcal{R}_{\vec{K}'}(1|-i\vec{K}'\cdot\vec{v}_1)$ (which contains the initial value function $f_{\vec{K}'}(1|t=0)$) be a sufficiently smooth function of $\vec{K}'$ that the second term of (7.2-6) vanish (after the integration over $\vec{K}'$) in the limit of large times. We note here explicitly that in order for this to occur there must be a <u>spectrum</u> of unstable modes. Then:

$$\begin{aligned}
&\lim_{t\to\infty}\vec{K}\cdot\frac{\partial}{\partial\vec{v}_1}\int d\vec{v}_j\,L(ij)\,\mathcal{P}_{\vec{K}_1}(1|t)\,f_{\vec{K}_1}(1|t=0)\,\mathcal{P}_{\vec{K}_j}(j|t)\,f_{\vec{K}_j}(j|t=0)\simeq\\
&\simeq-\frac{i}{(2\pi)^3}\,\vec{K}\cdot\frac{\partial}{\partial\vec{v}_1}\int d\vec{K}'\,\vec{K}'\,\Psi(K')\cdot\frac{\partial}{\partial\vec{v}_1}\,\mathcal{R}_{\vec{K}'}(1)\,\widetilde{\mathcal{R}}_{-\vec{K}'}\,e^{2\gamma_{\vec{K}'}t}\\
&\equiv\vec{K}\cdot\frac{\partial}{\partial\vec{v}_1}\,C(1|t)
\end{aligned}\tag{7.2-7}$$

where we have used:

$$\begin{aligned}
p_{\vec{K}} &= i\omega_{\vec{K}}+\gamma_{\vec{K}}\\
p_{-\vec{K}} &= -i\omega_{\vec{K}}+\gamma_{\vec{K}}
\end{aligned}\tag{7.2-8}$$

The relation (7.2-8) between $p_{\vec{K}}$ and $p_{-\vec{K}}$ follows from the definition of $\varepsilon(\vec{K},p)$.

The function $C(1|t)$, which depends only upon the velocity $\vec{v}_1$ and time, is a fundamental unit of our solution. In order to write $C(1|t)$ in a more explicit form we note that the residue of $\left(\varepsilon(\vec{K},p)\right)^{-1}$ at the pole $p=p_{\vec{K}}$ (assumed to be a simple pole) is $\left(\partial\varepsilon/\partial p\right)^{-1}_{p=p_{\vec{K}}} \equiv \left(\partial\varepsilon/\partial p_{\vec{K}}\right)^{-1}$ so that

$$R_{\vec{K}'}(1) = \frac{i\vec{K}'\psi(K')\cdot\frac{\partial\varphi}{\partial\vec{v}_1}\int d\vec{v}_1\,\frac{f_{\vec{K}'}(1|t=0)}{p_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}}{(p_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1)\left(\frac{\partial\varepsilon}{\partial p_{\vec{K}'}}\right)} \tag{7.2-9}$$

and (using $\varepsilon(\vec{K},p_{\vec{K}})=0$)

$$\tilde{R}_{\vec{K}'} = \frac{\int d\vec{v}_1\,\frac{f_{\vec{K}'}(1|t=0)}{p_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}}{\frac{\partial\varepsilon}{\partial p_{\vec{K}'}}} \tag{7.2-10}$$

If the relations (7.2-9) and (7.2-10) are substituted into (7.2-7) we find

$$C(1|t) = \frac{1}{(2\pi)^3}\frac{\partial}{\partial\vec{v}_1}\cdot\int d\vec{K}'\,\frac{\vec{K}'\vec{K}'\cdot\frac{\partial\varphi}{\partial\vec{v}_1}}{p_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}\,\psi^2(K')\,\frac{\left|\int d\vec{v}_1\,\frac{f_{\vec{K}'}(1|t=0)}{p_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}\right|^2}{\left|\frac{\partial\varepsilon}{\partial p_{\vec{K}'}}\right|^2}\,e^{2\gamma_{\vec{K}'}t} \tag{7.2-11}$$

$$\equiv \int d\vec{K}'\,C_{\vec{K}'}(1)\,e^{2\gamma_{\vec{K}'}t}$$

The asymptotic behavior of the second term in the solution for $f_{\vec{K}_1}(1|t)$ is determined by the asymptotic behavior of the product of $C(1|\tau)$ and $P_{\vec{K}_i}(i|\tau) f_{\vec{K}_i}(i|t=0)$. Thus

$$\lim_{t\to\infty}\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij)\, P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau)\, f^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \simeq$$
$$\simeq \int d\vec{K}'\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\, R_{\vec{K}_i}(i)\, C_{\vec{K}'}(1)\, e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})t} \tag{7.2-12}$$

To evaluate the convolution integral we take a Laplace transform in time and rewrite (7.2-12) as

$$\int d\vec{K}'\, \frac{S_{\vec{K}_1}(1,i|p)}{p-(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})}\, R_{\vec{K}_i}(i)\, C_{\vec{K}'}(1) \tag{7.2-13}$$

The method of residues is used to take the inverse Laplace transform of (7.2-13). We have noted in Chapter 6 that the unstable poles of the operator $S_{\vec{K}_1}(1,i|p)$ grow in time as $e^{\rho_{\vec{K}_1}t}$ and $te^{\rho_{\vec{K}_1}t}$. However, in the limit of large times the poles of $S_{\vec{K}_1}(1,i|p)$ are dominated by the pole at $p=\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}$ which grows approximately as $e^{3\gamma t}$ (γ is some average growth rate in the $\Delta\vec{K}$ interval), and we find

$$\lim_{t\to\infty}\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij)\, P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau)\, f^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \simeq \tag{7.2-14}$$
$$\simeq \int d\vec{K}'\, S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})\, R_{\vec{K}_i}(i)\, C_{\vec{K}'}(1)\, e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})t}$$

We note that the right-hand side of (7.2-14) does not vanish in the limit as $t \to 0$ as does the left-hand side. If the terms which arise from the poles of $S_{\vec{K}_1}(1,i\,|\,p)$ had been included in (7.2-14), the right-hand side would also vanish. It should be remembered that the relation (7.2-14) is correct only asymptotically in time.

The asymptotic behavior of the third and fourth terms of the solution for $f_{\vec{K}_1}(1\,|\,t)$ are found in exactly the same way as was done above. We see from (6.5-1) that the third term has three parts, each of which is similar in form to the second term just discussed. The third term may be written as:

$$\int_0^t d\tau\, S_{\vec{K}_1}(1,i\,|\,t-\tau)\int d\vec{v}_j\, L(ij)\int d\vec{K}''\, e^{(p_{\vec{K}_1}+p_{\vec{K}_i}+p_{\vec{K}_j}+2\gamma_{\vec{K}''})\tau}\,\times$$
$$\times\Big\{R_{\vec{K}_i}(i)R_{\vec{K}_j}(j)S_{\vec{K}_1}(1,\ell\,|\,p_{\vec{K}_1}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(\ell)\,C_{\vec{K}''}(1)$$
$$+R_{\vec{K}_1}(1)R_{\vec{K}_j}(j)S_{\vec{K}_i}(i,\ell\,|\,p_{\vec{K}_i}+2\gamma_{\vec{K}''})R_{\vec{K}_i}(\ell)\,C_{\vec{K}''}(i) \qquad (7.2\text{-}15)$$
$$+R_{\vec{K}_1}(1)R_{\vec{K}_i}(i)\,S_{\vec{K}_j}(j,\ell\,|\,p_{\vec{K}_j}+2\gamma_{\vec{K}''})R_{\vec{K}_j}(\ell)\,C_{\vec{K}''}(j)\Big\}$$

We note that the operator $L(ij)$ replaces $\vec{K}_i$ by $\vec{K}'$ and $\vec{K}_j$ by $-\vec{K}'$ so that in (7.2-15) $p_{\vec{K}_i}+p_{\vec{K}_j} = 2\gamma_{\vec{K}'}$. If a Laplace transform is used to evaluate the convolution integral in (7.2-15), we find the asymptotic behavior from the pole at $p_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}$ (the operator $S_{\vec{K}}(1,i\,|\,p)$ replaces $\vec{K}_i$ by $\vec{K}_1$) with the result

$$\lim_{t\to\infty}\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\int d\vec{v}_j\, L(ij)\times$$

$$\times\Big\{P^2_{\vec{K}_i\vec{K}_j}(ij|\tau)f^2_{\vec{K}_i\vec{K}_j}(ij|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_1}(1,i|\tau-\tau')\int d\vec{v}_m L(lm)P^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(ilm|\tau')f^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(ilm|t=0)$$

$$+P^2_{\vec{K}_i\vec{K}_j}(ij|\tau)f^2_{\vec{K}_i\vec{K}_j}(ij|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_i}(i,\ell|\tau-\tau')\int d\vec{v}_m L(im)P^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(ilm|\tau')f^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(ilm|t=0)$$

$$+P_{\vec{K}_i\vec{K}_i}(ii|\tau)f^2_{\vec{K}_i\vec{K}_i}(ii|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_j}(j,\ell|\tau-\tau')\int d\vec{v}_m L(jm)P^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(jlm|\tau')f^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(jlm|t=0)\Big\}=$$

$$\simeq\int d\vec{K}'d\vec{K}''\, S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})L_{\vec{K}'}(1)\,e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t}\times$$

$$\times\Big\{R_{\vec{K}_1}(i)\tilde{R}_{-\vec{K}'}\,S_{\vec{K}'}(1,\ell|\rho_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}'}(\ell)\,C_{\vec{K}''}(1)\tag{7.2-16}$$

$$+R_{\vec{K}'}(1)\tilde{R}_{-\vec{K}'}S_{\vec{K}_1}(i,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(\ell)\,C_{\vec{K}''}(i)$$

$$+R_{\vec{K}'}(1)R_{\vec{K}_1}(i)\tilde{S}_{-\vec{K}'}(j,\ell|\rho_{\vec{K}'}+2\gamma_{\vec{K}''})R_{-\vec{K}'}(\ell)C_{\vec{K}''}(j)\Big\}$$

where we have defined (compare with the definition (4.2-6) of $L(ij)$)

$$L_{\vec{K}}(1)\equiv-\frac{i}{(2\pi)^3}\,\vec{K}\,\psi(K)\cdot\frac{\partial}{\partial\vec{v}_1}\tag{7.2-17}$$

The fourth term of (6.5-1) contains the factor

$$\lim_{\tau\to\infty}\int_0^\tau d\tau'\Big\{\int d\vec{v}_j L(ij)P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau-\tau')\int d\vec{v}_m L(\ell m)P^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|\tau')f^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|t=0)$$

$$+\int d\vec{v}_m L(\ell m)P^3_{\vec{K}_\ell\vec{K}_m\vec{K}_i}(\ell mi|\tau-\tau')\int d\vec{v}_j L(ij)P^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m)f^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|t=0)\Big\}$$

$$\simeq \int d\vec{K}' d\vec{K}'' P_{\vec{K}_i}(i|\tau) f_{\vec{K}_i}(i|t=0) \int_0^\tau d\tau' \Big\{ C_{\vec{K}'}(1) e^{2\gamma_{\vec{K}'}\tau} C_{\vec{K}''}(\ell) e^{2\gamma_{\vec{K}''}\tau'}$$

$$+ C_{\vec{K}''}(\ell) e^{2\gamma_{\vec{K}''}\tau} C_{\vec{K}'}(1) e^{2\gamma_{\vec{K}'}\tau'} \Big\}$$

$$\simeq \int d\vec{K}' d\vec{K}'' P_{\vec{K}_i}(i|\tau) f_{\vec{K}_i}(i|t=0)\, C_{\vec{K}'}(1) C_{\vec{K}''}(\ell) e^{(2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\tau} \left(\frac{1}{2\gamma_{\vec{K}'}} + \frac{1}{2\gamma_{\vec{K}''}}\right) \tag{7.2-18}$$

The convolution theorem for Laplace transforms is also used to rewrite the fourth term of the solution for $f_{\vec{K}_1}(1|t)$. The asymptotic behavior is determined by the pole at $p = p_{\vec{K}_1} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''}$, and we find

$$\lim_{t\to\infty} \int_0^t d\tau\, S^{(1)}_{\vec{K}_1}(1,i|\ell|t-\tau) \int_0^\tau d\tau' \Big\{ \int d\vec{v}_j L(1j) P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau-\tau') \int d\vec{v}_m L(\ell m) P^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|\tau') f^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|t=0)$$

$$+ \int d\vec{v}_m L(\ell m) P^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|\tau-\tau') \int d\vec{v}_j L(1j) P^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|\tau') f_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|t=0) \Big\} \tag{7.2-19}$$

$$\simeq \int d\vec{K}' d\vec{K}'' S^{(1)}_{\vec{K}_1}(1,i|\ell|p_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\, e^{(p_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t} P_{\vec{K}_1}(i) C_{\vec{K}'}(1) C_{\vec{K}''}(\ell) \left(\frac{1}{2\gamma_{\vec{K}'}} + \frac{1}{2\gamma_{\vec{K}''}}\right)$$

The above methods may also be used to evaluate many of the terms in the solution for $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$. One new problem appears in the third term of the solution (6.5-2), where we find a convolution in time of the product of the two operators $S_{\vec{K}_1}(1,i|t-\tau) S_{\vec{K}_2}(2,\ell|t-\tau)$ and a function $F_{\vec{K}_1\vec{K}_2}(12,i\ell|\tau)$. The Laplace transform of the product of the two $S_{\vec{K}}(p)$ operators is

$$\int_0^t dt\, e^{-pt} S_{\vec{K}_1}(1,i|t) S_{\vec{K}_2}(2,\ell|t) = \frac{1}{2\pi i}\int_C dp'\, S_{\vec{K}_1}(1,i|p-p') S_{\vec{K}_2}(2,\ell|p') \qquad (7.2\text{-}20)$$

where the contour C passes to the right of the singularities of $S_{\vec{K}_2}(2,\ell|p')$ and to the left of the singularities of $S_{\vec{K}_1}(1,i|p-p')$. The Laplace transform of $F_{\vec{K}_1\vec{K}_2}(12,i\ell|t)$ is of the general form

$$\int_0^\infty e^{-pt} F_{\vec{K}_1\vec{K}_2}(12,i\ell|t)\,dt = F_{\vec{K}_1\vec{K}_2}(12,i\ell|p) = \frac{G_{\vec{K}_1\vec{K}_2}(12,i\ell|p)}{p-p_0} \qquad (7.2\text{-}21)$$

where we have separated from $F_{\vec{K}_1\vec{K}_2}(12,i\ell|p)$ the pole ($p = p_0$) which lies furthest to the right in the p - plane. The long-time behavior of $F_{\vec{K}_1\vec{K}_2}(12,i\ell|t)$ is determined by the pole at $p = p_0$ so that the convolution of the two $S_{\vec{K}}$ operators and $F_{\vec{K}_1\vec{K}_2}(12,i\ell|t)$ becomes, asymptotically in time

$$\lim_{t\to\infty}\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau) S_{\vec{K}_2}(2,\ell|t-\tau) F_{\vec{K}_1\vec{K}_2}(12,i\ell|\tau) \simeq$$

$$\simeq \frac{1}{2\pi i}\int_C dp'\, S_{\vec{K}_1}(1,i|p_0-p') S_{\vec{K}_2}(2,\ell|p') G_{\vec{K}_1\vec{K}_2}(12,i\ell|p_0) e^{p_0 t} \qquad (7.2\text{-}22)$$

We discuss in Section 7.3 and Appendix D the evaluation of convolution integrals of the type appearing on the right of (7.2-22).

To summarize, the asymptotic forms of the solutions for $f_{\vec{K}_1}(1|t)$ and $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$ are written below.

$$\lim_{t\to\infty} f_{\vec{K}_1}(1|t) \simeq R_{\vec{K}_1}(1)e^{\rho_{\vec{K}_1}t} + \int d\vec{K}'\, S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})R_{\vec{K}_1}(i)\, C_{\vec{K}'}(i)\, e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})t}$$

$$+\int d\vec{K}'d\vec{K}''\, e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t}\Big\{ S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})L_{\vec{K}'}(1)\times$$

$$\times\begin{bmatrix} R_{\vec{K}_1}(i)\tilde{R}_{-\vec{K}'}S_{\vec{K}'}(1,\ell|\rho_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}'}(\ell)\,C_{\vec{K}''}(1) \\ +R_{\vec{K}'}(1)\tilde{R}_{-\vec{K}'}S_{\vec{K}_1}(i,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(\ell)\,C_{\vec{K}''}(i) \\ +R_{\vec{K}'}(1)R_{\vec{K}_1}(i)\tilde{S}_{-\vec{K}'}(j,\ell|\rho_{-\vec{K}'}+2\gamma_{\vec{K}''})R_{-\vec{K}'}(\ell)\,C_{\vec{K}''}(j) \end{bmatrix} \qquad (7.2\text{-}23)$$

$$+ S^{(1)}_{\vec{K}_1}(1,i|\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(i)\,C_{\vec{K}'}(1)\,C_{\vec{K}''}(\ell)\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\Big\}$$

$$+\;\cdot\;\cdot\;\cdot$$

$$\lim_{t\to\infty} f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t) \simeq R_{\vec{K}_1}(1)\,R_{\vec{K}_2}(2)\,e^{(\rho_{\vec{K}_1}+\rho_{\vec{K}_2})t}$$

$$+R_{\vec{K}_2}(2)e^{\rho_{\vec{K}_2}t}\Big[\int d\vec{K}'\,e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})t}\,S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})R_{\vec{K}_1}(i)C_{\vec{K}'}(1)$$

$$+\int d\vec{K}'d\vec{K}''e^{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t}\Big\{S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})L_{\vec{K}'}(1)\times$$

$$\times\begin{bmatrix} R_{\vec{K}_1}(i)\tilde{R}_{-\vec{K}'}S_{\vec{K}'}(1,\ell|\rho_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}'}(\ell)\,C_{\vec{K}''}(1) \\ +R_{\vec{K}'}(1)\tilde{R}_{-\vec{K}'}S_{\vec{K}_1}(i,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(\ell)\,C_{\vec{K}''}(i) \\ +R_{\vec{K}'}(1)R_{\vec{K}_1}(i)\tilde{S}_{-\vec{K}'}(j,\ell|\rho_{-\vec{K}'}+2\gamma_{\vec{K}''})R_{-\vec{K}'}(\ell)\,C_{\vec{K}''}(j) \end{bmatrix}$$

$$+ S^{(1)}_{\vec{K}_1}(1,i|\ell|p_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}_1}(i)\,C_{\vec{K}'}(1)\,C_{\vec{K}''}(\ell)\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\Big\}\Big]$$

$$+R_{\vec{K}_1}(1)e^{p_{\vec{K}_1}t}\Big[\int d\vec{K}'\,e^{(p_{\vec{K}_2}+2\gamma_{\vec{K}'})t}\,S_{\vec{K}_2}(2,\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}'})R_{\vec{K}_2}(\ell)\,C_{\vec{K}'}(2)$$

$$+\int d\vec{K}'d\vec{K}''\,e^{(p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t}\Big\{S_{\vec{K}_2}(2,i|p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})L_{\vec{K}'}(2)\times$$

$$\times\begin{bmatrix} R_{\vec{K}_2}(i)\widetilde{R}_{-\vec{K}'}\,S_{\vec{K}'}(2,\ell|p_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}'}(\ell)\,C_{\vec{K}''}(2) \\ +R_{\vec{K}'}(2)\widetilde{R}_{-\vec{K}'}\,S_{\vec{K}_2}(i,\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}''})R_{\vec{K}_2}(\ell)\,C_{\vec{K}''}(i) \\ +R_{\vec{K}'}(2)R_{\vec{K}_2}(i)\widetilde{S}_{-\vec{K}'}(j,\ell|p_{-\vec{K}'}+2\gamma_{\vec{K}''})R_{-\vec{K}'}(\ell)\,C_{\vec{K}''}(j) \end{bmatrix}$$

$$+S^{(1)}_{\vec{K}_2}(2,i|\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})R_{\vec{K}_2}(i)\,C_{\vec{K}'}(2)\,C_{\vec{K}''}(\ell)\left(\frac{1}{2\gamma_{\vec{K}''}}+\frac{1}{2\gamma_{\vec{K}'}}\right)\Big\}\Big] \tag{7.2-24}$$

$$+\frac{1}{2\pi i}\int_C dp'\int d\vec{K}'d\vec{K}''\,e^{(p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})t}\times$$

$$\times\begin{bmatrix} S_{\vec{K}_1}(1,i|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')P_{\vec{K}_2}(2|p')R_{\vec{K}_1}(i)C_{\vec{K}'}(1)S_{\vec{K}_2}(2,\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}''})R_{\vec{K}_2}(\ell)C_{\vec{K}''}(2) \\ +P_{\vec{K}_1}(1|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')S_{\vec{K}_2}(2,\ell|p')R_{\vec{K}_1}(\ell)C_{\vec{K}''}(2)S_{\vec{K}_1}(1,i|p_{\vec{K}_1}+2\gamma_{\vec{K}'})R_{\vec{K}_1}(i)C_{\vec{K}'}(1) \\ +S_{\vec{K}_1}(1,i|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')S_{\vec{K}_2}(2,\ell|p')\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)R_{\vec{K}_1}(i)C_{\vec{K}'}(1)R_{\vec{K}_2}(\ell)C_{\vec{K}''}(2) \end{bmatrix}$$

$$+\ \cdot\ \cdot\ \cdot$$

7.3 Properties of the Solutions

Some of the properties of the solutions (7.2-23) and (7.2-24) for $f_{\vec{K}_1}(1|t)$ and $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$ are determined below. The solution for the single particle function is shown to obey the linearized Vlasov equation (5.3-6) (i.e., the "quasi-linear" version). To obtain this result the solution (7.2-23) is differentiated with respect to time and the following property of the operators (listed in Table I) is used.

$$\frac{\partial}{\partial t} S^{(n)}_{\vec{K}_1}(1,i|j,\ell,\cdots,n|t) + i\vec{K}_1\cdot\vec{v}_1 S^{(n)}_{\vec{K}_1}(1,i|j,\ell,\cdots,n|t) =$$

$$= iD_{\vec{K}_1}(1)\,\varphi(1)\int d\vec{v}_1\, S^{(n)}_{\vec{K}_1}(1,i|j,\ell\cdots,n|t) + iD_{\vec{K}_1}(1)\int d\vec{v}_n\, S^{(n-1)}_{\vec{K}_1}(n,i|j,\ell,\cdots,n-1|t) \tag{7.3-1}$$

A straightforward application of the relation (7.3-1) to the terms of (7.2-23) for $f_{\vec{K}_1}(1|t)$ yields

$$\left(\frac{\partial}{\partial t} + i\vec{K}_1\cdot\vec{v}_1\right) f_{\vec{K}_1}(1|t) + iD_{\vec{K}_1}(1)\,\varphi(1)\int d\vec{v}_1\, f_{\vec{K}_1}(1|t) =$$

$$= iD_{\vec{K}_1}(1)\int_0^t d\tau\int d\vec{v}_i\, P_{\vec{K}_1}(i|t-\tau)\int d\vec{v}_j\, L(1j)\, P^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau)\, f^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)$$

$$+ iD_{\vec{K}_1}(1)\int_0^t d\tau\int d\vec{v}_i\, P_{\vec{K}_1}(i|t-\tau)\int d\vec{v}_j\, L(1j)\times$$

$$\times\begin{bmatrix} P^2_{\vec{K}_i\vec{K}_j}(ij|\tau)\, f^2_{\vec{K}_i\vec{K}_j}(ij|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_1}(1,\ell|\tau-\tau')\int d\vec{v}_m\, L(1m)\, P^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|\tau')\, f^3_{\vec{K}_1\vec{K}_\ell\vec{K}_m}(1\ell m|t=0) \\ + P^2_{\vec{K}_1\vec{K}_j}(1j|\tau)\, f^2_{\vec{K}_1\vec{K}_j}(1j|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_i}(i,\ell|\tau-\tau')\int d\vec{v}_m\, L(im)\, P^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|\tau')\, f^3_{\vec{K}_i\vec{K}_\ell\vec{K}_m}(i\ell m|t=0) \\ + P^2_{\vec{K}_1\vec{K}_i}(1i|\tau)\, f^2_{\vec{K}_1\vec{K}_i}(1i|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_j}(j,\ell|\tau-\tau')\int d\vec{v}_m\, L(jm)\, P^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|\tau')\, f^3_{\vec{K}_j\vec{K}_\ell\vec{K}_m}(j\ell m|t=0) \end{bmatrix}$$

$$+ iD_{\vec{K}_1}(1)\int_0^t d\tau \int d\vec{v}_l\, S_{\vec{K}_1}(l,i|t-\tau)\int d\vec{v}_j\, L(ij)\, \times$$

$$\times \begin{bmatrix} P^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)\int_0^\tau d\tau' \int d\vec{v}_m\, L(lm)\, P^{2}_{\vec{K}_l\vec{K}_m}(lm|\tau') f^{2}_{\vec{K}_l\vec{K}_m}(lm|t=0) \\ + \int d\vec{v}_m\, L(lm) P^{3}_{\vec{K}_i\vec{K}_l\vec{K}_m}(ilm|\tau) f^{3}_{\vec{K}_i\vec{K}_l\vec{K}_m}(ilm|t=0)\int_0^\tau d\tau'\, P^{2}_{\vec{K}_1\vec{K}_j}(1j|\tau') f^{2}_{\vec{K}_1\vec{K}_j}(1j|t=0) \end{bmatrix}$$

$$+ \; . \; . \; . \qquad (7.3\text{-}2)$$

We now show that the terms on the right-hand side of (7.3-2) are equal in the limit $t \to \infty$ to the product of the functions $iD_{\vec{K}_1}(1)\left(f_0(1|t) - \varphi(1)\right)$ and $\int d\vec{v}_1 f_{\vec{K}_1}(1|t)$.

The spatially-independent function $f_0(1|t)$ is obtained from equation (2-9) for $\mathcal{F}^{N,0}(t)$. We integrate both sides of (2-9) over the velocities N-1, substitute the expression (7.2-24) for $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$ into the right-hand side, and integrate with respect to time to obtain

$$f_0(1|t) - \varphi(1) = \int d\vec{v}_2\, L(12)\Big\{\int_0^t d\tau\, P^{2}_{\vec{K}\vec{K}_2}(12|\tau) f^{2}_{\vec{K}_1\vec{K}_2}(12|t=0)$$

$$+ \int_0^t d\tau \Big[P_{\vec{K}_2}(2|\tau) f_{\vec{K}_2}(2|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_1}(1,i|\tau-\tau')\int d\vec{v}_j\, L(ij) P^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) f^{3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0) \qquad (7.3\text{-}3)$$

$$+ P_{\vec{K}_1}(1|\tau) f_{\vec{K}_1}(1|t=0)\int_0^\tau d\tau'\, S_{\vec{K}_2}(2,i|t-\tau)\int d\vec{v}_j\, L(2j) P^{3}_{\vec{K}_2\vec{K}_i\vec{K}_j}(2ij|\tau) f^{3}_{\vec{K}_2\vec{K}_i\vec{K}_j}(2ij|t=0)\Big]$$

$$+ \; . \; . \; .$$

The integral of the function $f_{\vec K_1}(1|t)$ over the velocity $\vec v_1$ is from (6.5-1)

$$\int d\vec v_1 f_{\vec K_1}(1|t) = \int d\vec v_1 P_{\vec K_1}(1|t) f_{\vec K_1}(1|t=0) + \int_0^t d\tau \int d\vec v_1 S_{\vec K_1}(1,i|t-\tau) \int d\vec v_j L(ij) P^3_{\vec K_1 \vec K_i \vec K_j}(1ij|\tau) f^3_{\vec K_1 \vec K_i \vec K_j}(1ij|t=0) + \cdots \qquad (7.3\text{-}4)$$

The relations (7.3-3) and (7.3-4) are used to write the product $iD_{\vec K_1}(1)\left(f_0(1|t) - \varphi(1)\right)\int d\vec v_1 f_{\vec K_1}(1|t)$, and the result is subtracted from (7.3-2) to obtain the following equation.

$$\left(\frac{\partial}{\partial t} + i\vec K_1 \cdot \vec v_1\right) f_{\vec K_1}(1|t) - iD_{\vec K_1}(1) f_0(1|t) \int d\vec v_1 f_{\vec K_1}(1|t) =$$

$$= iD_{\vec K_1}(1) \int_0^t d\tau \int d\vec v_j L(ij) P^2_{\vec K_i \vec K_j}(ij|\tau) f^2_{\vec K_i \vec K_j}(ij|t=0) \times$$

$$\times \left[\begin{array}{l} \int d\vec v_i P_{\vec K_i}(i|t-\tau) \int_0^\tau d\tau' S_{\vec K_1}(i,\ell|\tau-\tau') \int d\vec v_m L(im) P^3_{\vec K_i \vec K_\ell \vec K_m}(i\ell m|\tau') f^3_{\vec K_i \vec K_\ell \vec K_m}(i\ell m|t=0) \\ + \int d\vec v_\ell S_{\vec K_1}(\ell,i|t-\tau) P_{\vec K_i}(i|\tau) f_{\vec K_i}(i|t=0) \int_0^\tau d\tau' \, C(\ell|\tau') \end{array} \right]$$

$$+ iD_{\vec K_1}(1) \int_0^t d\tau \int d\vec v_j L(ij) \int d\vec v_\ell S_{\vec K_1}(\ell,i|t-\tau) P_{\vec K_i}(i|\tau) f_{\vec K_i}(i|t=0) C(\ell|\tau) \int_0^\tau d\tau' P^2_{\vec K_i \vec K_j}(ij|\tau') f^2_{\vec K_i \vec K_j}(ij|t=0)$$

$$- \left(iD_{\vec K_1}(1) \int d\vec v_j L(ij) \int_0^t d\tau P^2_{\vec K_i \vec K_j}(ij|\tau) f^2_{\vec K_i \vec K_j}(ij|t=0) \right) \times \qquad (7.3\text{-}5)$$

$$\times \left(\int_0^t d\tau \int d\vec v_1 S_{\vec K_1}(1,i|t-\tau) \int d\vec v_j L(ij) P^3_{\vec K_1 \vec K_i \vec K_j}(1ij|\tau) f^3_{\vec K_1 \vec K_i \vec K_j}(1ij|t=0) \right)$$

$$+ \cdot \cdot \cdot$$

The terms on the right-hand side of (7.3-5) are evaluated by the methods of Section 7.2. In the limit as $t \to \infty$, we find

$$\text{rhs } (7.3\text{-}5) = iD_{\vec K_1}(1) \int d\vec K' d\vec K'' \, e^{(p_{\vec K_1} + 2\gamma_{\vec K'} + 2\gamma_{\vec K''})t} \times$$

$$\times \Big\{ C_{\vec K'}(1) \tilde P_{\vec K_1}(1|p_{\vec K_1} + 2\gamma_{\vec K'} + 2\gamma_{\vec K''}) S_{\vec K_1}(1,i|p_{\vec K_1} + 2\gamma_{\vec K''}) R_{\vec K_1}(i) C_{\vec K''}(1)$$

$$+ C_{\vec{K}'}(1)\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\tilde{S}_{\vec{K}_1}(1,i\,|\,\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\mathcal{R}_{\vec{K}_1}(i)\,C_{\vec{K}''}(1)$$

$$-\left(C_{\vec{K}'}(1)\right)\left(\tilde{S}_{\vec{K}_1}(1,i\,|\,\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\mathcal{R}_{\vec{K}_1}(i)\,C_{\vec{K}''}(1)\right)\Big\} \qquad (7.3\text{-}6)$$

$+ \quad . \quad . \quad .$

In order to simplify (7.3-6) we introduce the explicit forms for the operators $\mathcal{P}_{\vec{K}_1}(1|\rho)$ and $S_{\vec{K}_1}(1,i|\rho)$ (see Table I). We note that

$$\frac{\displaystyle\int \frac{d\vec{v}_i}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_i}}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})}\,(\vec{K}_i\to\vec{K}_1)\,\mathcal{R}_{\vec{K}_i}(i) =$$

$$= \frac{1}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})}\int\frac{d\vec{v}_i}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_i}\,\frac{\mathcal{D}_{\vec{K}_1}(i)\,\varphi(i)}{\rho_{\vec{K}_1}+i\vec{K}_1\cdot\vec{v}_i}\;\frac{\displaystyle\int d\vec{v}_i\,\frac{f_{\vec{K}_1}(i|t=0)}{\rho_{\vec{K}_1}+i\vec{K}_1\cdot\vec{v}_i}}{(\partial\mathcal{E}/\partial\rho_{\vec{K}_1})}$$

$$= \frac{\displaystyle\int d\vec{v}_i\,\frac{\mathcal{D}_{\vec{K}_1}(i)\varphi(i)}{\rho_{\vec{K}_1}+i\vec{K}_1\cdot\vec{v}_i}-\int d\vec{v}_i\,\frac{\mathcal{D}_{\vec{K}_1}(i)\,\varphi(i)}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_i}}{2\gamma_{\vec{K}''}\;\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})}\;\frac{\displaystyle\int d\vec{v}_i\,\frac{f_{\vec{K}_1}(i|t=0)}{\rho_{\vec{K}_1}+i\vec{K}_1\cdot\vec{v}_i}}{(\partial\mathcal{E}/\partial\rho_{\vec{K}_1})} \qquad (7.3\text{-}7)$$

$$= \frac{\displaystyle\int d\vec{v}_i\,\frac{f_{\vec{K}_1}(i|t=0)}{\rho_{\vec{K}_1}+i\vec{K}_1\cdot\vec{v}_i}}{2\gamma_{\vec{K}''}(\partial\mathcal{E}/\partial\rho_{\vec{K}_1})}$$

where we have used the definition (3.2-24) for $\mathcal{E}(\vec{K},\rho)$ and the condition that $\mathcal{E}(\vec{K},\rho_{\vec{K}})=0$. Equation (7.3-7) can be used to rewrite the first term on the right-hand side of (7.3-6) in the following way.

$$\int d\vec{v}_1\, P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\, S_{\vec{K}_1}(1,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\, R_{\vec{K}_1}(\ell) =$$

$$= \frac{\int \frac{d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})} \left\{ \frac{i D_{\vec{K}_1}(1)}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1} + \frac{i D_{\vec{K}_1}(1)\varphi(1) \int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1)\, \mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})} \right\} \frac{\tilde{R}_{\vec{K}_1}}{2\gamma_{\vec{K}''}}$$

$$= \left\{ \frac{\int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1} - \int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{2\gamma_{\vec{K}'}\; \mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})} \right.$$

$$\left. + \frac{\left(\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}) - \mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\right) \int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{2\gamma_{\vec{K}'}\; \mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\; \mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})} \right\} \frac{R_{\vec{K}_1}}{2\gamma_{\vec{K}''}}$$

$$= \left\{ - \frac{\int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})} + \frac{\int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})} \right\} \frac{\tilde{R}_{\vec{K}_1}}{(2\gamma_{\vec{K}'})(2\gamma_{\vec{K}''})} \qquad (7.3\text{-}8)$$

One can show from a relation similar to (7.3-7) that

$$\frac{\int \frac{i D_{\vec{K}_1}(1)\, d\vec{v}_1}{\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1}}{\mathcal{E}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})} \tilde{R}_{\vec{K}_1} = -(2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}) \int d\vec{v}_1\, S_{\vec{K}_1}(1,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\, R_{\vec{K}_1}(\ell) \qquad (7.3\text{-}9)$$

Equation (7.3-7) is used to rewrite the second term on the right-hand side of (7/3-8) and to obtain

$$\int d\vec{v}_1\, P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\, S_{\vec{K}_1}(1,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\, R_{\vec{K}_1}(\ell) =$$

$$= -\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right) \int d\vec{v}_1\, S_{\vec{K}_1}(1,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})\, R_{\vec{K}_1}(\ell) + \frac{1}{2\gamma_{\vec{K}'}} \int d\vec{v}_1\, S_{\vec{K}_1}(1,\ell|\rho_{\vec{K}_1}+2\gamma_{\vec{K}''})\, R_{\vec{K}_1}(\ell) \qquad (7.3\text{-}10)$$

The two terms (7.3-10) exactly cancel the second and third terms on the right-hand side of (7.3-6). The above methods may be used to reduce other terms on the right-hand side of (7.3-6) and to show that each term is exactly cancelled by some other. We conclude that the right-hand side of (7.3-5) vanishes and write

$$\left(\frac{\partial}{\partial t} + i\vec{K}_1\cdot\vec{v}_1\right) f_{\vec{K}_1}(1|t) - i\vec{K}\,\psi(K)\cdot\frac{\partial f_0(1|t)}{\partial \vec{v}_1}\int d\vec{v}_1\, f_{\vec{K}_1}(1|t) = 0 \qquad (7.3\text{-}11)$$

This is the desired result.

As a further property of our solution, we now demonstrate that two particles remain statistically independent in the limit as $t \to \infty$ (we assumed in Chapter 5 only that they were independent at $t = 0$). The solution (7.2-23) for $f_{\vec{K}_2}(2|t)$ is multiplied by the solution for $f_{\vec{K}_1}(1|t)$ and the product subtracted from the solution (7.2-24) for $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t)$ to obtain

$$\begin{aligned}
f^{2,2}_{\vec{K}_1\vec{K}_2}&(12|t) - f_{\vec{K}_1}(1|t) f_{\vec{K}_2}(2|t) = \\
&= \int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\, C(1|\tau)\, P_{\vec{K}_i}(i|\tau)\, f_{\vec{K}_i}(i|t=0)\, P_{\vec{K}_2}(2|t-\tau) \int_0^\tau d\tau'\, S_{\vec{K}_2}(2,\ell|\tau-\tau')\, C(2|\tau')\, P_{\vec{K}_\ell}(\ell|\tau')\, f_{\vec{K}_\ell}(\ell|t=0) \\
&+ \int_0^t d\tau\, S_{\vec{K}_2}(2,\ell|t-\tau)\, C(2|\tau)\, P_{\vec{K}_\ell}(\ell|\tau)\, f_{\vec{K}_\ell}(\ell|t=0)\, P_{\vec{K}_1}(1|t-\tau) \int_0^\tau d\tau'\, S_{\vec{K}_1}(1,i|\tau-\tau')\, C(2|\tau')\, P_{\vec{K}_i}(i|\tau')\, f_{\vec{K}_i}(i|t=0) \\
&+ \int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\, S_{\vec{K}_2}(2,\ell|t-\tau)\, P^2_{\vec{K}_i\vec{K}_\ell}(i\ell|\tau)\, f^2_{\vec{K}_i\vec{K}_\ell}(i\ell|t=0)\left[C(1|\tau)\int_0^\tau d\tau'\, C(2|\tau') + C(2|\tau)\int_0^\tau d\tau'\, C(1|\tau')\right] \\
&- \left(\int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\, C(1|\tau)\, P_{\vec{K}_i}(i|\tau)\, f_{\vec{K}_i}(i|t=0)\right)\left(\int_0^t d\tau\, S_{\vec{K}_2}(2,\ell|t-\tau)\, C(2|\tau)\, P_{\vec{K}_\ell}(\ell|\tau)\, f_{\vec{K}_\ell}(\ell|t=0)\right) \\
&+ \cdots
\end{aligned} \qquad (7.3\text{-}12)$$

If $f^{(2,2)}_{\vec{K}_1\vec{K}_2}(12|t) = f_{\vec{K}_1}(1|t) f_{\vec{K}_2}(2|t)$ then the particles 1 and 2 are statistically independent (see Chapter 2). The methods of Section 7.2 are used to rewrite (7.3-12) in the limit $t \to \infty$ in the following way

$$
\begin{aligned}
& f_{\vec{K}_1\vec{K}_2}(12|t) - f_{\vec{K}_1}(1|t) f_{\vec{K}_2}(2|t) = \int d\vec{K}' d\vec{K}'' \, e^{(\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''})t} \times \\
& \times \Big\{ \frac{1}{2\pi i} \int_C d\rho' \Big[S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho') R_{\vec{K}_i}(i) C_{\vec{K}'}(1) P_{\vec{K}_2}(2|\rho') S_{\vec{K}_2}(2,\ell|\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''}) R_{\vec{K}_\ell}(\ell) C_{\vec{K}''}(2) \\
& \quad + P_{\vec{K}_1}(1|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho') S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1} + 2\gamma_{\vec{K}'}) R_{\vec{K}_i}(i) C_{\vec{K}'}(1) S_{\vec{K}_2}(2,\ell|\rho') R_{\vec{K}_\ell}(\ell) C_{\vec{K}''}(2) \\
& \quad + \left(\frac{1}{2\gamma_{\vec{K}'}} + \frac{1}{2\gamma_{\vec{K}''}} \right) S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho') R_{\vec{K}_i}(i) C_{\vec{K}'}(1) S_{\vec{K}_2}(2,\ell|\rho') R_{\vec{K}_\ell}(\ell) C_{\vec{K}''}(2) \Big] \\
& \quad - S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1} + 2\gamma_{\vec{K}'}) R_{\vec{K}_i}(i) C_{\vec{K}'}(1) S_{\vec{K}_2}(2,\ell|\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''}) R_{\vec{K}_\ell}(\ell) C_{\vec{K}''}(2) \Big\}
\end{aligned}
\tag{7.3-13}
$$

$+ \cdots$

The sum of the three terms which involve convolution integrals on the right-hand side of (7.3-13) equals the fourth term which contains no convolution integral.

We illustrate the calculation (for more details see Appendix D) by considering the simplest part of each term in (7.3-13). The operators $P_{\vec{K}_1}(1|\rho)$ and $S_{\vec{K}_1}(1,i|\rho)$ consist of two terms (see Table I). If we define the lower case operators $p_{\vec{K}_1}(1|\rho)$ and $s_{\vec{K}_1}(1,i|\rho)$ to be the first part of the operators $P_{\vec{K}_1}(1|\rho)$ and $S_{\vec{K}_1}(1,i|\rho)$, respectively, then

$$\rho_{\vec{K}_1}(1|p) = \frac{1}{p+i\vec{K}_1\cdot\vec{v}_1}$$

$$\mathcal{D}_{\vec{K}_1}(1,i|p) = \frac{iD_{\vec{K}_1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1}\,\frac{\displaystyle\int\frac{d\vec{v}_i}{p+i\vec{K}_i\cdot\vec{v}_i}}{\varepsilon(\vec{K}_1,p)}\,(\vec{K}_i\rightarrow\vec{K}_1) \tag{7.3-14}$$

The simplest part of the first term on the right-hand side of (7.3-13) is then

$$\begin{aligned}\frac{1}{2\pi i}\int_C dp'\,\mathcal{D}_{\vec{K}_1}(1,i|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')\mathcal{R}_{\vec{K}_i}(i)\,C_{\vec{K}'}(1)\,\times\\ \times\rho_{\vec{K}_2}(2|p')\,\mathcal{D}_{\vec{K}_2}(2,\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}''})\mathcal{R}_{\vec{K}_\ell}(\ell)\,C_{\vec{K}''}(2)\end{aligned} \tag{7.3-15}$$

We use a relation similar to (7.3-7) to write

$$\mathcal{D}_{\vec{K}_1}(1,i|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')\mathcal{R}_{\vec{K}_i}(i) = \frac{iD_{\vec{K}_1}(1)\;\widetilde{\mathcal{R}}_{\vec{K}_1}}{(p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p'+i\vec{K}_1\cdot\vec{v}_1)(p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')}$$

$$\mathcal{D}_{\vec{K}_2}(2,\ell|p_{\vec{K}_2}+2\gamma_{\vec{K}''})\mathcal{R}_{\vec{K}_\ell}(\ell) = \frac{iD_{\vec{K}_2}(2)\,\widetilde{\mathcal{R}}_{\vec{K}_2}}{(p_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)(2\gamma_{\vec{K}''})} \tag{7.3-16}$$

The relations (7.3-16) are used to rewrite (7.3-15) as

$$\frac{1}{2\pi i}\int_C dp'\frac{iD_{\vec{K}_1}(1)\widetilde{\mathcal{R}}_{\vec{K}_1}C_{\vec{K}'}(1)\;iD_{\vec{K}_2}(2)\widetilde{\mathcal{R}}_{\vec{K}_2}C_{\vec{K}''}(2)}{(p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p'+i\vec{K}_1\cdot\vec{v}_1)(p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p')(p'+i\vec{K}_2\cdot\vec{v}_2)(p_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)(2\gamma_{\vec{K}''})} \tag{7.3-17}$$

The integration over p' is performed by closing the contour to the left at infinity. The integrand vanishes as $1/p'^3$ along the path of integration at infinity, so the only contribution to the integral is from the pole $p' = -i\vec{K}_2\cdot\vec{v}_2$ inside the contour. We find that (7.3-17) becomes

$$\frac{iD_{\vec K_1}(1)\tilde R_{\vec K_1} C_{\vec K'}(1)\quad iD_{\vec K_2}(2)\tilde R_{\vec K_2} C_{\vec K''}(2)}{(\rho_{\vec K_1}+\rho_{\vec K_2}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_1\cdot\vec v_1+i\vec K_2\cdot\vec v_2)(\rho_{\vec K_2}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_2\cdot\vec v_2)(\rho_{\vec K_2}+2\gamma_{\vec K''}+i\vec K_2\cdot\vec v_2)(2\gamma_{\vec K''})} \tag{7.3-18}$$

The same methods are used to determine the equivalent parts of the second and third terms on the right-hand side of (7.3-13). We find:

$$\frac{iD_{\vec K_1}(1)\tilde R_{\vec K_1} C_{\vec K'}(1)\quad iD_{\vec K_2}(2)\tilde R_{\vec K_2} C_{\vec K''}(2)}{(\rho_{\vec K_1}+\rho_{\vec K_2}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_1\cdot\vec v_1+i\vec K_2\cdot\vec v_2)(\rho_{\vec K_1}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_1\cdot\vec v_1)(\rho_{\vec K_1}+2\gamma_{\vec K'}+i\vec K_1\cdot\vec v_1)(2\gamma_{\vec K'})} \tag{7.3-19}$$

and

$$-\left(\frac{1}{2\gamma_{\vec K'}}+\frac{1}{2\gamma_{\vec K''}}\right)\frac{iD_{\vec K_1}(1)\tilde R_{\vec K_1} C_{\vec K'}(1)\quad iD_{\vec K_2}(2)\tilde R_{\vec K_2} C_{\vec K''}(2)}{(\rho_{\vec K_1}+\rho_{\vec K_2}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_1\cdot\vec v_1+i\vec K_2\cdot\vec v_2)(\rho_{\vec K_2}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_2\cdot\vec v_2)(\rho_{\vec K_2}+i\vec K_2\cdot\vec v_2)} \tag{7.3-20}$$

$$+\left(\frac{1}{2\gamma_{\vec K'}}+\frac{1}{2\gamma_{\vec K''}}\right)\frac{iD_{\vec K_1}(1)\tilde R_{\vec K_1} C_{\vec K'}(1)\quad iD_{\vec K_2}(2)\tilde R_{\vec K_2} C_{\vec K''}(2)}{(\rho_{\vec K_1}+2\gamma_{\vec K'}+2\gamma_{\vec K''}+i\vec K_1\cdot\vec v_1)(2\gamma_{\vec K'}+2\gamma_{\vec K''})(\rho_{\vec K_2}+i\vec K_2\cdot\vec v_2)}$$

The sum of the three quantities (7.3-18)-(7.3-20) reduces after some algebraic manipulation to the following

$$\frac{iD_{\vec K_1}(1)\tilde R_{\vec K_1} C_{\vec K'}(1)\; iD_{\vec K_2}(2)\tilde R_{\vec K_2} C_{\vec K''}(2)}{(2\gamma_{\vec K'})(2\gamma_{\vec K''})(\rho_{\vec K_1}+2\gamma_{\vec K'}+i\vec K_1\cdot\vec v_1)(\rho_{\vec K_2}+2\gamma_{\vec K''}+i\vec K_2\cdot\vec v_2)} \tag{7.3-21}$$

The quantity (7.3-21) is exactly cancelled by the first part of the fourth term on the right-hand side of (7.3-13).

The above techniques can be used to evaluate other parts of the terms in (7.3-13). We show in Appendix D that some parts of the first three terms cancel. The inverse Laplace transform of the remaining parts may be easily evaluated by the method of residues. We find, after some algebra, that the first three terms of (7.3-13) exactly cancel the fourth. The same methods can be used to reduce other terms of the expression for $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t) - f_{\vec{K}_1}(1|t) f_{\vec{K}_2}(2|t)$. Once again, corresponding terms are found to cancel one another. We conclude that $f^{2,2}_{\vec{K}_1\vec{K}_2}(12|t) = f_{\vec{K}_1}(1|t) f_{\vec{K}_2}(2|t)$, and that no correlation between the particles 1 and 2 arises if the initial correlation is zero.

7.4 Order of Magnitude Analysis

An order of magnitude analysis is used to estimate the conditions for which the asymptotic solution (7.2-23) for $f_{\vec{K}}(1|t)$ is both a valid and a useful approximation to the solution of the complete hierarchy of equations (2-17) in the limit $\mathcal{E} \to 0$. We have noted that the Landau solution, $\mathcal{P}_{\vec{K}}(1|t) f_{\vec{K}}(1|t=0)$, has a part which, for an unstable plasma, grows exponentially and, after a time, dominates the remaining parts (which are either exponentially damped or oscillate as $e^{i\vec{K}\cdot\vec{v}_1 t}$). If γ represents an average growth rate of the disturbance then one can characterize by $1/\gamma$ the time at which $\mathcal{R}_{\vec{K}}(1)\, e^{p_{\vec{K}} t}$ becomes a good approximation to the Landau solution. Thus, we need only require that $t > \frac{1}{\gamma}$ for the asymptotic form of each term in the solution for $f_{\vec{k}}(1|t)$ to be a good approximation to the exact term.

However, in order for the asymptotic solution to be a <u>useful</u> approximation, the adjustment through wave-wave interactions of the plasma to the presence of the disturbance must not begin on any significant scale until some time after $t \sim \frac{1}{\gamma}$. In other words, the terms of (7.2-23) which represent corrections to the Landau solution must be sufficiently small that they can be neglected for times $t < \frac{1}{\gamma}$. If such is not the case then part of the reaction of the plasma through wave-wave coupling takes place during the short times $t < \frac{1}{\gamma}$. The form of the solution cannot be greatly simplified for these times as the mode-coupling mechanism does not dominate over the initial transients in the plasma.

The complete hierarchy of equations (2-17) contains, in the limit $\varepsilon \to 0$, a term of the form $\mathcal{M}^{\nu}(f^{N,\nu+1})$ which we have discarded in writing the hierarchy (5.3-4). We have anticipated in Chapter 5 that the contribution of this term to the solution would, for a certain class of problems, be small for some times of interest. To confirm this view, we solve the complete hierarchy (2-17) in the limit $\varepsilon \to 0$ and use order of magnitude arguments to show that the terms which appear in addition to those shown in (7.2-23) are small for some "intermediate" interval of time.

7.4.1 Diffusion Terms

We consider the integral of $f_{\vec{K}}(1|t)$ over the velocity $\vec{v}_1$, a quantity related to the electric field $\vec{E}_{\vec{K}}(t)$. The first term of the solution (7.2-23) becomes

$$\int d\vec{v}_1 \, \frac{i D_{\vec{K}}(1)\, \varphi(1)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} \, \tilde{R}_{\vec{K}} \, e^{p_{\vec{K}} t} = \tilde{R}_{\vec{K}} \, e^{p_{\vec{K}} t} \tag{7.4.1-1}$$

andcan be characterized by the order of magnitude

$$\tilde{R}_{\vec{K}} \, e^{p_{\vec{K}} t} \sim \sigma \, e^{\gamma t} \tag{7.4.1-2}$$

The parameter σ comes from the initial value function $f_{\vec{K}}(1|t=0)$ present in $\tilde{R}_{\vec{K}}$. In order to characterize the order of magnitude of the second term of the solution (7.4.1-2) we use the relation (7.3-7) to write

$$\tilde{S}_{\vec{K}}(1,i|p_{\vec{K}} + 2\gamma_{\vec{K}'})\, R_{\vec{K}}(i) = \frac{\tilde{R}_{\vec{K}}}{2\gamma_{\vec{K}'}} \, \frac{\int \frac{i D_{\vec{K}}(1)\, d\vec{v}_1}{p_{\vec{K}} + 2\gamma_{\vec{K}'} + i\vec{K}\cdot\vec{v}_1}}{\varepsilon(p_{\vec{K}} + 2\gamma_{\vec{K}'})} \tag{7.4.1-3}$$

If $\gamma_{\vec{K}'}$ is small we may replace

$$\begin{aligned} p_{\vec{K}} + 2\gamma_{\vec{K}'} + i\vec{K}\cdot\vec{v}_1 \quad &\text{by} \quad p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1 \\ \varepsilon(p_{\vec{K}} + 2\gamma_{\vec{K}'}) \quad &\text{by} \quad 2\gamma_{\vec{K}'} \frac{\partial \varepsilon}{\partial p_{\vec{K}}} \end{aligned} \tag{7.4.1-4}$$

and write the first terms of the solution for $\vec{E}_{\vec{K}}(t)$ as

$$\vec{E}_{\vec{K}}(t) \simeq i\vec{K}\,\Psi(K)\tilde{R}_{\vec{K}}\,e^{\rho_{\vec{K}}t}\left\{1 + \int d\vec{K}'d\vec{v}_1 \frac{iD_{\vec{K}}(1)C_{\vec{K}'}(1)}{\rho_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}\frac{e^{2\gamma_{\vec{K}'}t}}{(2\gamma_{\vec{K}'})^2\frac{\partial\varepsilon}{\partial\rho_{\vec{K}}}} + \cdots\right\} \quad (7.4.1\text{-}5)$$

The second term within the brackets is simplified by substituting (7.2-11) for $C_{\vec{K}'}(1)$ and using the Plemelj formula (4.3-15) to evaluate the integral over $\vec{K}'$ in the limit $\gamma_{\vec{K}'} \to 0$. Taking note that, aside from the factor $(\rho_{\vec{K}'} + i\vec{K}'\cdot\vec{v}_1)^{-1}$, the integrand of (7.4.1-5) is an even function of $\vec{K}'$ we find

$$\int d\vec{K}'C_{\vec{K}'}(1)e^{2\gamma_{\vec{K}'}t} = \frac{\partial}{\partial\vec{v}_1}\cdot\left(\int d\vec{K}'\,\delta(\vec{K}'\cdot\vec{v}_1-\omega_{\vec{K}'})\frac{2\pi\vec{K}'\vec{K}'\,\Psi^2(K')e^{2\gamma_{\vec{K}'}t}\left|\lim\limits_{\gamma_{\vec{K}'}\to 0}\int d\vec{v}_1\frac{f_{\vec{K}'}(1|t=0)}{\rho_{\vec{K}'}+i\vec{K}'\cdot\vec{v}_1}\right|^2}{\left|\lim\limits_{\gamma_{\vec{K}'}\to 0}\frac{\partial\varepsilon}{\partial\rho_{\vec{K}'}}\right|^2}\right)\cdot\frac{\partial\Phi}{\partial\vec{v}_1} \quad (7.4.1\text{-}6)$$

The velocity dependence of (7.4.1-6) is dominated (in the limit of large times) by the exponential factor $e^{2\gamma_{\vec{K}'}t}$. We have assumed that the unstable modes are confined to a small region $\Delta\vec{K}$ of wave number space. The initial growth rate $\gamma(\vec{v}_1)$ (considered now to be a function of velocity) is a positive function of $\vec{v}_1$ only in the velocity range $\Delta\vec{V}$ which corresponds to the velocities of the unstable modes in the wave number range $\Delta\vec{K}$. Thus, the exponential $e^{2\gamma(\vec{v}_1)t}$ and therefore the quantity within the brackets of (7.4.1-6) becomes a peaked function of velocity with a width $\Delta\vec{V}$ and a maximum near the initially most unstable mode. The amplitude of (7.4.1-6) outside $\Delta\vec{V}$ is very small (of $\mathcal{O}(\sigma^2)$).

The order of magnitude of (7.4.1-6) inside $\Delta\vec{V}$ is characterized by ($\vec{V}$ is in $\Delta\vec{V}$)

$$\left(\int d\vec{K}' C_{\vec{K}'}(1)\, e^{2\gamma_{\vec{K}'}t}\right)_{\vec{v}=\vec{V}} \sim \sigma^2 e^{2\gamma(\vec{V})t} \phi(\vec{V}) \qquad (7.4.1\text{-}7)$$

Thus, the second term of (7.4.1-5) may be written

$$\int d\vec{K}' d\vec{v}_1 \frac{i D_{\vec{K}}(1) C_{\vec{K}'}(1)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} \frac{e^{2\gamma_{\vec{K}'}t}}{(2\gamma_{\vec{K}'})^2 \frac{\partial \mathcal{E}}{\partial p_{\vec{K}}}} \sim \sigma^2 e^{2\gamma(\vec{V})t} \phi(\vec{V}) \qquad (7.4.1\text{-}8)$$

where γ is some average growth rate. The growth rate (for small $\gamma_{\vec{K}}$) is found from the Landau solution to be[29] (see also (3.2-25))

$$\gamma_{\vec{K}} = -\pi |K| \frac{\left(\partial\Phi/\partial u\right)_{u=\frac{\omega_K}{K}}}{\oint du \, \frac{\partial^2\Phi/\partial u^2}{u - \omega_K/K}} \qquad (7.4.1\text{-}9)$$

If we assume that $\mathcal{O}(\Phi(V)) \sim \mathcal{O}\left(\partial\Phi/\partial u\right)_{u=V}$ and that

$$\frac{\partial \mathcal{E}}{\partial p_{\vec{K}}} \sim \psi(K) \oint \frac{i\vec{K}\cdot\frac{\partial\phi}{\partial \vec{v}_1}}{(p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1)^2} d\vec{v}_1 \qquad (7.4.1\text{-}10)$$

(see the discussion following (8-17)) then the ratio $\psi(\kappa)\,\phi(\vec{v}) / \partial\epsilon/\partial p_{\vec{\kappa}}$ has the same order of magnitude as the growth rate γ, and the magnitude of (7.4.1-5) may be written (for $t > \frac{1}{\gamma}$)

$$E\,(t) \sim \sigma\, e^{\gamma t}\left\{1 + \left(\frac{\sigma^2}{\gamma}\, e^{2\gamma t}\right) + \cdots\right\} \qquad (7.4.1\text{-}11)$$

All additional terms of the solution for $E_{\vec{\kappa}}(t)$ can be characterized by an order of magnitude which is some power of the quantity ($\frac{\sigma^2}{\gamma}\, e^{2\gamma t}$).

We have argued that the asymptotic form of each term is a good approximation for times $t > \frac{1}{\gamma}$. We see from (7.4.1-11) that if (σ^2/γ) is a small quantity then the terms of $\mathcal{O}(\frac{\sigma^2}{\gamma}\, e^{2\gamma t})$ (which we henceforth refer to as the "diffusion" terms because, as we show in Chapter 8, they lead to a diffusion of the distribution function $f_0(1|t)$ in velocity space) do not contribute significantly to the solution for $\vec{E}_{\vec{\kappa}}(t)$ until some time $t > \frac{1}{\gamma}$. However, if (σ^2/γ) is not small then the diffusion terms begin to contribute significantly to the solution before they can be approximated by their asymptotic form (i.e., $\frac{\sigma^2}{\gamma}\, e^{2\gamma t} \sim 1$ for $t < \frac{1}{\gamma}$). In the latter case, part, if not all, of the readjustment of the plasma through wave-wave interactions takes place during the initial stages of development of the disturbance. In order to determine the nature of the solution for short times $(t < \frac{1}{\gamma})$ it is necessary to include the contributions from the poles of $S_{\vec{\kappa}}(1, i|p)$ in the evaluation of the convolution integral (7.2-14). Further, the free-streaming $e^{-i\vec{k}\cdot\vec{v}_1 t}$ and additional collective modes must be included in the time dependence of $\mathcal{P}_{\vec{k}}(1|t)\, f_{\vec{k}}(1|t=0)$ (see the discussion of (7.2-1)). The short-time behavior of the solution becomes very complicated and the complete readjustment of the plasma to the disturbance due to wave-wave coupling can not be followed. We assume henceforward that the initial amplitude σ of the initial perturbation is sufficiently small that $\sigma^2/\gamma \ll 1$.

The wave-coupling terms in (7.4.1-5) begin to exert a significant effect upon the solution at a time τ which may be characterized by

$$\frac{\sigma^2}{\gamma} e^{2\gamma\tau} \sim 1 \qquad (7.4.1\text{-}12)$$

However, we note from (7.4.1-12) that at time τ , $\sigma e^{\gamma\tau} \sim \gamma^{1/2}$, and the electric field has an order of magnitude (see (7.4.1-11))

$$E(\tau) \sim \gamma^{1/2} \qquad (7.4.1\text{-}13)$$

in agreement with the conclusions of Frieman, Bodner and Rutherford[43], and Aamodt and Drummond.[39] The initial growth rate characterizes the amplitude of the electric field at the time the wave-wave interaction terms begin to control the development of the disturbance. When (7.4.1-13) is combined with the condition that $\sigma^2/\gamma \ll 1$ we find

$$\frac{\sigma^2}{E^2(\tau)} << 1 \tag{7.4.1-14}$$

We conclude from (7.4.1-14) that the initial amplitude of the disturbance must be small compared with that at the time τ in order for the asymptotic form of the solution to describe the initial stages of the adjustment of the plasma to the presence of a disturbance.

7.4.2 Redistribution Terms

There are many wave-coupling terms which have not been included in the asymptotic solution (7.2-23) for $f_{\vec{K}}(1|t)$. In order to estimate the contribution of these terms we consider the complete equation (2-10) for $f^{N,1}(1|t)$ and take a Fourier transform in the spatial variable $\vec{x}_1$ to obtain

$$\left(\frac{\partial}{\partial t} + \mathcal{H}_{\vec{K}_1}(1)\right) f^{N,1}_{\vec{K}_1}(1|t) = \frac{1}{N}\sum_j{}' M(1j) f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|t) + \frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum{}' L(ij) f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t) \tag{7.4.2-1}$$

Equation (7.4.2-1) may be solved in terms of the propagators $\mathcal{P}_{\vec{K}}(1|t)$.

$$\begin{aligned} f^{N,1}_{\vec{K}_1}(1|t) &= \mathcal{P}_{\vec{K}_1}(1|t) f^{N,1}_{\vec{K}_1}(1|t=0) + \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_j{}' M(1j) f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|\tau) \\ &+ \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_{i<j}\sum{}' L(ij) f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau) \end{aligned} \tag{7.4.2-2}$$

If we substitute in the second term on the right-hand side of (7.4.2-2) the solution of the equation for $f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|\tau)$ we find

$$\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_j{}' M(1j) f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|\tau) =$$

$$= \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_j{}' M(1j)\mathcal{P}^2_{\vec{K}_1\vec{K}_j}(1j|\tau) f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|t=0) + \cdots \tag{7.4.2-3}$$

which becomes after an integration over the velocities $\vec{v}_2 \ldots \vec{v}_N$

$$\int (d\vec{v})^{N-1}\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_j{}' M(1j) f^{N,2}_{\vec{K}_1\vec{K}_j}(1j|\tau) =$$

$$= \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\int d\vec{v}_j\, M(1j)\mathcal{P}^2_{\vec{K}_1\vec{K}_j}(1j|\tau) f^{2,2}_{\vec{K}_1\vec{K}_j}(1j|t=0) + \cdots \tag{7.4.2-4}$$

The operator $\int d\vec{v}_j M(1j)$, when followed by a function $f_{\vec{K}_1\vec{K}_j}(1j)$ of the velocities $\vec{v}_1$ and $\vec{v}_j$ and the wave numbers $\vec{K}_1$ and $\vec{K}_j$ becomes, from (5.3-1)

$$\int d\vec{v}_j\, M(1j) f_{\vec{K}_1\vec{K}_j}(1j) = -\frac{i}{(2\pi)^3}\int d\vec{K}'(\vec{K}_1-\vec{K}')\,\Psi(|\vec{K}_1-\vec{K}'|)\cdot\frac{\partial}{\partial\vec{v}_1}\begin{pmatrix}\vec{K}_1\to\vec{K}_1-\vec{K}'\\ \vec{K}_j\to\vec{K}'\end{pmatrix}\int d\vec{v}_j\, f_{\vec{K}_1\vec{K}_j}(1j)$$

$$\equiv \int d\vec{K}'\, M_{\vec{K}_1-\vec{K}',\vec{K}'}(1,j)\int d\vec{v}_j\, f_{\vec{K}_1\vec{K}_j}(1j) \tag{7.4.2-5}$$

where an integration by parts has been used to write (7.4.2-5).

The asymptotic behavior of the term shown on the right-hand side of (7.4.2-4) is determined by the factor

$P^{2}_{\vec{K}_i\vec{K}_j}(ij|T) f^{2,2}_{\vec{K}_i\vec{K}_j}(ij|t=0)$. The methods of Section 7.2 are used to evaluate the convolution integral in the limit of large times. We can continue to substitute for terms on the right-hand side of (7.4.2-2) and obtain a complete solution (in the limit $\varepsilon \to 0$) for the single-particle function. The leading terms are

$$\lim_{t\to\infty} \vec{E}_{\vec{K}}(t) = i\vec{K}\Psi(K)\tilde{R}_{\vec{K}} e^{p_{\vec{K}}t}\left\{1 + \int d\vec{K}' S_{\vec{K}}(1,1|p_{\vec{K}}+2\gamma_{\vec{K}'}) R_{\vec{K}}(1) C_{\vec{K}'}(1) e^{(p_{\vec{K}}+2\gamma_{\vec{K}'})t} + \cdots\right\}$$
$$+ \int d\vec{K}' P_{\vec{K}}(1|p_{\vec{K}-\vec{K}'}+p_{\vec{K}'}) M_{\vec{K}-\vec{K}_i,\vec{K}'}(ij) R_{\vec{K}_i}(1)\tilde{R}_{\vec{K}_j} e^{(p_{\vec{K}-\vec{K}'}+p_{\vec{K}'})t} + \cdots \quad (7.4.2\text{-}6)$$

The solution (7.4.2-6) contains all wave-wave interactions, including those which lead to a spreading of energy throughout the wave spectrum.

We observe that the coupling between waves may be divided into two categories. Interactions between the waves $\vec{K}'$ and $-\vec{K}'$, represented by the operator $L(ij)$, are found to lead to a diffusion of the distribution function $f_0(1|t)$ in velocity space; interactions between the waves $\vec{K}-\vec{K}'$ and $\vec{K}'$, represented by the operator $M(ij)$, are found to lead to a redistribution of the wave energy. We therefore refer to $L(ij)$ as the "diffusion" operator and $M(ij)$ as the "redistribution" operator. Every term of the solution for $f_{\vec{K}}(1|t)$ but the first contains some combination of these operators.

The order of magnitudes of the first terms of (7.4.2-6) have been discussed. The product $R_{\vec{K}_1}(1)\,\tilde{R}_{\vec{K}_j}$ in the third term is proportional to the square of the amplitude σ of the initial perturbation. We once again approximate

$$\mathcal{E}(p_{\vec{K}-\vec{K}'}+p_{\vec{K}'}) \sim O\left(\gamma \frac{\partial \mathcal{E}}{\partial p_K}\right)$$

and

$$\int d\vec{K}'\,d\vec{v}_1 \frac{M_{\vec{K}_1-\vec{K}_1',\vec{K}'}(1j)}{p_{\vec{K}_1-\vec{K}'}+p_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1}\,\frac{e^{(p_{\vec{K}_1-\vec{K}'}+p_{\vec{K}'})t}}{\mathcal{E}(p_{\vec{K}_1-\vec{K}'}+p_{\vec{K}'})}\,R_{\vec{K}_1}(1)\,\tilde{R}_{\vec{K}_j} \sim O\left(\frac{\sigma^2 \varphi(\vec{v})e^{2\gamma t}}{\gamma \frac{\partial \mathcal{E}}{\partial p_{\vec{K}}}}\right)$$

$$\sim O(\sigma^2 e^{2\gamma t}) \qquad (7.4.2\text{-}7)$$

to write the order of magnitude of the terms on the right-hand side of (7.4.2-6) as (see (7.4.1-11))

$$\lim_{t\to\infty} E(t) \sim \sigma e^{\gamma t}\left[1 + \left(\frac{\sigma^2}{\gamma} e^{2\gamma t}\right) + \cdots\right] + \sigma^2 e^{2\gamma t} + \cdots \qquad (7.4.2\text{-}8)$$

We note from (7.4.2-8) that the relative importance of the redistribution and diffusion terms is determined by the growth rate γ. If $\gamma < 1$ then the diffusion terms dominate the redistribution terms, and if $\gamma > 1$ the latter terms dominate the former. We may interpret this result by noting that the redistribution of wave energy through the

Thus, the asymptotic solution (7.2-23) for $f_{\vec{k}}(1|t)$ is not valid for all times $t > \frac{1}{\gamma}$. It is valid only in an "intermediate" interval of time which we may characterize by

$$\text{intermediate times: } \frac{1}{\gamma} < t < T \tag{7.4.2-10}$$

The length of the intermediate interval is inversely proportional to the growth rate. If γ is very small then the amplitude of the disturbance is limited to small values ($\mathcal{O}(\gamma^{1/2})$) by the diffusion of $f_0(1|t)$ in velocity space, and the redistribution of wave energy takes place so slowly that its influence is not felt until well after diffusion has been completed. However, with increasing values of γ the energy redistribution becomes stronger and may exert a significant effect upon the behavior of the plasma while diffusion is still in progress.

CHAPTER 8

SIMPLIFICATION OF THE SOLUTION IN THE LIMIT OF SMALL INITIAL GROWTH RATES

The growth rate $\gamma_{\vec{k}}$ (normalized with respect to the plasma frequency) determines the relative importance of the diffusion and redistribution terms in the solution for $f_{\vec{k}}(1|t)$. If $\gamma < 1$, the diffusion terms provide the dominant correction to the Landau result for an intermediate interval of time which has been discussed. We show in this Chapter that if the growth rate is sufficiently small that terms of $\mathcal{O}(\gamma)$ may be neglected compared to terms of $\mathcal{O}(1)$ then the solution, for these intermediate times, reduces to a form which is in essential agreement with that obtained from quasi-linear theory.

We found in Chapter 7 that the solution (7.2-23) (without the redistribution terms) for the single particle distribution function satisfied the linearized Vlasov equation (7.3-11) and that the two-particle function, if it could be factored initially, remained factored. The spatially-homogeneous function $f_0(1|t)$ and the disturbance $f_{\vec{k}}(1|t)$ then obey, for intermediate times, the following set of coupled equations.

$$\frac{\partial f_{\vec{K}}(1|t)}{\partial t} + i\vec{K}\cdot\vec{v}_1 f_{\vec{K}}(1|t) - i\vec{K}\,\psi(K)\cdot\frac{\partial f_0(1|t)}{\partial \vec{v}_1}\int d\vec{v}_1 f_{\vec{K}}(1|t) = 0$$

$$\frac{\partial f_0(1|t)}{\partial t} = \int d\vec{v}_j\, L(1j) f_{\vec{K}_1}(1|t) f_{\vec{K}_j}(j|t) \tag{8-1}$$

The relations (8-1) may be combined to write

$$\frac{\partial f_0(1|t)}{\partial t} - \int d\vec{K}\, L_{\vec{K}}(1)\frac{\left|\int d\vec{v}_1 f_{\vec{K}}(1|t)\right|^2}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} i\mathcal{D}_{\vec{K}}(1) f_0(1|t) =$$

$$= -\int d\vec{K}\, L_{\vec{K}}(1)\frac{e^{p_{\vec{K}}t}\left(\int d\vec{v}_1 f_{-\vec{K}}(1|t)\right)\frac{\partial}{\partial t}\left(e^{-p_{\vec{K}}t} f_{\vec{K}}(1|t)\right)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} \tag{8-2}$$

An order of magnitude analysis is used to show that the term on the right-hand side of (8-2) is of $\mathcal{O}(\gamma)$ compared to the terms on the left. If the solution (7.2-23) for $f_{\vec{K}}(1|t)$ is substituted into the right-hand side of (8-2) we find

$$-\int d\vec{K}\, L_{\vec{K}}(1)\frac{e^{p_{\vec{K}}t}\left(\int d\vec{v}_1 f_{-\vec{K}}(1|t)\right)\frac{\partial}{\partial t}\left(e^{-p_{\vec{K}}t} f_{\vec{K}}(1|t)\right)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} =$$

$$= -\int d\vec{K}\, L_{\vec{K}}(1)\frac{e^{2\gamma_{\vec{K}}t}|\tilde{R}_{\vec{K}}|^2}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1}\int d\vec{K}'\, e^{2\gamma_{\vec{K}'}t} P_{\vec{K}}(1|p_{\vec{K}} + 2\gamma_{\vec{K}'})\, i\mathcal{D}_{\vec{K}}(1) C_{\vec{K}'}(1) + \cdots \tag{8-3}$$

The right-hand side of (8-3), as a function of velocity, is very small (of $\mathcal{O}(\sigma^4)$) outside of the velocity range centered about the most unstable mode. The order of magnitude of (8-3) inside $\Delta\vec{V}$ can be characterized by

$$\text{rhs (8-3)} \sim \frac{e^{4\gamma t}\sigma^4\gamma^2}{\gamma} = \gamma^3\left(\frac{\sigma^2}{\gamma}e^{2\gamma t}\right) \tag{8-4}$$

The correction term (8-4) increases exponentially with time. At the time τ when the diffusion terms first become important (from (7.4-11), $\frac{\sigma^2}{\gamma}e^{2\gamma\tau} \sim 1$) the correction (8-4) is of $\mathcal{O}(\gamma^3)$. However, at time τ the second term on the left-hand side of (8-2) can be characterized by the order of magnitude (we use $|E_K(\tau)|^2 \sim \gamma$ from (7.4-12))

$$\int d\vec{K}\, L_{\vec{K}}(1)\, \frac{\left|\int d\vec{v}_1 f_{\vec{K}}(1|t)\right|^2}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1}\, iD_{\vec{K}}(1) f_0(1|t) \sim \mathcal{O}(\gamma^2) \tag{8-5}$$

If γ is sufficiently small then the term on the right-hand side of (8-2) may be neglected compared with the terms on the left, and we find

$$\frac{\partial f_0(1|t)}{\partial t} = \int d\vec{K}\, L_{\vec{K}}(1)\, \frac{\left|\int d\vec{v}_1 f_{\vec{K}}(1|t)\right|^2}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1}\, iD_{\vec{K}}(1) f_0(1|t) \tag{8-6}$$

As a matter of convenience we have included in the denominator of the integrand of (8-5) the initial values of the frequency and growth rate rather than the time-dependent quantities associated with the quasi-linear solution (see equation (3.5) of ref. 23). The difference between $(-i\omega_{\vec{K}} + \gamma_{\vec{K}} + i\vec{K}\cdot\vec{v}_1)^{-1}$ and $(-i\omega_{\vec{K}}(t) + \gamma_{\vec{K}}(t) + i\vec{K}\cdot\vec{v}_1)^{-1}$ in the integral over $\vec{K}$ is of $\mathcal{O}(\gamma)$ and so is a correction which is of the same order of magnitude as the term discarded from the right-hand side of (8-2) (see also the discussion following (8-11)).

Equation (8-6) describes the diffusion of the spatially-homogeneous part of the distribution function in velocity space (there is a velocity derivative in the operator $L_{\vec{K}}(1)$). The diffusion coefficient depends upon $\left|\int d\vec{v}_1 f_{\vec{K}}(1|t)\right|^2$ which we may calculate from the solution (7.2-23) for $f_{\vec{K}}(1|t)$. An equation for $\left|\int d\vec{v}_1 f_{\vec{K}}(1|t)\right|^2$ is found by integrating (7.2-23) over the velocity $\vec{v}_1$, multiplying the result by $\int d\vec{v}_1 f_{\vec{K}}(1|t)$ and differentiating with respect to time to obtain (multiply by $K^2 \psi^2(K)$ to obtain the square of the electric field)

$$\frac{\partial}{\partial t}\left|E_{\vec{K}}(t)\right|^2 = 2\gamma_{\vec{K}}\left|E_{\vec{K}}(t)\right|^2$$

$$+K^2\psi^2(K)\left|\tilde{R}_{\vec{K}}\right|^2 e^{2\gamma_{\vec{K}}t}\int d\vec{K}' e^{2\gamma_{\vec{K}'}t}\left(\tilde{P}_{\vec{K}}(1|\rho_{\vec{K}}+2\gamma_{\vec{K}'})\,iD_{\vec{K}}(1)+\tilde{P}_{-\vec{K}}(1|\rho_{-\vec{K}}+2\gamma_{\vec{K}'})\,iD_{\vec{K}}(1)\right)C_{\vec{K}'}(1)+\cdots \tag{8-7}$$

The second term on the right-hand side of (8-7) contains the operator $\tilde{P}_{\vec{K}}(1|\rho_{\vec{K}}+2\gamma_{\vec{K}'})\,iD_{\vec{K}}(1)$ which we may expand about the point $\rho_{\vec{K}}$ in powers of the small quantity $2\gamma_{\vec{K}'}$.

$$\tilde{P}_{\vec{K}}(1|\rho_{\vec{K}}+2\gamma_{\vec{K}'})\,iD_{\vec{K}}(1) = \frac{1}{2\gamma_{\vec{K}'}}T_{\vec{K}}(1) + T'_{\vec{K}}(1) + \cdots \tag{8-8}$$

We have defined the new operators

$$T_{\vec{K}}(1) \equiv \frac{\int d\vec{v}_1 \frac{iD_{\vec{K}}(1)}{\rho_{\vec{K}}+i\vec{K}\cdot\vec{v}_1}}{\partial\varepsilon/\partial\rho_{\vec{K}}} \tag{8-9}$$

$$T_{\vec{k}}'(1) \equiv \frac{\partial}{\partial p_{\vec{k}}} T_{\vec{k}}(1) \tag{8-10}$$

To obtain the expansion (8-8) it has been necessary to write

$$\frac{1}{p_{\vec{k}}+2\gamma_{\vec{k}'}+i\vec{k}\cdot\vec{v}_1} = \frac{1}{p_{\vec{k}}+i\vec{k}\cdot\vec{v}_1} + 2\gamma_{\vec{k}'}\frac{1}{(p_{\vec{k}}+i\vec{k}\cdot\vec{v}_1)^2} + \cdots \tag{8-11}$$

The relation (8-11) does not converge for values of velocity for which $|\vec{k}\cdot\vec{v}_1 - \omega_{\vec{k}}| < |2\gamma_{\vec{k}'}|$. However, the series (8-11) always appears with some functions of velocity $iD_{\vec{k}}(1)\,F(\vec{v}_1)$ in an integral over the velocity $\vec{v}_1$, (for instance, $T_{\vec{k}}(1|p_{\vec{k}}+2\gamma_{\vec{k}'})\,iD_{\vec{k}}(1)\,F(\vec{v}_1)$). If each term, but the first, is integrated by parts (the function $F(\vec{v}_1)$ vanishes as $\vec{v}_1 \to \infty$) the n^{th} term of the series becomes the n^{th} term in a Taylor expansion of $iD_{\vec{k}}(1)F(\vec{v}_1)$ in powers of (γ_K/iK). This expansion will converge for functions $F(\vec{v}_1)$ which are analytic in the neighborhood of the real $\vec{v}_1$ axis. We require the growth rate to be sufficiently small that we need keep only the first term in the expansion.

We may use (8-8) to approximate the second term of (8-7) as

$$|\tilde{R}_{\vec{k}}|^2 e^{2\gamma_{\vec{k}}t}\left(T_{\vec{k}}(1) + T_{\vec{k}}(1)\right)\int d\vec{k}'\,\frac{1}{2\gamma_{\vec{k}'}}\,e^{2\gamma_{\vec{k}'}t}\,C_{\vec{k}'}(1) \tag{8-12}$$

Similar approximations may be used to evaluate the remaining terms of (8-7) with the following result

$$\frac{\partial}{\partial t}\left|\vec{E}_{\vec{K}}(t)\right|^2 \simeq \left(2\gamma_{\vec{K}} + \left(T_{\vec{K}}(1) + T_{-\vec{K}}(1)\right)\left(f_0(1|t) - \varphi(1)\right)\right)\left|\vec{E}_{\vec{K}}(t)\right|^2 \qquad (8\text{-}13)$$

The first correction term to (8-13) is

$$\left(T'_{\vec{K}}(1) + T'_{-\vec{K}}(1)\right) \frac{\partial f_0(1|t)}{\partial t} \left|\vec{E}_{\vec{K}}(t)\right|^2 \qquad (8\text{-}14)$$

which is of $\mathcal{O}(\gamma)$ compared to the terms on the right-hand side of (8-13)(from (8-5) and (8-6) $\left(\frac{\partial f_0/\partial t}{f_0}\right)_{\vec{v}=\vec{V}} \sim \gamma$).

In order to simplify the relation (8-13) we note from (8-6) that the most significant change in $f_0(1|t)$ with time takes place in the small region $\Delta\vec{V}$ centered about the wave velocity of the most unstable mode (see also the discussion following (7.4-6)). The quantity $\left(f_0(1|t) - \varphi(1)\right)$ is small outside of $\Delta\vec{V}$. We use the Plemelj formula to write, in the limit of small γ

$$\int d\vec{v}_1 \frac{i D_{\vec{K}}(1)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1}\left(f_0(1|t) - \varphi(1)\right) = \oint\!\!\!\!\mathrm{P}\, d\vec{v}_1 \frac{i D_{\vec{K}}(1)}{p_{\vec{K}} + i\vec{K}\cdot\vec{v}_1}\left(f_0(1|t) - \varphi(1)\right)$$

$$+ i\pi\,\Psi(K)\left[\frac{\partial}{\partial u_1}\left(F_0(1|t) - \Phi(1)\right)\right]_{u_1 = \frac{\omega_K}{K}} \qquad (8\text{-}15)$$

where (see also (3.2-25))

$$F_0(1|t) = \int d\vec{v}_\perp \, f_0(u_1, \vec{v}_\perp | t) \tag{8-16}$$

We note that the denominator of the principal value integral in (8-15) is an odd function of the velocity $\vec{v}_1$ inside $\Delta\vec{V}$ whereas the numerator is an even function. We therefore expect the principal value integral to be small, certainly no larger than the second term of (8-15).

The Plemelj formula may also be used to evaluate $\partial\mathcal{E}/\partial\rho_{\vec{K}}$. The ratio of the real and imaginary parts is

$$\frac{\mathcal{R}e\left(\frac{\partial\mathcal{E}}{\partial\rho_{\vec{K}}}\right)}{\mathcal{I}m\left(\frac{\partial\mathcal{E}}{\partial\rho_{\vec{K}}}\right)} = -\frac{\frac{\pi}{K^2}\left(\frac{\partial^2\Phi(1)}{\partial u_1^2}\right)_{u_1=\frac{\omega_K}{K}}}{\oint du_1 \frac{\partial^2\Phi(1)/\partial u_1^2}{u_1 - \omega_K/K}} \tag{8-17}$$

If the first and second derivatives of the distribution function $\Phi(1)$ at the velocity $u_1 = \omega_K/K$ are considered to be of the same order of magnitude then the ratio (8-7) is, from (7.4.1-9), proportional to the growth rate γ. We infer from the condition that γ be small that the real part of $\partial\mathcal{E}/\partial\rho_{\vec{K}}$ is small compared with the imaginary part.

The operator $T_{\vec{K}}(1)$ appears in summation with its complex conjugate in equation (8-14). We require, then, only the real part of the operator $T_{\vec{K}}(1)$. To approximate the real part we use the imaginary part of the numerator and the imaginary part of the denominator of (8-9) to write (the error is of $\mathcal{O}(\gamma)$)

$$\left(T_{\vec{K}}(1)+T_{-\vec{K}}(1)\right)\left(f_0(1|t)-\varphi(1)\right) = 2\,\frac{\pi|K|\left[\frac{\partial}{\partial u_1}\left(F_0(1|t)-\Phi(1)\right)\right]_{u_1=\frac{\omega_K}{K}}}{\oint du_1\,\frac{\partial^2\Phi(1)/\partial u_1^2}{u_1-\frac{\omega_K}{K}}} \tag{8-18}$$

The result (8-18) is substituted into (8-15) and the definition (7.4.1-9) of the Landau growth rate $\gamma_{\vec{K}}$ used to obtain

$$\frac{\partial}{\partial t}\left|\vec{E}_{\vec{K}}(t)\right|^2 \simeq 2\gamma_{\vec{K}}(t)\left|\vec{E}_{\vec{K}}(t)\right|^2 \tag{8-19}$$

where

$$\gamma_{\vec{K}}(t) = \frac{\pi|K|\left[\frac{\partial}{\partial u_1}F_0(1|t)\right]_{u_1=\frac{\omega_K}{K}}}{\oint du_1\,\frac{\partial^2\Phi(1)/\partial u_1^2}{u_1-\frac{\omega_K}{K}}} \tag{8-20}$$

The relations (8-6) and (8-19) describe the self-limiting of the instability at some maximum amplitude. If the plasma is strictly one-dimensional (the infinite magnetic field problem of Aamodt and Drummond[39]) then, according to the equations (8-6) and (8-19), the instability grows until a spectrum of waves is established, and the disturbance remains at equilibrium thereafter. However, if the plasma can support a three-dimensional disturbance then the disturbance, after reaching its maximum amplitude, gradually begins to decay, as discussed by Bernstein and Engelmann.[21]

The approximate results (8-6) and (8-19) have been obtained from the exact solution (7.2-23) of the linearized Vlasov equation (7.3-11) by an expansion in powers of γ . The growth rate is assumed to be sufficiently small that terms of $O(\gamma)$ may be neglected compared with terms of $O(1)$. This assumption is less restrictive than that made in Chapter 7 where the redistribution terms were discarded on the basis that they were an order $\gamma^{1/2}$ smaller than the magnitude of $\vec{E}_{\vec{K}}(t)$. The redistribution terms represent the dominant correction to the results of this Chapter.

CHAPTER 9

APPROXIMATE CALCULATION

An approximate calculation is made to determine the variation of the electric field energy with time. The advantage that the approximate method enjoys over the numerical calculations used until now is that the basic parameters of the system can be varied and the correspon-ding growth and self-limiting of the disturbance easily determined. The approximate calculation is limited to the energy in the most unstable mode. However, the results are shown to have a more general application.

The discussion is limited to the case of a one-dimensional plasma (the infinite magnetic field problem of ref. 39). An approximate solution for the electric field during an intermediate interval of time has been obtained in Chapters 7 and 8. The relations required for the present discussion are obtained from Chapter 8 by integrating equations (8-6) and (8-19) for $f_0(1|t)$ and $E_K(t)$ with respect to time. If the solution for $f_0(1|t)$ is substituted into the definition (8-20) of $\gamma_K(t)$ (the Plemelj formula is used to integrate over K) we find

$$|E_K(t)|^2 = |E_K(0)|^2 e^{2\int_0^t d\tau\, \gamma_K(\tau)} \tag{9-1}$$

$$\gamma_K(t) = \gamma_K + \frac{1}{2}\left(\frac{\partial^2}{\partial u^2}|E(u,t)|^2\right)_{u=\frac{\omega_K}{K}} \tag{9-2}$$

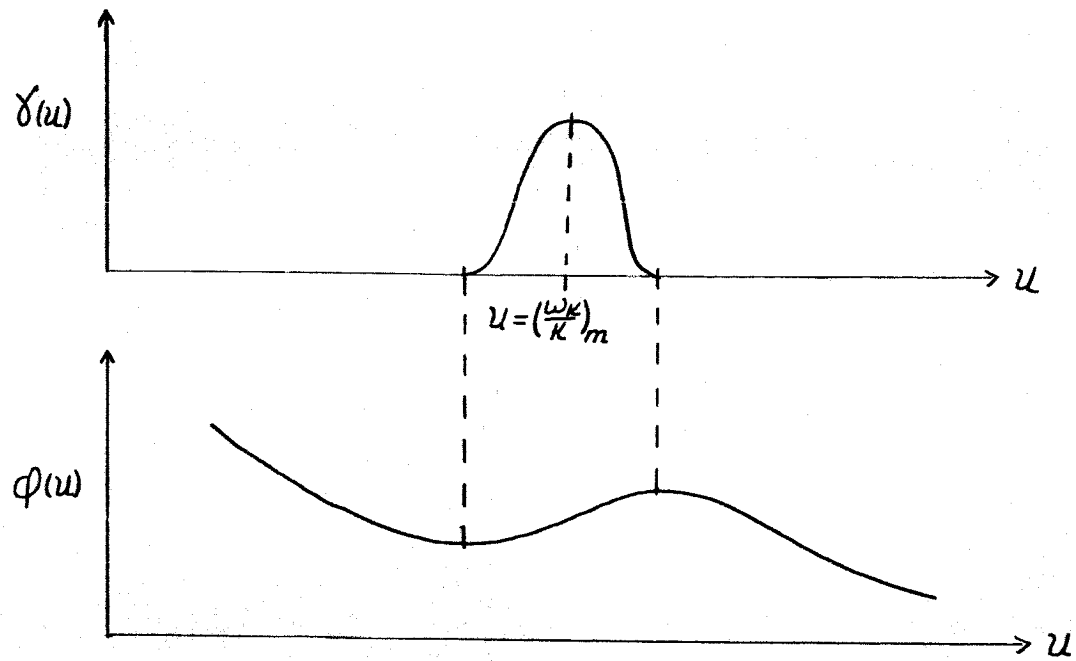
$|E_K(0)|^2$ is the initial value of $|E_K(t)|^2$ and $E(u|t) \equiv \left(E_K(t)\right)_{\frac{\omega_K}{K}=u}$. The two relations (9-1) and (9-2) may now be combined to write ($\gamma(u|t) \equiv \left(\gamma_K(t)\right)_{\frac{\omega_K}{K}=u}$)

$$\gamma_K(t) = \gamma_K + \frac{1}{2}\left(\frac{\partial^2}{\partial u^2}|E(u,0)|^2 e^{2\int_0^t d\tau\, \gamma(u|\tau)}\right)_{u=\frac{\omega_K}{K}} \tag{9-3}$$

We argue, as in Chapter 7 (see discussion following equ. (7.4.1-6)) that the time-asymptotic behavior of the quantity within the brackets of (9-3) is dominated by the exponential, not by the initial value $|E(u,0)|^2$, and write

$$\gamma_K(t) \simeq \gamma_K + \frac{1}{2}\left(\left[\left(2\int_0^t d\tau\, \frac{\partial \gamma(u|\tau)}{\partial u}\right)^2 + 2\int_0^t d\tau\, \frac{\partial^2 \gamma(u|\tau)}{\partial u^2}\right]|E(u,0)|^2 e^{2\int_0^t d\tau\, \gamma(u|\tau)}\right)_{u=\frac{\omega_K}{K}} \tag{9-4}$$

The right-hand side of (9-4) may be simplified. The growth rate $\gamma(u|t)$, considered as a function of velocity, is initially distributed in a way similar to that shown in the sketch below (we show only the unstable modes).



We designate by u_m the velocity at which $\gamma(u)$ (and therefore the slope $\partial\phi(u)/\partial u$) has a maximum positive value. The curvature $\partial^2\gamma/\partial u_m^2$ is negative. We note from the Drummond and Pines calculation (Fig. 4 of ref. 23) that the maximum positive slope within the bump of the distribution function $f_0(1|t)$ remains, as a function of time, near the same velocity ($u_m \sim 4.7$ for ref. 23). The wave mode which initially has the maximum growth rate continues to have the maximum growth rate for all times (see also Fig. 5 of ref. 23 where the maximum in the curve $\frac{1}{8\pi}|E(u|t)|^2$ is shown to remain at approximately the same velocity for all times). Thus, we may rewrite (9-4) ($\partial\gamma/\partial u_m = 0$)

$$\gamma_m(t) \simeq \gamma_m + \left(\int_0^t d\tau \; \frac{\partial^2 \gamma(u|\tau)}{\partial u_m^2}\right) |E_m(t)|^2 \qquad (9\text{-}5)$$

where $\gamma_m(t)$) and $E_m(t)$; are the growth rate and electric field of the most unstable mode.

The integral of (9-5) over time is evaluated by expanding the growth rate about its initial value.

$$\begin{aligned} \int_0^t d\tau \, \gamma_m(\tau) &\simeq \gamma_m t + \int_0^t d\tau \left(\int_0^\tau d\tau' \, \frac{\partial^2 \gamma(u|0)}{\partial u_m^2}\right) |E_m(0)|^2 \, e^{2\gamma_m \tau} \\ &\simeq \gamma_m t + \frac{\partial^2 \gamma(u|0)/\partial u_m^2}{2\gamma_m} \, t \, |E_m(t)|^2 \end{aligned} \qquad (9\text{-}6)$$

The result (9-6) is substituted back into equation (9-2) for $|E_K(t)|^2$ to obtain

$$|E_m(t)|^2 \simeq |E_m(0)|^2 e^{2\gamma_m t} \exp\left(\frac{\partial^2 \gamma(u|0)/\partial u_m^2}{2\gamma_m} \, t \, |E_m(t)|^2\right) \qquad (9\text{-}7)$$

For convenience, the function $F(t)$ of time is defined as

$$F(t) \equiv \exp 2\gamma_m t \left\{1 + \frac{\partial^2 \gamma(u|0)/\partial u_m^2}{(2\gamma_m)^2} |E_m(0)|^2 F(t)\right\} \qquad (9\text{-}8)$$

Then

$$|E_m(t)|^2 = |E_m(0)|^2 F(t) \qquad (9\text{-}9)$$

$\mathcal{F}(t)$ depends upon the initial conditions on the problem.

The asymptotic value of $\mathcal{F}(t)$ is found by taking the log of both sides of (9-8) to write

$$2\gamma_m t = \frac{\ln \mathcal{F}(t)}{1 + \frac{\partial^2 \gamma(u|0)/\partial u_m^2}{(2\gamma_m)^2} |E_m(0)|^2 \mathcal{F}(t)} \tag{9-10}$$

In the limit of large times the denominator of (9-10) must go to zero (if $\mathcal{F}(t)$ is to remain finite).

$$\lim_{t\to\infty} \mathcal{F}(t) \sim - \frac{(2\gamma_m)^2}{\frac{\partial^2\gamma(u|0)}{\partial u_m^2} |E_m(0)|^2} \tag{9-11}$$

The asymptotic value of $\mathcal{F}(t)$ is positive by the condition that $\partial^2\gamma(u|0)/\partial u_m^2$ is negative. The asymptotic value of the electric field, which becomes

$$\lim_{t\to\infty} |E_m(t)|^2 \sim - \frac{(2\gamma_m)^2}{\partial^2\gamma(u|0)/\partial u_m^2} \tag{9-12}$$

is independent of the initial perturbation and varies with the characteristics of the initial bump on the distribution function. The larger the value of γ_m the higher the bump, and the smaller the curvature $\partial^2\gamma(u|0)/\partial u_m^2$ the wider the bump. The result (9-12) indicates that in general the larger the area underneath the bump the larger will be the equilibrium energy of the initially most unstable mode. Note, further, that if we consider $\mathcal{O}(\partial^2\gamma/\partial u_m^2) \sim \mathcal{O}(\gamma_m)$ then from (9-12)

$$\lim_{t \to \infty} |E_m(t)|^2 \sim \gamma_m \qquad (9\text{-}13)$$

in agreement with the results of Section 7.4.2.

The relation (9-7) has been used to plot in Fig. 1 the energy in the most unstable mode as a function of time. Included also are the results of a numerical calculation by Drummond and Pines.[23] The initial amplitude $\mathcal{E}_m(o)$ and the initial growth rate γ_m for the approximate calculation have been matched with the corresponding values of ref. 23. The maximum amplitude has been normalized to unity. Although the approximate calculation takes a longer time to reach the final equilibrium state than does the numerical calculation (see Fig. 1), the similarity of the two curves for times t less than 10^4 plasma periods should be noted. The approximate method appears to predict correctly the time interval in which the most rapid readjustment of the growth rate of the most unstable mode takes place.

The results of Fig. 1 have a wider application. The stabilization time of the most unstable mode is a characteristic time for all the modes in the plasma. Drummond and Pines have shown that even a mode which is initially stable and becomes unstable during the development of the distur-

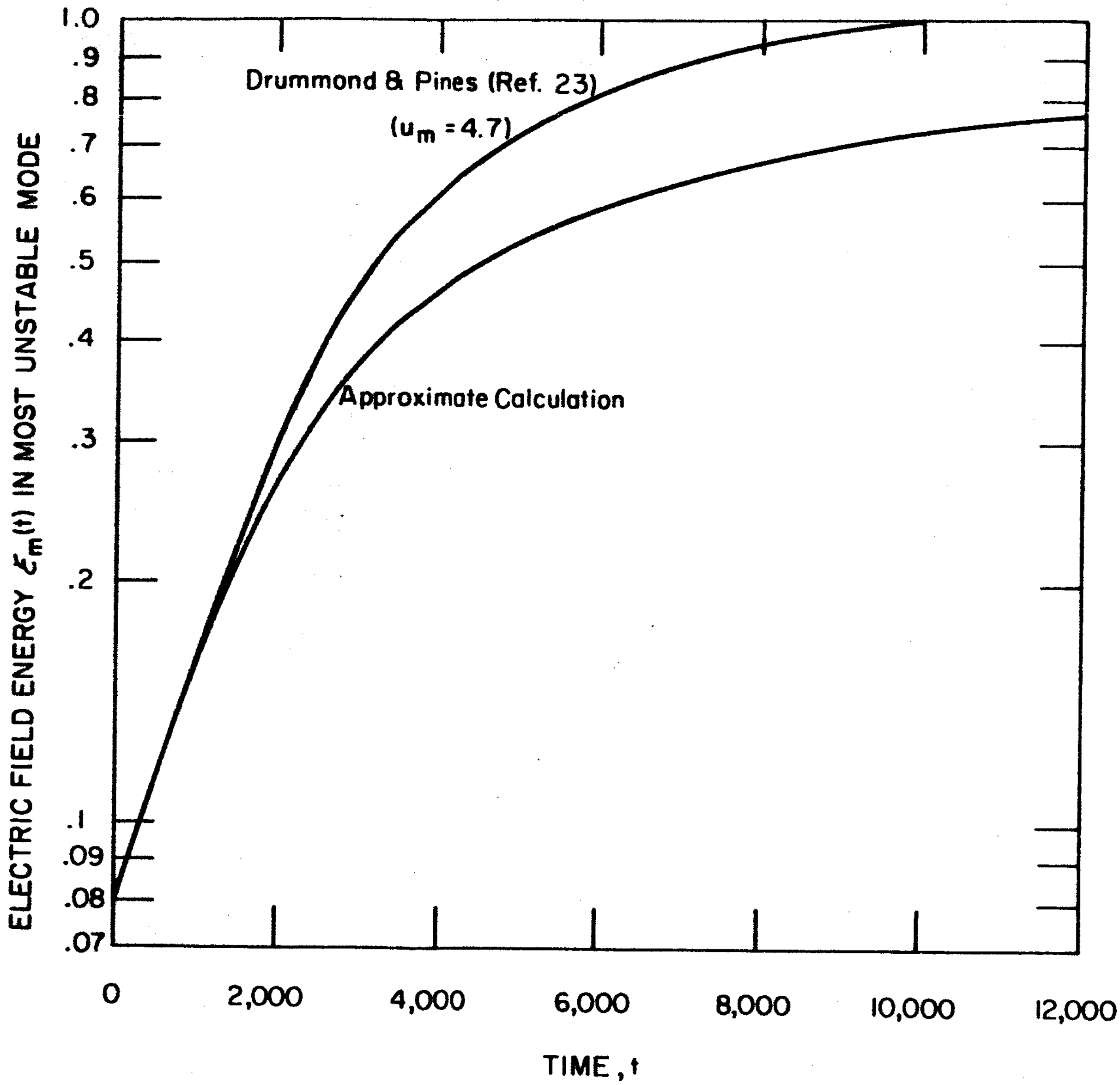


Fig. 1. Variation of the energy in the most unstable disturbance with time.

bance finally levels off at an equilibrium amplitude in approximately the same time that it takes the most unstable mode to attain equilibrium. Thus, (for the case shown in Fig. 1) the time $t \sim 10^4$ plasma periods is characteristic of the time in which <u>all</u> the unstable modes have undergone most readjustment of their growth rates.

The advantage the approximate method enjoys over the numerical calculation is that the parameters of the system can be varied and the resulting behavior easily determined. As an example, we note from the discussion in Chapter 7 that the smaller the initial growth rate the later in the development of the disturbance does the redistribution of energy in the wave spectrum become important. The approximate calculation can be used to estimate the magnitude of γ for which the redistribution terms begin to significantly alter the solution <u>just</u> as the diffusion of $f_0(1|t)$ in velocity space nears completion.

If the ratio $\left(\frac{\partial^2 \gamma(u|0)/\partial u_m^2}{(2\gamma_m)^2} \, |E_m(0)|^2 \right)$ of the initial to the final values of the energy is considered a constant (0.08 in Fig. 1) then $F(t)$ is a function only of the quantity $\gamma_m t$. The period Δt of greatest readjustment of the plasma to the disturbance ($\Delta t \sim 10^4$ in Fig. 1) then scales directly with γ_m. The initial growth rate is approximately $\gamma_m \simeq 3.3 \times 10^{-4}$, and

$$\gamma_m \Delta t \sim \text{constant} \sim 3.3 \qquad (9\text{-}14)$$

We have denoted by τ the time at which the diffusion terms first become important (from (7.4.1-12), $\frac{\sigma^2}{\gamma} e^{2\gamma_m \tau} \sim 1$) and by T the time at which the redistribution terms first become significant (from (7.4.2-9), $T \sim \frac{1}{2\gamma} \ln\left(\frac{E_m(max)}{\sigma^2}\right)$). The initial amplitude σ may be eliminated from the relations for τ and T to obtain

$$2\gamma_m (T-\tau) \sim \ln\left(\frac{E_m(max)}{\sigma^2}\right) \tag{9-15}$$

If the coupling between different wave modes is to become important just as the velocity diffusion nears completion then we require that $T-\tau \sim \Delta t$, and from (9-14)

$$\ln\left(\frac{E_m(max)}{\sigma^2}\right) \sim 6.6 \tag{9-16}$$

To eliminate $E_m(max)$ from (9-16) we note from Chapter 7 that $\gamma_m \sim E_m^2(\tau)$. The result $E_m(max)/E_m(\tau) \sim 10$ is then used from Fig. 1 to rewrite (9-16)

$$\ln\left(\frac{10}{\gamma_m^{1/2}}\right) \sim 6.6 \tag{9-17}$$

which we may solve for γ_m .

$$\gamma_m \sim 10^{-4} \tag{9-18}$$

If the growth rate (normalized with respect to the plasma frequency) is smaller than 10^{-4} then the diffusion can be expected to be nearly completed before redistribution of the wave energy begins to take place. However, if γ_m is greater than 10^{-4} then significant spreading of the wave energy begins while diffusion is still in progress.

CHAPTER 10

CONCLUDING REMARKS

The approach to the kinetic theory of plasmas which has been presented lies between the approaches which employ diagram methods and the approaches based upon the BBGKY hierarchy. The former start from a direct solution of the Liouville equation and eliminate excess information after the solution has been obtained. The number of different coordinates is enormous and diagrammatic methods are required to simplify the terms. On the other hand, the BBGKY approach, which disregards excess information from the start, leads, in the collisionless limit, to a non-linear equation which has defied nearly all attempts at an exact solution. The method discussed here eliminates only part of the excess information at the start; all N velocity coordinates are retained in the formulation. As a consequence, the equations of the theory are linear and the differential and integral operators are independent of time. However, the number of independent coordinates is not so large as to require the use of any diagrammatic techniques.

The present method appears to be particularly useful for studying plasma in the "collisionless" regime. In the limit $\mathcal{E} \to 0$, the hierarchy of linear equations reduces to a form which can be solved in detail. The solution for any distribution function, which can be written entirely in terms of the initial conditions on the problem, includes all wave-wave interactions. The form of the solution can be simplified by assuming that initially the distribution function may be factored into a product of single-particle functions. It follows directly from the solution that the distribution functions then remain factored at later times.

We have considered in detail the problem of an (initially) small amplitude disturbance in a weakly-unstable plasma. In the limit of small growth rates an expansion in powers of γ can be used to evaluate the fundamental elements of the solution, i.e. the singular velocity integrals. If only the first term in the expansion is retained the resulting approximate solution for the single-particle distribution function is found to be in essential agreement, for an "intermediate" interval of time, with the results of quasi-linear theory.

However, there are many problems of practical interest where γ is not small, and the approximations of Chapters 7 and 8 are no longer valid. In particular, problems with $\gamma \sim 1$ have been discussed in connection with shock waves in plasma guns.[46] The general solution for the single-particle distribution function, obtained by the methods of

Chapters 5-7, is still valid for these values of the growth rate as it includes all wave-coupling effects and has been obtained without the use of any perturbation methods.

However, the solution in its general form is very complex, and some simplifications must be made. For a strongly-unstable plasma, the solution could be simplified by considering the problem of two interpenetrating streams of cold gas. This model has been discussed by Parker[47] in connection with shock fronts in astrophysical problems. The distribution function for each stream becomes, in the limit that the temperature approaches zero, a delta function centered at the velocity of the stream. In this limit, the fundamental units of the solution, the singular velocity integrals, could be evaluated and a simplified form of each term thus obtained. However, the solution would still involve an infinite number of terms, and these would have to be rewritten in closed form before a complete understanding of the solution could be obtained. The solution would be compared directly with the results obtained from the computer experiments of Buneman[48,49] and Dawson.[50] It is planned to investigate this possibility as an extension of the present work.

APPENDIX A

"TWO PARTICLE" PROBLEM

The self-adjoint nature of the operator matrix $\underset{\sim}{V}$ in the equation for the vector $\underset{\sim}{f}(t)$ has been discussed in Chapter 3. In this Appendix, we study in detail the special case N = 2 and obtain explicit solutions for the eigenvectors. The eigenvalues are degenerate in that two eigenvectors are found to correspond to each eigenvalue. Since the eigenvectors are orthogonal and form a complete set, the solution for $\underset{\sim}{h}(t)$ may be expressed as a sum of these eigenvectors with appropriate coefficients. The coefficients which at time t are related in a simple way to the coefficients, at t = 0, are shown to oscillate harmonically in time at their characteristic frequency. It should be stressed that this problem has no real physical significance since the equation for $f^{N,1}(1|t)$ was derived on the basis that terms of $\mathcal{O}(\frac{1}{N})$ could be neglected. The discussion below is meant simply to illustrate some of the mathematical remarks made in Chapter 3 about the properties of the self-adjoint operator $\underset{\sim}{V}$.

The operator $\underset{\sim}{V}$ for the special case N = 2 is reproduced below.

$$\underset{\sim}{V} = \begin{pmatrix} \vec{K}\cdot\vec{v}_1 & -\frac{1}{2}\vec{K}\,\psi(K)\cdot\left(\frac{\partial}{\partial\vec{v}_1}-\frac{\partial}{\partial\vec{v}_2}\right) \\ -\frac{1}{2}\vec{K}\,\psi(K)\cdot\left(\frac{\partial}{\partial\vec{v}_2}-\frac{\partial}{\partial\vec{v}_1}\right) & \vec{K}\cdot\vec{v}_2 \end{pmatrix} \qquad \text{(A-1)}$$

The only component of velocity which appears in this operator is the component parallel to $\vec{K}$. If we denote by u_1 and u_2 the components of $\vec{v}_1$ and $\vec{v}_2$ parallel to the direction $\vec{K}$ and assume that the intermolecular potential is the Coulomb potential ($\psi(K) = \frac{1}{K^2}$) the operator $\underset{\sim}{V}$ becomes

$$\underset{\sim}{V} = \begin{pmatrix} K u_1 & -\frac{1}{2K}\left(\frac{\partial}{\partial u_1}-\frac{\partial}{\partial u_2}\right) \\ -\frac{1}{2K}\left(\frac{\partial}{\partial u_2}-\frac{\partial}{\partial u_1}\right) & K u_2 \end{pmatrix} \qquad \text{(A-2)}$$

The eigenvalue problem we wish to study is formulated as

$$\underset{\sim}{V}\,\underset{\sim}{h}_\lambda = \lambda\,\underset{\sim}{h}_\lambda \qquad \text{(A-3)}$$

where λ is the eigenvalue and $\underset{\sim}{h}$ is the corresponding (two-dimensional) eigenvector

$$h = \begin{pmatrix} h(1) \\ h(2) \end{pmatrix} \qquad \text{(A-4)}$$

If we introduce the new independent variables ξ and η

(A.5)

$$\xi = K(u_1 - u_2)$$
$$\eta = K(u_1 + u_2) \qquad \text{(A-5)}$$

the equations (A-3) become the following

$$\left(\tfrac{1}{2}\xi + \eta - \lambda\right) h(1) - \frac{\partial}{\partial \xi} h(2) = 0$$
$$\frac{\partial}{\partial \xi} h(1) + \left(\eta - \tfrac{1}{2}\xi - \lambda\right) h(2) = 0 \qquad \text{(A-6)}$$

Note that these equations do not contain derivatives with respect to η . Since η is proportional to the total momentum of the (two-particle) system, and is therefore a constant of the motion, we expect it to enter our solution only as a parameter.

The equations (A-6) are to be solved simultaneously for the functions $h(1)$ and $h(2)$. It is convenient to introduce the new dependent variables

$$H^+ = h(1) + h(2)$$
$$H^- = h(1) - h(2) \qquad \text{(A-7)}$$

and rewrite the equations (A-6) as

$$(\eta - \lambda) H^+ + \left(\tfrac{1}{2}\xi + \frac{\partial}{\partial \xi}\right) H^- = 0$$
$$\left(\tfrac{1}{2}\xi - \frac{\partial}{\partial \xi}\right) H^+ + (\eta - \lambda)\, H = 0 \qquad \text{(A-8)}$$

We eliminate the quantity H^- from these equations by multiplying the first by ($\eta-\lambda$), operating on the second by ($\frac{1}{2}\xi + \frac{\partial}{\partial\xi}$) and subtracting.

$$\left[\frac{\partial^2}{\partial\xi^2} - \frac{\xi^2}{4} + \left((\eta-\lambda)^2 - \frac{1}{2}\right)\right] H^+ = 0 \tag{A-9}$$

The solution to equation (A-9) (for m real) is

$$H^+ = D_m(\xi) \tag{A-10}$$

where $D(\xi)$ is the Weber function (see Morse and Feshbach[51]) and

$$m = (\eta-\lambda)^2 - 1 \tag{A-11}$$

Although the function $D_m(\xi)$ is defined for all real values of m, the condition that the eigenfunctions be normalizable restricts us to values of m which are zero or an integer. If m is not zero or an integer the asymptotic behavior of the function $D_m(\xi)$ as $\xi \to \pm\infty$ is

$$D_m(\xi) \to \begin{cases} \xi^m e^{-\xi^2/4} & \text{as} \quad \xi \to +\infty \\ \dfrac{\sqrt{2\pi}}{\Gamma(-m)} \dfrac{e^{\xi^2/4}}{(-\xi)^{m+1}} & \text{as} \quad \xi \to -\infty \end{cases} \tag{A-12}$$

These choices for m must be rejected because of the singularity in the Weber functions as $\xi \to -\infty$. However, if m is zero or a positive integer, the Weber functions are simply related to the Hermite polynomials

$$\mathcal{D}_m(\xi) = \frac{1}{\sqrt{2^m}}\, e^{-\xi^2/4}\, H_m\!\left(\frac{\xi}{\sqrt{2}}\right) \tag{A-13}$$

where the exponential $e^{-\xi^2/4}$ assures proper behavior at the limits $\xi \to \pm\infty$. The Weber functions are orthogonal. We have

$$\int_{-\infty}^{\infty} d\xi\; \mathcal{D}_m(\xi)\mathcal{D}_{m'}(\xi) = \sqrt{2\pi}\; m!\; \delta_{mm'} \tag{A-14}$$

where $\delta_{mm'}$ is the Kronecker delta. Furthermore, the Hermite polynomials $H_m\!\left(\frac{\xi}{\sqrt{2}}\right)$ form a complete set. Thus, it is possible to represent an arbitrary function as a sum of the quantities $\mathcal{D}_m(\xi)$ with appropriate coefficients.

If the solution (A-10) for H^+ is substituted into the second of the equations (A-8) then

$$H^- = \frac{1}{\eta - \lambda}\left(\frac{1}{2}\xi - \frac{\partial}{\partial \xi}\right)\mathcal{D}_m(\xi) \tag{A-15}$$

The following relation for the Weber functions

$$\frac{1}{2}\xi\mathcal{D}_m(\xi) - \frac{\partial}{\partial\xi}\mathcal{D}_m(\xi) = \mathcal{D}_{m+1}(\xi) \tag{A-16}$$

may be used to write

$$H^- = \pm\frac{1}{\left|\sqrt{m+1}\right|}\,\mathcal{D}_{m+1}(\xi) \tag{A-17}$$

Then $\pm$ in (A-17) follows from the fact that the square root of a quantity ($(m+1)$ in this case) may be positive or negative. The result that two solutions H^- exist for each value of m leads to a degeneracy of the eigenvalues; two eigenvectors are obtained for each eigenvalue.

The same symmetry must exist between the elements of the eigenvector $\underset{\sim}{h}$ as between the elements of the vector $\underset{\sim}{f}(t)$. The second element of $\underset{\sim}{h}$ is therefore related to the first by an interchange of the velocities u_1 and u_2. We introduce the variables u_1+u_2 and u_1-u_2 and rewrite the function $h(1)$ as the sum of a term which is even in u_1-u_2 and a term which is odd in u_1-u_2.

$$\begin{aligned} h(1) &= h_e(u_1-u_2) + h_o(u_1-u_2) \\ h(2) &= h_e(u_1-u_2) - h_o(u_1-u_2) \end{aligned} \tag{A-18}$$

The subscripts e and o denote, respectively, functions which are even and odd in the velocity difference u_1-u_2. We conclude from (A-18) that the function H^+ is even and the function H^- is odd in the variable ξ. The Weber functions have the property that $\mathcal{D}_m(\xi)$ is an even function of ξ if m is zero or an even positive integer and an odd function of ξ if m is an odd integer. The condition that H^+ be an even function of ξ then restricts m to values which are zero or even positive integers (see (A-10) and (A-17)).

The eigenvectors $\underset{\sim}{h}_m$ become (for m even)

$$\underset{\sim}{h}_m^{+} = A \begin{pmatrix} \mathcal{D}_m(\xi) + \frac{1}{|\sqrt{m+1}|}\mathcal{D}_{m+1}(\xi) \\ \mathcal{D}_m(\xi) - \frac{1}{|\sqrt{m+1}|}\mathcal{D}_{m+1}(\xi) \end{pmatrix}$$

$$\underset{\sim}{h}_m^{-} = A \begin{pmatrix} \mathcal{D}_m(\xi) - \frac{1}{|\sqrt{m+1}|}\mathcal{D}_{m+1}(\xi) \\ \mathcal{D}_m(\xi) + \frac{1}{|\sqrt{m+1}|}\mathcal{D}_{m+1}(\xi) \end{pmatrix} \tag{A-19}$$

where the superscript + or - follows from the sign in the first element of the vector $\underset{\sim}{h}_m$. The constant A is to be chosen from the normalization of the eigenvectors. Since $\mathcal{D}_{m+1}(\xi)$ is always an odd function of the velocity difference $u_1 - u_2$, the eigenvector $\underset{\sim}{h}_m^{-}$ is related to the eigenvector h_m^{+} by an interchange of the velocities u_1 and u_2.

The orthogonality of two eigenvectors h_m and $h_{m'}$ follows from the orthogonality of the Weber functions (see (A-14)). If we consider the vector inner product of two "plus" eigenvectors, then

$$\int_{-\infty}^{\infty} d\xi \left\{ \underset{\sim}{h}_m^{+}, \underset{\sim}{h}_{m'}^{+} \right\} = 2A^2 \int_{-\infty}^{\infty} d\xi \left(\mathcal{D}_m(\xi)\mathcal{D}_{m'}(\xi) + \frac{1}{|\sqrt{m+1}||\sqrt{m'+1}|}\mathcal{D}_{m+1}(\xi)\mathcal{D}_{m'+1}(\xi) \right)$$

$$= 4A^2 \sqrt{2\pi}\; m!\; \delta_{mm'} \tag{A-20}$$

The identical result is obtained from the vector inner product of two "minus" eigenvectors. There remains the case of the product of a "plus" and a "minus" eigenvector.

$$\int_{-\infty}^{\infty} d\xi \left\{ h_m^{+}, h_{m'}^{-} \right\} = 2A^2 \int_{-\infty}^{\infty} d\xi \left(\mathcal{D}_m(\xi)\mathcal{D}_{m'}(\xi) - \frac{1}{|\sqrt{m+1}||\sqrt{m'+1}|}\mathcal{D}_{m+1}(\xi)\mathcal{D}_{m'+1}(\xi) \right)$$

$$= 0 \tag{A-21}$$

where we have once again used the orthogonality relations (A-14). If the eigenvectors are to be normalized to unity we find from (A-20)

$$A^2 = \frac{1}{4\sqrt{2\pi}\ m!} \tag{A-22}$$

An arbitrary vector $\underset{\sim}{f}(t)$ may be written in terms of these eigenvectors in the following way

$$\underset{\sim}{f}(t) = \sum_{m\ even} \mathcal{B}_m^+(\eta,t)\underset{\sim}{h}_m^+(\xi) + \sum_{m\ even} \mathcal{B}_m^-(\eta,t)\underset{\sim}{h}_m^-(\xi) \tag{A-23}$$

where we have noted explicitly that the coefficients $\mathcal{B}_m$ are functions of η and t. The coefficients are found by taking the vector inner product of (A-23) with an eigenvector, integrating over the variable ξ and using the orthogonality relations (A-20) and (A-21)

$$\mathcal{B}_m^+(\eta,t) = \int_{-\infty}^{\infty} d\xi \left\{\underset{\sim}{f}(t), \underset{\sim}{h}_m^+(\xi)\right\} \quad , \quad \mathcal{B}_m^-(\eta,t) = \int_{-\infty}^{\infty} d\xi \left\{\underset{\sim}{f}(t), \underset{\sim}{h}_m^-(\xi)\right\} \tag{A-24}$$

where we have assumed that the eigenvectors have been normalized to unity.

To determine the time dependence of the coefficients we substitute the expression (A-23) into the differential euqation for $\underset{\sim}{f}(t)$

$$\frac{\partial}{\partial t}\underset{\sim}{f}(t) + i\underset{\sim}{V}\,\underset{\sim}{f}(t) = 0 \tag{A-25}$$

The orthogonality of the eigenvectors and the relation (A-3) are used to write

$$\frac{\partial}{\partial t}\mathcal{B}^{\pm}(\eta,t) + i\lambda^{\pm}\mathcal{B}^{\pm}(\eta,t) = 0 \tag{A-26}$$

The solution to (A-26) is

$$\mathcal{B}^{\pm}(\eta,t) = \mathcal{B}^{\pm}(\eta)\,e^{-i\lambda^{\pm}t} \tag{A-27}$$

where

$$\lambda^{\pm} = \eta \mp |\sqrt{m+1}| \tag{A-28}$$

The oscillation frequencies of the normal modes are found to vary with the total momentum (proportional to η) of the two particles. The coefficients $\mathcal{B}_m^{+}(\eta)$ and $\mathcal{B}_m^{-}(\eta)$ may be found from the initial value of the vector $\underset{\sim}{f}$ by (A-23). The solution for all later times is then found by substituting (A-27) into (A-23).

It is interesting to note that the independent variable ξ is the product of the wave number K and the velocity difference $u_1 - u_2$. If the eigenvectors are viewed as functions of velocity then the "spread" of each eigenvector in velocity space will vary with the wave number K . In the short wavelength ($K \to \infty$) limit, the eigenvectors become concentrated near the origin $u_1 - u_2 = 0$ because of the factor $e^{-\xi^2/4}$. We expect that as $K \to \infty$ more and more of these functions will be needed to represent some function with a finite spread in the velocity variable $u_1 - u_2$. Since each mode oscillates at its own frequency, the larger the number of modes, the larger the number of frequencies that are present and the more quickly we can expect the coherence of the disturbance to disappear by a phase mixing process.

APPENDIX B

CORRECTIONS TO THE LANDAU SOLUTION FOR $f_{\vec{k}}^{N,1}(1|t)$

The solution of the homogeneous equation for $f_{\vec{k}}^{N,1}(1|t)$, when integrated over all velocities except $\vec{v}_1$, is found to agree with the result obtained by Landau only after some terms of $\mathcal{O}(\frac{1}{N})$ have been discarded. The solution as written in equation (3.2-20) is a series in ascending powers of the quantity $L(\vec{k},p)$. The number of terms that must be discarded increases as one goes to higher powers of $L(\vec{k},p)$. Eventually, the number of "correction" terms becomes so large that despite the fact that each is only of $\mathcal{O}(\frac{1}{N})$ their total contribution can be as great as that of the terms retained. We take a term by term inverse Laplace transform of the solution for $f_{\vec{k}}(1|t)$ and estimate the time t at which the ν th term in the series becomes of $\mathcal{O}(1)$. If the ν th term is the one in which the correction terms have the same contribution as those retained then t characterizes the time at which the correction terms begin to exert a significant influence upon the solution.

We consider the dominant contribution of these correction terms. The first element in the third term of the solution (3.2-15) for the vector $\underset{\sim}{f}^{N,1}(p)$ is

$$\sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \sum_\ell \frac{\frac{i}{N}D_{\vec{k}}(j\ell)}{p+i\vec{k}\cdot\vec{v}_j} \frac{f_{\vec{k}}^{N,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell} =$$

$$= \sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \sum_{\ell\neq 1} \frac{\frac{i}{N}D_{\vec{k}}(j\ell)}{p+i\vec{k}\cdot\vec{v}_j} \frac{f_{\vec{k}}^{N,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell} + \sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \frac{\frac{i}{N}D_{\vec{k}}(j1)}{p+i\vec{k}\cdot\vec{v}_j} \frac{f_{\vec{k}}^{N,1}(1|t=0)}{p+i\vec{k}\cdot\vec{v}_1} \tag{B-1}$$

where we have separated the correction terms from the "Landaü" part of the solution. The correction terms are distinguished by the repetition of the index 1 in the operators $D_{\vec{k}}(ij)$. There are (N-1) such terms, each of which is of $\mathcal{O}(\frac{1}{N^2})$. The fourth term in the solution for $f_{\vec{k}}^{N,1}(1|p)$ contains three different terms where there is a repeated index. These are

$$\sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \frac{\frac{i}{N}D_{\vec{k}}(j1)}{p+i\vec{k}\cdot\vec{v}_j} \sum_\ell \frac{\frac{i}{N}D_{\vec{k}}(1\ell)}{p+i\vec{k}\cdot\vec{v}_1} \frac{f_{\vec{k}}^{N,1}(\ell|t=0)}{p+i\vec{k}\cdot\vec{v}_\ell}$$

$$\sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \sum_\ell \frac{\frac{i}{N}D_{\vec{k}}(j\ell)}{p+i\vec{k}\cdot\vec{v}_j} \frac{\frac{i}{N}D_{\vec{k}}(\ell 1)}{p+i\vec{k}\cdot\vec{v}_\ell} \frac{f_{\vec{k}}^{N,1}(1|t=0)}{p+i\vec{k}\cdot\vec{v}_1} \tag{B-2}$$

$$\sum_j \frac{\frac{i}{N}D_{\vec{k}}(1j)}{p+i\vec{k}\cdot\vec{v}_1} \sum_\ell \frac{\frac{i}{N}D_{\vec{k}}(j\ell)}{p+i\vec{k}\cdot\vec{v}_j} \frac{\frac{i}{N}D_{\vec{k}}(\ell j)}{p+i\vec{k}\cdot\vec{v}_\ell} \frac{f_{\vec{k}}^{N,1}(j|t=0)}{p+i\vec{k}\cdot\vec{v}_j}$$

There are six different types of times with one repeated index in the fifth term of $f_{\vec{k}}^{N,1}(1|p)$. In addition, there are some terms with two repeated indices. Every time an index is repeated the number of summations decreases by one. In the fifth term of $f_{\vec{k}}^{N,1}(1|p)$ there are N^4 terms (of $\mathcal{O}(\frac{1}{N^3})$) with one repeated index while there are only N^3 terms with two repeated indices. The terms with one repeated index are assumed to represent the dominant correction to the Landau solution.

There are, in general, $\frac{1}{2}(\nu-1)(\nu-2)N^{\nu-1}$ terms with one repeated index in the ν th term of the solution for $f_{\vec{K}}^{N,1}(1|p)$. Since each term is of $\mathcal{O}\left(\frac{1}{N^{\nu}}\right)$ the correction terms will have a total contribution of $\mathcal{O}\left(\frac{1}{2N}(\nu-1)(\nu-2)\right)$ compared with the "Landau" terms which have a contribution of $\mathcal{O}(1)$. These two types of terms contribute equally to the solution when $\nu \sim \sqrt{N}$.

In order to estimate the time at which the ν th term becomes of $\mathcal{O}(1)$ we consider the solution for $f_{\vec{K}}(1|p)$ in the form (3.2-20) and assume that the function $\phi(\vec{v})$ may be represented by a delta function in velocity space. The quantity $L(\vec{K},p)$ becomes

$$L(\vec{K},p) = \int \frac{i\vec{K}\psi(K)\cdot\frac{\partial\phi(1)}{\partial\vec{v}_1}}{p+i\vec{K}\cdot\vec{v}_1}\,d\vec{v}_1 = -\frac{K^2\psi(K)}{p^2} \tag{B-3}$$

If the intermolecular potential is assumed to be the Coulomb potential ($\psi(K) = \frac{1}{K^2}$) then the series in powers of $L(\vec{K},p)$ (equation (3.2-20)) becomes

$$1 + L(\vec{K},p) + L^2(\vec{K},p) + \cdots = 1 - \frac{1}{p^2} + \frac{1}{p^4} - \cdots \tag{B-4}$$

If we take a term by term inverse Laplace transform (the series (B-4) is multiplied by the quantity $\frac{i\vec{K}\psi(K)\cdot\frac{\partial\phi}{\partial\vec{v}_1}}{p+i\vec{K}\cdot\vec{v}_1}\int d\vec{v}_1 \frac{f_{\vec{K}}(1|t=0)}{p+i\vec{K}\cdot\vec{v}_1}$)

we obtain a series solution in which the ν^{th} term contains the factor $t^{2\nu-1}/(2\nu-1)!$ (the inverse Laplace transform of $1/p^n$ is $t^{n-1}/(n-1)!$). This quantity which is small initially will be of $\mathcal{O}(1)$ when

$$t^{2\nu-1} \sim (2\nu-1)! \tag{B-5}$$

If Stirling's approximation is used for the factorial we find in the limit of large ν

$$(2\nu-1)\ln t \sim (2\nu-1)\ln(2\nu-1)$$

or

$$t \sim 2\nu \tag{B-6}$$

We have shown above that the correction terms will have the same order of magnitude as those retained when $\nu \sim \sqrt{N}$. Comparing this result with (B-6) we see that the correction terms can be expected to contribute significantly to the solution for $f_{\vec{K}}(1|t)$ when $t \sim \sqrt{N}$. When we remember that time has been normalized with respect to the plasma frequency we see that the solution for $f_{\vec{K}}(1|t) = \int (d\vec{v})^{N-1} f_{\vec{K}}^{N,1}(1|t)$ agrees with the Landau solution for times which are less than $\sqrt{N}$ plasma periods.

APPENDIX C

MULTIPLE OPERATORS

C.1 Introduction

We have shown in Chapter 5 that the term $\mathcal{P}_K(1|p)\,\gamma^{N,1}(1)$ reduces, upon an integration over (N-1) velocities, to different forms which depend upon the nature of the function $\gamma^{N,1}(1)$. The reduced function $\gamma^{S,1}(1) = \int (d\vec{v})^{N-S}\,\gamma^{N,1}(1)$ is required to be symmetric in the velocities in the set $\{S-1\}$. However, this symmetry may be obtained from a summation of $S-1$ functions of the form $h_1^{S,1}(1|i)$ or from a double sum over i and j in the set $\{S-1\}$ of the functions $h_2^{S,1}(1|ij)$, and so on. The function $h_1^{S,1}(1|i)$ is not, in general, symmetric to an interchange of the index i with any other index in the set $\{S-1\}$, and $h_2^{S,1}(1|i,j)$ is not symmetric to interchange of i and j with any other index in the set $\{S-1\}$. We can imagine a hierarchy of these functions where the required symmetry of each number $\gamma_n^{S,1}(1)$ is obtained from n summations over the set $\{S-1\}$.

$$\gamma_n^{S,1}(1) = \sum_i \sum_j^{\{S-1\}} \cdots \sum_n h_n^{S,1}(1|ij\cdots n) \tag{C.1-1}$$

We consider only the lowest members of the hierarchy in this Appendix. The extension of the results to other members may be accomplished entirely by the methods which are presented.

We consider two problems. First, the hierarchy of functions $\gamma_n^{N,1}(1)$ is generalized in Section C.2 to include functions of the form $\gamma_n^{N,2}(12)$ and $\gamma_n^{N,3}(123)$ which are symmetric in the indices 1,2 and the indices 1,2,3, respectively. We consider the product of two $\mathcal{P}_{\vec{K}}(t)$ operators with the function $\gamma_n^{N,2}(12)$ and of three $\mathcal{P}_{\vec{K}}(t)$ operators with the function $\gamma_n^{N,3}(123)$. The forms to which these terms reduce upon an integration over the extra velocities is determined. The extension of these results to ν $\mathcal{P}_{\vec{K}}(t)$ operators and the function $\gamma_n^{N,\nu}(\{\nu\})$ is straightforward. The functions $\gamma_n^{S,\nu}(\{\nu\})$ are assumed in Section C.2 to possess a given set of properties. The second problem is to show that the functions $\gamma_n^{N,\nu}(\{\nu\})$ which arise in the solutions for $f(1)$ and $f^{2,2}(12)$ have these stated properties. This is the business of Section C.3.

C.2 Reduction with Multiple Operators

The hierarchy of functions $\gamma_n^{N,1}(1)$ discussed in Chapter 5 is generalized to include functions with more than one spatial variable. The generalized functions $\gamma_n^{N,\nu}(\{\nu\}|t)$ are, in general, functions of time. The reduced functions $\gamma_n^{S,\nu}(\{\nu\}|t)$ obtained from $\gamma_n^{N,\nu}(\{\nu\}|t)$ by an integration over $N-S$ velocities in the set $\{N-\nu\}$ have the following properties.

a) The function $\gamma_0^{S,\nu}(\{\nu\}|t)$ is symmetric in all velocities in the set $\{S-\nu\}$ and may be written as a product of the function $h_0^{\nu,\nu}(\{\nu\}|t)$ and $S-\nu$ functions of velocity $\varphi(\vec{v})$.

$$\gamma_0^{S,\nu}(\{\nu\}|t) = h_0^{\nu,\nu}(\{\nu\}|t) \prod_j^{\{S-\nu\}} \varphi(j) \qquad \text{(C.2-1)}$$

The functions $\varphi(j)$ are independent of time.

b) The function $\gamma_1^{S,\nu}(\{\nu\}|t)$ may be written as the sum of $S-\nu$ terms

$$\gamma_1^{S,\nu}(\{\nu\}|t) = \sum_i^{\{S-\nu\}} h_1^{S,\nu}(\{\nu\}|i|t) \qquad \text{(C.2-2)}$$

where $h_1^{S,\nu}(\{\nu\}|i|t)$ is not symmetric to the interchange of i with any other index in the set $\{S-\nu\}$. This function may in general be factored in the following way

$$h_1^{S,\nu}(\{\nu\}|i|t) = h_1^{\nu+1,\nu}(\{\nu\}|i|t) \prod_{j\neq i}^{\{S-\nu\}} \varphi(j) \qquad \text{(C.2-3)}$$

The function $h_1^{\nu+1,\nu}(\{\nu\}|i|t)$ has the property that

$$\int d\vec{v}_i\, h_1^{\nu+1,\nu}(\{\nu\}|i|t) = 0 \qquad \text{(C.2-4)}$$

c) The function $\gamma_2^{S,\nu}(\{\nu\}|t)$ may be written as a double sum

$$\gamma_2^{S,\nu}(\{\nu\}|t) = \sum_{i \neq \ell}^{\{S-\nu\}}\sum h_2^{S,\nu}(\{\nu\}|i,\ell|t) \qquad \text{(C.2-5)}$$

where $h_2^{S,\nu}(\{\nu\}|i,\ell|t)$ is not symmetric to the interchange of the indices i and ℓ, or to the interchange of either of these indices with another index in the set $\{S-\nu\}$. The function $h_2^{S,\nu}(\{\nu\}|i,\ell|t)$ may be factored as

$$h_2^{S,\nu}(\{\nu\}|i,\ell|t) = h_2^{\nu+2,\nu}(\{\nu\}|i,\ell|t)\prod_{j \neq i,\ell}^{\{S-\nu\}} \varphi(j) \qquad \text{(C.2-6)}$$

This function also has the property

$$\int d\vec{v}_i\, h_2^{\nu+2,\nu}(\{\nu\}|i,\ell|t) = \int d\vec{v}_\ell\, h_2^{\nu+2,\nu}(\{\nu\}|i,\ell|t) = 0 \qquad \text{(C.2-7)}$$

The extension of these results to other functions $\gamma_n^{S,\nu}(\{\nu\}|t)$ of the hierarchy is straightforward.

<u>C.2.1</u>. The first example to be studied is

$$\int (d\vec{v})^{N-2} \rho_{\vec{K}_1}(1|t)\,\rho_{\vec{K}_2}(2|t)\,\gamma_0^{N,2}(12) \qquad \text{(C.2.1-1)}$$

The notation $(d\vec{v})^{N-2}$ denotes the element $d\vec{v}_3\, d\vec{v}_4 \cdots d\vec{v}_N$

in velocity space. The function $\gamma_0^{N,2}(12)$ is not in this case a function of time. Since the $\mathcal{P}_{\vec{K}}(1|t)$ operator is known explicitly only in terms of the Laplace transform variable p, we take a Laplace transform of (C.2.1-1) and use the convolution theorem to write

$$\int_0^t d\tau\, e^{-pt} \int (d\vec{v})\, \mathcal{P}_{\vec{K}_1}(1|\tau)\, \mathcal{P}_{\vec{K}_2}(2|\tau)\, \gamma_0^{N,2}(12) =$$

$$= \frac{1}{2\pi i} \int_{C_1} dp_1 \int (d\vec{v})^{N-2}\, \mathcal{P}_{\vec{K}_1}(1|p_1)\, \mathcal{P}_{\vec{K}_2}(2|p-p_1)\, \gamma_0^{N,2}(12) \qquad \text{(C.2.1-2)}$$

The operator $\mathcal{P}_{\vec{K}_2}(2|p-p_1)$ is, from (3.3-5)

$$\mathcal{P}_{\vec{K}_2}(2|p-p_1) = \frac{1}{p - p_1 + i\vec{K}_2\cdot\vec{v}_2}$$

$$+ \sum_i \frac{\frac{i}{N} D_{\vec{K}_2}(2i)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}\, \frac{(2 \leftrightarrow i)}{p-p_1+i\vec{K}_2\cdot\vec{v}_i} \qquad \text{(C.2.1-3)}$$

$$+ \sum_i \frac{\frac{i}{N} D_{\vec{K}_2}(2i)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2} \sum_j \frac{\frac{i}{N} D_{\vec{K}_2}(ij)}{p-p_1+i\vec{K}_2\cdot\vec{v}_i}\, \frac{(2 \leftrightarrow j)}{p-p_1+i\vec{K}_2\cdot\vec{v}_j} + \cdots$$

The products of these two operators $\mathcal{P}_{\vec{K}_1}(1|p_1)$ and $\mathcal{P}_{\vec{K}_2}(2|p-p_1)$ may be written as

$$\mathcal{P}_{\vec{K}_1}(1|p_1)\, \mathcal{P}_{\vec{K}_2}(2|p-p_1) = \sum_{\alpha,\beta=0}^{\infty}\sum I_{\alpha\beta} \qquad \text{(C.2.1-4)}$$

where

$$I_{00} = \frac{1}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1}\,\frac{1}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_2}$$

$$I_{10} = \sum_i \frac{\frac{i}{N} D_{\vec{K}_1}(1i)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1}\,\frac{(1\leftrightarrow i)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_i}\,\frac{1}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_2}$$

$$I_{01} = \frac{1}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1}\sum_i \frac{\frac{i}{N} D_{\vec{K}_2}(2i)}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_2}\,\frac{(2\leftrightarrow i)}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_i}$$

In general

$$I_{\alpha\beta} = \Bigg[\underbrace{\sum_i \frac{\frac{i}{N} D_{\vec{K}_1}(1i)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1} \;\cdot\;\cdot\;\cdot\;\cdot\; \sum_m \frac{\frac{i}{N} D_{\vec{K}_1}(\ell m)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_\ell}}_{\alpha \text{ summations}}\, \frac{(1\leftrightarrow m)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_m}\Bigg]\times \qquad \text{(C.2.1-5)}$$

$$\times\Bigg[\underbrace{\sum_i \frac{\frac{i}{N} D_{\vec{K}_2}(2i)}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_2} \;\cdot\;\cdot\;\cdot\; \sum_n \frac{\frac{i}{N} D_{\vec{K}_2}(mn)}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_m}}_{\beta \text{ summations}}\, \frac{(2\leftrightarrow n)}{\rho - \rho_1 + i\vec{K}_2\cdot\vec{v}_n}\Bigg]$$

If the factored form (C.2-1) of the function $\gamma_0^{\nu,2}(12)$ is used, then

$$\int (d\vec{v})^{N-2} I_{00}\, \gamma_0^{N,2}(12) = \frac{h_0^{2,2}(12)}{(p_1+i\vec{K}_1\cdot\vec{v}_1)(p-p_1+i\vec{K}_2\cdot\vec{v}_2)}$$

$$\int (d\vec{v})^{N-2} I_{10}\, \gamma_0^{N,2}(12) = \frac{i\mathcal{D}_{\vec{K}_1}(1)\,\phi(1)}{p_1+i\vec{K}_1\cdot\vec{v}_1}\int d\vec{v}_1 \frac{h_0^{2,2}(12)}{(p_1+i\vec{K}_1\cdot\vec{v}_1)(p-p_1+i\vec{K}_2\cdot\vec{v}_2)}$$

$$\int (d\vec{v})^{N-2} I_{01}\, \gamma_0^{N,2}(12) = \frac{1}{p_1+i\vec{K}_1\cdot\vec{v}_1}\,\frac{i\mathcal{D}_{\vec{K}_2}(2)\,\phi(2)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}\int d\vec{v}_2 \frac{h_0^{2,2}(12)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}$$

$$\int (d\vec{v})^{N-2} I_{\alpha\beta}\, \gamma^{N,2}(12) = \frac{i\mathcal{D}_{\vec{K}_1}(1)\,\phi(1)}{p_1+i\vec{K}_1\cdot\vec{v}_1}\,\frac{i\mathcal{D}_{\vec{K}_2}(2)\,\phi(2)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}\,\times \tag{C.2.1-6}$$

$$\times\int d\vec{v}_1\, d\vec{v}_2 \frac{h_0^{2,2}(12)}{(p_1+i\vec{K}_1\cdot\vec{v}_1)(p-p_1+i\vec{K}_2\cdot\vec{v}_2)}\left(L(\vec{K}_1,p_1)\right)^{\alpha-1}\left(L(\vec{K}_2,p-p_1)\right)^{\beta-1}$$

It is not difficult to show from these results that

$$\int (d\vec{v})^{N-2} \sum_{\alpha,\beta=0}^{\infty} I_{\alpha\beta}\, \gamma_0^{N,2}(12) = P_{\vec{K}_1}(1|p_1)\, P_{\vec{K}_2}(2|p-p_1)\, h_0^{2,2}(12) \tag{C.2.1-7}$$

where the Landaue operator $P_{\vec{K}_1}(1|p_1)$ has been defined in Chapter 3 (3.3-7). We take an inverse Laplace transform to establish the final result

$$\int (d\vec{v})^{N-2}\, \mathcal{P}_{\vec{K}_1}(1|t)\, \mathcal{P}_{\vec{K}_2}(2|t)\, \gamma_0^{N,2}(12) = P_{K_1}(1|t)\, P_{K_2}(2|t)\, h_0^{2,2}(12) \tag{C.2.1-8}$$

C.2.2. The next member of the hierarchy is $\gamma_1^{N,2}(12|t)$. The product of two $\mathcal{P}_{\vec{K}}(t)$ operators and $\gamma_1^{N,2}(12|t)$ always appears as a convolution integral of the following form.

$$\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\, \mathcal{P}_{\vec{K}_2}(2|t-\tau)\, \gamma_1^{N,2}(12|\tau) \qquad \text{(C.2.2-1)}$$

A Laplace transform is taken and the convolution theorem used to rewrite (C.2.2-1) as

$$\frac{1}{2\pi i}\int_{C_1} d\rho_1\, \mathcal{P}_{\vec{K}_1}(1|\rho_1)\, \mathcal{P}_{\vec{K}_2}(2|\rho-\rho_1)\, \gamma_1^{N,2}(12|\rho) \qquad \text{(C.2.2-2)}$$

If (C.2.2-2) is integrated over the velocities $\vec{v}_2 \ldots \vec{v}_N$ and expressions of the form (C.2.1-3) used for the $\mathcal{P}_{\vec{K}}(\rho)$ operators the terms of (C.2.2-2) may be written, as in section (C.2.1)

$$\int (d\vec{v})^{N-2} \sum_{\alpha,\beta=0}^{\infty}\sum I_{\alpha\beta}\; h_1^{N,2}(12|\rho) \qquad \text{(C.2.2-3)}$$

The first term of (C.2.2-3) ($\alpha=\beta=0$) vanishes by the condition (C.2-4) on the function $\gamma_1^{N,2}(12|\rho)$. We find from the relation (C.2-3)

$$\int (d\vec{v})^{N-2} I_{10}\, \gamma_1^{N,2}(12|\rho) = \frac{i\mathcal{D}_{\vec{K}_1}(1)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1}\int \frac{d\vec{v}_i}{\rho_1 + i\vec{K}_1\cdot\vec{v}_i}\,\frac{h_1^{3,2}(i2|1|\rho)}{\rho-\rho_1+i\vec{K}_2\cdot\vec{v}_2}$$

$$\int (d\vec{v})^{N-2} I_{20}\, \gamma_1^{N,2}(12|\rho) = \frac{i\mathcal{D}_{\vec{K}_1}(1)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1}\int \frac{i\mathcal{D}_{\vec{K}_1}(j)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_j}\,\frac{d\vec{v}_i\, d\vec{v}_j}{\rho_1 + i\vec{K}_1\cdot\vec{v}_i} \times \qquad \text{(C.2.2-4)}$$

$$\times\; \frac{\varphi(j)\,h_1^{3,2}(i2|1|\rho) + \varphi(1)\,h_1^{3,2}(i2|j|\rho)}{\rho-\rho_1+i\vec{K}_2\cdot\vec{v}_2}$$

In general

$$\int (d\vec{v})^{N-2} I_{\alpha\beta}\, \gamma_1^{N,2}(12|p) =$$

$$= \frac{iD_{\vec{K}_1}(1)}{p_1+i\vec{K}_1\cdot\vec{v}_1}\,\frac{iD_{\vec{K}_2}(2)\varphi(2)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}\int\frac{iD_{\vec{K}_1}(j)}{p_1+i\vec{K}_1\cdot\vec{v}_j}\,\frac{d\vec{v}_i}{p_1+i\vec{K}_1\cdot\vec{v}_i}\,\frac{d\vec{v}_j\,d\vec{v}_\ell}{p-p_1+i\vec{K}_2\cdot\vec{v}_\ell}\,\times$$

$$\times\left(L(\vec{K}_1,p_1)\right)^{\alpha-2}\left(L(\vec{K}_2,p-p_1)\right)^{\beta-1}\left[\varphi(j)h_1^{3,2}(i\ell|1|p)+(\alpha-1)\varphi(1)h_1^{3,2}(i\ell|j|p)\right]$$

$$+\frac{iD_{\vec{K}_1}(1)\varphi(1)}{p_1+i\vec{K}_1\cdot\vec{v}_1}\,\frac{iD_{\vec{K}_2}(2)}{p-p_2+i\vec{K}_2\cdot\vec{v}_2}\int\frac{iD_{\vec{K}_2}(j)}{p-p_1+i\vec{K}_2\cdot\vec{v}_j}\,\frac{d\vec{v}_i}{p_1+i\vec{K}_1\cdot\vec{v}_i}\,\frac{d\vec{v}_j\,d\vec{v}_\ell}{p-p_1+i\vec{K}_2\cdot\vec{v}_\ell}\,\times \qquad (C.2.2\text{-}5)$$

$$\times\left(L(\vec{K}_1,p_1)\right)^{\alpha-1}\left(L(\vec{K}_2,p-p_1)\right)^{\beta-2}\left[\varphi(j)h_1^{3,2}(i\ell|2|p)+(\beta-1)\varphi(2)h_1^{3,2}(i\ell|j|p)\right]$$

The sum of the terms (C.2.2-3) can be shown to be equal to

$$P_{\vec{K}_1}(1|p_1)S_{\vec{K}_2}(2,i|p-p_1)h_1^{3,2}(1,i|2|p)+S_{\vec{K}_1}(1,i|p_1)P_{\vec{K}_2}(2|p-p_1)h_1^{3,2}(i2|1|p) \qquad (C.2.2\text{-}6)$$

and we find from the inverse Laplace transform of (C.2.2-6)

$$\int_0^t d\tau\int(d\vec{v})^{N-2}\,\mathcal{P}_{\vec{K}_1}(1|t-\tau)\,\mathcal{P}_{\vec{K}_2}(2|t-\tau)\,\gamma_1^{N,2}(12|\tau) =$$

$$=\int_0^t d\tau\Big(P_{\vec{K}_1}(1|t-\tau)\,S_{\vec{K}_2}(2,i|t-\tau)\,h_1^{3,2}(1i|2|\tau) \qquad (C.2.2\text{-}7)$$

$$+S_{\vec{K}_1}(1,i|t-\tau)\,P_{\vec{K}_2}(2|t-\tau)\,h_1^{3,2}(i2|1|\tau)\Big)$$

C.2.3. The next member of the hierarchy is $\gamma_2^{N,2}(12|t)$. Once again this function always appears with the $\mathcal{P}_{\vec{K}}(t)$ operators in a convolution integral.

$$\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\,\mathcal{P}_{\vec{K}_2}(2|t-\tau)\,\gamma_2^{N,2}(12|\tau) \tag{C.2.3-1}$$

We take a Laplace transform of (C.2.3-1), integrate over the velocities $\vec{v}_3 \cdots \vec{v}_N$ and use the form (C.2.1-3) for the operators to write

$$\int (d\vec{v})^{N-2} \int_0^\infty dt\, e^{-pt} \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\,\mathcal{P}_{\vec{K}_2}(2|t-\tau)\,\gamma_2^{N,2}(12|\tau) =$$

$$= \frac{1}{2\pi i}\int_{C_1} dp_1 \int (d\vec{v}_1)^{N-2} \sum_{\alpha,\beta=0}^{\infty} I_{\alpha\beta}\, \gamma_2^{N,2}(12|p) \tag{C.2.3-2}$$

The first three terms ($\alpha=0, \beta=0$; $\alpha=1, \beta=0$; $\alpha=0, \beta=1$) of (C.2.3-2) are zero by the condition (C.2-7). For $\alpha = 0$, $\beta \geq 2$, we find

$$\beta \geq 2: \quad \int (d\vec{v})^{N-2}\, I_{0\beta}\, \gamma_2^{N,2}(12|p) =$$

$$= \frac{i\mathcal{D}_{\vec{K}_1}(1)}{p_1 + i\vec{K}_1\cdot\vec{v}_1}\,\frac{i\mathcal{D}_{\vec{K}_2}(2)}{p-p_1+i\vec{K}_2\cdot\vec{v}_2}\int d\vec{v}_i\, d\vec{v}_j\, d\vec{v}_\ell\, \frac{i\mathcal{D}_{\vec{K}_2}(j)}{p-p_1+i\vec{K}_2\cdot\vec{v}_j}\,\frac{i\mathcal{D}_{\vec{K}_2}(\ell)}{p-p_1+i\vec{K}_2\cdot\vec{v}_\ell}\, \times$$

$$\times \frac{\left(L(\vec{K}_2, p-p_1)\right)^{\beta-2}}{p-p_1+i\vec{K}_2\cdot\vec{v}_j} \left\{ \begin{aligned} &(\beta-1)\varphi(\ell)\left(h_2^{4,2}(1i|2j|p) + h_2^{4,2}(1i|j,2|p)\right) \\ &+(\beta-1)(\beta-2)\varphi(2)\, h_2^{4,2}(1i|j,\ell|p) \end{aligned} \right\} \tag{C.2.3-3}$$

The terms for $\beta = 0$, $\alpha \geq 2$ may be obtained from (C.2.3-3) by interchanging ρ_1 and $\rho - \rho_1$, α and β , and the indices 1 and 2. The general term for $\alpha, \beta \geq 1$ is not shown; its form may be determined by the methods discussed above. When these results are combined we find

$$\int_0^t d\tau \int (d\vec{v})^{N-2} \mathcal{P}_{\vec{K}_1}(1|t-\tau)\, \mathcal{P}_{\vec{K}_2}(2|t-\tau)\, \gamma_2^{N,2}(12|\tau) =$$

$$\begin{aligned} &= \int_0^t d\tau \Big[\mathcal{P}_{\vec{K}_1}(1|t-\tau)\, S^{(1)}_{\vec{K}_2}(2,i|j|t-\tau) \Big(h_2^{4,2}(1,i|2,j|\tau) + h_2^{4,2}(1,i|j,2|\tau) \Big) \\ &+ S_{\vec{K}_1}(1,i|t-\tau)\, S_{\vec{K}_2}(2,j|t-\tau) \Big(h_2^{4,2}(ij|1,2|\tau) + h_2^{4,2}(ij|2,1|\tau) \Big) \\ &+ S^{(1)}_{\vec{K}_1}(1,i|j|t-\tau)\, \mathcal{P}_{\vec{K}_2}(2|t-\tau) \Big(h_2^{4,2}(i2|1,j|\tau) + h_2^{4,2}(i2|j,1|\tau) \Big) \Big] \end{aligned} \qquad \text{(C.2.3-4)}$$

These same methods may be used to extend these results to other members of the hierarchy of functions $\gamma_n^{N,2}(12)$.

<u>C.2.4</u>. The problem of three $\mathcal{P}_{\vec{K}}(t)$ operators is handled in much the same way as was the case of the two operators. We consider the simplest case of the function $\gamma_0^{N,3}(123)$. This function is in all cases independent of time and the problem is formulated as

$$\mathcal{P}_{\vec{K}_1}(1|t)\, \mathcal{P}_{\vec{K}_2}(2|t)\, \mathcal{P}_{\vec{K}_3}(3|t)\, \gamma_0^{N,3}(123) \qquad \text{(C.2.4-1)}$$

We take a Laplace transform and use the convolution theorem to write (C.2.4-1) as a double convolution integral. Substitute the form (C.2.1-3) for the operators $\mathcal{P}_{\vec{K}}(p)$ and integrate over the velocities (N-3) to write

$$\int_0^t dt\, e^{-pt} \int (d\vec{v})^{N-3} \mathcal{P}_{\vec{K}_1}(1|t)\mathcal{P}_{\vec{K}_2}(2|t)\mathcal{P}_{\vec{K}_3}(3|t)\, \gamma_0^{N,3}(123) =$$

(C.2.4-2)

$$= \frac{1}{(2\pi i)^2}\int_{C_1} dp_1 \int_{C_2} dp_2 \int (d\vec{v})^{N-3} \sum_{\alpha,\beta,\gamma=0}^{\infty}\sum\sum I_{\alpha\beta\gamma}\, \gamma_0^{N,3}(123)$$

where the element in velocity space is now $(d\vec{v})^{N-3} = d\vec{v}_4 \cdots d\vec{v}_N$. The first index α is associated with $\mathcal{P}_{\vec{K}_1}(1|p_1)$, the second β with $\mathcal{P}_{\vec{K}_2}(2|p_2)$ and the third γ with $\mathcal{P}_{\vec{K}_3}(3|p-p_1-p_2)$. We have

$$\int (d\vec{v})^{N-3} I_{000}\, \gamma_0^{N,3}(123) =$$

$$= \frac{h_0^{3,3}(123)}{(p_1+i\vec{K}_1\cdot\vec{v}_1)(p_2+i\vec{K}_2\cdot\vec{v}_2)(p-p_1-p_2+i\vec{K}_3\cdot\vec{v}_3)}$$

$$\int (d\vec{v})^{N-3} I_{100}\, \gamma_0^{N,3}(123) =$$

$$= \frac{i D_{\vec{K}_1}(1)\varphi(1)}{p_1+i\vec{K}_1\cdot\vec{v}_1}\int d\vec{v}_1 \frac{h_0^{3,3}(123)}{(p_1+i\vec{K}_1\cdot\vec{v}_1)(p_2+i\vec{K}_2\cdot\vec{v}_2)(p-p_1-p_2+i\vec{K}_3\cdot\vec{v}_3)}$$

In general, for $\alpha, \beta, \gamma \neq 0$, we find

$$\int (d\vec{v})^{N-3} I_{\alpha\beta\gamma}\, \gamma_0^{N,3}(123) = \frac{iD_{\vec{K}_1}(1)\varphi(1)}{\rho_1 + i\vec{K}_1\cdot\vec{v}_1} \frac{iD_{\vec{K}_2}(2)\varphi(2)}{\rho_2 + i\vec{K}_2\cdot\vec{v}_2} \frac{iD_{\vec{K}_3}(3)\varphi(3)}{\rho - \rho_1 - \rho_2 + i\vec{K}_3\cdot\vec{v}_3} \times$$

$$\times \int d\vec{v}_1 d\vec{v}_2 d\vec{v}_3 \frac{(L(\vec{K}_1,\rho_1))^{\alpha-1}(L(\vec{K}_2,\rho_2))^{\beta-1}(L(\vec{K}_3,\rho-\rho_1-\rho_2))^{\gamma-1} h_0^{3,3}(123)}{(\rho_1 + i\vec{K}_1\cdot\vec{v}_1)(\rho_2 + i\vec{K}_2\cdot\vec{v}_2)(\rho-\rho_1-\rho_2 + i\vec{K}_3\cdot\vec{v}_3)} \qquad \text{(C.2.4-3)}$$

The terms (C.2.4-3), when summed, are equal to the product of three $\mathcal{P}_{\vec{K}}(\rho)$ operators. We find, upon taking an inverse Laplace transform

$$\int (d\vec{v})^{N-3} \mathcal{P}_{\vec{K}_1}(1|t)\mathcal{P}_{\vec{K}_2}(2|t)\mathcal{P}_{\vec{K}_3}(3|t)\, \gamma_0^{N,3}(123) =$$

$$= \mathcal{P}_{\vec{K}_1}(1|t)\,\mathcal{P}_{\vec{K}_2}(2|t)\,\mathcal{P}_{\vec{K}_3}(3|t)\, h_0^{3,3}(123) \qquad \text{(C.2.4-4)}$$

The extension of these results to the next function $\gamma_1^{N,3}(123|t)$ is accomplished by combining the above methods with those of section C.2.2. We find a sum of three terms.

$$\int_0^t d\tau \int (d\vec{v})^{N-3} \mathcal{P}_{\vec{K}_1}(1|t-\tau)\mathcal{P}_{\vec{K}_2}(2|t-\tau)\mathcal{P}_{\vec{K}_3}(3|t-\tau)\, \gamma_1^{N,3}(123|\tau) =$$

$$= \int_0^t d\tau\, S_{\vec{K}_1}(1,i|t-\tau)\mathcal{P}_{\vec{K}_2}(2|t-\tau)\mathcal{P}_{\vec{K}_3}(3|t-\tau)\, h_1^{4,3}(i23|1|\tau)$$

$$+ \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\, S_{\vec{K}_2}(2,i|t-\tau)\mathcal{P}_{\vec{K}_3}(3|t-\tau)\, h_1^{4,3}(1i3|2|\tau) \qquad \text{(C.2.4-5)}$$

$$+ \int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\,\mathcal{P}_{\vec{K}_2}(2|t-\tau)\, S_{\vec{K}_3}(3,i|t-\tau)\, h_1^{4,3}(12i|3|\tau)$$

The reduction of other functions $\gamma_n^{N,3}(123|\tau)$ may be determined by these same methods.

C.2.5. We will find in section C.3 of this appendix that the above results must be extended to cases for which the velocity integration is over only (N-S) velocities rather than (N-2) as in sections C.2.1 - C.2.3 and (N-3) as in section C.2.4. The simplest problem is

$$\int (d\vec{v})^{N-S}\, \mathcal{P}_{\vec{K}_1}(1|t)\, \gamma_0^{N,1}(1) \tag{C.2.5-1}$$

We take a Laplace transform in time and substitute the form (C.2.1-3) for the operator $\mathcal{P}_{\vec{K}_1}(1|p)$. The first term is

$$\int (d\vec{v})^{N-S} \frac{\gamma_0^{N,1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1} = \left(\prod_j^{\{S-1\}} \varphi(j)\right) \frac{h_0^{1,1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1} \tag{C.2.5-2}$$

where we have used the result (C.2-1) The set $\{S-1\}$ includes all those indices in the set $\{S\}$ except the index 1. The second term becomes

$$\int (d\vec{v})^{N-S} \sum_i \frac{\frac{i}{N}\mathcal{D}_{\vec{K}_1}(1i)}{p+i\vec{K}_1\cdot\vec{v}_1}\, \frac{\gamma_0^{N,1}(i)}{p+i\vec{K}_1\cdot\vec{v}_i} =$$

$$= \left(\prod_j^{\{S-1\}} \varphi(j)\right) \frac{i\mathcal{D}_{\vec{K}_1}(1)\varphi(1) \int d\vec{v}_1 \frac{h_0^{1,1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1}}{p+i\vec{K}_1\cdot\vec{v}_1} \tag{C.2.5-3}$$

where we have dropped S terms of $\mathcal{O}(\frac{1}{N})$. We see from (C.2.5-1) and (C.2.5-2) that we are simply producing the terms of the operator $\mathcal{P}_{\vec{K}_1}(1|\rho)$ multiplied by the quantity $(\prod_j^{\{S-1\}} \varphi(j))$. This indeed is the case, and we find after an inverse Laplace transform

$$\int (d\vec{v})^{N-S} \mathcal{P}_{\vec{K}_1}(1|t)\, \gamma_0^{N,1}(1) = \prod_j^{\{S-1\}} \varphi(j)\, \mathcal{P}_{\vec{K}_1}(1|t)\, h_0^{1,1}(1) \qquad \text{(C.2.5-4)}$$

C.2.6. The next term to be considered is

$$\int (d\vec{v})^{N-S} \mathcal{P}_{\vec{K}_1}(1|t)\, \gamma_1^{N,1}(1) \qquad \text{(C.2.6-1)}$$

The primary difference between (C.2.6-1) and the result obtained (for S = 1) in Chapter 6 of the text is that there are now some terms present which vanished earlier. For instance, the first term of (C.2.6-1) is, after a Laplace transform has been taken

$$\int (d\vec{v})^{N-S} \frac{\gamma_1^{N,1}(1)}{\rho + i\vec{K}_1\cdot\vec{v}_1} = \sum_j^{\{S-1\}} \frac{h_1^{S,1}(1|j)}{\rho + i\vec{K}_1\cdot\vec{v}_1} \qquad \text{(C.2.6-2)}$$

These terms vanish if we integrate over the velocities in the set $\{S-1\}$, in agreement with the result of Chapter 6. The second term becomes

$$\int (d\vec{v})^{N-S} \sum_i \frac{\frac{i}{N} D_{\vec{K}_1}(1i)}{p+i\vec{K}_1\cdot\vec{v}_1} \frac{\gamma_1^{N,1}(i)}{p+i\vec{K}_1\cdot\vec{v}_i} =$$

$$= \frac{i D_{\vec{K}_1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1} \sum_i^{\{S\}} \left(\prod_{\omega\neq i}^{\{S\}} \varphi(\omega) \right) \int d\vec{v}_j \frac{h_1^{2,1}(j|i)}{p+i\vec{K}_1\cdot\vec{v}_j} \qquad \text{(C.2.6-3)}$$

The nth term of the expansion consists of two different terms

$$\frac{i D_{\vec{K}_1}(1)}{p+i\vec{K}_1\cdot\vec{v}_1} \sum_i^{\{S\}} \left(\prod_{\omega\neq i}^{\{S\}} \varphi(\omega) \right) \int d\vec{v}_j \frac{h_1^{2,1}(j|i)}{p+i\vec{K}_1\cdot\vec{v}_j} \left(L(\vec{K}_1,p) \right)^{n-2}$$

$$+ (n-2) \left(\prod_{\omega\neq 1}^{\{S\}} \varphi(\omega) \right) \frac{i D_{\vec{K}_1}(1)\varphi(1)}{p+i\vec{K}_1\cdot\vec{v}_1} \int d\vec{v}_i \, d\vec{v}_j \frac{i D_{\vec{K}_1}(i)}{p+i\vec{K}_1\cdot\vec{v}_i} \frac{h_1^{2,1}(j|i)}{p+i\vec{K}_1\cdot\vec{v}_j} \left(L(\vec{K}_1,p) \right)^{n-3} \qquad \text{(C.2.6-4)}$$

The first group of terms of (C.2.6-4) may be separated into those which contribute to the second term in the operator $P_{\vec{K}_1}(1|p)$ $(i\neq 1)$ and one $(i=1)$ which contributes to the first term in the operator $S_{\vec{K}_1}(1,i|p)$. The second term of (C.2.6-4) contributes to the second term of the operator $S_{\vec{K}_1}(1,i|p)$. If these results are combined we find, after an inverse Laplace transform

$$\int (d\vec{v})^{N-S} \mathcal{P}_{\vec{K}_1}(1|t)\, \gamma_1^{N,1}(1) = \prod_{w}^{\{S-1\}} \varphi(w)\, S_{\vec{K}_1}(1,i|t)\, h_1^{2,1}(i|1)$$

$$+ \sum_{i}^{\{S-1\}} \left(\prod_{w \neq i}^{\{S-1\}} \varphi(w) \right) P_{\vec{K}_1}(1|t)\, h_1^{2,1}(1|i) \qquad \text{(C.2.6-5)}$$

We note in passing that the second term of (C.2.6-5) contains a summation over the index i in the set $\{S-1\}$. Each term of this summation is not symmetric to an interchange of the velocity $\vec{v}_i$ with the other velocities in the set $\{S-1\}$. The proper symmetry is obtained only after the summation over the index i. Further, each term of the summation vanishes upon an integration over the velocity $\vec{v}_i$.

C.2.7. These results may be extended to the next function $\gamma_2^{N,1}(1)$ of the hierarchy. Three terms are found in addition to the one shown in Chapter 6. Two involve a single sum and the third a double sum over the set $\{S-1\}$. The first term of the operator $\mathcal{P}_{\vec{K}_1}(1|p)$ is

$$\int (d\vec{v})^{N-S} \frac{\gamma_2^{N,1}(1)}{p + i\vec{K}_1 \cdot \vec{v}_1} = \sum_{\ell \neq m}^{\{S-1\}} \sum \left(\prod_{w \neq \ell, m}^{\{S-1\}} \varphi(w) \right) \frac{h_2^{3,1}(1|\ell,m)}{p + i\vec{K}_1 \cdot \vec{v}_1} \qquad \text{(C.2.7-1)}$$

The second term becomes

$$\sum_{m}^{\{S-1\}} \Big(\prod_{w \neq m}^{\{S-1\}} \varphi(w) \Big) \frac{i \mathcal{D}_{\vec{K}_1}(1)}{p + i \vec{K}_1 \cdot \vec{v}_1} \int d\vec{v}_i \, \frac{h_2^{3,1}(i|1,m) + h_2^{3,1}(i|m,1)}{p + i \vec{K}_1 \cdot \vec{v}_i}$$

$$+ \sum_{\ell \neq m}^{\{S-1\}} \sum \Big(\prod_{w \neq \ell, m}^{\{S-1\}} \varphi(w) \Big) \frac{i \mathcal{D}_{\vec{K}_1}(1)}{p + i \vec{K}_1 \cdot \vec{v}_1} \int d\vec{v}_i \, \frac{h_2^{3,1}(i|\ell,m)}{p + i \vec{K}_1 \cdot \vec{v}_i} \qquad \text{(C.2.7-2)}$$

Note that if the relations (C.2.7-1) and (C.2.7-2) are integrated over the velocities $\{S-1\}$ all terms vanish.

There are three different contributions to the third term of the expansion of $\mathcal{P}_{\vec{K}}(1|p)$.

$$\Big(\prod_{w \neq 1}^{\{S\}} \varphi(w) \Big) \frac{i \mathcal{D}_{\vec{K}_1}(1)}{p + i \vec{K}_1 \cdot \vec{v}_1} \int d\vec{v}_i \, d\vec{v}_j \, \frac{i \mathcal{D}_{\vec{K}_1}(j)}{p + i \vec{K}_1 \cdot \vec{v}_j} \, \frac{h_2^{3,1}(i|1,j) + h_2^{3,1}(i|j,1)}{p + i \vec{K}_1 \cdot \vec{v}_i}$$

$$+ \sum_{m}^{\{S-1\}} \Big(\prod_{w \neq m}^{\{S-1\}} \varphi(w) \Big) \frac{i \mathcal{D}_{\vec{K}_1}(1)}{p + i \vec{K}_1 \cdot \vec{v}_1} \int d\vec{v}_i \, d\vec{v}_j \, \frac{i \mathcal{D}_{\vec{K}_1}(j)}{p + i \vec{K}_1 \cdot \vec{v}_j} \, \frac{1}{p + i \vec{K}_1 \cdot \vec{v}_i} \times$$

$$\times \left\{ \begin{aligned} & \varphi(j) \Big(h_2^{3,1}(i|1,m) + h_2^{3,1}(i|m,1) \Big) \\ & + \varphi(1) \Big(h_2^{3,1}(i|j,m) + h_2^{3,1}(i|m,j) \Big) \end{aligned} \right\} \qquad \text{(C.2.7-3)}$$

$$+ \sum_{\ell \neq m}^{\{S-1\}} \sum \Big(\prod_{w \neq \ell, m}^{\{S-1\}} \varphi(w) \Big) \frac{i \mathcal{D}_{\vec{K}_1}(1)}{p + i \vec{K}_1 \cdot \vec{v}_1} \int d\vec{v}_i \, \frac{h_2^{3,1}(i|\ell,m)}{p + i \vec{K}_1 \cdot \vec{v}_i} \, L(\vec{K}_1, p)$$

The first term of (C.2.7-3) is similar to the first term in the expansion of the operator $S^{(1)}_{\vec{K}_1}(1,i|j|\rho)$ (see (6.2-29)). Indeed, the fourth and fifth terms in the expansion of $\mathcal{P}_{\vec{K}_1}(1|\rho)$ contribute other parts of this operator. Each term is multiplied by the factor $\prod_{w}^{\{S-1\}} \varphi(w)$ and is symmetric in the velocities in the set $\{S-1\}$. The second term of (C.2.7-3) and the first term of (C.2.7-2) are the first two elements in the expansion of the operator $S_{\vec{K}_1}(1,i|\rho)$. These terms all have one index in the set $\{S-1\}$ which is not symmetric to interchange with the other indices in that set. Finally, the last terms of (C.2.7-2) and (C.2.7-3), when combined with (C.2.7-1) form the first three elements in the expansion of $\mathcal{P}_{\vec{K}_1}(1|\rho)$. Each member has two asymmetric indices in the set $\{S-1\}$. These results are combined to write

$$\int (d\vec{v})^{N-S}\, \mathcal{P}_{\vec{K}_1}(1|t)\, \gamma_2^{N,1}(1) =$$

$$= \Big(\prod_{w}^{\{S-1\}} \varphi(w)\Big) S^{(1)}_{\vec{K}_1}(1,i|j|t)\Big(h_2^{3,1}(i|1,j) + h_2^{3,1}(i|j,1)\Big)$$

$$+ \sum_{\ell}^{\{S-1\}} \Big(\prod_{w\neq\ell}^{\{S-1\}} \varphi(w)\Big) S_{\vec{K}_1}(1,i|t)\Big(h_2^{3,1}(i|1,\ell) + h_2^{3,1}(i|\ell,1)\Big) \qquad \text{(C.2.7-4)}$$

$$+ \sum_{\ell\neq m}^{\{S-1\}}\sum \Big(\prod_{w\neq\ell,m}^{\{S-1\}} \varphi(w)\Big) \mathcal{P}_{\vec{K}_1}(1|t)\, h_2^{3,1}(1|\ell,m)$$

The extension of these results to multiple operators and the functions $\gamma_n^{N,2}(12)$ and $\gamma_n^{N,3}(123)$ is accomplished by the methods discussed in the first parts of this section.

C.3 Source of the Functions $\gamma_n^{N,\nu}(\{\nu\})$.

We consider in this section the source of the functions $\gamma_n^{N,\nu}(\{\nu\})$ which have been simply defined and then used in Section 2 of this appendix. The initial value functions $f^{N,\nu}(\{\nu\}|t=0)$ are all functions of the form $\gamma_0^{N,\nu}(\{\nu\})$ since, when we integrate over the velocities $\{N-S\}$ we obtain a single term which is completely symmetric in the velocities $\{S-\nu\}$. The second term in the solution (5.4-4) for $f_{\vec{K}_1}^{N,1}(1|t)$ is

$$\int_0^t d\tau\, \mathcal{P}_{\vec{K}_1}(1|t-\tau)\frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum L(ij)\mathcal{P}_{\vec{K}_1}(1|\tau)\mathcal{P}_{\vec{K}_i}(i|\tau)\mathcal{P}_{\vec{K}_j}(j|\tau)f_{\vec{K}_1\vec{K}_i\vec{K}_j}^{N,3}(1ij|t=0) \qquad \text{(C.3-1)}$$

We have shown in Chapter 5 that $\frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau)f^{N,3}_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|t=0)$ is a function of the form $\gamma_1^{N,1}(1|\tau)$. A simple extension of these arguments can be used to show that the factor

$$\frac{1}{N}\sum_{i<j}^{\{N-2\}}\sum L(ij)\,\mathcal{P}_{\vec{K}_1}(1|\tau)\mathcal{P}_{\vec{K}_2}(2|\tau)\mathcal{P}_{\vec{K}_i}(i|\tau)\mathcal{P}_{\vec{K}_j}(j|\tau)f^{N,4}_{\vec{K}_1\cdots\vec{K}_j}(12ij|t=0) \qquad \text{(C.3-2)}$$

in the second term of the solution for $f^{N,2}_{\vec{K}_1\vec{K}_2}(12|t)$ is a function of the form $\gamma_1^{N,2}(12)$.

The third term in the solution for $f^{N,1}_{\vec{K}_1}(1|t)$ contains the following factor

$$\int_0^\tau d\tau'\frac{1}{N}\sum_{i<j}^{\{N-1\}}\sum L(ij)\,\mathcal{P}^3_{\vec{K}_1\vec{K}_i\vec{K}_j}(1ij|\tau-\tau')\frac{1}{N}\sum_{\ell<m}^{\{N-3\}}\sum L(\ell m)\,\mathcal{P}^5_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|\tau')f^{N,5}_{\vec{K}_1\cdots\vec{K}_m}(1ij\ell m|t=0) \qquad \text{(C.3-3)}$$

In order to determine the functional form of (C.3-3) we integrate over the velocities $\{N-S\}$ to obtain

$$\sum_{i}^{\{S-1\}}\int_0^{\tau} d\tau' \int (d\vec{v})^{N-S} L(i\,S+1) \mathcal{P}^{3}_{\vec{R}_1\vec{R}_i\vec{R}_{S+1}}(1i\,S+1|\tau-\tau') \frac{1}{N}\sum_{\ell<m}^{\{N-3\}}\sum L(\ell m)\mathcal{P}^{5}_{\vec{R}_1\cdots\vec{R}_m}(1i\,S+1\,\ell m|\tau') f^{N,5}_{\vec{R}_1\cdots\vec{R}_m}(1i\,S+1\,\ell m|t=0) \qquad (C.3-4)$$

The factor $\frac{1}{N}\sum\sum_{\ell<m}^{\{N-3\}} L(\ell m)\mathcal{P}^{5}_{\vec{R}_1\cdots\vec{R}_m}(1i\,S+1\,\ell m|\tau') f^{N,5}_{\vec{R}_1\cdots\vec{R}_m}(1i\,S+1\,\ell m|t=0)$ is a function of the form $\gamma_1^{N,3}(1i\,S+1)$. To illustrate we integrate the former function over the velocities $N-n$ (where $n > S+1$). If both ℓ and m are in the set $\{N-n\}$ then the term will vanish. The only non-zero terms are those for which the index ℓ is in the set $\{n-3\}$. The sum over the index m will contribute $n-3$ terms for m in the set $\{n-3\}$ and $N-n+3$ terms for m in the set $\{N-n+3\}$. If we require that $n << N$ the contribution of the first group of terms is very small, and we find (see also (6.2-5) for a similar argument)

$$\begin{aligned}\int (d\vec{v})^{N-n}\frac{1}{N}\sum\sum_{\ell<m}^{\{N-3\}} L(\ell m)\mathcal{P}^{5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|\tau') f^{N,5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|t=0) \simeq \\ \simeq \sum_{\ell}^{\{n-3\}}\int d\vec{v}_m\, L(\ell m) P^{5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|\tau') f^{n+1,5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|t=0)\end{aligned} \qquad (C.3-5)$$

which we may rewrite as

$$\simeq \sum_{\ell}^{\{n-3\}}\left(\int d\vec{v}_m\, L(\ell m) P^{5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|\tau') f^{5,5}_{\vec{R}_1\cdots\vec{R}_m}(1ij\ell m|t=0)\right)\left(\prod_{w\neq\ell}^{\{n-3\}} \varphi(w)\right) \qquad (C.3-6)$$

which is a function of the form $\gamma_1^{n,3}(1ij)$. The results of of sections C.2.4 and C.2.6 can be used to establish the following

$$\int_0^\tau d\tau' \int (d\vec{v})^{N-S} \mathcal{P}^{3}_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1 i S+1|\tau-\tau') \frac{1}{N} \sum_{\ell<m}^{\{N-3\}}\sum L(\ell m) \mathcal{P}^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{N,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0) =$$

$$= \Big(\prod_{w}^{\{S-3\}} \varphi(w)\Big)\Big\{\int_0^\tau d\tau' S_{\vec{K}_i}(1,\ell|\tau-\tau') P^{2}_{\vec{K}_i\vec{K}_{S+1}}(i S+1|\tau-\tau') \int d\vec{v}_m L(\ell m) P^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0)$$

$$+\int_0^\tau d\tau' S_{\vec{K}_i}(i,\ell|\tau-\tau') P^{2}_{\vec{K}_1\vec{K}_{S+1}}(1 S+1|\tau-\tau') \int d\vec{v}_m L(i m) P^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0)$$

$$+\int_0^\tau d\tau' S_{\vec{K}_{S+1}}(S+1,\ell|\tau-\tau') P^{2}_{\vec{K}_1\vec{K}_i}(1 i|\tau-\tau') \int d\vec{v}_m L(S+1\, m) P^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0)\Big\} \tag{C.3-7}$$

$$+\sum_{\ell}^{\{S-3\}} \Big(\prod_{w\neq\ell}^{\{S-3\}} \varphi(w)\Big) \int_0^\tau d\tau' P^{3}_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1 i S+1|\tau-\tau') \int d\vec{v}_m L(\ell m) P^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0)$$

The result (C.3-7) when used in (C.3-4) indicates that (C.3-4) is made up of two types of terms, some of the form $h_1^{S,1}(1|i)$ and others of the form $h_2^{S,1}(1|i,j)$. We find

$$\int_0^\tau d\tau' \sum_i^{\{S-1\}} \int (d\vec{v})^{N-S} L(i\, S+1) \mathcal{P}^{3}_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1 i S+1|\tau-\tau') \frac{1}{N} \sum_{\ell<m}^{\{N-3\}}\sum L(\ell m) \mathcal{P}^{5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|\tau') f^{N,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i S+1 \ell m|t=0) =$$

$$(\text{C.3-8})$$

$$= \sum_i^{\{S-1\}} h_1^{S,1}(1|i) + \sum_{i\neq j}^{\{S-1\}}\sum h_2^{S,1}(1|i,j)$$

where

$$h_1^{S,1}(1|i) = \left(\prod_{\omega\neq i}^{\{S-1\}} \phi(\omega)\right) \int d\vec{v}_{S+1}\, L(i\; s{+}1) \times \tag{C.3-9}$$

$$\times \left[\begin{aligned} &\int_0^{\tau} d\tau'\, S_{\vec{K}_1}(1,\ell|\tau-\tau')\, P^2_{\vec{K}_i\vec{K}_{S+1}}(i\; s{+}1|\tau-\tau') \int d\vec{v}_m\, L(\ell m)\, P^5_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|\tau')\, f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|t=0) \\ &+\int_0^{\tau} d\tau'\, S_{\vec{K}_i}(i,\ell|\tau-\tau')\, P^2_{\vec{K}_1\vec{K}_{S+1}}(1\; s{+}1|\tau-\tau') \int d\vec{v}_m\, L(im)\, P^5_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|\tau')\, f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|t=0) \\ &+\int_0^{\tau} d\tau'\, S_{\vec{K}_{S+1}}(s{+}1,\ell|\tau-\tau')\, P^2_{\vec{K}_1\vec{K}_i}(1 i|\tau-\tau') \int d\vec{v}_m\, L(s{+}1\; m)\, P^5_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|\tau')\, f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|t=0) \end{aligned} \right]$$

$$h_2^{S,1}(1|i;\ell) = \prod_{\omega\neq i\ell}^{\{S-1\}} \phi(\omega) \int d\vec{v}_{S+1}\, L(i\; s{+}1) \int_0^{\tau} d\tau'\, P^3_{\vec{K}_1\vec{K}_i\vec{K}_{S+1}}(1 i\; s{+}1|\tau-\tau') \int d\vec{v}_m\, L(\ell m)\, P^5_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|\tau')\, f^{5,5}_{\vec{K}_1\cdots\vec{K}_m}(1 i\; s{+}1\; \ell m|t=0)$$

We see that quantities with two unsymmetric indices arise in the third term of the solution for $f_{\vec{K}}(1|t)$. Similarly, terms with four unsymmetric indices will arise in the fourth term of this solution, and so on. We have by no means exhausted the combinations of operators and functions that arise in the reduction of the solutions for the single and two-particle distribution functions. However, all the terms can be evaluated by a straightforward application of the methods presented and discussed above.

APPENDIX D

LAPLACE CONVOLUTION INTEGRALS

We consider in this Appendix the evaluation of some of the Laplace convolution integrals discussed in Chapter 7. The algebraic identity (D-1) is used to simplify the form of many of the terms.

$$\int \frac{d\vec{v}_1}{(\rho + i\vec{K}\cdot\vec{v}_1)(\rho_{\vec{K}_1} + 2\gamma_{\vec{K}} + i\vec{K}_1\cdot\vec{v}_1)} = \frac{1}{\rho - (\rho_{\vec{K}_1} + 2\gamma_{\vec{K}})}\left(\int \frac{d\vec{v}_1}{\rho_{\vec{K}_1} + 2\gamma_{\vec{K}} + i\vec{K}\cdot\vec{v}_1} - \int \frac{d\vec{v}_1}{\rho + i\vec{K}\cdot\vec{v}_1}\right) \tag{D-1}$$

As a specific example note that if the integrand of (D-1) contains the function $i\mathcal{D}_{\vec{K}}(1)\ \varphi(1)$ then we may write

$$\int d\vec{v}_1 \frac{i\mathcal{D}_{\vec{K}}(1)\ \varphi(1)}{(\rho + i\vec{K}\cdot\vec{v}_1)(\rho_{\vec{K}} + 2\gamma_{\vec{K}} + i\vec{K}\cdot\vec{v}_1)} = \frac{\mathcal{E}(\vec{K},\rho) - \mathcal{E}(\vec{K},\rho_{\vec{K}} + 2\gamma_{\vec{K}})}{\rho - (\rho_{\vec{K}} + 2\gamma_{\vec{K}})} \tag{D-2}$$

We note that there is no pole at the point $\rho = \rho_{\vec{K}} + 2\gamma_{\vec{K}}$ on the right-hand side of (D-2). However, it is convenient to treat each term on the right-hand side of (D-2) separately, in which case each has a pole at this point. We are at liberty to choose on which side of the contour C this pole is to lie, provided we are consistent in our choice. The relation (D-1) is the only one needed to reduce the terms to forms where they can be evaluated by a simple application of the method of residues.

The following three terms from equation (7.3-13) for the two-particle correlation function are to be considered

$$
\begin{aligned}
\frac{1}{2\pi i}\int_C d\rho \Big\{ & S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)R_{\vec{K}_1}(i)\,C_{\vec{K}'}(1)\times \\
& \times P_{\vec{K}_2}(2|\rho)S_{\vec{K}_2}(2,\ell|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})R_{\vec{K}_2}(\ell)C_{\vec{K}''}(2) \\
& + P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})R_{\vec{K}_1}(i)\,C_{\vec{K}'}(1)\times \\
& \times S_{\vec{K}_2}(2,\ell|\rho)R_{\vec{K}_2}(\ell)C_{\vec{K}''}(2) \\
& + \left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right) S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)R_{\vec{K}_1}(i)\,C_{\vec{K}'}(1)\times \\
& \times S_{\vec{K}_2}(2,\ell|\rho)R_{\vec{K}_2}(\ell)C_{\vec{K}''}(2)
\end{aligned}
\tag{D-3}
$$

With a repeated application of the relation (D-1) one can show that these terms, referred to as the first, second and third terms, respectively, reduce to the following forms.

FIRST TERM:

$$
\begin{aligned}
& \frac{1}{2\gamma_{\vec{K}''}}\,\frac{P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)}{\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho}\Bigg\{\frac{1}{(\rho+i\vec{K}_2\cdot\vec{v}_2)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)} \\
& + \frac{iD_{\vec{K}_2}(2)\,\varphi(2)}{\rho-(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})}\left[-\frac{\tilde{P}_{\vec{K}_2}(2|\rho)}{\rho+i\vec{K}_2\cdot\vec{v}_2}+\frac{\tilde{P}_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})}{\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2}\right]\Bigg\}\times \\
& \times D_{\vec{K}_1}(1)\,C_{\vec{K}'}(1)\,\tilde{R}_{\vec{K}_1}\,iD_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\,\tilde{R}_{\vec{K}_2}
\end{aligned}
\tag{D-4}
$$

SECOND TERM:

$$\frac{1}{2\gamma_{\vec{K}'}}\frac{\mathcal{P}_{\vec{K}_2}(2|p)}{p-p_{\vec{K}_2}}\Bigg\{\frac{1}{(p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p+i\vec{K}_1\cdot\vec{v}_1)(p_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)}$$

$$+\frac{iD_{\vec{K}_1}(1)\,\varphi(1)}{p_{\vec{K}_2}+2\gamma_{\vec{K}''}-p}\left[-\frac{\widetilde{\mathcal{P}}_{\vec{K}_1}(1|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p)}{p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p+i\vec{K}_1\cdot\vec{v}_1}+\frac{\widetilde{\mathcal{P}}_{\vec{K}_1}(1|p_{\vec{K}_1}+2\gamma_{\vec{K}'})}{p_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1}\right]\Bigg\}\times \tag{D-5}$$

$$\times iD_{\vec{K}_1}(1)\,C_{\vec{K}'}(1)\,\widetilde{R}_{\vec{K}_1}\,iD_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\,\widetilde{R}_{\vec{K}_2}$$

THIRD TERM:

$$\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\frac{\mathcal{P}_{\vec{K}_1}(1|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p)}{p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p}\;\frac{\mathcal{P}_{\vec{K}_2}(2|p)}{p-p_{\vec{K}_2}}\times$$

$$\times (i)^2 D_{\vec{K}_1}(1)\,C_{\vec{K}'}(1)\,\widetilde{R}_{\vec{K}_1}\,D_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\,\widetilde{R}_{\vec{K}_2} \tag{D-6}$$

We have made use of the definitions of the operators $\mathcal{P}_{\vec{K}_1}(1|p)$ and $\widetilde{\mathcal{P}}_{\vec{K}_1}(1|p)$ to write

$$\mathcal{P}_{\vec{K}_1}(1|p)=\frac{1}{p+i\vec{K}_1\cdot\vec{v}_1}+\frac{iD_{\vec{K}_1}(1)\,\varphi(1)}{p+i\vec{K}_1\cdot\vec{v}_1}\,\widetilde{\mathcal{P}}_{\vec{K}_1}(1|p) \tag{D-7}$$

The contour C passes to the right of the poles of $\mathcal{P}_{\vec{K}_2}(2|p)$ and to the left of the poles of $\mathcal{P}_{\vec{K}_1}(1|p_{\vec{K}_1}+p_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-p)$. Three new poles have been introduced into the p-plane by the use of the relations (D-1) and (D-2). We choose to put the contour C to the right of the new poles at $p=p_{\vec{K}_2}$ and $p=p_{\vec{K}_2}+2\gamma_{\vec{K}''}$ and to the left of the new pole at

$\rho = \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''}$. The simplest part of each term has been considered in Chapter 7 of the text. Thus, if we take the first part of the operators $\mathcal{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)$ and $\mathcal{P}_{\vec{K}_2}(2|\rho)$ (see (D-7)) and the first terms within the brackets $\{\ \}$ of (D-4) and (D-5) we find

$$\frac{(i)^2 \mathcal{D}_{\vec{K}_1}(1) C_{\vec{K}'}(1) \widetilde{R}_{\vec{K}_1} \mathcal{D}_{\vec{K}_2}(2) C_{\vec{K}''}(2) \widetilde{R}_{\vec{K}_2}}{(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho+i\vec{K}_2\cdot\vec{v}_2)} \times$$

$$\times\left\{\frac{1}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)} + \frac{1}{2\gamma_{\vec{K}'}(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)}\right. \quad \text{(D-8)}$$

$$\left.+\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\frac{1}{(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)}\right\}$$

The inversion contour is closed to the left. The integrand vanishes at least as ($1/\rho^3$) in the limit of large ρ so that the only contributions to the integral come from the poles $\rho = -i\vec{K}_2\cdot\vec{v}_2$ and $\rho = \rho_{\vec{K}_2}$. The inverse Laplace transform of (D-8) becomes

$$\frac{(i)^2 \mathcal{D}_{\vec{K}_1}(1) C_{\vec{K}'}(1) \widetilde{R}_{\vec{K}_1} \mathcal{D}_{\vec{K}_2}(2) C_{\vec{K}''}(2) \widetilde{R}_{\vec{K}_2}}{(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1+i\vec{K}_2\cdot\vec{v}_2)} \times$$

$$\times\left\{\frac{1}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)}\right.$$

$$+\frac{1}{2\gamma_{\vec{K}'}(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)} \quad \text{(D-9)}$$

$$\left.+\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\frac{1}{(2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''})(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}+i\vec{K}_1\cdot\vec{v}_1)}\right\}$$

which reduces after some algebraic manipulation to

$$\frac{(i)^2 D_{\vec{K}_1}(1)\, C_{\vec{K}'}(1)\, \widetilde{R}_{\vec{K}_1}\, D_{\vec{K}_2}(2)\, C_{\vec{K}''}(2)\, \widetilde{R}_{\vec{K}_2}}{(2\gamma_{\vec{K}'})(2\gamma_{\vec{K}''})(\rho_{\vec{K}_1} + 2\gamma_{\vec{K}'} + i\vec{K}\cdot\vec{v}_1)(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''} + i\vec{K}\cdot\vec{v}_2)} \tag{D-10}$$

The next group of terms to be combined are those in (D-4)-(D-6) which come from the second part of the operator $\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho)$ and the first part of the operator $\widetilde{P}_{\vec{K}_2}(2|\rho)$. These terms are grouped in the following way

$$\frac{i D_{\vec{K}_1}\varphi(1)}{2\pi i}\int_C d\rho \left\{ \frac{\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho)}{\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho + i\vec{K}\cdot\vec{v}_1} \times \right.$$
$$\times \left[\frac{1}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho)(\rho + i\vec{K}_2\cdot\vec{v}_2)(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''} + i\vec{K}_2\cdot\vec{v}_2)} \right.$$
$$- \frac{1}{2\gamma_{\vec{K}'}(\rho + i\vec{K}_2\cdot\vec{v}_2)(\rho - \rho_{\vec{K}_2})(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''} - \rho)}$$
$$\left. + \left(\frac{1}{2\gamma_{\vec{K}'}} + \frac{1}{2\gamma_{\vec{K}''}}\right) \frac{1}{(\rho + i\vec{K}_2\cdot\vec{v}_2)(\rho - \rho_{\vec{K}_2})(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho)} \right] \tag{D-11}$$
$$\left. + \frac{\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1} + 2\gamma_{\vec{K}''})}{2\gamma_{\vec{K}'}(\rho + i\vec{K}_2\cdot\vec{v}_2)(\rho - \rho_{\vec{K}_2})(\rho_{\vec{K}_2} + 2\gamma_{\vec{K}''} - \rho)(\rho_{\vec{K}_1} + 2\gamma_{\vec{K}'} - \rho)} \right\} \times$$
$$\times (i)^2 D_{\vec{K}_1}(1)\, C_{\vec{K}'}(1)\, \widetilde{R}_{\vec{K}_1}\, D_{\vec{K}_2}(2)\, C_{\vec{K}''}(2)\, \widetilde{R}_{\vec{K}_2}$$

The poles of $\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1} + \rho_{\vec{K}_2} + 2\gamma_{\vec{K}'} + 2\gamma_{\vec{K}''} - \rho)$ lie to the right of the contour C. The inversion contour for the first three terms of (D-11) is therefore closed to the left. The integrand of (D-11) vanishes along the contour at infinity so that the only contributions to the first three terms of (D-11) come from the poles at $\rho = -i\vec{K}_2\cdot\vec{v}_2$, $\rho = \rho_{\vec{K}_2}$ and $\rho = \rho_{\vec{K}_2} + 2\gamma_{\vec{K}''}$. These terms become after some algebraic reduction

$$\frac{(i)^3 D_{\vec{K}_1}(1)\varphi(1)\tilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})D_{\vec{K}_1}(1)C_{\vec{K}'}(1)\tilde{R}_{\vec{K}_1}D_{\vec{K}_2}(2)C_{\vec{K}''}(2)\tilde{R}_{\vec{K}_2}}{(2\gamma_{\vec{K}'})(2\gamma_{\vec{K}''})(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)} \qquad \text{(D-12)}$$

The fourth term of (D-11) is evaluated by closing the inversion contour to the right. All the poles of this term lie to the left of the contour C, and there are none inside the curve when closed to the right. This term vanishes, and (D-12) is the complete contribution of the terms in (D-11).

The next group of terms are those which come from the first part of the operator $\tilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)$ and the second part of the operator $\tilde{P}_{\vec{K}_2}(2|\rho)$. These terms are

$$\begin{aligned}
&\frac{i D_{\vec{K}_2}(2)\varphi(2)}{2\pi i}\int_C d\rho \Bigg\{\frac{\tilde{P}_{\vec{K}_2}(2|\rho)}{\rho+i\vec{K}_2\cdot\vec{v}_2}\Bigg[\frac{1}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}-\rho)} \\
&+\frac{1}{2\gamma_{\vec{K}'}(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)} \qquad \text{(D-13)} \\
&+\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\frac{}{(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)}\Bigg] \\
&+\frac{\tilde{P}_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)(\rho-\rho_{\vec{K}_2}-2\gamma_{\vec{K}''})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)}\Bigg\}\times \\
&\times (i)^2 D_{\vec{K}_1}(1)C_{\vec{K}'}(1)\tilde{R}_{\vec{K}_1}D_{\vec{K}_2}(2)C_{\vec{K}''}(2)\tilde{R}_{\vec{K}_2}
\end{aligned}$$

The first three terms of (D-13) may be combined with the result

$$\frac{1}{2\pi i}\int_C d\rho \, \frac{iD_{\vec{K}_2}\varphi(2)\widetilde{P}_{\vec{K}_2}(2|-\rho)\, iD_{\vec{K}_1}(1)C_{\vec{K}'}(1)\widetilde{R}_{\vec{K}_1}\; iD_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\widetilde{R}_{\vec{K}_2}}{2\gamma_{\vec{K}'}(\rho+i\vec{K}_2\cdot\vec{v}_2)(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}-\rho)(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)} \tag{D-14}$$

The poles of $\widetilde{P}_{\vec{K}_2}(2|\rho)$ lie to the left of the contour C so that the first three terms of (D-13) are evaluated by closing the Laplace contour to the right. Once again the integrand vanishes along the contour at infinity. There are no poles inside the contour so that the integral (D-14) is zero. The only contribution to (D-13) comes from the fourth term. The inversion contour is closed to the left. The only pole inside the contour is the one at $\rho = \rho_{\vec{K}_2} + 2\gamma_{\vec{K}''}$ and we find

$$\frac{iD_{\vec{K}_2}(2)\varphi(2)\widetilde{P}_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})\, iD_{\vec{K}_1}(1)C_{\vec{K}'}(1)\widetilde{R}_{\vec{K}_1}\; iD_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\widetilde{R}_{\vec{K}_2}}{(2\gamma_{\vec{K}'})(2\gamma_{\vec{K}''})(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)} \tag{D-15}$$

The last group of terms consists of those which come from the second part of the two operators $\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)$ and $\widetilde{P}_{\vec{K}_2}(2|\rho)$. We rearrange the order of (D-4)-(D-6) slightly to write these terms in the following way.

$$\frac{iD_{\vec{K}_1}(1)\varphi(1)\,iD_{\vec{K}_2}(2)\varphi(2)}{2\pi i}\int_C d\rho\,\times$$

$$\times\Bigg\{\frac{P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)}{\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1}\,\frac{P_{\vec{K}_2}(2|\rho)}{\rho+i\vec{K}_2\cdot\vec{v}_2}\Bigg[-\frac{1}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)(\rho-\rho_{\vec{K}_2}-2\gamma_{\vec{K}''})}$$

$$-\frac{1}{2\gamma_{\vec{K}'}(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}-\rho)}+\left(\frac{1}{2\gamma_{\vec{K}'}}+\frac{1}{2\gamma_{\vec{K}''}}\right)\frac{1}{(\rho-\rho_{\vec{K}_2})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)}\Bigg]$$

$$+\frac{\tilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})\tilde{P}_{\vec{K}_2}(2|\rho)}{2\gamma_{\vec{K}'}(\rho-\rho_{\vec{K}_2})(\rho+i\vec{K}_2\cdot\vec{v}_2)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}-\rho)(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)} \tag{D-16}$$

$$+\frac{\tilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)\,\tilde{P}_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''})}{2\gamma_{\vec{K}''}(\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)(\rho-\rho_{\vec{K}_2}-2\gamma_{\vec{K}''})(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}_2\cdot\vec{v}_2)}\Bigg\}\times$$

$$\times\, iD_{\vec{K}_1}(1)\,C_{\vec{K}'}(1)\tilde{R}_{\vec{K}_1}\;iD_{\vec{K}_2}(2)\,C_{\vec{K}''}(2)\tilde{R}_{\vec{K}_2}$$

The inverse Laplace transform of the first group of terms (those within the brackets [ ]) is complicated by the presence of the poles of $P_{\vec{K}_2}(2|\rho)$ on the left-hand side of the contour C and the poles of $P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+\rho_{\vec{K}_2}+2\gamma_{\vec{K}'}+2\gamma_{\vec{K}''}-\rho)$ on the right-hand side of C. We are unable to evaluate each term conveniently by closing the contour either to the left or to the right. However, these terms vanish when added together and do not contribute to the integral. The next to last term of (D-16) is evaluated by closing the contour to the right. There are no poles inside the contour, and this term contributes nothing. The last term may be evaluated by closing the contour to the left. The only pole inside the contour is the one at $\rho=\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}$, and we find

$$\frac{(i)^4 D_{\vec{K}_1}(1)\varphi(1) D_{\vec{K}_2}(2)\varphi(2)\widetilde{P}_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'})\widetilde{P}_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}) D_{\vec{K}_1}(1) C_{\vec{K}'}(1)\widetilde{R}_{\vec{K}_1} D_{\vec{K}_2}(2) C_{\vec{K}_2}(2)\widetilde{R}_{\vec{K}_2}}{(2\gamma_{\vec{K}'})(2\gamma_{\vec{K}''})(\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}+i\vec{K}_1\cdot\vec{v}_1)(\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}+i\vec{K}\cdot\vec{v}_2)} \quad \text{(D-17)}$$

If the contributions (D-10), (D-12), (D-15) and (D-17) are added together the convolution integral (D-3) may be written in the following way

$$\left(\frac{1}{2\gamma_{\vec{K}'}} P_{\vec{K}_1}(1|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}) D_{\vec{K}_1}(1) C_{\vec{K}'}(1)\widetilde{R}_{\vec{K}_1}\right)\left(\frac{1}{2\gamma_{\vec{K}''}} P_{\vec{K}_2}(2|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}) D_{\vec{K}_2}(2) C_{\vec{K}''}(2)\widetilde{R}_{\vec{K}_2}\right) \quad \text{(D-18)}$$

which is exactly cancelled by the term

$$S_{\vec{K}_1}(1,i|\rho_{\vec{K}_1}+2\gamma_{\vec{K}'}) R_{\vec{K}_1}(i) C_{\vec{K}'}(1) S_{\vec{K}_2}(2,\ell|\rho_{\vec{K}_2}+2\gamma_{\vec{K}''}) R_{\vec{K}_2}(\ell) C_{\vec{K}''}(2) \quad \text{(D-19)}$$

from the product $f_{\vec{K}_1}(1) f_{\vec{K}_2}(2)$ of two single particle functions.

TABLE I

SUMMARY OF OPERATORS

A) Explicit form of the operators

$$P_{\vec{K}}(1|p) = \frac{1}{p+i\vec{K}\cdot\vec{v}_1} + \frac{iD_{\vec{K}}(1)\varphi(1)\int\frac{d\vec{v}_1}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)}$$

$$S_{\vec{K}}(1,i|p) = \left(\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}}}{(p+i\vec{K}\cdot\vec{v})\,\varepsilon(\vec{K},p)}\right)\frac{\int\frac{d\vec{v}_i}{p+i\vec{K}\cdot\vec{v}_i}}{\varepsilon(\vec{K},p)}\;(\vec{K}_i\to\vec{K})$$

$$S^{(1)}_{\vec{K}}(1,i|j|p) = \left(\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1} + \frac{iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\frac{iD_{\vec{K}}(1)}{p+i\vec{K}\cdot\vec{v}_1}}{(p+i\vec{K}\cdot\vec{v}_1)\,\varepsilon(\vec{K},p)}\right)\times$$

$$\times\frac{\int\frac{d\vec{v}_i}{p+i\vec{K}\cdot\vec{v}_i}}{\varepsilon(\vec{K},p)}\;\frac{\int d\vec{v}_j\frac{iD_{\vec{K}}(j)}{p+iK\cdot v_j}}{\varepsilon(\vec{K},p)}\begin{pmatrix}\vec{K}_i\to\vec{K}\\ \vec{K}_j\to\vec{K}\end{pmatrix}$$

$$S^{(2)}_{\vec{K}}(1,i|j,\ell|p) = S^{(1)}_{\vec{K}}(1,i|j|p)\,\frac{\int d\vec{v}_\ell\frac{iD_{\vec{K}}(\ell)}{p+i\vec{K}\cdot\vec{v}_\ell}}{\varepsilon(\vec{K},p)}\;(\vec{K}_\ell\to\vec{K})$$

In general:

$$S^{(n)}_{\vec{K}}(1,i|j,\cdots,m,n|p) = S^{(n-1)}_{\vec{K}}(1,i|j,\cdots,m|p)\,\frac{\int d\vec{v}_n\frac{iD_{\vec{K}}(n)}{p+i\vec{K}\cdot\vec{v}_n}}{\varepsilon(\vec{K},p)}\;(\vec{K}_n\to\vec{K})$$

B) The operators satisfy the following equations

$$\left(\frac{\partial}{\partial t} + i\vec{K}\cdot\vec{v}_1 - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\right) P_{\vec{K}}(1|t) = 0$$

$$P_{\vec{K}}(1|t=0) = 1$$

$$\left(\frac{\partial}{\partial t} + i\vec{K}\cdot\vec{v}_1 - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\right) S_{\vec{K}}(1,i|t) = \int d\vec{v}_i\, P_{\vec{K}}(i|t)\, iD_{\vec{K}}(1)$$

$$S_{\vec{K}}(1,i|t=0) = 0$$

$$\left(\frac{\partial}{\partial t} + i\vec{K}\cdot\vec{v}_1 - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\right) S^{(1)}_{\vec{K}}(1,i|j|t) = i\int d\vec{v}_j\, S_{\vec{K}}(j,i|t)\, D_{\vec{K}}(1)$$

$$S^{(1)}_{\vec{K}}(1,i|j|t=0) = 0$$

In general

$$\left(\frac{\partial}{\partial t} + i\vec{K}\cdot\vec{v}_1 - iD_{\vec{K}}(1)\varphi(1)\int d\vec{v}_1\right) S^{(n)}_{\vec{K}}(1,i|j,\cdots,m,n|t) =$$

$$= i\int d\vec{v}_n\, S^{(n-1)}_{\vec{K}}(1,i|j,\cdots,m|t)\, D_{\vec{K}}(1)$$

$$S^{(n)}_{\vec{K}}(1,i|j,\cdots,m,n|t=0) = 0$$

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