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A STUDY OF ELECTRODYNAMICS INVOLVING MOVING MEDIA

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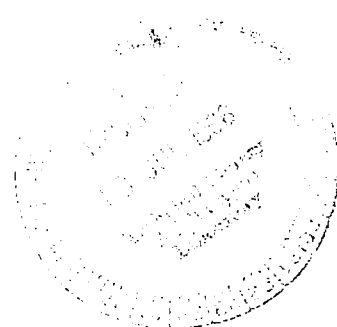
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ABSTRACT

The electromagnetic field problems involving moving isotropic and lossless medium have been studied, based upon Minkowski's theory of electrodynamics of moving media. The analysis of the problem is facilitated by the use of the auxiliary functions known as potentials, which are introduced in a manner similar to that used for the stationary medium. Two problems are considered: the field expansions in and the parameters of the waveguide filled with medium moving uniformly along the direction of the axis of the guide, and the radiation field of a line source located above and parallel to a moving dielectric half-space.

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CHAPTER I
INTRODUCTION

The extension of Maxwell's theory from media at rest to those in motion was originally studied in the latter part of the nineteenth century. The covariance of the equations of electrodynamics under the Lorentz transformation was first proved by Lorentz [1] and Poincare. [2] Einstein [3] formulated the special theory of relativity in 1905. However, their work was confined to the question of the isolated electron and did not cover the case of ponderable bodies in general.

The problem of the electrodynamics of moving media was first formulated correctly in 1908 by Minkowski. [4] Despite the fact that his work was done almost sixty years ago, this subject has received very little attention; recently, however, there has been a revival of interest in this topic, principally as a result of the work of C. T. Tai. [5] Two other works on this subject which should be mentioned are those of Fano, Chu, and Adler, [6] and Boffi. [7] These authors have each presented a formulation of the electrodynamics of moving media that is apparently different from that of Minkowski. It has been shown by Tai, [8] however, that all

three of these formulations, as well as some other possible ones, are mathematically equivalent. For the case in which the velocity of the medium is small compared with the speed of light, the Maxwell-Minkowski equations can be simplified. Several studies were made under this assumption. Compton and Tai[9] have derived the dyadic Green's function for an infinite moving medium. Collier and Tai[10] discussed guided waves in moving media. A problem dealing with the reflection and refraction of a plane wave at the boundary of a semi-infinite moving medium was also investigated. [11]

The exact theory with no restriction upon the magnitude of the velocity was developed by Tai[12] in connection with the radiation problem in a moving isotropic medium. By transforming the wave equation into a conventional form and then solving it by means of an operational method due to Levine and Schwinger, a compact result was obtained. Another exact formulation for the same problem has been developed independently by Lee and Papas. [13] They derived the differential equations for potentials in the moving media from those which are well known in the stationary case through the Lorentz transformation, and then solved the transformed wave equations using the Green's function technique. Their method

was also extended to the case in which the moving medium is dispersive. [14]

In this study, we consider some additional problems in the electrodynamics of moving media, for the case in which there is no restriction on the velocity of the medium. In Chapter II, the Maxwell-Minkowski equations for the electromagnetic fields are presented and some suitable potential functions are introduced in a way analogous to that commonly used for stationary media. In Chapter III the work of Collier and Tai on the propagation of guided waves in moving media is extended to the case of arbitrary velocity. Chapter IV presents the theory of the radiation of a line source over a moving dielectric half-space. Four cases are considered: an electric line source and a magnetic line source each in two orientations, parallel to and perpendicular to the direction of the velocity of the medium. The moving medium is assumed to be lossless, isotropic, and to have an index of refraction greater than unity. A solution for the Maxwell-Minkowski equations is constructed in the form of a Fourier integral. The integral is evaluated for the far field by deforming the original contour into the steepest descent path. In the process, an additional branch cut integral is encountered, but this is found to give a negligible contribution to the far fields.

CHAPTER II
MAXWELL'S EQUATIONS AND WAVE EQUATIONS
ASSOCIATED WITH MOVING MEDIUM

In this chapter the wave equations as well as the equations satisfied by the potential functions in the moving isotropic medium will be derived which reduce to those of the stationary medium as a special case.

Maxwell-Minkowski Equations and the Transformation of the Field Vectors

According to the special theory of relativity, the Maxwell's equations must be covariant under the Lorentz transformation. In other words, the Maxwell equations have the same form in all inertial coordinate frames. Hence, for any medium moving or stationary we have

$$(1) \qquad \nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$(2) \qquad \nabla \times \overline{H} = J + \frac{\partial \overline{D}}{\partial t}$$

$$(3) \qquad \nabla \cdot \overline{D} = \rho$$

$$(4) \qquad \nabla \cdot \overline{B} = 0 \; .$$

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We shall use primed quantities to denote field variables which are measured in an initial frame K' and unprimed quantities to denote the field variables in an initial frame K. In particular, we assume all the electromagnetic sources and the observers to be stationary in the K frame and the medium which is moving at a uniform velocity $\bar{v}$ with respect to the source and the observers to be stationary with respect to the K' frame. If we assume that the two inertial frames K and K' have the same orientation (i.e., the x, y, z axes are respectively parallel to the x', y', z' axes) and are coincident at t = t' = 0, then the field variables defined in the two frames transform according to the following relations: [15]

(5a) $\quad \bar{E}' = \gamma(\bar{E} + \bar{v} \times \bar{B}) + (1-\gamma)\dfrac{\bar{E} \cdot \bar{v}}{v^2}\bar{v}$

(6a) $\quad \bar{B}' = \gamma\left(\bar{B} - \dfrac{1}{c^2}\bar{v} \times \bar{E}\right) + (1-\gamma)\dfrac{\bar{B} \cdot \bar{v}}{v^2}\bar{v}$

(7a) $\quad \bar{H}' = \gamma(\bar{H} - \bar{v} \times \bar{D}) + (1-\gamma)\dfrac{\bar{H} \cdot \bar{v}}{v^2}\bar{v}$

(8a) $\quad \bar{D}' = \gamma\left(\bar{D} + \dfrac{1}{c^2}\bar{v} \times \bar{H}\right) + (1-\gamma)\dfrac{\bar{D} \cdot \bar{v}}{v^2}\bar{v}$

where

$$\gamma = 1/\sqrt{1 - \beta^2}$$

$$\beta = v/c$$

$$c = 1/\sqrt{\mu_o \epsilon_o} = \text{velocity of light in free space.}$$

If the velocity $\bar{v}$ is directed in the y-direction Eqs. (5a), (6a), (7a), and (8a) can be written in a more compact form as follows:[16]

(5b)
$$\bar{E}' = \bar{\bar{Y}} \cdot (\bar{E} + \bar{v} \times \bar{B})$$

(6b)
$$\bar{B}' = \bar{\bar{Y}} \cdot \left(\bar{B} - \frac{1}{c^2} \bar{v} \times \bar{E} \right)$$

(7b)
$$\bar{H}' = \bar{\bar{Y}} \cdot (\bar{H} - \bar{v} \times \bar{D})$$

(8b)
$$\bar{D}' = \bar{\bar{Y}} \cdot \left(\bar{D} + \frac{1}{c^2} \bar{v} \times \bar{H} \right)$$

where the tensor $\bar{\bar{Y}}$ is defined as

$$\bar{\bar{Y}} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix}.$$

For an isotropic linear medium which is stationary with respect to the K' frame the constitutive relations between the primed field vectors are then given by

(9)
$$\bar{D}' = \epsilon \bar{E}'$$

(10)
$$\bar{B}' = \mu \bar{H}'$$

where ϵ and μ denote, respectively, the permittivity and permeability

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of the medium when it is stationary. We assume the medium to be lossless. Expressing $\overline{E}'$, $\overline{D}'$, $\overline{H}'$ and $\overline{B}'$ in terms of $\overline{E}$, $\overline{D}$, $\overline{H}$ and $\overline{B}$ we find with the aid of Eqs. (5b), (6b), (7b), and (8b) the constitutive relations in the K frame which are

$$(11) \qquad \overline{D} + \frac{1}{c^2}\, \overline{v} \times \overline{H} = \epsilon(\overline{E} + \overline{v} \times \overline{B})$$

$$(12) \qquad \overline{B} - \frac{1}{c^2}\, \overline{v} \times \overline{E} = \mu(\overline{H} - \overline{v} \times \overline{D}) \ .$$

By solving for $\overline{B}$ and $\overline{D}$ in terms of $\overline{E}$ and $\overline{H}$ with $\overline{v} = v\,\hat{y}$, one finds [17]

$$(13) \qquad \overline{D} = \epsilon\, \overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H}$$

$$(14) \qquad \overline{B} = \mu\, \overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E}$$

where

$$\overline{\Omega} = \frac{(n^2 - 1)\, \beta}{(1 - n^2\beta^2)\, c}\ \hat{y}$$

$$n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$$

$$\overline{\overline{\alpha}} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$a = \frac{1 - \beta^2}{1 - n^2\beta^2}\qquad {}^* \quad .$$

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*This useful notation is due to Prof. C. T. Tai of the University of Michigan.

Substitution of (13) and (14) into (1) and (2) yields the Maxwell-Minkowski equations for moving isotropic medium. They are

$$(15) \qquad \nabla \times \overline{E} = - \frac{\partial}{\partial t} \left[\mu \overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E} \right]$$

$$(16) \qquad \nabla \times H = J + \frac{\partial}{\partial t} \left[\epsilon \overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H} \right] .$$

For harmonically oscillating fields with a time convention $e^{-i\omega t}$ (15) and (16) may be converted into

$$(17) \qquad (\nabla + i\omega \overline{\Omega}) \times \overline{E} = i\omega\mu \overline{\overline{\alpha}} \cdot \overline{H}$$

$$(18) \qquad (\nabla + i\omega \overline{\Omega}) \times \overline{H} = \overline{J} - i\omega\epsilon \overline{\overline{\alpha}} \cdot \overline{E}$$

The time factor $e^{-i\omega t}$ is understood and $\overline{J}$ is considered to be the current source term in a lossless moving medium.

The Wave Equations

The wave equations for $\overline{E}$ and $\overline{H}$ can be obtained by eliminating respectively $\overline{E}$ or $\overline{H}$ between (17) and (18). They are

$$(19) \qquad (\nabla + i\omega \overline{\Omega}) \times \left[\overline{\overline{\alpha}}^{-1} \cdot (\nabla + i\omega \overline{\Omega}) \times \overline{E} \right] - k^2 \overline{\overline{\alpha}} \cdot \overline{E} = i\omega\mu \overline{J}$$

$$(20) \qquad (\nabla + i\omega \overline{\Omega}) \times \left[\overline{\overline{\alpha}}^{-1} \cdot (\nabla + i\omega \overline{\Omega}) \times \overline{H} \right] - k^2 \overline{\overline{\alpha}} \cdot \overline{H}$$
$$= (\nabla + i\omega \overline{\Omega}) \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{J})$$

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where $\overline{\overline{\alpha}}^{-1}$ denotes the inverse of $\overline{\overline{\alpha}}$ and $k^2 = \omega^2 \mu\epsilon$. By virtue of (13) and (14), Maxwell's equations can be written as

$$(17) \qquad (\nabla + i\omega\overline{\Omega}) \times \overline{E} = i\omega\mu\,\overline{\overline{\alpha}} \cdot \overline{H}$$

$$(18) \qquad (\nabla + i\omega\overline{\Omega}) \times \overline{H} = -i\omega\epsilon\,\overline{\overline{\alpha}} \cdot \overline{E} + \overline{J}$$

$$(21) \qquad \nabla \cdot (\epsilon\,\overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H}) = \rho$$

$$(22) \qquad \nabla \cdot (\mu\,\overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E}) = 0 \quad .$$

Expanding (17) and (21) and eliminating the term including $\overline{H}$ we have instead of (21) the following relation

$$(23) \qquad (\nabla + i\omega\overline{\Omega}) \cdot (\epsilon\,\overline{\overline{\alpha}} \cdot \overline{E}) = \overline{\Omega} \cdot \overline{J} + \rho \quad .$$

Similarly,

$$(24) \qquad (\nabla + i\omega\overline{\Omega}) \cdot (\mu\,\overline{\overline{\alpha}} \cdot \overline{H}) = 0 \,.$$

These equations are seen to be similar to those for the stationary medium, except for the substitution of the operator

$$(25) \qquad D_1 = \nabla + i\omega\overline{\Omega}$$

for the nabla operator ∇. If the vector field functions $\overline{E}_1$ and $\overline{H}_1$ are defined such that

$$(26) \qquad \overline{E}_1 = \overline{\overline{\alpha}} \cdot \overline{E}$$

and

(27) $\qquad \bar{H}_1 = \bar{\bar{\alpha}} \cdot \bar{H}$

then $\bar{E}_1$ and $\bar{H}_1$ satisfy the equation

(28a) $\qquad D_1 \times [\bar{\bar{\alpha}}^{-1} \cdot D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{E}_1)] - k^2 \bar{E}_1 = i\omega\mu\bar{J}$

(28b) $\qquad D_1 \times [\bar{\bar{\alpha}}^{-1} \cdot D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{H}_1)] - k^2 \bar{H}_1 = D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{J})$.

It is not difficult to show by writing out the operators in cartesian coordinates that

(29) $\qquad D_1 \times [\bar{\bar{\alpha}}^{-1} \cdot D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{E}_1)] = \dfrac{1}{a} [D_a(D_1 \cdot \bar{E}_1) - (D_a \cdot D_1) \bar{E}_1]$

where the operator ∇_a and D_a are defined by

(30) $\qquad \nabla_a = \hat{x} \dfrac{\partial}{\partial x} + \hat{y} \dfrac{1}{a} \dfrac{\partial}{\partial y} + \hat{z} \dfrac{\partial}{\partial z} = \dfrac{1}{a} \bar{\bar{\alpha}} \cdot \nabla$

(31) $\qquad D_a = \nabla_a + \dfrac{i\omega\bar{\Omega}}{a}$.

In view of Eqs. (28), (29), (23) and (24) the differential equations for $\bar{E}_1$ and $\bar{H}_1$ are then

(32) $\qquad (D_a \cdot D_1) \bar{E}_1 + k^2 a \bar{E}_1 = -i\omega\mu a\bar{J} + \dfrac{D_a(\bar{\Omega} \cdot \bar{J} + \rho)}{\epsilon}$

(32b) $\qquad (D_a \cdot D_1) \bar{H}_1 + k^2 a \bar{H}_1 = -a D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{J})$

Potential Functions

The equations satisfied by the electric and magnetic vector and scalar potentials are found by proceeding in the same way as for a stationary medium. [18]   Since from (24) and (25)

(24a) $D_1 \cdot (\overline{\overline{\alpha}} \cdot \overline{H}) = 0$

we may write (taking advantage of the fact that $D_1 \cdot D_1 \times \overline{W} \equiv 0$ for any vector $\overline{W}$)

(33a) $\mu \overline{\overline{\alpha}} \cdot \overline{H}^e = D_1 \times \overline{A}$

where the superscript e denotes that $\overline{H}^e$ is associated with fields of the electric type (transverse magnetic TM). Substituting (33a) into (17) we obtain

(34a) $D_1 \times (\overline{E}^e - i\omega \overline{A}) = 0$.

In view of the above equation we introduce the electric scalar potential function U. Since $D_1 \times D_1 U \equiv 0$, we set

(35a) $\overline{E}^e - i\omega \overline{A} = -D_1 U$.

Applying the operator "$D_1 \times$" to (33a) we have

(36a) $D_1 \times (\mu \overline{H}^e) = D_1 \times [\overline{\overline{\alpha}}^{-1} \cdot D_1 \times \overline{A}]$.

Substituting (36a) in (18) and making use of (35a) gives

(37a) $\quad D_1 \times [\bar{\bar{\alpha}}^{-1} \cdot D_1 \times \bar{A}] = k^2 \bar{\bar{\alpha}} \cdot \bar{A} + i\omega\mu\epsilon \, \bar{\bar{\alpha}} \cdot D_1 U + \mu \bar{J}$.

If we define another vector function $\bar{A}_1$ such that

(38a) $\quad \bar{A}_1 = \bar{\bar{\alpha}} \cdot \bar{A}$

then $\bar{A}_1$ satisfies the following equation

(39a) $\quad D_1 \times [\bar{\bar{\alpha}}^{-1} \cdot D_1 \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1)] = k^2 \bar{A}_1 + i\omega\mu\epsilon \, \bar{\bar{\alpha}} \cdot D_1 U + \mu \bar{J}$.

We may now impose upon $\bar{A}_1$ and U the supplementary condition

(40a) $\quad D_1 \cdot \bar{A}_1 = i\omega\mu\epsilon \, a^2 U$.

As a result of (29) and (40a), (39a) becomes

(41a) $\quad (D_a \cdot D_1) \bar{A}_1 + k^2 a \bar{A}_1 = -a\mu \bar{J}$.

Substituting (35a) in (23) and making use of (40a) one obtains the differential equation for U as

(42a) $\quad D_1 \cdot (\bar{\bar{\alpha}} \cdot D_1 U) + k^2 a^2 U = - \dfrac{(\bar{\Omega} \cdot \bar{J} + \rho)}{\epsilon}$.

The explicit expressions of (41a) and (42a) are

(43a) $\quad \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{1}{a} \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} + \dfrac{2i\omega\Omega}{a} \dfrac{\partial}{\partial y} - \dfrac{\omega^2\Omega^2}{a} \right.$

$\qquad\qquad\qquad \left. + k^2 a \right] \bar{A}_1 = -a\mu \bar{J}$

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(44a) $$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a} \frac{\partial}{\partial y} - \frac{\omega^2\Omega^2}{a} + k^2 a \right] U = - \frac{(\overline{\Omega} \cdot \overline{J} + \rho)}{a\epsilon} \, .$$

Since $\overline{E}^e$ and $\overline{H}^e$ are derivable from A and U, we have, from (33a), (35a) and (38a)

(45a) $$\overline{E}^e = i\omega\overline{A} - D_1 U = i\omega \overline{\overline{\alpha}}^{-1} \cdot \overline{A}_1 - \frac{1}{i\omega\mu\epsilon \, a^2} D_1 (D_1 \cdot \overline{A}_1)$$

(46a) $$\overline{H}^e = \frac{1}{\mu} \overline{\overline{\alpha}}^{-1} \cdot D_1 \times \overline{A} = \frac{1}{\mu} \overline{\overline{\alpha}}^{-1} \cdot D_1 \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{A}_1) \, .$$

Assuming $\overline{J} = 0$ and $\rho = 0$ and there is a magnetic current source $\overline{M}$ (17) becomes

(17a) $$(\nabla + i\omega\overline{\Omega}) \times \overline{E} = i\omega\mu \overline{\overline{\alpha}} \cdot \overline{H} - \overline{M} \, .$$

A similar procedure is followed to find the equations satisfied by the potential functions $\overline{F}$ and V associated with the fields $\overline{E}^m$ and $\overline{H}^m$ of the magnetic type (transverse electric, TE). A summary of these results is given below. The equation numbers correspond to those in deriving the electric type fields as shown above. $\overline{E}^m$ and $\overline{H}^m$ are evaluated in terms of $\overline{F}$ and V as

(45b) $$\overline{E}^m = - \frac{1}{\epsilon} \overline{\overline{\alpha}}^{-1} \cdot (D_1 \times \overline{F}) = - \frac{1}{\epsilon} \overline{\overline{\alpha}}^{-1} \cdot \left[D_1 \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{F}_1) \right]$$

(46b) $$\overline{H}^m = i\omega\overline{F} - D_1 V = i\omega \overline{\overline{\alpha}}^{-1} \cdot \overline{F}_1 - D_1 V$$

$\overline{F}$ and $\overline{F}_1$ are related by

(38b) $$\overline{F}_1 = \overline{\overline{\alpha}} \cdot \overline{F} \, .$$

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The supplementary condition imposed on $\overline{F}_1$ and V is

$$(40b) \qquad D_1 \cdot \overline{F}_1 = i\omega\mu\epsilon\, a^2 V$$

$\overline{F}_1$ and V satisfy the following differential equations

$$(41b) \qquad (D_a \cdot D_1)\, \overline{F}_1 + k^2 a\, \overline{F}_1 = -a\epsilon\overline{M}$$

and

$$(42b) \qquad D_1 \cdot (\overline{\overline{\alpha}} \cdot D_1 V) + k^2 a^2 V = -\frac{\overline{\Omega} \cdot \overline{M}}{\mu} \qquad .$$

The explicit expressions for (41b) and (42b) are

$$(43b) \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{1}{a}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial y} - \frac{\omega^2\Omega^2}{a} + k^2 a\right]\overline{F}_1 = -a\epsilon\overline{M}$$

and

$$(44b) \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{1}{a}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial y} - \frac{\omega^2\Omega^2}{a} + k^2 a\right]V = -\frac{\overline{\Omega} \cdot \overline{M}}{a\mu} \qquad .$$

All four of the potential functions $\overline{A}_1$, $\overline{F}_1$, U and V satisfy the same equation as that of $\overline{E}_1$ and $\overline{H}_1$ in the source-free region. By combining (45) and (46) one finds the total $\overline{E}$ and $\overline{H}$ fields in terms of the four potentials. These are

$$(45) \qquad E = i\omega\overline{A} - D_1 U - \frac{1}{\epsilon}\overline{\overline{\alpha}}^{-1} \cdot (D_1 \times \overline{F})$$

$$= i\omega\overline{\overline{\alpha}}^{-1} \cdot \overline{A}_1 - \frac{1}{i\omega\mu\epsilon\, a^2} D_1(D_1 \cdot \overline{A}_1) - \frac{1}{\epsilon}\overline{\overline{\alpha}}^{-1}$$

$$\cdot [D_1 \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{F}_1)]$$

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$$(46) \quad \overline{H} = \frac{1}{\mu} \, \overline{\overline{\alpha}}^{-1} \cdot D_1 \times \overline{A} + i\omega\overline{F} - D_1 V$$

$$= \frac{1}{\mu} \, \overline{\overline{\alpha}}^{-1} \cdot D_1 \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{A}_1) + i\omega\overline{\overline{\alpha}}^{-1} \cdot \overline{F}_1$$

$$- \frac{1}{i\omega\mu\epsilon \, a^2} \, D_1(D_1 \cdot \overline{F}_1).$$

CHAPTER III
PROPAGATION IN WAVEGUIDES

In this chapter we shall consider the problems of electro-magnetic waves in the interior of a cylindrical or rectangular waveguide which is filled with a homogeneous, isotropic and loss-less medium with constitutive parameters μ and ϵ moving at uniform velocity $\overline{v} = \hat{z}v$ along the axis of the guide. The wave-guide is assumed to have perfectly conducting walls and to be infinitely long. The field solution in the guide can be divided into two basic modes, TE and TM. TM modes have no axial component of magnetic field, and the field components can be derived from a vector potential $\overline{A} = \hat{z}A$. TE modes have no axial component of electric field and the fields may be derived from a vector potential $\overline{F} = \hat{z}F$.

Equation (43) which the potentials $\overline{A}$ and $\overline{F}$ just mentioned have to satisfy can be written for the problems discussed in this chapter as

$$(47) \qquad \left[\nabla_t^2 + \frac{1}{a} \frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a} \frac{\partial}{\partial z} - \frac{\omega^2\Omega^2}{a} + k^2 a \right] \left\{ \begin{array}{c} A_1 \\ F_1 \end{array} \right\} = 0$$

where

$$\overline{A}_1 = \hat{z} A_1 = \overline{\overline{\alpha}} \cdot \overline{A} = \overline{A} = \hat{z} A \text{ and } \overline{F}_1 = \hat{z} F_1 = \overline{\overline{\alpha}} \cdot \overline{F} = \overline{F} = \hat{z} F.$$

The operator ∇_t^2 is the transverse part of the ∇^2 operator. In the u_1, u_2, z coordinate system with scale factors h_1, h_2, and unity, Eq. (47) becomes

$$(48) \quad \left[\frac{1}{h_1 h_2} \frac{\partial}{\partial u_1} \frac{h_2}{h_1} \frac{\partial}{\partial u_1} + \frac{1}{h_1 h_2} \frac{\partial}{\partial u_2} \frac{h_1}{h_2} \frac{\partial}{\partial u_2} + \frac{1}{a} \frac{\partial^2}{\partial z^2} \right.$$
$$\left. + \frac{2i\omega\Omega}{a} \frac{\partial}{\partial z} - \frac{\omega^2 \Omega^2}{a} + k^2 a \right] \left\{ \begin{array}{c} A_1 \\ F_1 \end{array} \right\} = 0 \; .$$

Since $\overline{A}_1 = \overline{A}$ and $\overline{F}_1 = \overline{F}$ we will formulate the problem in terms of $\overline{A}$ and $\overline{F}$ directly.

The Rectangular Waveguide

The appropriate solution of A and F which satisfies the boundary conditions for the waveguide configuration shown in Fig. 1 is

$$(49) \quad \overline{A} = \hat{z} A = \hat{z} A_o \sin k_x x \, \sin k_y y \, e^{ihz} \qquad m = 1, 2, 3, \cdots$$
$$= \hat{z} A_o \sin \frac{m\pi}{x_o} x \sin \frac{\ell \pi}{y_o} y \; e^{ihz} \qquad \ell = 1, 2, 3, \cdots$$

$$(50) \quad \overline{F} = \hat{z} F = \hat{z} F_o \cos k_x x \, \cos k_y y \, e^{ihz} \qquad m = 0, 1, 2, \cdots$$
$$= \hat{z} F_o \cos \frac{m\pi}{x_o} x \cos \frac{\ell \pi}{y_o} y \; e^{ihz} \qquad \begin{array}{l} \ell = 0, 1, 2, \cdots \\ m \neq \ell \neq 0 \; . \end{array}$$

Substituting (49) or (50) into (47) or (48) gives the following equation relating the propagation constants

$$(51) \quad -(k_x^2 + k_y^2) - \frac{h^2}{a} - \frac{2\omega\Omega h}{a} - \frac{\omega^2 \Omega^2}{a} + k^2 a = 0 \; .$$

The above equation can be solved to give

(52) $$h = -\omega\Omega \pm \sqrt{k^2 a^2 - k_c^2 a}$$

where $k_c^2 = k_x^2 + k_y^2 = \left(\dfrac{m\pi}{x_o}\right)^2 + \left(\dfrac{\ell\pi}{y_o}\right)^2$. Each set of integers m and ℓ

corresponds to a given mode which will be designated as the $TM_{m\ell}$

(or $TE_{m\ell}$) modes.
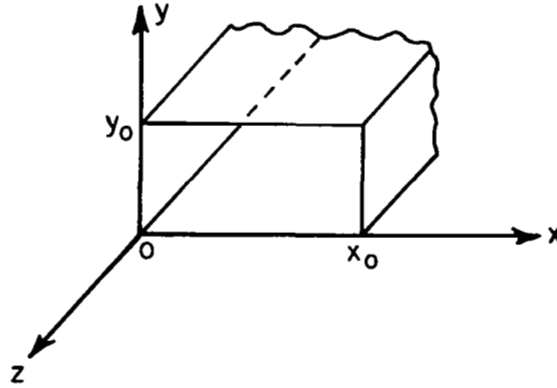
Fig. 1--The rectangular waveguide

The TM modes may be obtained from A by means of Eq. (45a)

and (46a) where we have to interchange the y and z coordinates

because the medium here is moving in the + z direction. Thus

(53a)
$$\overline{E} = i\omega\overline{A} - (\nabla + i\omega\overline{\Omega})\,\frac{(\nabla + i\omega\overline{\Omega}) \cdot \overline{A}}{i\omega\mu\epsilon\,a^2}$$

$$= -\hat{x}\,\frac{A_o(h+\omega\Omega)k_x}{\omega\mu\epsilon\,a^2}\,\cos k_x x \,\sin k_y y \, e^{ihz}$$

$$-\hat{y}\,\frac{A_o(h+\omega\Omega)k_y}{\omega\mu\epsilon\,a^2}\,\sin k_x x\,\cos k_y y\,e^{ihz}$$

$$+\hat{z}\,A_o\left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon\,a^2}\right]\sin k_x x\,\sin k_y y\,e^{ikz}$$

(53b) $\quad \overline{H} = \dfrac{1}{\mu} \overline{\overline{\alpha}}^{-1} \cdot [(\nabla + i\omega\overline{\Omega}) \times \overline{A}]$

$\qquad = \hat{x} \dfrac{A_0 k_y}{\mu a} \sin k_x x \cos k_y y \, e^{ihz}$

$\qquad - \hat{y} \dfrac{A_0 k_x}{\mu a} \cos k_x x \sin k_y y \, e^{ihz} \quad .$

In a similar way the TE modes may be obtained from $\overline{F}$ by means of Eqs. (45b) and (46b) giving

(54a) $\quad \overline{E} = \dfrac{-1}{\epsilon} \overline{\overline{\alpha}}^{-1} \cdot [(\nabla + i\omega\overline{\Omega}) \times \overline{F}]$

$\qquad = \hat{x} \dfrac{F_0 k_y}{\epsilon \, a} \cos k_x x \sin k_y y \, e^{ihz}$

$\qquad - \hat{y} \dfrac{F_0 k_x}{\epsilon \, a} \sin k_x x \cos k_y y \, e^{ihz}$

(54b) $\quad \overline{H} = i\omega \overline{F} - (\nabla + i\omega\overline{\Omega}) \dfrac{(\nabla + i\omega\overline{\Omega}) \cdot \overline{F}}{i\omega\mu\epsilon \, a^2}$

$\qquad = \hat{x} \dfrac{F_0 k_x (h + \omega\Omega)}{\omega\mu\epsilon \, a^2} \sin k_x x \cos k_y y \, e^{ihz}$

$\qquad + \hat{y} \dfrac{F_0 k_y (h + \omega\Omega)}{\omega\mu\epsilon \, a^2} \cos k_x x \sin k_y y \, e^{ihz}$

$\qquad + \hat{z} \, F_0 \left[i\omega + \dfrac{(h + \omega\Omega)}{i\omega\mu\epsilon \, a^2} \right] \cos k_x x \cos k_y y \, e^{ihz} \quad .$

Cylindrical Waveguides

The proper form of $\overline{A}$ and $\overline{F}$ for the cylindrical waveguide shown in Fig. 2 may be written as

$$(55) \qquad \overline{A} = \hat{z}\, A = \hat{z}\, A_o J_m(k_c r) \,{\cos \atop \sin}\, m\phi \; e^{ihz}$$

$$(56) \qquad \overline{F} = \hat{z}\, F = \hat{z}\, F_o J_m(k_c r) \,{\cos \atop \sin}\, m\phi \; e^{ihz}$$

where $J_m(k_c r)$ is the Bessel function of integer order m. A and F satisfy (48) which for this problem is

$$(57) \qquad \left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial z} - \frac{\omega^2\Omega^2}{a} + k^2 a\right]\left\{{A \atop F}\right\} = 0.$$
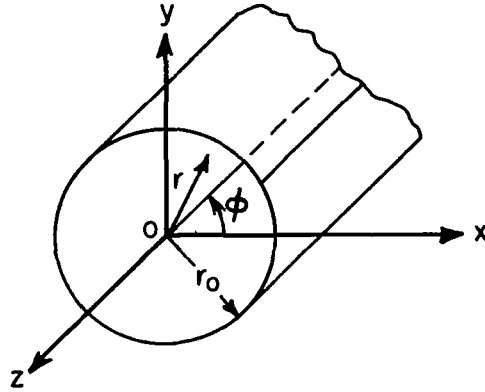

Fig. 2--The cylindrical waveguide

From (55)(or (56)) and (57) we obtain the relation

$$(58) \qquad k_c^2 = k^2 a - \frac{\omega^2 \Omega^2}{a} - \frac{2\omega\Omega h}{a} - \frac{h^2}{a} .$$

The TM modes may be derived from A by means of Eqs. (45a) and (46a) in the form

(59a)
$$\overline{E} = i\omega\overline{A} - (\nabla + i\omega\overline{\Omega}) \frac{(\nabla + i\omega\overline{\Omega}) \cdot \overline{A}}{i\omega\mu\epsilon\, a^2}$$

$$= -\hat{r} \frac{A_0(h+\omega\Omega)}{\omega\mu\epsilon\, a^2} \frac{dJ_m(k_c r)}{dr} \frac{\cos}{\sin} m\phi\, e^{ihz}$$

$$- \hat{\phi} \frac{m A_0(h+\omega\Omega)}{\omega\mu\epsilon\, a^2} \times \frac{1}{r} J_m(k_c r) \frac{-\sin}{\cos} m\phi\, e^{ihz}$$

$$+ \hat{z} A_0 \left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon\, a^2} \right] J_m(k_c r) \frac{\cos}{\sin} m\phi\, e^{ihz}$$

(59b)
$$\overline{H} = \hat{r} \frac{m A_0}{\mu a} \times \frac{1}{r} J_m(k_c r) \frac{-\sin}{\cos} m\phi\, e^{ihz}$$

$$- \hat{\phi} \frac{A_0}{\mu a} \frac{dJ_m(k_c r)}{dr} \frac{\cos}{\sin} m\phi\, e^{ihz} \quad .$$

The boundary condition at $r = r_o$ requires

(60)
$$J_m(k_c r_o) = 0$$

which determines the allowed values of k_c. There are an infinite number of solutions which will be enumerated as $\rho_m\,{}_\ell$. Hence k_c can assume only those values

(61)
$$k_{c, m\ell} = \frac{\rho_{m\ell}}{r_o}$$

and the corresponding modes will be labeled $TM_{m\ell}$ where the first subscript refers to the number of cyclic variations with ϕ, and the second subscript refers to the ℓth root of the Bessel function.

The TE modes may be derived by means of Eqs. (45b) and (46b) as

(62a)
$$\bar{E} = -\hat{r} \frac{m F_o}{\epsilon \, a r} J_m(k_c r) \begin{matrix} -\sin \\ \cos \end{matrix} m\phi \, e^{ihz}$$

$$+ \hat{\phi} \frac{F_o}{\epsilon \, a} \frac{d}{dr} J_m(k_c r) \begin{matrix} \cos \\ \sin \end{matrix} m\phi \, e^{ihz}$$

(62b)
$$\bar{H} = -\hat{r} \frac{F_o(h + \omega\Omega)}{\omega\mu\epsilon \, a^2} \frac{d}{dr} J_m(k_c r) \begin{matrix} \cos \\ \sin \end{matrix} m\phi \, e^{ihz}$$

$$-\hat{\phi} \frac{m F_o(h + \omega\Omega)}{\omega\mu\epsilon \, a^2} \times \frac{1}{r} J_m(k_c r) \begin{matrix} -\sin \\ \cos \end{matrix} m\phi \, e^{ihz}$$

$$+ \hat{z} \, F_o \left[i\omega + \frac{(h + \omega\Omega)^2}{i\omega\mu\epsilon \, a^2} \right] J_m(k_c r) \begin{matrix} \cos \\ \sin \end{matrix} m\phi \, e^{ihz} \, .$$

The boundary condition at $r = r_o$ requires

(63)
$$\frac{d}{dr} J_m(k_c r) \Big|_{r=r_o} = 0 \, .$$

There are also infinitely many solutions for (63) which will be designated by $\rho'_{m\ell}$ and k_c is given by

(64)
$$k_{c, m\ell} = \frac{\rho'_{m\ell}}{r_o} \, .$$

The corresponding modes will be labeled $TE_{m\ell}$. Equation (58) can be used to solve for h to give

(65)
$$h = -\omega\Omega \pm \sqrt{k^2 a^2 - k_c^2 a}$$

where

$$k_c = \frac{p_{m\ell}}{r_o} \quad \text{(TM modes)}$$

$$k_c = \frac{p'_{m\ell}}{r_o} \quad \text{(TE modes)} .$$

Waveguide Parameters

The formula and the conclusions given in this section apply to both rectangular and cylindrical waveguides with few exceptions which will be specified individually. k_c will assume the value given in (52) for the rectangular waveguides and the value given in (61) or (64) for cylindrical waveguides.

The propagation constant is given in (52) and (65) as

$$h = -\omega\Omega \pm \sqrt{k^2 a^2 - k_c^2 a} .$$

When $n\beta < 1$ the cut off occurs for

$$k^2 a \leq k_c^2$$

i. e.,

$$(66) \qquad k_o \sqrt{\frac{n^2(1-\beta^2)}{1-n^2\beta^2}} \leq k_c$$

where $k_o = \omega\sqrt{\mu_o \epsilon_o}$ $n = \sqrt{\frac{\mu\epsilon}{\mu_o \epsilon_o}}$, and the cut off frequency is

$$(67) \qquad f_c = \frac{k_c}{2\pi\sqrt{\mu_o \epsilon_o}\sqrt{\frac{n^2(1-\beta^2)}{1-n^2\beta^2}}} .$$

When f is less than f_c wave may propagate in the -z direction with phase velocity $v_p = 1/\Omega$ and an exponentially varying magnitude. When f is slightly greater than f_c there is no attenuation, but waves can propagate in the -z direction only (two waves with different phase velocities) unless f is large enough such that the following relation is satisfied

$$(68) \qquad k^2 a^2 - k_c^2 a \geq \omega^2 \Omega^2$$

Equation (68) can be manipulated to the form

$$(69) \qquad f \geq f_+ = \frac{k_c}{2\pi \sqrt{\mu_o \epsilon_o} \sqrt{\dfrac{n^2 - \beta^2}{1 - \beta^2}}} \quad .$$

For frequencies greater than f_+ waves can propagate in either direction without attenuation but with different phase velocities. If $\overline{v} = 0$ then $\beta = 0$ and we have

$$(70) \qquad f_+ = f_c = \frac{k_c}{2\pi \sqrt{\mu \epsilon}}$$

which is the usual cut off frequency in the stationary case. When $n\beta > 1$ a will be negative while $-\omega\Omega$ is positive. In this case there is no cut off phenomenon at all. Propagation (with two different phase velocities) will be possible in the +z direction only, unless the following relation is true

(71) $\quad k^2 a^2 - k_c^2 a \geq (-\omega\Omega)^2 .$

Equation (71) can be simplified to give

(72) $\quad f \leq f_- = \dfrac{k_c}{2\pi\sqrt{\mu_o\epsilon_o}\sqrt{\dfrac{n^2-\beta^2}{1-\beta^2}}}$.
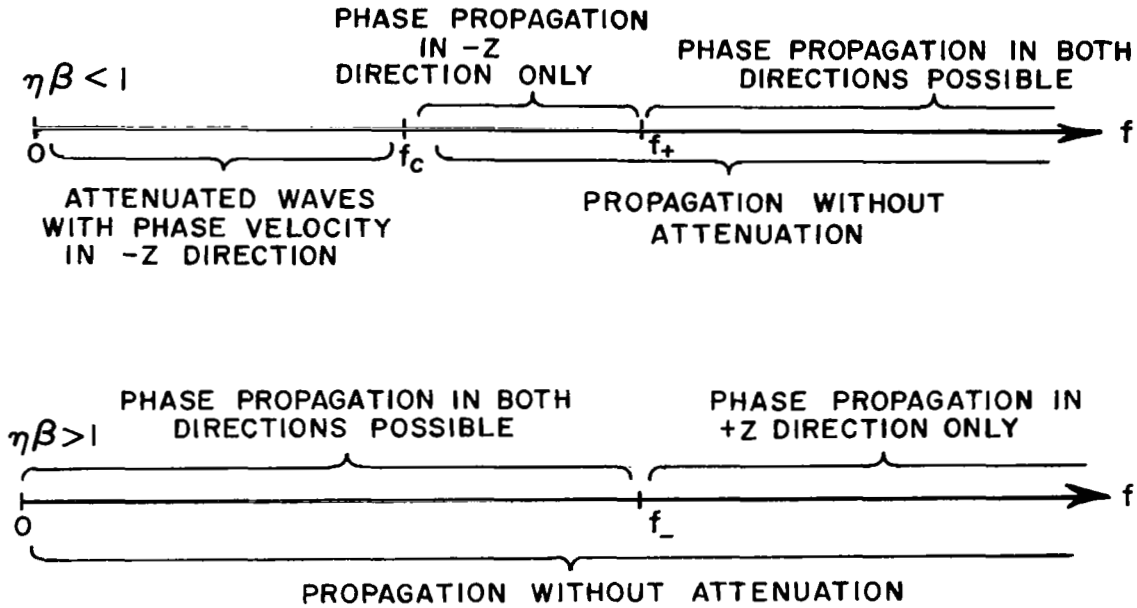
The summary of these results is exhibited in Fig. 3.

Fig. 3--Frequency ranges for wave propagation in the waveguide with the medium in it moving in +z direction

There are an infinite number of modes which can exist in the waveguide but for a given frequency only a finite number of them can propagate freely if the velocity $\overline{v} = \hat{z}v$ is small such that $n\beta < 1$.

Therefore, when $n\beta < 1$ several parameters can be expressed in terms of the cut off frequency. These are

$$(73) \qquad k_c = 2\pi f_c \sqrt{\mu\epsilon}\, a$$

$$(74a) \qquad h = -\omega\Omega \pm a\omega\sqrt{\mu\epsilon}\left[1-\left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}}$$

$$= \frac{\omega}{(1-n^2\beta^2)\,c}\left\{(1-n^2)\,\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}}\right\} \quad (f > f_c)$$

$$(74b) \qquad h = -\omega\Omega \pm ia\omega_c\sqrt{\mu\epsilon}\left[1-\left(\frac{f}{f_c}\right)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{(1-n^2\beta^2)\,c}\left\{\omega(1-n^2)\,\beta \pm i\omega_c n(1-\beta^2)\left[1-\left(\frac{f}{f_c}\right)^2\right]^{\frac{1}{2}}\right\} \quad (f < f_c)$$

where $\omega_c = 2\pi f_c$. The guide phase velocity and guide wavelength are respectively

$$(75) \qquad v_g = \omega/h = (1-n^2\beta^2)\,c/\left\{(1-n^2)\,\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$(76) \qquad \lambda_g = 2\pi/h = (1-n^2\beta^2)\lambda_o/\left\{(1-n^2)\,\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}}\right\}$$

where $c = 1/\sqrt{\mu_o\epsilon_o}$ and λ_o is the free space wavelength. The $TM_{m\ell}$ characteristic wave impedance is

$$(77) \qquad Z^{TM}_{m\ell} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \quad \text{(rectangular waveguides)}$$

$$= \frac{E_r}{H_\phi} = -\frac{E_\phi}{H_r} \quad \text{(cylindrical waveguides)}$$

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$$= \frac{h + \omega\Omega}{\omega\epsilon\, a}$$

$$= \frac{\pm\sqrt{k^2 a^2 - k_c^2 a^2}}{\omega\epsilon\, a} \qquad (n\beta > 1 \ \text{ or } \ n\beta < 1)$$

$$= \pm\, \eta \left[1 - \left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}} \qquad (n\beta < 1 \ \text{ and } \ f > f_c)$$

$$= \pm\, i\, \frac{\omega_c}{\omega}\, \eta \left[1 - \left(\frac{f}{f_c}\right)^2\right]^{\frac{1}{2}} \qquad (n\beta < 1 \ \text{ and } \ f < f_c)$$

where $\eta = \sqrt{\dfrac{\mu}{\epsilon}}$ is the intrinsic impedance of the medium. Similarly, the $TE_{m\ell}$ characteristic wave impedance is

(78)
$$Z_{m\ell}^{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \qquad \text{(rectangular waveguides)}$$

$$= \frac{E_r}{H_\phi} = -\frac{E_\phi}{H_r} \qquad \text{(cylindrical waveguides)}$$

$$= \frac{\omega\mu a}{h + \omega\Omega}$$

$$= \pm\, \frac{\omega\mu a}{\sqrt{k^2 a^2 - k_c^2 a}} \qquad (n\beta > 1 \ \text{ or } \ n\beta < 1)$$

$$= \pm\, \eta\, [1 - (f_c/f)^2]^{-\frac{1}{2}} \qquad (n\beta < 1 \ \text{ and } \ f > f_c)$$

$$= \pm\, i\, \frac{\omega}{\omega_c}\, \eta\, [1 - (f/f_c)^2]^{-\frac{1}{2}} \qquad (n\beta < 1 \ \text{ and } \ f < f_c)\, .$$

It is interesting to note that the product $Z_{m\ell}^{TM}\, Z_{m\ell}^{TE} = \eta^2 = \mu/\epsilon$ at all frequencies and all velocities of motion of the medium. $Z_{m\ell}^{TM}$ and

$Z_{m\ell}^{TE}$ as given in (77) and (78) when $n\beta < 1$ are of the same form as those when the medium in the waveguide is not moving. The power flow in the rectangular waveguide for TM modes is

$$(79) \qquad P = \frac{1}{2} \text{ Re} \int_0^{x_0} \int_0^{y_0} \overline{E} \times \overline{H}^* \cdot d\overline{S}$$

$$= \frac{+}{-} \frac{x_0 y_0}{2\epsilon_{om}\epsilon_{o\ell}} |A_0|^2 k_c^2 \frac{\sqrt{k^2 a^2 - k_c^2 a^2}}{\omega\mu^2\epsilon a^3} \qquad (n\beta > 1 \text{ or } n\beta < 1)$$

$$= \frac{+}{-} \frac{x_0 y_0 |A_0|^2 k_c^2}{2\,\epsilon_{om}\epsilon_{o\ell}} \times \frac{[1-(f_c/f)^2]^{\frac{1}{2}}}{a^2\mu\sqrt{\mu\epsilon}} \qquad (n\beta < 1, \ f > f_c)$$

where $\epsilon_{o\ell}$ is defined as equal to 1 when $\ell = 0$ and equal to 2 when $\ell > 0$. For TE modes it is

$$(80) \qquad P = \frac{+}{-} \frac{x_0 y_0 |F_0|^2 k_c^2}{2\epsilon_{o\ell}\,\epsilon_{om}} \times \frac{\sqrt{k^2 a^2 - k_c^2 a}}{\omega\mu\epsilon^2 a^3} \qquad (n\beta > 1 \text{ or } n\beta < 1)$$

$$= \frac{+}{-} \frac{x_0 y_0 |F_0|^2 k_c^2}{2\epsilon_{o\ell}\,\epsilon_{om}} \times \frac{[1-(f_c/f)^2]^{\frac{1}{2}}}{a^2\epsilon\sqrt{\mu\epsilon}} \qquad (n\beta < 1, \ f > f_c) \ .$$

In cylindrical waveguides the corresponding expressions are

$$(81) \qquad P = \frac{+}{-} \frac{\pi |A_0|^2 r_0^2}{2\epsilon_{om}} \left[\frac{d}{dr} J_m(k_c r) \Big|_{r=r_0} \right]^2 \frac{\sqrt{k^2 a^2 - k_c^2 a}}{\omega\mu^2\epsilon a^3}$$

$$(n\beta > 1 \text{ or } n\beta < 1)$$

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$$= \pm \frac{\pi |A_o|^2 r_o^2}{2\epsilon_{om}} \left[\frac{d}{dr} J_m(k_c r) \Big|_{r=r_o} \right]^2 \frac{\left[1 - \left(\frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}}}{a^2 \mu \sqrt{\mu\epsilon}}$$

$$(n\beta < 1, \; f > f_c)$$

for TM modes and

$$(82) \quad P = \pm \frac{\pi |F_o|^2 r_o^2 k_c^2}{2\epsilon_{om}} \left(1 - \frac{m^2}{k_c^2 r_o^2} \right) J_m^2 (k_c r_o) \frac{\sqrt{k^2 a^2 - k_c^2 a}}{\omega \mu \epsilon^2 a^3}$$

$$(n\beta > 1 \text{ or } n\beta < 1)$$

$$= \pm \frac{\pi |F_o|^2 r_o^2 k_c^2}{2\epsilon_{om}} \left(1 - \frac{m^2}{k_c^2 r_o^2} \right) J_m^2 (k_c r_o) \frac{\left[1 - \left(\frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}}}{a^2 \epsilon \sqrt{\mu\epsilon}}$$

$$(n\beta < 1, \; f > f_c)$$

for TE modes. Although the phase propagates in the way shown in Fig. 3, the power flow for the two waves in the guide are of the same magnitude and in opposite directions.

When the velocity $\bar{v}$ approaches zero or when the constitutive parameters of the medium are equal to those of free space, Ω will approach zero and a will approach one; the expressions and results obtained reduce to the familiar ones for media at rest.

CHAPTER IV

ELECTRIC AND MAGNETIC LINE SOURCES LOCATED
OVER A SEMI-INFINITE MOVING HALF-SPACE

The geometry of the problem which will be considered in

this chapter is illustrated in Fig. 4. J (or M) is a line source
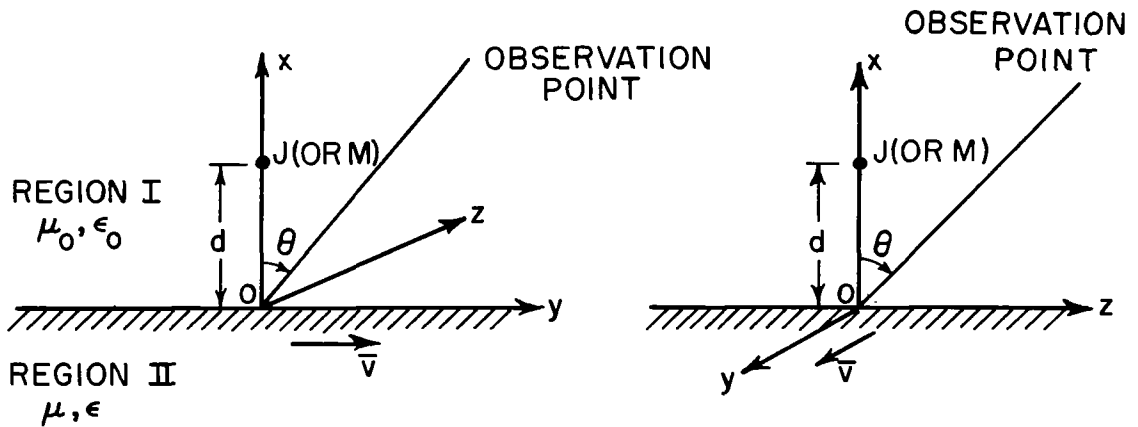
Fig. 4--Line source located over a semi-infinite half-space

of electric (or magnetic) current located at x = d above the plane

x = 0. The line source is either parallel to the y-axis or the z-axis.

The half-space x > 0 is assumed to be free-space. The half-space

x < 0 is assumed to be a dielectric with permittivity ϵ and permea-

bility μ and it moves at constant velocity v in the positive y-direction.

The purpose of this study is to find out the effect of the motion of this

region on the radiation pattern. Analytically the problem is quite similar to that of a line source above a grounded dielectric slab as given by Tai, [19] Barone, [20] and Whitmer. [21] The Fourier transform method has been used to construct a solution for the field in integral form. The resultant integral is solved by the saddle point method for the far field. A branch cut integration must be included in several cases which will be discussed later on. The time dependence is assumed to be of the form $e^{-i\omega t}$.

Electric Line Source
Parallel to Z-Axis

An electric line source located at x = d, y = 0 and parallel to the z-axis can be represented as

(83)
$$J = \hat{z}\, I_o\, \delta(x-d)\, \delta(y)\ .$$

Since the current has no variation in the z-direction, the radiated field is also independent of z. The z-component of the magnetic field is zero, and hence the electromagnetic fields may be derived from vector potentials $\overline{A}_T$ in region I and $\overline{A}_B$ in region II having only z-components. The electric and magnetic fields in region I are given by

(84a)
$$\overline{E} = i\omega \overline{A}_T - \frac{\nabla \nabla \cdot \overline{A}_T}{i\omega \mu_o \epsilon_o}$$

$$\overline{H} = \frac{1}{\mu_o}\, \nabla \times \overline{A}_T$$

or

(84b) $\qquad E_x = 0 \qquad H_x = \dfrac{1}{\mu_o} \dfrac{\partial A_T}{\partial y}$

$\qquad\qquad\quad E_y = 0 \qquad H_y = \dfrac{-1}{\mu_o} \dfrac{\partial A_T}{\partial x}$

$\qquad\qquad\quad E_z = i\omega A_T \quad H_z = 0$

where $\overline{A}_T = A_T \hat{z}$ and A_T satisfies the wave equation

(85) $\qquad \nabla^2 A_T + k_o^2 A_T = -\mu_o I_o\, \delta(x-d)\, \delta(y)$

and suitable boundary conditions as discussed below.

Let the Fourier transform of A_T be denoted by g_1

(86) $\qquad g_1(x, k_y) = \displaystyle\int_{-\infty}^{\infty} A_T(x, y)\, e^{-ik_y y}\, d_y$

then the Fourier integral

(87) $\qquad A_T(x, y) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} g_1(x, k_y)\, e^{ik_y y}\, dk_y$

is a solution of Eq. (85) provided that g_1 satisfies the equation

(88) $\qquad \dfrac{\partial^2 g_1}{\partial x^2} + (k_o^2 - k_y^2) g_1 = -\mu_o I_o\, \delta(x-d).$

A suitable form for the function g_1 is

(89a) $\qquad g_{11}(x, k_y) = c_1 e^{i\ell(x-d)} \qquad\qquad\qquad (x \geq d)$

(89b) $\qquad g_{12}(x, k_y) = c_2 \left[e^{-i\ell(x-d)} + R\, e^{i\ell(x+d)} \right] \quad (o \leq x \leq d)$

where $\ell^2 = k_o^2 - k_y^2$ and c_1, c_2 and R are constants to be determined.

At $x = d$, g_1 is continuous but $\frac{dg_1}{dx}$ is discontinuous. Integrating

(88) from $x = d_-$ to $x = d_+$ gives

$$\frac{dg_1}{dx}\bigg|_{d_-}^{d_+} = -\mu_o I_o$$

and this specifies the discontinuity of the derivative at the source.

The boundary conditions at $x = d$ are now readily found to give

(90a) $\qquad g_{11} = \left(\frac{-\mu_o I_o}{2i\ell}\right)[\, 1 + R\, e^{i2\ell d}]\, e^{i\ell(x-d)} \qquad (x \geq d)$

(90b) $\qquad g_{12} = \left(\frac{-\mu_o I_o}{2i\ell}\right)[\, e^{-i\ell(x-d)} + R\, e^{i\ell(x+d)}]\, (o \leq x \leq d)\,.$

In region II let

(91) $\qquad \overline{v} = \hat{y}\, v \qquad\qquad \overline{\Omega} = \hat{y}\Omega$

$\qquad\qquad \overline{A}_B = \hat{z}\, A_B$

where $\overline{A}_B$ is the vector potential and $\overline{\Omega}$ is defined as before. Introducing the vector function $\overline{A}_{B1}$ such that,

(92) $\qquad \overline{A}_{B1} = \hat{z}\, A_{B1} = \overline{\overline{\alpha}} \cdot \overline{A}_B = a\overline{A}_B = \hat{z}\, a\, A_B$

then A_{B1} has to satisfy Eq. (43a) which in this case is

(93) $\qquad \left[\frac{\partial^2}{\partial x^2} + \frac{1}{a}\frac{\partial^2}{\partial y^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial y} - \frac{\omega^2\Omega^2}{a} + k^2 a\right] A_{B1} = 0\,.$

The Fourier integral

(94) $$A_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_y) \, e^{ik_x x + ik_y y} \, dk_y$$

is a solution of (93) where k_x is given by

(95) $$k_x^2 = k^2 a - \frac{\omega^2 \Omega^2}{a} - \frac{2\omega\Omega}{a} k_y - \frac{1}{a} k_y^2$$

and f_1 has to be determined from the boundary condition at $x = 0$.

For transverse wave numbers ℓ and k_x, the branches that lead to attenuated waves so the Fourier integrals converge must be chosen. This requires that we choose the roots such that,

(96a) $$I_m \ell \geq 0$$

and

(96b) $$I_m k_x \leq 0$$

also the conditions $\mathrm{Re}\,\ell \geq 0$ and $\mathrm{Re}\,k_x \leq 0$ correspond to outward propagating waves.

The fields in region II are given from Eqs. (45a) and (46a) by

(97a) $$\overline{E} = i\omega\,\overline{A}_B - D_1 U = i\omega\,\overline{\overline{\alpha}}^{-1} \cdot \overline{A}_{B1} - D_1 U$$

$$= \frac{i\omega\overline{A}_{B1}}{a} - \frac{(\nabla + i\omega\overline{\Omega})\left[(\nabla + i\omega\overline{\Omega}) \cdot \overline{A}_{B1}\right]}{i\omega\,\mu\epsilon\,a^2}$$

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(97b)
$$\bar{H} = \frac{1}{\mu} \bar{\bar{\alpha}}^{-1} \cdot (\nabla + i\omega\bar{\Omega}) \times \bar{A}_B$$

$$= \frac{1}{\mu} \bar{\bar{\alpha}}^{-1} \cdot [(\nabla + i\omega\bar{\Omega}) \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_{B_1})]$$

or

$$E_x = 0 \qquad H_x = \frac{1}{\mu a^2} \left[\frac{\partial A_{B_1}}{\partial y} + i\omega\Omega A_{B_1} \right]$$

$$E_y = 0 \qquad H_y = \frac{-1}{\mu a} \frac{\partial A_{B_1}}{\partial x}$$

$$E_z = \frac{i\omega A_{B_1}}{a} \qquad H_z = 0 \ .$$

The boundary conditions of continuous tangential electric and magnetic fields at $x = 0$ demand the following relations:

(98a)
$$f_1 = \frac{-\mu_o a I_o}{2i\ell} e^{i\ell d} (1+R)$$

(98b)
$$f_1 = \frac{\mu a I_o}{2ik_x} e^{i\ell d} (1-R)$$

which gives the solutions

(99a)
$$R = \frac{\mu \ell + \mu_o k_x}{\mu \ell - \mu_o k_x}$$

(99b)
$$f_1 = \frac{i I_o a \mu \mu_o e^{i\ell d}}{\mu \ell - \mu_o k_x} \ .$$

Therefore the final expressions for the field components in the region $x \geq d$ are given by the integrals of the form,

$$\text{(100a)} \quad \overline{E} = \hat{z} E_z = \hat{z} \, i\omega A_T = \hat{z} \, \frac{-\omega\mu_o I_o}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell} \left[1 + \frac{\mu\ell + \mu_o k_x}{\mu\ell - \mu_o k_x} e^{i2\ell d} \right] \cdot e^{i\ell(x-d)+ik_y y} \, dk_y$$

$$\text{(100b)} \quad H_x = \frac{-I_o}{4\pi} \int_{-\infty}^{\infty} \frac{k_y}{\ell} \left[1 + \frac{\mu\ell + \mu_o k_x}{\mu\ell - \mu_o k_x} e^{i2\ell d} \right] \cdot e^{i\ell(x-d)+ik_y y} \, dk_y$$

$$\text{(100c)} \quad H_y = \frac{I_o}{4\pi} \int_{-\infty}^{\infty} \left[1 + \frac{\mu\ell + \mu_o k_x}{\mu\ell - \mu_o k_x} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_y y} \, dk_y.$$

The branch points in the k_y-plane can be located by solving the equations.

$$\text{(101a)} \quad k_o^2 - k_y^2 = 0$$

$$\text{(101b)} \quad k^2 a - \frac{\omega^2 \Omega^2}{a} - \frac{2\omega\Omega}{a} k_y - \frac{1}{a} k_y^2 = 0 .$$

The results are

$$\text{(102a)} \quad k_{y1} = \pm k_o$$

$$\text{(102b)} \quad k_{y2} = -\omega\Omega \pm ka = k_o \frac{n+\beta}{1+n\beta} \quad \text{and} \quad -k_o \frac{n-\beta}{1-n\beta} .$$

When $n\beta < 1$ one of the branch point shown in (102b) is on the right side of point k_o while the other one on the left side of point $-k_o$.

When $n\beta > 1$ both are located on the right side of point k_o. When $n = 1$ the branch points given in (102b) are the same as those given in (102a). This is shown in Fig. 8.

The proper positions of the branch cuts can be found as follows. Assuming that there is small loss in the free space the wave numbers k_o, ℓ and k_x become

(103)
$$k_o = k_o' + ik_o''$$

(104)
$$\ell^2 = (k_o' + ik_o'')^2 - (k_y' + ik_y'')^2 = (k_o'^2 - k_o''^2 - k_y'^2 + k_y''^2)$$
$$+ i2(k_o' k_o'' - k_y' k_y'')$$

(105)
$$k_x^2 = (k_o' + ik_o'')^2 \frac{n^2 - \beta^2}{1 - \beta^2} - 2(k_o' + ik_o'') \frac{(n^2 - 1)\beta}{1 - \beta^2} (k_y' + ik_y'')$$
$$- \frac{1 - n^2\beta^2}{1 - \beta^2} (k_y' + ik_y'')^2$$
$$= (k_o'^2 - k_o''^2) \frac{n^2 - \beta^2}{1 - \beta^2} - 2 \frac{(n^2 - 1)\beta}{1 - \beta^2} (k_o' k_y' - k_o'' k_y'')$$
$$- \frac{1 - n^2\beta^2}{1 - \beta^2} (k_y'^2 - k_y''^2)$$
$$+ i2 \left[k_o' k_o'' \left(\frac{n^2 - \beta^2}{1 - \beta^2} \right) - (k_o'' k_y' + k_o' k_y'') \frac{(n^2 - 1)\beta}{1 - \beta^2} \right.$$
$$\left. - k_y' k_y'' \left(\frac{1 - n^2\beta^2}{1 - \beta^2} \right) \right].$$

The branch cuts that separate the proper and improper branches occur along the curves $\text{Im}\,\ell = 0$ and $\text{Im}\,k_x = 0$ which are portions of the hyperbolas determined by the equations.

(106)
$$k_y' \, k_y'' - k_o' \, k_o'' = 0$$

and

(107a)
$$k_o' \, k_o'' \, \frac{n^2 - \beta^2}{1 - \beta^2} - \frac{(n^2 - 1)}{1 - \beta^2} \, (k_o'' \, k_y' + k_o' \, k_y'')$$

$$- \frac{(1 - n^2 \beta^2)}{1 - \beta^2} \, k_y' \, k_y'' = 0$$

i. e. ,

(107b)
$$\left[k_y' + \frac{(n^2 - 1)\,\beta}{1 - n^2 \beta^2} \, k_o' \right] \left[k_y'' + \frac{(n^2 - 1)\,\beta}{1 - n^2 \beta^2} \, k_o'' \right] = k_o' \, k_o'' \, \frac{n^2 (1 - \beta^2)^2}{(1 - n^2 \beta^2)^2} \, .$$

The branch cuts due to transverse wave number ℓ are shown in Fig. 5. They run from the branch points toward the imaginary axis. In the limit as $k_o'' \to 0$ the cut becomes part of the real axis between $-k_o$ and k_o and the imaginary axis. In the crosshatched region both the real and imaginary parts are positive provided we choose the positive root or branch for ℓ. The original contour c_o must lie entirely in this region to represent outgoing waves.

Fig. 5--Proper branch cut due to ℓ
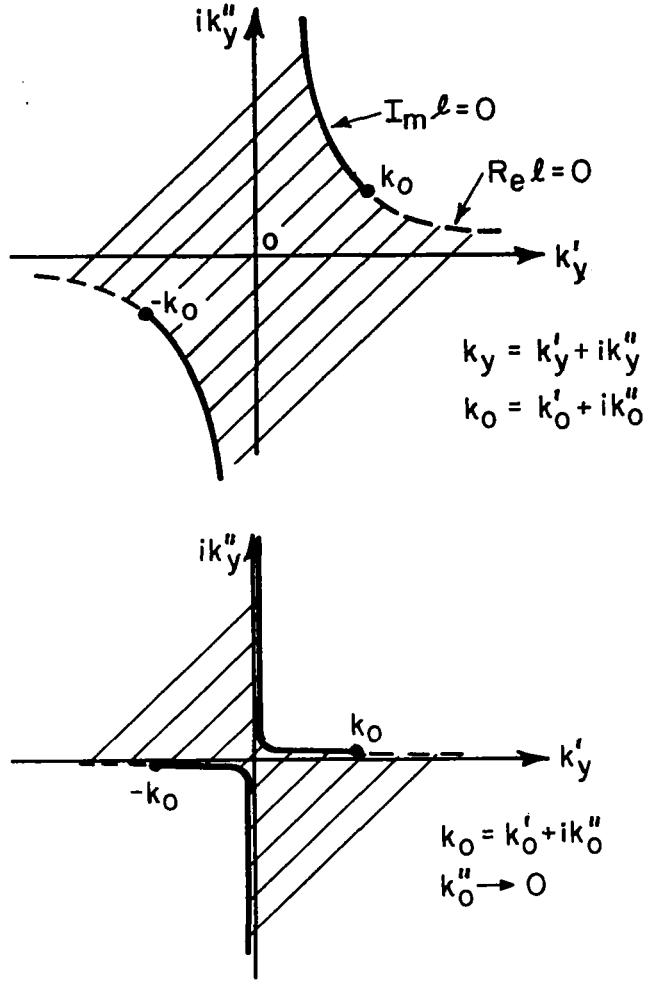
The branch cuts due to the transverse wave number k_x are shown in Fig. 6 when $n\beta < 1$ and in Fig. 7 when $n\beta > 1$. The centers of the hyperbola are in the third quadrant for $n\beta < 1$ and in the first quadrant when $n\beta > 1$. In the limit when k_o'' reduces to
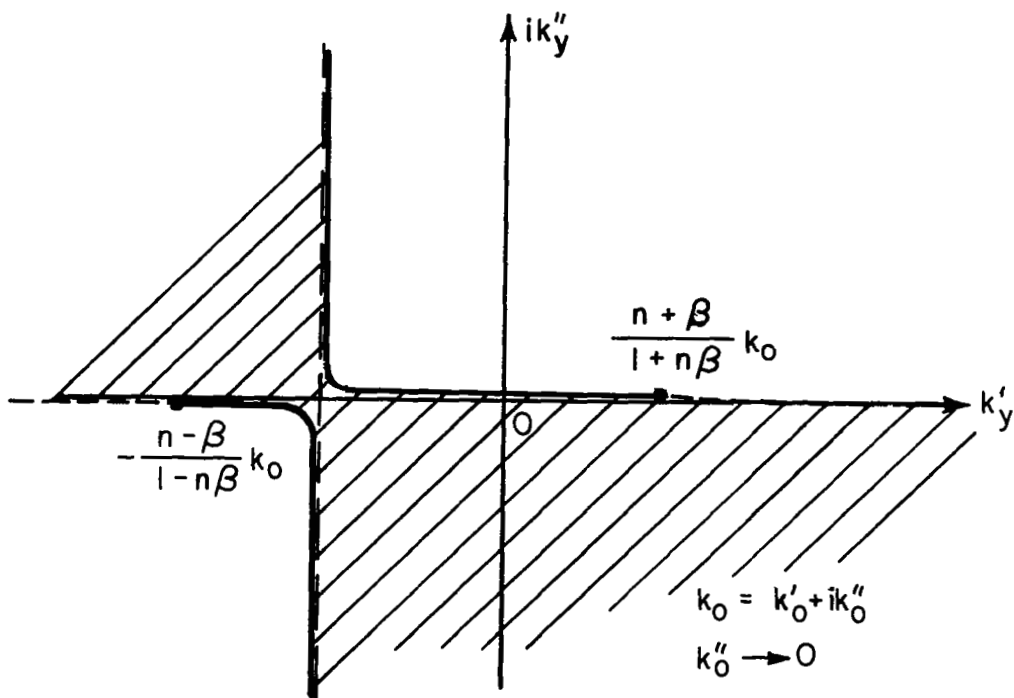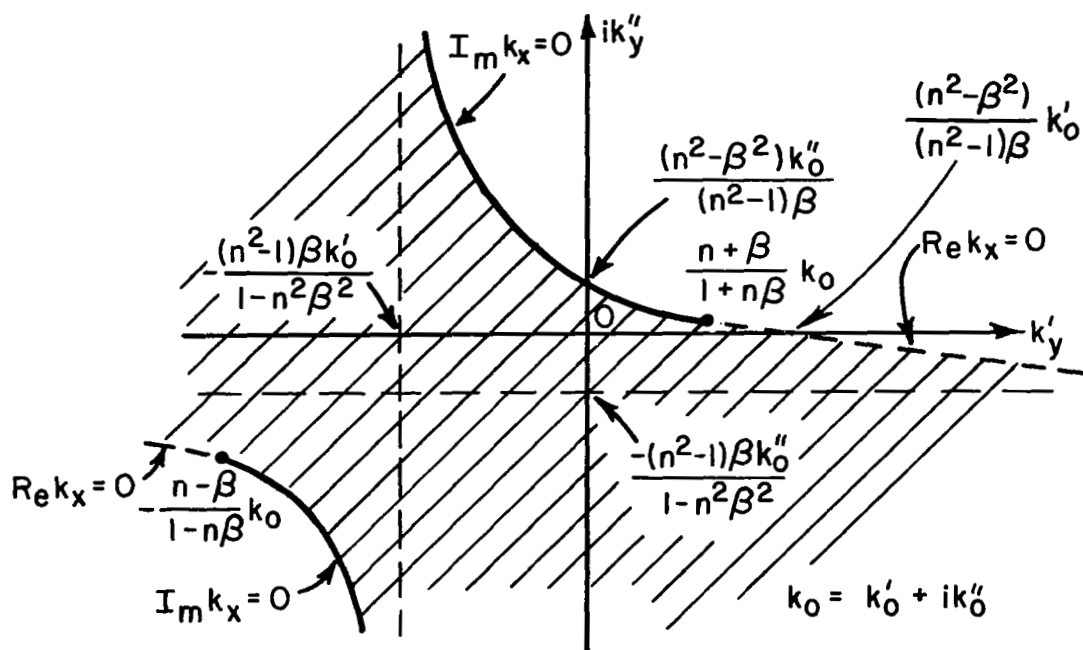
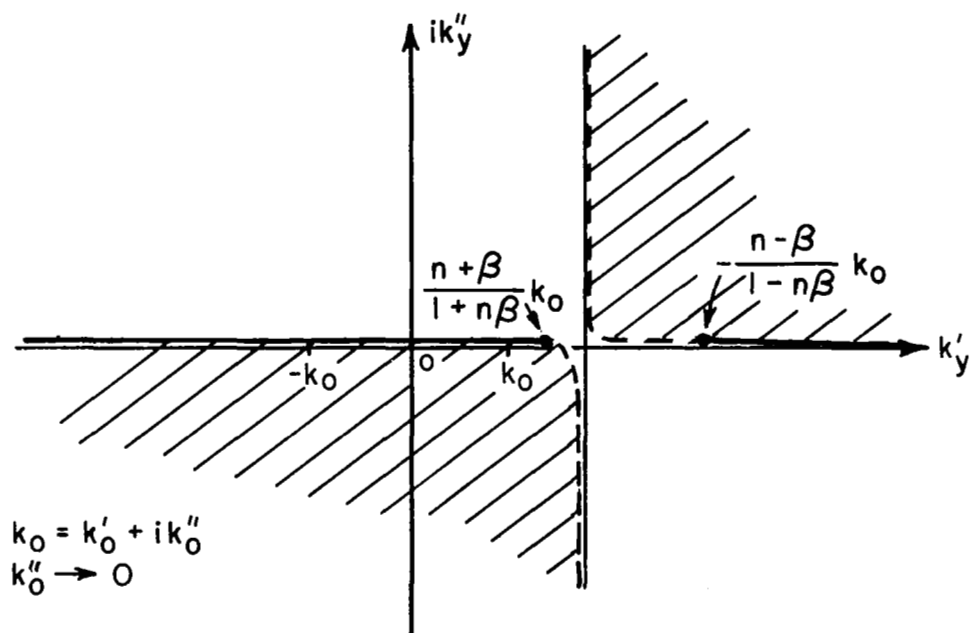Fig. 6--Proper branch cut due to k_x when $n\beta < 1$
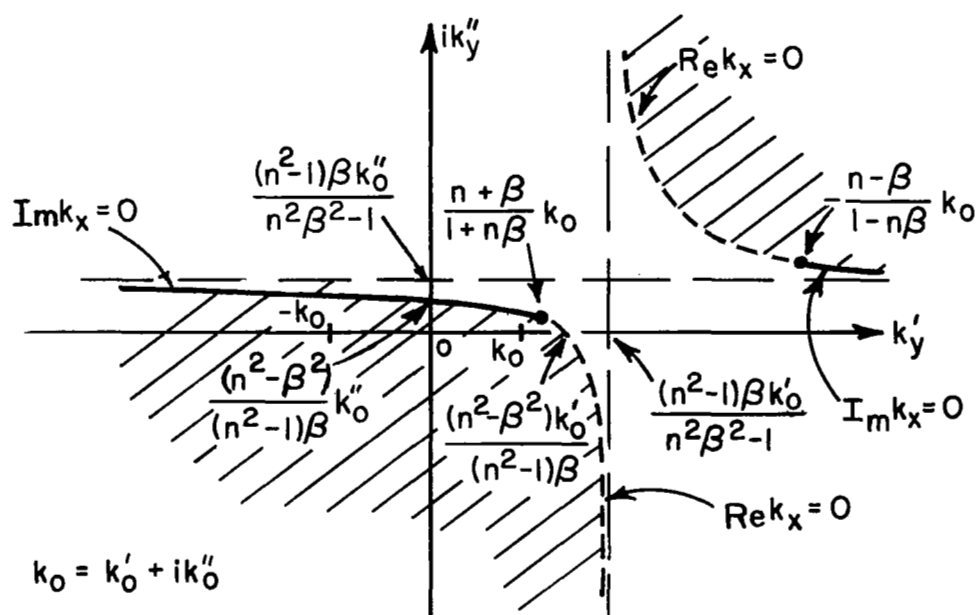
Fig. 7--Proper branch cut due to k_x when $n\beta > 1$

zero the cut becomes the section of the real axis from $-\dfrac{n-\beta}{1-n\beta}\, k_0$

to $\dfrac{n+\beta}{1+n\beta}\, k_0$ and the line $k_y' = -\dfrac{(n^2-1)\,\beta}{1-n^2\beta^2}\, k_0$ when $n\beta < 1$. This

line moves further toward left as $n\beta$ approaches 1. When $n\beta > 1$

the branch cuts become the sections of the real axis from $\dfrac{n-\beta}{n\beta-1}\, k_0$

to $+\infty$ and from $\dfrac{n+\beta}{1+n\beta}\, k_0$ to $-\infty$. In the crosshatched regions the

real and imaginary parts of k_x are both negative provided we choose

the negative root or branch. The entire original contour c_0 must

lie in this region also. As the branch cuts may be chosen quite

arbitrarily, as long as they do not intersect the contour c_0, we fix

their positions in the way shown in Fig. 8. This choice will be con-

venient for estimating the branch cut integrals, as discussed later

on.

The location of the poles are determined by requiring

(108a) $\mu\ell = \mu_0 k_x$.

As $\ell^2 = k_0^2 - k_y^2$ and $k_x^2 = k^2 a - \dfrac{\omega^2 \Omega^2}{a} - \dfrac{2\omega\Omega}{a}\, k_y - \dfrac{1}{a}\, k_y^2$. Equation (108a)

can be solved by first squaring both sides. The roots obtained have

to be examined to find out on which sheet of the Riemann surface

they are located. Thus,

$$\mu^2 \ell^2 = \mu_0^2 \, k_x^2$$

i. e. ,

$n\beta < 1$

$-\dfrac{n-\beta}{1-n\beta} k_o$

$-k_o$

O

k_o

$\dfrac{n+\beta}{1+n\beta} k_o$

C_o

ik_y''

k_y'

$n\beta > 1$

$-k_o$

O

k_o

$\dfrac{n+\beta}{1+n\beta} k_o$

$\dfrac{n-\beta}{n\beta-1} k_o$

C_o

ik_y''

k_y'
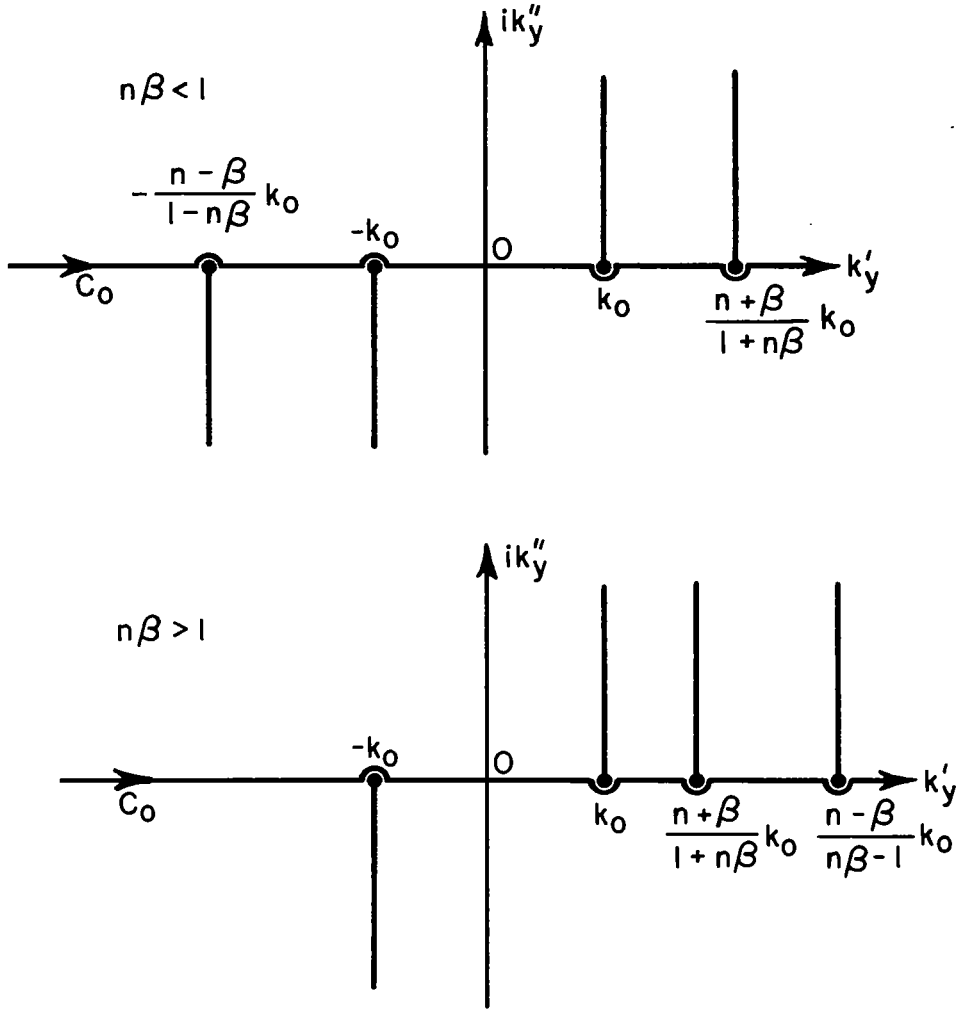
Fig. 8--Relative positions of branch points
and branch cuts

$$(108\mathrm{b}) \qquad \left(\mu^2 - \frac{\mu_o^2}{a}\right) k_y^2 - \frac{2b\,k_o\,\mu_o^2}{a} k_y + \mu_o^2 n^2 k_o^2 a - \frac{\mu_o^2 b^2 k_o^2}{a} - \mu^2 k_o^2 = 0$$

where $\quad b = \dfrac{(n^2-1)\,\beta}{1-n^2\beta^2} = c\Omega.$ The solutions are,

(109a) $\qquad k_y = k_0 \left[\dfrac{b\mu_0^2}{a} \pm A^{\frac{1}{2}} \right] \Big/ \left(\mu^2 - \dfrac{\mu_0^2}{a} \right)$

where

$$A = \left[\dfrac{\mu^2 \mu_0^2}{a}(b^2 - 1) + \mu_0^2 n^2 (\mu_0^2 - \mu^2 a) + \mu^4 \right] .$$

For the special case where $\mu = \mu_0$, Eq. (109a) reduces to

(109b) $\qquad k_y = \dfrac{k_0}{\beta} \geq k_0$

and the pole is of order 2.

On the proper sheet of the Riemann surface we require that
$\mathrm{Im}\,\ell \geq 0$ and $\mathrm{Im}\,k_x \leq 0$. So the poles which are solutions of Eq. (108a)
are on the sheets either proper for ℓ and improper for k_x or vice
versa.

When $v = 0$, Eqs. (100) reduce to that of reflection from a
semi-infinite dielectric medium. [22]   When $\mu = \mu_0$ and $\epsilon = \epsilon_0$ R
reduces to zero and Eqs. (100) represents the field radiated from
a line source in free-space. Hence the motion of the bottom region
which has the same constitutive parameters as the upper region
will not produce any effect on the radiation field.

Evaluation of the Contour Integral

The integrals shown in Eqs. (100) are too complicated to be evaluated rigorously and approximate method must be used. The saddle point method is useful for an asymptotic estimation of integrals of this type when there is a large parameter involved. To apply this method of integration it is convenient to transform the integral into a complex ϕ-plane defined as

(110a) $\qquad k_y = k_o \sin \phi$

(110b) $\qquad dk_y = k_o \cos \phi \, d\phi$

(110c) $\qquad \ell = \pm \sqrt{k_o^2 - k_y^2} = k_o \cos \phi$

(110d) $\qquad \phi = \sigma + i\eta$

(110e) $\qquad y = r \sin \theta$

(110f) $\qquad x-d = r \cos \theta \, .$

The integral (100a) becomes

(111) $\qquad E_z = \dfrac{-\omega \mu_o I_o}{4\pi} \displaystyle\int_{c_o} P(\phi) \, e^{ik_o r \cos(\phi-\theta)} \, d\phi$

where

$k_y = k'_y + ik''_y$

ik''_y

C_{s1}

$k_0 \csc \theta_2$

C_0

$-k_0$

$k_0 \tan \theta_1$

$-\dfrac{n-\beta}{1-n\beta} k_0$

C_{s2}

θ_1

θ_2

k'_y

O

θ_1

θ_2

$k_0 \sin \theta_1$

k_0

$\dfrac{n+\beta}{1+n\beta} k_0$

$k_0 \csc \theta_1$

$k_0 \sin \theta_2$

$k_0 \tan \theta_2$

$\dfrac{\pi}{2} + i\infty$

$\theta_2 - \dfrac{\pi}{2} + i\infty$

$\theta_1 - \dfrac{\pi}{2} + i\infty$

C_{s2}

C_0

C_{s1}

$i\eta$

$k_y = k_0 \sin \phi$

$\phi = \sigma + i\eta$

$\eta = \cosh^{-1}\left(\dfrac{n-\beta}{1-n\beta}\right)$

$\eta = \cosh^{-1}\left(\dfrac{n+\beta}{1+n\beta}\right)$

$\eta = -\cosh^{-1}\left(\dfrac{n-\beta}{1-n\beta}\right)$

$\eta = -\cosh^{-1}\left(\dfrac{n+\beta}{1+n\beta}\right)$

$-\pi$

$-\dfrac{\pi}{2}$

θ_2

O θ_1

$\dfrac{\pi}{2}$

π

σ

C_{s2}

C_0

C_{s1}

$\dfrac{\pi}{2} - i\infty$

$\dfrac{\pi}{2} + i\infty$

$\theta_2 + \dfrac{\pi}{2} - i\infty$
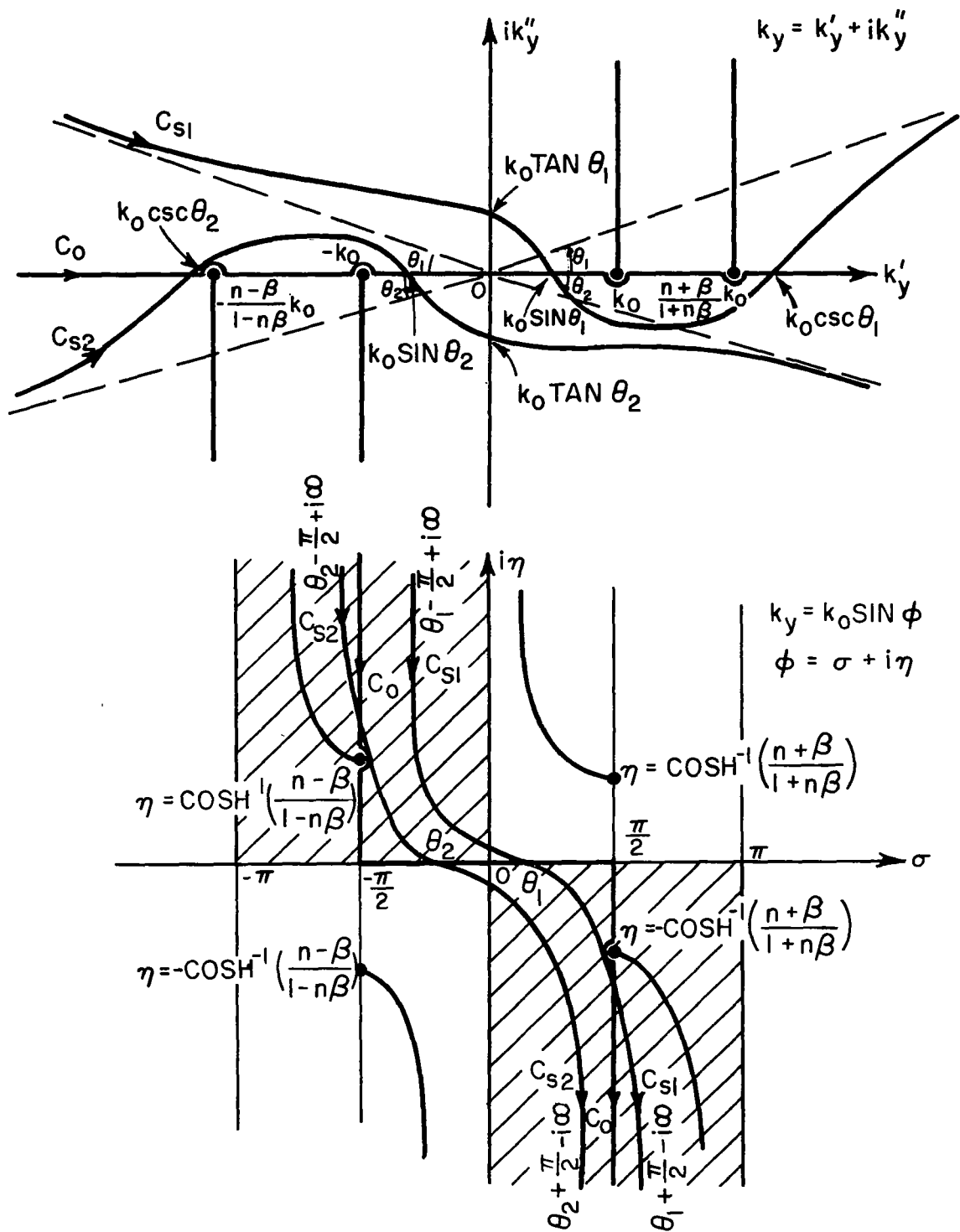
$\theta_1 + \dfrac{\pi}{2} - i\infty$

Fig. 9a--Path of integration in k_y-plane and ϕ-plane
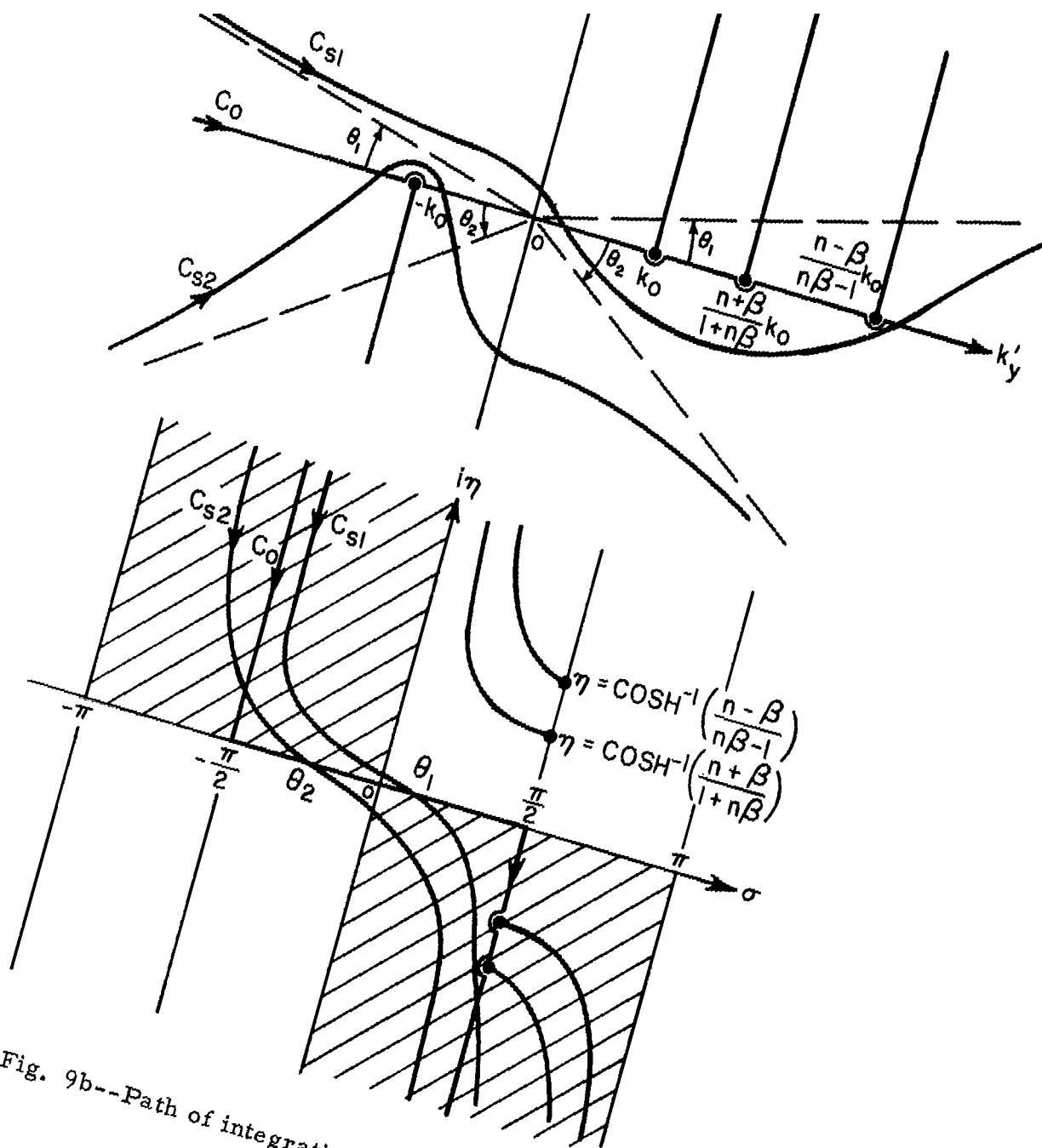when $n\beta < 1$

Fig. 9b--Path of integration in k_y-plane and ϕ-plane when $n\beta > 1$

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$$(111a) \qquad P(\phi) = 1 + \frac{\mu \cos \phi - \mu_0 \left[n^3 a - \dfrac{b^2}{a} - \dfrac{2b}{a} \sin \phi - \dfrac{1}{a} \sin^2 \phi \right]^{\frac{1}{2}}}{\mu \cos \phi + \mu_0 \left[n^2 a - \dfrac{b^2}{a} - \dfrac{2b}{a} \sin \phi - \dfrac{1}{a} \sin^2 \phi \right]^{\frac{1}{2}}}$$

$$e^{i2d\,k_0 \cos \phi} \, .$$

Equation (110) represents a mapping of the complex k_y-plane into a stripe of the complex ϕ-plane. The transformed path of integration c_0 and the branch cuts in ϕ-plane are shown in Fig. 9. The branch cuts which separate the sheet for which Im $\ell \geq 0$ from the sheet for which Im $\ell \leq 0$ are no longer cuts in ϕ-plane. Because of the choice of positive sign in (110c) the hatched region in Fig. 9 corresponds to the proper sheet for ℓ. The branch points associated with k_x in the ϕ-plane are transformed from (102b) as the solution of the equation

$$(112a) \qquad \sin(\sigma + i\eta) = \frac{n + \beta}{1 + n\beta} \qquad \text{and} \quad - \frac{n - \beta}{1 - n\beta}$$

and the poles from (109) as

$$(112b) \qquad \sin(\sigma_p + \eta_p) = \left[\frac{b \mu_0^2}{a} \pm A^{\frac{1}{2}} \right] \Big/ \left(\mu^2 - \frac{\mu_0^2}{a} \right)$$

$$(112c) \qquad \qquad = 1/\beta \qquad (\text{if } \mu = \mu_0) \, .$$

Their positions on the ϕ-plane as relative to those branch points are shown in Fig. 10 when $\mu = \mu_0$. As was mentioned earlier, those poles given by (109) and (112b, c) are located on sheets which are
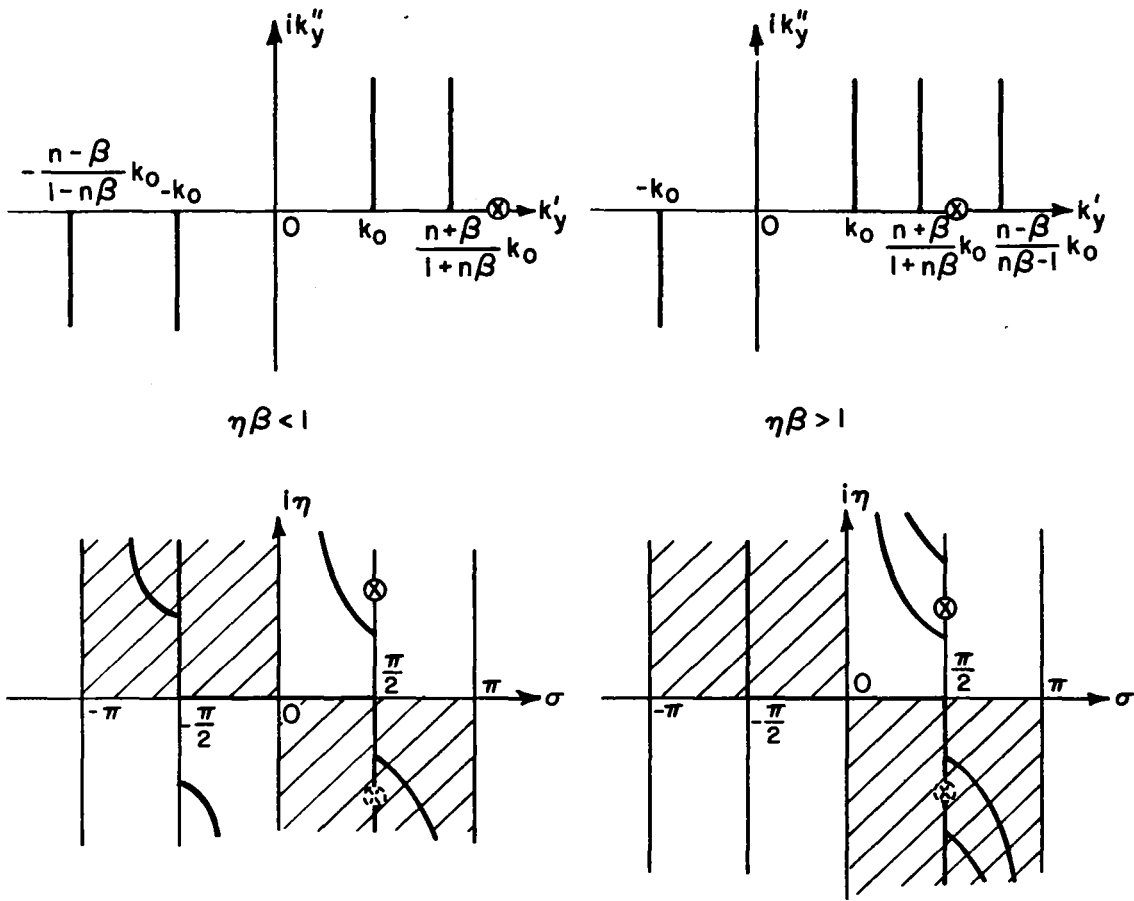
48

Fig. 10--Positions of poles relative to branch
points when $\mu = \mu_o$

proper for one of the transverse wave number and improper for the

other one, so the poles in the hatched region of ϕ-plane are on the

bottom sheet while the poles in the non-hatched region are on the

top sheet.

The saddle point of the exponential term occurring in Eq. (111)

is found by setting

$$\frac{d}{d\phi} \cos(\phi-\theta) = 0$$

which gives $\phi = \theta$. The steepest descent contour is determined by $\cos(\sigma-\theta) \cosh \eta = 1$ which is denoted by c_s in Fig. 9 where c_{s1} is the contour corresponding to positive observation angle θ_1 and c_{s2} corresponding to a negative observation angle θ_2.

The main task now is to deform the original path c_o into the steepest descent contour c_s and then perform the integration along c_s.

In deforming the contour c_o into contour c_s passing through the saddle point $\phi = \theta$ some of the poles of $P(\phi)$ given in (109) or (112b, c) may be encountered and the contour may be intercepted by the branch cut. So (111) may be written as

$$(113) \qquad E_z = - \frac{\omega \mu_o I_o}{4\pi} \int_{c_s} P(\phi) e^{ik_o r \cos(\phi-\theta)} d\phi$$

$$- \frac{i\omega \mu_o I_o}{2} \sum_p F(\phi_p) e^{ik_o r \cos(\phi_p-\theta)} + K$$

where ϕ_p is the value of ϕ at the pole as given by (112b, c), $F(\phi_p)$ has a magnitude equal to the residue of $P(\phi)$ at the pole and a sign chosen to correspond to the position of the pole. K is the branch cut integral.

If $P(\phi)$ does not have a pole in the vicinity of the saddle point the integral along the steepest descent contour is readily evaluated by expanding $P(\phi)$ in a Taylor series about θ as

$$(114) \qquad P(\phi) = P(\theta) + \sum_{m=1}^{\infty} P^m(\theta) \frac{(\phi-\theta)^m}{m!}$$

where

$$P^m(\phi) = \frac{d^m P(\phi)}{d\phi^m}\bigg|_{\phi=\theta} .$$

Along the steepest descent contour c_s near θ, let $\phi-\theta = \rho e^{iw}$. Then

$$f(\phi) = ik_o r \cos(\phi-\theta) = f(\theta) + \frac{\partial f}{\partial \phi}\bigg|_{\phi=\theta} (\phi-\theta) + \frac{1}{2} \frac{\partial^2 f}{\partial \phi^2}\bigg|_{\phi=\theta} (\phi-\theta)^2$$

$$+ \cdots$$

$$\simeq f(\theta) + \frac{1}{2} \frac{\partial^2 f}{\partial \phi^2}\bigg|_{\phi=\theta} (\phi-\theta)^2$$

$$= ik_o r - \frac{1}{2} ik_o r \rho^2 \cos 2w + \frac{1}{2} k_o r \rho^2 \sin 2w$$

$$= ik_o r - \frac{1}{2} k_o r \rho^2$$

and

$$w = \pi - \frac{\pi}{4} \qquad \phi = \theta + \rho e^{i\frac{3\pi}{4}} \qquad d\phi = e^{i\frac{3\pi}{4}} d\rho \text{ in the 2nd quadrant}$$

$$w = -\frac{\pi}{4} \qquad \phi = \theta + \rho e^{-i\frac{\pi}{4}} \qquad d\phi = e^{-i\frac{\pi}{4}} d\rho \text{ in the 4th quadrant.}$$

Hence the integral term in (113) becomes

$$(115) \qquad \frac{-\omega \mu_0 I_0}{4\pi} \int_{c_s} P(\phi)\, e^{ik_0 r \cos(\phi - \theta)}\, d\phi$$

$$= -\frac{\omega \mu_0 I_0}{4\pi} \int_{c_s} \sum_{m=0}^{\infty} P^m(\theta)\, \frac{(\phi - \theta)^m}{m!}\, e^{ik_0 r \cos(\phi - \theta)}\, d\phi$$

$$\cong -\frac{\omega \mu_0 I_0}{4\pi} \sum_{m=0}^{\infty} \left[\int_{\rho_1}^{0} \frac{P^m(\theta)}{m!}\, \rho^m\, e^{im\left(\frac{3\pi}{4}\right)} e^{ik_0 r - \frac{k_0 r}{2}\rho^2} \right.$$

$$\left. \cdot\, e^{i\left(\pi - \frac{\pi}{4}\right)}\, d\rho \right.$$

$$\left. + \int_{0}^{\rho_1} \frac{P^m(\theta)}{m!}\, \rho^m\, e^{im\left(-\frac{\pi}{4}\right)} e^{ik_0 r - \frac{k_0 r}{2}\rho^2}\, e^{-i\frac{\pi}{4}}\, d\rho \right]$$

The major contribution to the integral comes from a small range $0 < \rho < \rho_1$ along the contour c_s provided $k_0 r$ is sufficiently large. Combining the two terms in (115) gives

$$(116a) \qquad -\frac{\omega \mu_0 I_0}{4\pi}\, e^{ik_0 r - i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{P^m(\theta)}{m!}\, e^{-im\frac{\pi}{4}}\, [\, 1 + (-1)^m\,]$$

$$\cdot \int_{0}^{\rho_1} \rho^m\, e^{-\frac{1}{2}k_0 r \rho^2}\, d\rho\ .$$

Because of the rapid decay of the exponential, the integral from 0 to ρ_1 does not differ much from

$$\int_{0}^{\infty} \rho^m\, e^{-\frac{1}{2}k_0 r \rho^2}\, d\rho = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\left(\frac{k_0 r}{2}\right)^{\frac{m+1}{2}}} \qquad m > -1$$

where Γ is the gamma function. The leading term of (116a) is therefore

(116b)
$$E_z \overset{\sim}{\sim} - \frac{\omega \mu_o I_o}{2\sqrt{2\pi}\,(k_o r)^{\frac{1}{2}}} e^{\,i k_o r - i \frac{\pi}{4}} P(\theta) .$$

The residue of $P(\phi)$ at the pole ϕ_p is given by

(117)

$$F(\phi_p) = \left. \frac{\mu \cos\phi + \mu_o (n^2 a - \frac{b^2}{a} - \frac{2b}{a}\sin\phi - \frac{1}{a}\sin^2\phi)^{\frac{1}{2}}}{\frac{d}{d\phi}\left[\mu\cos\phi + \mu_o(n^2 a - \frac{b^2}{a} - \frac{2b}{a}\sin\phi - \frac{1}{a}\sin^2\phi)^{\frac{1}{2}}\right]}\right|_{\phi=\phi_p}$$

$$\left. + \frac{[\mu\cos\phi - \mu_o(n^2 a - \frac{b^2}{a} - \frac{2b}{a}\sin\phi - \frac{1}{a}\sin^2\phi)^{\frac{1}{2}}]\,e^{\,i2d k_o \cos\phi}}{\frac{d}{d\phi}[\mu\cos\phi + \mu_o(n^2 a - \frac{b^2}{a} - \frac{2b}{a}\sin\phi - \frac{1}{a}\sin^2\phi)^{\frac{1}{2}}]}\right|_{\phi=\phi_p}$$

$$= \frac{2\mu^2 \cos\phi_p\, e^{\,i2d k_o \cos\phi_p}}{-\mu^2\sin\phi_p + \frac{\mu_o^2}{a}(b + \sin\phi_p)}$$

(118)
$$= \frac{4a}{(a-1)}\tan\phi_p\,(1 + i2d k_o \cos\phi_p)\,e^{\,i2d k_o \cos\phi_p} \quad (\text{if } \mu = \mu_o).$$

If a pole ϕ_p happens to be near the saddle point $\phi = \theta$ the Taylor series expansion of $P(\phi)$ is no longer valid in a sufficiently large enough region around the point θ. Then a Laurent series must be used. When $\mu = \mu_o$ the pole is of order two and unless the observation angle θ is near 90 and β is nearly equal one this pole of order two will not be in the vicinity of the saddle point. In all other cases

the pole is a simple one. We have

(119) $P(\phi) = \dfrac{F(\phi_p)}{\phi - \phi_p} + P_1(\phi)$

where $P_1(\phi)$ is analytic in the region around θ and may be developed into a Taylor series. $F(\phi_p)$ is the residue of $P(\phi)$ at $\phi = \phi_p$ and $P_1(\phi)$ can be expanded in Taylor's series as,

$$P_1(\phi) = \sum_{m=0}^{\infty} a_m (\phi - \theta)^m$$

$$a_m = \frac{1}{m!} \frac{d^m}{d\phi^m} \left[P(\phi) - \frac{F(\phi_p)}{\phi - \phi_p} \right] .$$

The portion of the integral for the field involving $P_1(\phi)$ leads to the same form discussed before. The remaining integral to be evaluated is,

(120) $\dfrac{-\omega \mu_o I_o}{4\pi} \displaystyle\int_{c_s} \dfrac{F(\phi_p)}{\phi - \phi_p} e^{ik_o r \cos(\phi - \theta)} d\phi .$

Near the saddle point and along the contour c_s

$$\cos(\phi - \theta) \overset{\sim}{=} 1 - (\phi - \theta)^2 / 2 = 1 - \frac{1}{2} \rho^2 e^{i2w}$$

$$\phi - \phi_p = \phi - \theta - (\phi_p - \theta)$$

$$\phi - \phi_p = \rho e^{i\frac{3\pi}{4}} - (\phi_p - \theta) = -\rho e^{-i\frac{\pi}{4}} - (\phi_p - \theta)$$

$$d\phi = e^{i\frac{3\pi}{4}} d\rho \text{ in 2nd quadrant}$$

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$$\phi - \phi_p = \rho e^{-i\frac{\pi}{4}} - (\phi_p - \theta) \qquad\qquad d\phi = e^{-i\frac{\pi}{4}} d\rho \text{ in 4 th quadrant}$$

We get in place of expression (120),

$$(121) \qquad \frac{-\omega\mu_o I_o}{4\pi} F(\phi_p) e^{ik_o r} \left[\int_{\rho_1}^{0} \frac{e^{-\frac{k_o r \rho^2}{2}}}{\rho e^{i\frac{3\pi}{4}} - (\phi_p - \theta)} e^{i\frac{3\pi}{4}} d\rho \right.$$

$$\left. + \int_{0}^{\rho_1} \frac{e^{-\frac{k_o r \rho^2}{2}}}{\rho e^{-i\frac{\pi}{4}} - (\phi_p - \theta)} e^{-i\frac{\pi}{4}} d\rho \right]$$

$$\approx \frac{-\omega\mu_o I_o}{2\pi} F(\phi_p) e^{ik_o r + i\frac{\pi}{4}} (\phi_p - \theta) \int_{0}^{\infty} \frac{e^{-\frac{k_o r \rho^2}{2}}}{\rho^2 - i(\phi_p - \theta)^2} d\rho .$$

Making the change of variable $t = \rho^2$ in the integral

$$I = \int_{0}^{\infty} \frac{e^{-\frac{k_o r \rho^2}{2}}}{\rho^2 - i(\phi_p - \theta)^2} d\rho$$

and following the steps as shown by Oberhettinger [23] we have

$$(122) \qquad I = \frac{1}{2} \int_{0}^{\infty} \frac{e^{-\frac{k_o r t}{2}}}{t^{\frac{1}{2}} [t - i(\phi_p - \theta)^2]} dt$$

$$= \frac{\pi}{2} e^{-\frac{ik_o r}{2}(\phi_p - \theta)^2} [-i(\phi_p - \theta)^2]^{-\frac{1}{2}} \text{erfc}\left[-\frac{ik_o r}{2}(\phi_p - \theta)^2\right]^{\frac{1}{2}}$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt .$$

The error function may be expanded asymptotically for $x \to \infty$ as [24]

$$(123) \qquad \text{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} x^{-2m}$$

provided the phase of x is in the range between $-\frac{3\pi}{4}$ and $\frac{3\pi}{4}$. $\text{erf}(x)$ is an odd function of x so if the phase of x is not within this range the above asymptotic expansion can still be used through the use of the following relation.

$$(124) \qquad \text{erfc}(x) = 1 - \text{erf}(x) = 1 + \text{erf}(-x) = 2 - \text{erfc}(-x).$$

If the phase angle of $\left[-\frac{ik_o r}{2}(\phi_p - \theta)^2 \right]^{\frac{1}{2}}$ is in the range between $-\frac{3\pi}{4}$ and $\frac{3\pi}{4}$ integral (120) becomes

$$(125) \qquad \frac{-\omega \mu_o I_o}{2\pi} F(\phi_p) e^{ik_o r + i\frac{\pi}{4}} (\phi_p - \theta) \int_o^{\infty} \frac{e^{-\frac{k_o r \rho^2}{2}}}{\rho^2 - i(\phi_p - \theta)^2} d\rho$$

$$= -\frac{\omega \mu_o I_o}{4\sqrt{\pi}} F(\phi_p)(\phi_p - \theta) e^{ik_o r + i\frac{\pi}{4}} e^{-i\frac{k_o r}{2}(\phi_p - \theta)^2} \left[-i(\phi_p - \theta)^2 \right]^{-\frac{1}{2}}$$

$$\cdot \frac{e^{\frac{ik_o r}{2}(\phi_p - \theta)^2}}{\left[-\frac{ik_o r}{2}(\phi_p - \theta)^2 \right]^{\frac{1}{2}}}$$

$$\cdot \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left[-\frac{ik_o r}{2}(\phi_p - \theta)^2 \right]^{-m}$$

$$\begin{align}
(125)\text{ (cont.)} \quad =& \frac{-i\omega\,\mu_o I_o}{2\sqrt{\pi}} \times \frac{F(\phi_p)\,e^{ik_o r + i\frac{\pi}{4}}}{(2k_o r)^{\frac{1}{2}}(\phi_p - \theta)} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \\
& \cdot \left[-\frac{ik_o r}{2} (\phi_p - \theta)^2 \right]^{-m} \\
=& \frac{-i\omega\,\mu_o I_o\, F(\phi_p)}{2\sqrt{\pi}\,(2k_o r)^{\frac{1}{2}}(\phi_p - \theta)} \, e^{i\left(k_o r + \frac{\pi}{4}\right)} \left[1 + \frac{1}{ik_o r(\phi_p - \theta)^2} + \cdots \right].
\end{align}$$

This expression shows a significant modification of the radiation field when ϕ_p is located close to the saddle point θ.

When $n\beta < 1$ and if the angle of observation θ extends over one of the two limits (one on the right side and another on the left side of the x-axis given by $\theta_1 = \sin^{-1}\left(\frac{1+n\beta}{n+\beta}\right)$ and $\theta_2 = \sin^{-1}\left(-\frac{1-n\beta}{n-\beta}\right)$) the path of integration will be intercepted by a branch cut. To evaluate the integral in this case a branch cut integration must be included. Following Ott's [25] method, a complete contour can be constructed as shown by c_{sb1} or c_{sb2} in Fig. 11a where the dotted line denotes the part of the path which is traveled on the bottom sheet of the two Riemann surfaces. The corresponding path of integration in the k_y-plane is shown in the same figure. When $n\beta > 1$ and if the angle observation θ is in the range between $\theta = \sin^{-1}\left(\frac{1+n\beta}{n+\beta}\right)$ and $\theta = \sin^{-1}\left(\frac{n\beta-1}{n-\beta}\right)$ a branch cut integral must also be involved. The path of integration is shown in Fig. 11b. The branch cut integration which is now part of the total integration can
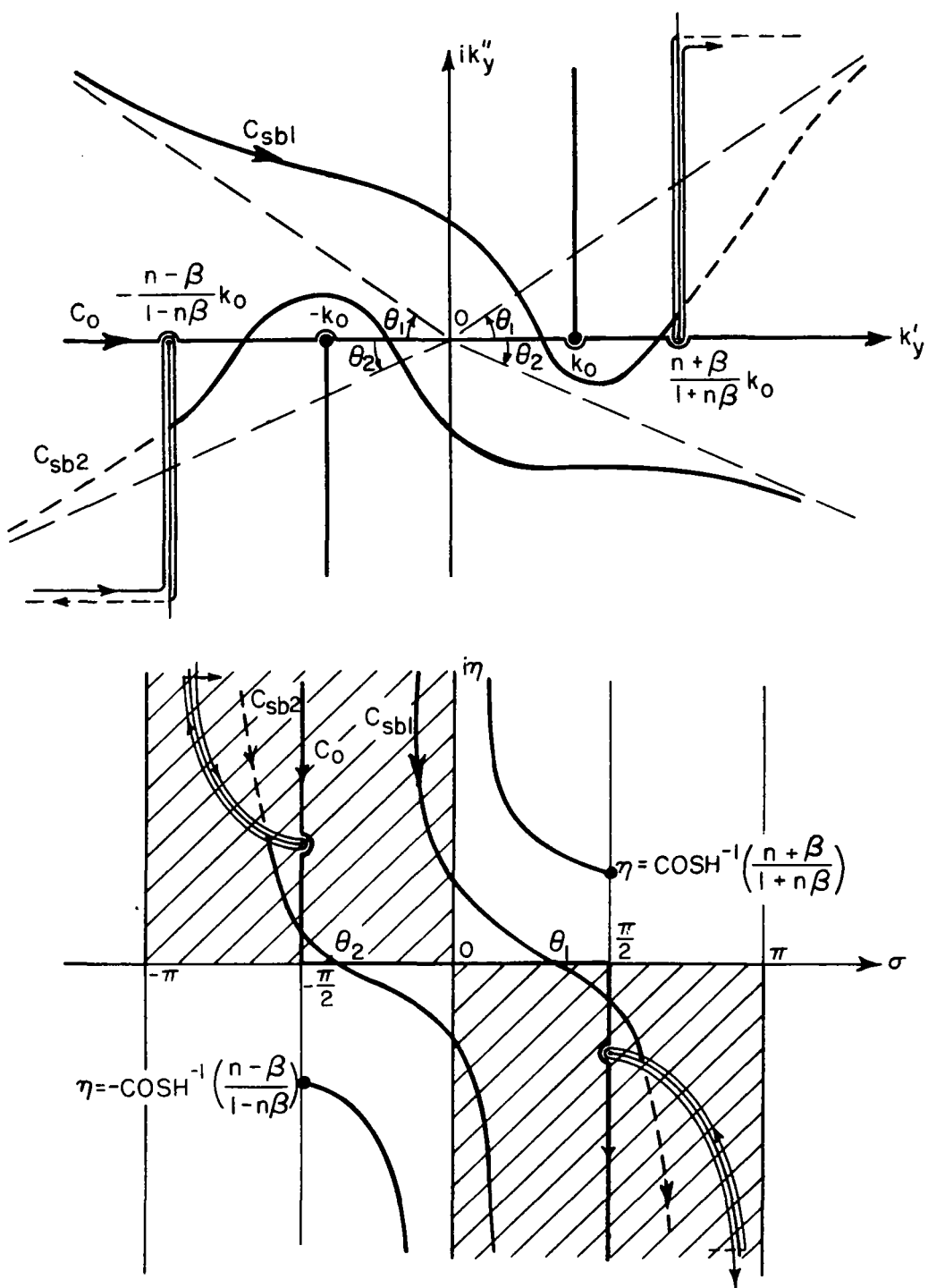
Fig. 11a--Path of integration when the steepest descent
path is intercepted by the branch cut and $n\beta < 1$

Fig. 11b--Path of integration when the steepest descent
path C_s is intercepted by the branch cut, $n\beta > 1$ and
$\sin^{-1}\left[(n\beta - 1)/(n - \beta)\right] < \theta < \sin^{-1}\left[(n\beta + 1)/(n + \beta)\right]$
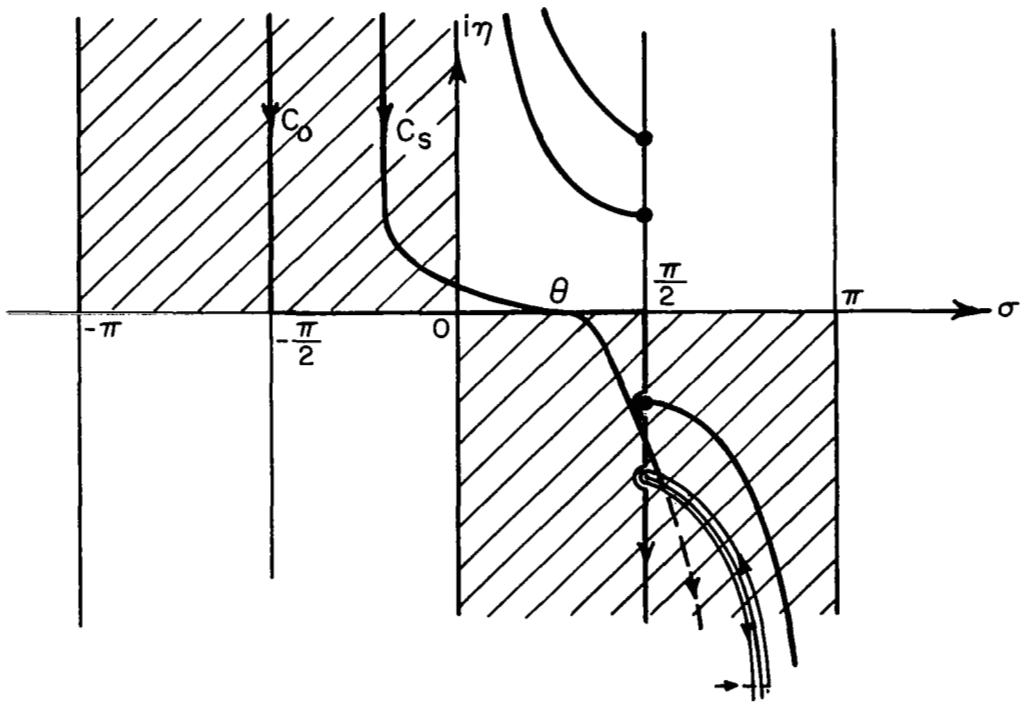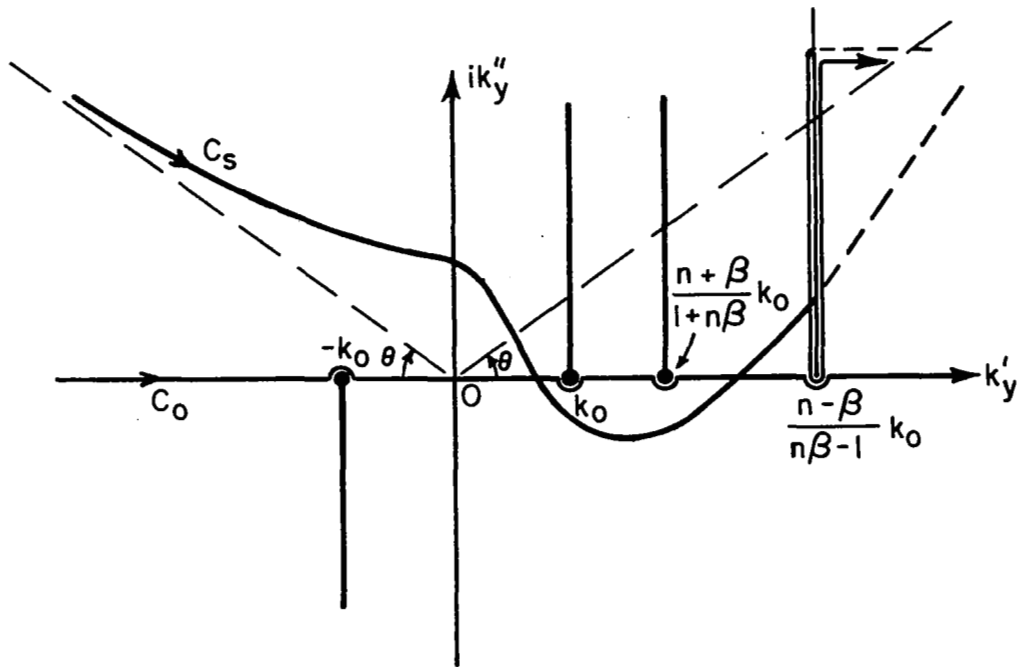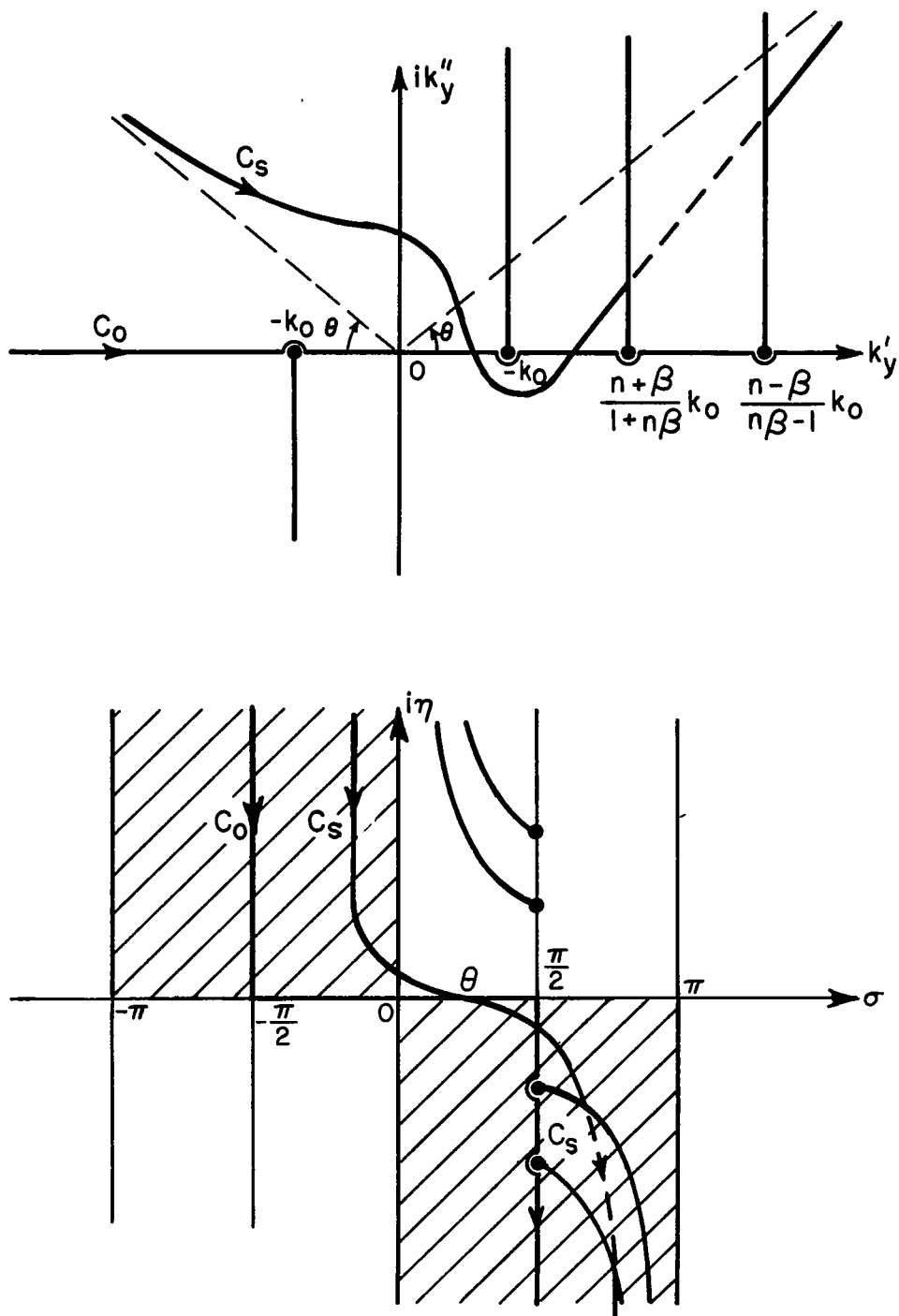
Fig. 11c--Path of integration when the steepest descent
path C_s is intercepted by the branch cut, $n\beta > 1$ and
$\theta > \sin^{-1}\left[(n\beta + 1)/(n + \beta)\right]$

be performed in the complex k_y-plane as follows:

Let a change of variable be made such that

(126) $k_y = k_{yo} + it/y$

where k_{yo} is one of the branch points given in (102b). In terms of the new variable t, the following approximations are permissible for large values of y.

$$\ell^2 = k_o^2 - k_y^2 = k_o^2 - \left(k_{yo} + \frac{it}{y}\right)^2 = k_o^2 - k_{yo}^2 + \frac{t^2}{y^2} - 2ik_{yo}\, t/y$$

$$\simeq k_o^2 - k_{yo}^2$$

$$k_x^2 = n^2 k_o^2 a - \frac{b^2 k_o^2}{a} - \frac{2bk_o}{a}\left(k_{yo} + i\frac{t}{y}\right) - \frac{1}{a}\left(k_{yo} + i\frac{t}{y}\right)^2$$

$$= -\frac{2bk_o}{a}\frac{it}{y} - \frac{2k_{yo}}{a}\frac{it}{y} + \frac{1}{a}\frac{t^2}{y^2}$$

$$\simeq -\frac{2}{a}\left(bk_o + k_{yo}\right)\left(\frac{it}{y}\right)$$

$$\frac{1}{\ell}\left[1 + \frac{\mu\ell + \mu_o k_x}{\mu\ell - \mu_o k_x}\, e^{i2\ell d}\right] = \frac{1}{\ell}\left[1 + \frac{1 + \dfrac{\mu_o k_x}{\mu\ell}}{1 - \dfrac{\mu_o k_x}{\mu\ell}}\, e^{i2\ell d}\right]$$

$$\simeq \frac{1}{\ell}\left[1 + \left(1 + 2\frac{\mu_o k_x}{\mu\ell}\right)e^{i2\ell d}\right].$$

Equation (100a) reduces to

$$(127) \quad E_{z(branch)} \approx \frac{-i\omega\mu_o I_o}{4\pi\ell y} e^{i\ell(x-d)} \int_\infty^{(+o)} \left[1 + \left(1 + 2\frac{\mu_o k_x}{\mu\ell} \right) e^{i2\ell d} \right]$$

$$\cdot e^{-t} e^{ik_{yo}y} dt$$

$$= \frac{-i\omega\mu_o^2 I_o}{2\pi\mu\ell^2 y} \left[\frac{2}{a}(bk_o + k_{yo}) \frac{i}{y} \right]^{\frac{1}{2}} e^{i\ell(x+d) + ik_{yo}y}$$

$$\cdot \int_\infty^{(+o)} e^{-t}(-t)^{\frac{1}{2}} dt$$

$$= \frac{\omega\mu_o^2 I_o}{2\sqrt{\pi}\,\mu^2(k_o^2 - k_{yo}^2)\, y^{3/2}} \left[\frac{i2}{a}(bk_o + k_{yo}) \right]^{\frac{1}{2}}$$

$$\cdot e^{-\sqrt{k_{yo}^2 - k_o^2}(x+d) + ik_{yo}y} \quad .$$

The above result is obtained by making use of the contour representation of the gamma function. [26] The branch cut integral given above contains an exponential damping factor and it is inversely proportional to the three half power of y. Hence it is always negligible as compared with those given by (116) or (125).

Those residue terms which have to be considered as shown in (113) when the contour was deformed contain an exponential factor

$$e^{ik_o r \cos(\phi_p - \theta)} = e^{ik_o r \cos(\sigma_p - \theta)\cosh\eta_p + k_o r \sin(\sigma_p - \theta)\sinh\eta_p}$$

and the real part of the exponent is always negative. Except when $\theta = 90°$ the residue wave is also negligible as compared to the space wave given by (116).

The same procedure can be used in evaluating the field components H_x and H_y as given in (100b) and (100c). The leading terms from the saddle point method of integration are

$$\text{(116c)} \qquad H_x \underset{\sim}{\sim} - \frac{k_o I_o}{4} \sqrt{\frac{2}{\pi}} \ \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \ \sin\theta \ P(\theta)$$

$$\text{(116d)} \qquad H_y \underset{\sim}{\sim} \frac{k_o I_o}{4} \sqrt{\frac{2}{\pi}} \ \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \ \cos\theta \ P(\theta)$$

where $P(\theta)$ is given in (111a). Converting into cylindrical coordinates we have

$$\text{(116e)} \qquad H_r \underset{\sim}{\sim} 0$$

$$\text{(116f)} \qquad H_\theta \underset{\sim}{\sim} H_y \cos\theta - H_x \sin\theta = \frac{k_o I_o}{2\sqrt{2\pi}} \ \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \ P(\theta)$$

$$\text{(116b)} \qquad E_z \underset{\sim}{\sim} \frac{-\omega \mu_o I_o}{2\sqrt{2\pi}\,(k_o r)^{\frac{1}{2}}} \ e^{ik_o r - i\frac{\pi}{4}} \ P(\theta)$$

and the Poynting vector is

$$\text{(116g)} \qquad \overline{S} = \frac{1}{2}\,\overline{E} \times \overline{H}^* = \hat{r}\,\frac{\omega \mu_o I_o^2}{16\,\pi r} \ |\,P(\theta)\,|^2$$

which has only a radial component.

Magnetic Line Source Parallel
to Z-Axis

For a magnetic line current source located at $x = d$, $y = 0$ and parallel to z-axis we write

(128) $\qquad \overline{M} = \hat{z} \, M_o \delta(x-d) \, \delta(y)$.

The radiated field has no variation with respect to z. As the z-component of the electric field is zero the total fields may be derived from electric vector potentials $\overline{F}_T$ in region I and $\overline{F}_B$ in region II having only z-components. Similar procedures are used in solving this problem as that for the electric line current source case given in the previous section. Only those important steps and results are given below.

The field components in upper region are derived from $\overline{F}_T$ as

(129a) $\qquad \overline{E} = \dfrac{-1}{\epsilon_o} \, \nabla \times \overline{F}_T$

(129b) $\qquad \overline{H} = i\omega \overline{F}_T - \dfrac{\nabla \nabla \cdot \overline{F}_T}{i\omega \mu_o \epsilon_o}$

or

$$E_x = \frac{-1}{\epsilon_o} \frac{\partial F_T}{\partial y} \qquad H_x = 0$$

$$E_y = \frac{1}{\epsilon_o} \frac{\partial F_T}{\partial x} \qquad H_y = 0$$

$$E_z = 0 \qquad\qquad H_z = i\omega F_T$$

where $\overline{F}_T = \hat{z} F_T$ and F_T satisfy the wave equation

$$(130) \qquad \nabla^2 F_T + k_o^2 F_T = -\epsilon_o M_o \delta(x-d)\, \delta(y)$$

and the suitable boundary conditions. The solution for (130) is

$$(131) \qquad F_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(x, k_y)\, e^{ik_y y}\, dk_y$$

where g_2 is given by

$$(132a) \qquad g_{21}(x, k_y) = \left(\frac{-\epsilon_o M_o}{2i\ell}\right) [\, 1 + R\, e^{i2\ell d}\,] \quad (x \geq d)$$

$$(132b) \qquad g_{22}(x, k_y) = \left(\frac{-\epsilon_o M_o}{2i\ell}\right) [\, e^{-i\ell(x-d)} + R\, e^{i\ell(x+d)}\,]\, (0 \leq x \leq d)$$

with $\ell^2 = k_o^2 - k_y^2$.

In the bottom region let

$$(133) \qquad \overline{v} = \hat{y}v \qquad \overline{\Omega} = \hat{y}\Omega$$

$$\overline{F}_B = \hat{z}\, F_B$$

and introduce the vector function such that

$$(134) \qquad \overline{F}_{B1} = \hat{z}\, F_{B1} = \overline{\overline{\alpha}} \cdot \overline{F}_B = a\overline{F}_B = \hat{z}\, a\, F_B$$

where F_{B1} satisfies Eq. (43b) which reduces here to

$$(135) \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{1}{a}\frac{\partial^2}{\partial y^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial y} - \frac{\omega^2\Omega^2}{a} + k^2 a\right] F_{B1} = 0\ .$$

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The field components in this region are derived from $\overline{F}_B$ by Eqs. (45b) and (46b)

(136a)
$$\overline{E} = \frac{-1}{\epsilon} \; \overline{\overline{\alpha}}^{-1} \cdot [\, (\nabla + i\omega\overline{\Omega}) \times \overline{F}_B] = \frac{-1}{\epsilon} \; \overline{\overline{\alpha}}^{-1}$$

$$\cdot \, [\, (\nabla + i\omega\Omega) \times (\overline{\overline{\alpha}}^{-1} \cdot \overline{F}_{B1})]$$

(136b)
$$\overline{H} = i\omega \, \overline{F}_B - D_1 V = i\omega \, \overline{\overline{\alpha}}^{-1} \cdot \overline{F}_{B1} - \frac{(\nabla + i\omega\,\overline{\Omega})\,[\,(\nabla + i\omega\overline{\Omega}) \cdot \overline{F}_{B1}]}{i\omega\,\mu\epsilon\,a^2}$$

or

$$E_x = \frac{-1}{\epsilon\,a^2} \left[\frac{\partial F_{B1}}{\partial y} + i\omega\Omega\,F_{B1} \right] \qquad H_x = 0$$

$$E_y = \frac{1}{\epsilon\,a} \; \frac{\partial F_{B1}}{\partial x} \qquad\qquad\qquad H_y = 0$$

$$E_z = 0 \qquad\qquad\qquad\qquad\qquad H_z = \frac{i\omega}{a} \; F_{B1} \, .$$

The Fourier integral

(137)
$$F_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_z(k_y)\, e^{ik_x x + ik_y y}\, dk_y$$

is a solution of (135) provided k_x is given as shown in Eq. (95). By matching boundary conditions at $x = 0$ we get

(138a)
$$R = \frac{\epsilon\ell + \epsilon_o k_x}{\epsilon\ell - \epsilon_o k_x}$$

(138b) $$f_2 = \frac{ia \epsilon_0 \epsilon\, M_0\, e^{i\ell d}}{\epsilon \ell - \epsilon_0 k_x} \quad .$$

The choice of the branches of ℓ and k_x are decided in the same way as for the electric line source case. The field components in the region $x \geq d$ are given by integrals of the form,

(139a) $$H_z = -\frac{\omega \epsilon_0 M_0}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell} [\,1 + R\, e^{i2\ell d}\,]\, e^{i\ell (x-d) + ik_y y}\, dk_y$$

(139b) $$E_x = \frac{M_0}{4\pi} \int_{-\infty}^{\infty} \frac{k_y}{\ell} [\,1 + R\, e^{i2\ell d}\,]\, e^{i\ell (x-d) + ik_y y}\, dk_y$$

(139c) $$E_y = \frac{-M_0}{4\pi} \int_{-\infty}^{\infty} [\,1 + R\, e^{i2\ell d}\,]\, e^{i\ell (x-d) + ik_y y}\, dk_y \quad .$$

The branch points are given in Eq. (102) and their relative positions shown in Fig. 8. The poles are found by solving the equation,

(140) $$\epsilon \ell = \epsilon_0 k_x \quad .$$

Squaring both sides of (140) we have the following results

(141a) $$k_y = k_0 \left[\frac{b\epsilon_0^2}{a} \pm B^{\frac{1}{2}} \right] \bigg/ \left(\epsilon^2 - \frac{\epsilon_0^2}{a} \right)$$

where

$$B = \left[\frac{\epsilon^2 \epsilon_0^2}{a} (b^2 - 1) + \epsilon_0^2 n^2 (\epsilon_0^2 - \epsilon^2 a) + \epsilon^4 \right]$$

and

(141b) $\qquad k_y = k_0/\beta \geq k_0 \qquad$ (if $\epsilon = \epsilon_0$) .

Applying the transformation shown in (110) Eq. (139a) changes to

(142) $\qquad H_z = \dfrac{-\omega\,\epsilon_0\,M_0}{4\pi} \displaystyle\int_{c_0} Q(\phi)\, e^{ik_0 r\cos(\phi-\theta)}\, d\phi$

where

(142a) $\qquad Q(\phi) = 1 + \dfrac{\epsilon\cos\phi - \epsilon_0\left(n^2 a - \dfrac{b^2}{a} - \dfrac{2b}{a}\sin\phi - \dfrac{1}{a}\sin^2\phi\right)^{\frac{1}{2}}}{\epsilon\cos\phi + \epsilon_0\left(n^2 a - \dfrac{b^2}{a} - \dfrac{2b}{a}\sin\phi - \dfrac{1}{a}\sin^2\phi\right)^{\frac{1}{2}}}$

$\cdot\, e^{i2dk_0\cos\phi}$.

The transformed path of integration c_0, the path of steepest descent, the saddle point and the branch points due to k_x are the same as those shown in Fig. 9. The poles are transformed from (141) as

(143a) $\qquad \sin\phi_p = \sin(\sigma_p + i\eta_p) = \left[\dfrac{b\epsilon_0^2}{a} \pm B^{\frac{1}{2}}\right] \Big/ \left(\epsilon^2 - \dfrac{\epsilon_0^2}{a}\right)$

(143b) $\qquad\qquad = \dfrac{1}{\beta} \qquad$ (if $\epsilon = \epsilon_0$) .

Because of the deformation of the path of integration the integral in (142) becomes

(144) $\qquad H_z = -\dfrac{\omega\,\epsilon_0 M_0}{4\pi} \displaystyle\int_{c_s} Q(\phi)\, e^{ik_0 r\cos(\phi-\theta)}\, d\phi$

$\qquad\qquad - \dfrac{i\omega\,\epsilon_0 M_0}{2} \displaystyle\sum_{p} G(\phi_p)\, e^{ik_0 r\cos(\phi_p-\theta)} + K$

where ϕ_p is given by (143), $G(\phi_p)$ is the residue of $Q(\phi)$ at ϕ_p and K is the branch cut integral. If there is no pole near the saddle point θ the first term in (144) can be evaluated approximately for far field as

$$(145a) \qquad H_z = - \frac{\omega \epsilon_o M_o}{4\pi} e^{ik_o r - i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{Q^m(\theta)}{m!} e^{-im\frac{\pi}{4}}$$

$$\cdot [1 + (-1)^m] \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\left(\frac{k_o r}{2}\right)^{\frac{m+1}{2}}}$$

where

$$Q^m(\theta) = \frac{d^m Q(\phi)}{d\phi^m} \Bigg|_{\phi=\theta} .$$

The leading term of (145a) is

$$(145b) \qquad \frac{-\omega \epsilon_o M_o}{2\sqrt{2\pi}(k_o r)^{\frac{1}{2}}} e^{ik_o r - i\frac{\pi}{4}} Q(\theta) .$$

The residue of $Q(\phi)$ at pole ϕ_p is

$$(146a) \qquad G(\phi_p) = \frac{2\epsilon^2 \cos\phi_p e^{i2d k_o \cos\phi_p}}{-\epsilon^2 \sin\phi_p + \frac{\epsilon_o^2}{2}(b + \sin\phi_p)}$$

$$(146b) \qquad = \left(\frac{4a}{a-1}\right) \tan\phi_p (1 + i2d k_o \cos\phi_p) e^{i2d k_o \cos\phi_p} \ (\text{if } \epsilon = \epsilon_o) .$$

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The branch cut integral K is given approximately by

$$(147) \quad H_{z(branch)} = \frac{\omega \epsilon_o^2 M_o}{2\sqrt{\pi} \epsilon^2 (k_o^2 - k_{yo}^2) y^3 \ ^2} \left[\frac{i2}{a} (bk_o + k_{yo}) \right]^{\frac{1}{2}}$$

$$\cdot e^{-\sqrt{k_{yo}^2 - k_o^2}(x+d) + ik_{yo}y}$$

where k_{yo} is the branch point given in (102b). The leading terms for E_x and E_y from saddle point method of integration are

$$(145b) \quad H_z \approx \frac{M_o k_o}{2\sqrt{2\pi}} \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \sin \theta \ Q(\theta)$$

$$(145c) \quad E_y \approx \frac{-M_o k_o}{2\sqrt{2\pi}} \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \cos \theta \ Q(\theta)$$

where $Q(\theta)$ is given by (142a). In terms of cylindrical coordinates the field components are

$$(145d) \quad E_r \approx 0$$

$$(145e) \quad E_\theta \approx -\frac{M_o k_o}{2\sqrt{2\pi}} \cdot \frac{e^{ik_o r - i\frac{\pi}{4}}}{(k_o r)^{\frac{1}{2}}} \ Q(\theta)$$

$$(145b) \quad H_y \approx \frac{-\omega \epsilon_o M_o}{2\sqrt{2\pi}(k_o r)^{\frac{1}{2}}} e^{ik_o r - i\frac{\pi}{4}} Q(\theta) .$$

The Poynting vector is

$$(145f) \quad \overline{S} \approx \frac{1}{2} \overline{E} \times \overline{H}^* = \hat{r} \ \frac{\omega \epsilon_o M_o^2}{16\pi r} \ |Q(\theta)|^2 .$$

Electric Line Source Parallel
to Y-Axis

The electric line source located at x = d, z = 0 and parallel to y-axis can be represented by

$$(146) \qquad \overline{J} = \hat{y} \, J_o \, \delta(x-d) \, \delta(z) \ .$$

As the current has no variation in y-direction the radiated field will be independent of y. The method of potentials is used in constructing the total electromagnetic fields. The fields will be expressed in terms of the magnetic and electric vector potentials $\overline{A}_T$ and $\overline{F}_T$ in the upper region and $\overline{A}_B$ and $\overline{F}_B$ in the lower region having only y-components. The field components in the upper region are given as

$$(147a) \qquad \overline{E} = i\omega A_T - \frac{\nabla\nabla\cdot\overline{A}_T}{i\omega\,\mu_o\epsilon_o} - \frac{1}{\epsilon_o}\,\nabla\times\overline{F}_T$$

$$(147b) \qquad \overline{H} = \frac{1}{\mu_o}\,\nabla\times\overline{A}_T + i\omega\overline{F}_T - \frac{\nabla\nabla\cdot\overline{F}_T}{i\omega\,\mu_o\epsilon_o}$$

or

$$E_x = \frac{1}{\epsilon_o}\,\frac{\partial F_T}{\partial z} \qquad\qquad H_x = \frac{-1}{\mu_o}\,\frac{\partial A_T}{\partial z}$$

$$E_y = i\omega A_T \qquad\qquad H_y = i\omega F_T$$

$$E_z = \frac{-1}{\epsilon_o}\,\frac{\partial F_T}{\partial x} \qquad\qquad H_z = \frac{1}{\mu_o}\,\frac{\partial A_T}{\partial x}$$

where $\overline{A}_T = \hat{y} A_T$ and $\overline{F}_T = \hat{y} F_T$. A_T and F_T satisfy respectively the wave equations

(148a)
$$\nabla^2 A_T + k_o^2 A_T = -\mu_o I_o \, \delta(x-d) \, \delta(z)$$

and

(148b)
$$\nabla^2 F_T + k_o^2 F_T = 0 \ .$$

The solutions are

(149a)
$$A_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(x, k_z) e^{ik_z z} \, dk_z$$

(149b)
$$F_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(k_z) e^{i\ell x + ik_z z} \, dk_z$$

where $\ell^2 = k_o^2 - k_z^2$ and g_1 is the same as that shown in (90). g_2 and R have to be determined by the boundary conditions at $x = 0$ which will be discussed later on.

In the bottom region we introduce the vector functions

(150a)
$$\overline{A}_{B1} = \hat{y} A_{B1} = \overline{\overline{\alpha}} \cdot \overline{A}_B = \overline{A}_B = \hat{y} A_B$$

and

(150b)
$$\overline{F}_{B1} = \hat{y} F_{B1} = \overline{\overline{\alpha}} \cdot \overline{F}_B = \overline{F}_B = \hat{y} F_B \ .$$

AB_1 and FB_1 have to satisfy the equation (43) which in this problem

is

(151) $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\omega^2 \Omega^2}{a} + k^2 a \right] \begin{Bmatrix} A_{B1} \\ F_{B1} \end{Bmatrix} = 0 .$$

The solutions are

(152a) $$A_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_z) \, e^{ik_x x + ik_z z} \, dk_z$$

and

(152b) $$F_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_z) \, e^{ik_x x + ik_z z} \, dk_z$$

where $k_x^2 = k^2 a - \dfrac{\omega^2 \Omega^2}{a} - k_z^2$. f_1 and f_2 are determined by the

boundary conditions at $x = 0$. The roots of ℓ and k_x will be chosen

in the same way as shown in (96). Using Eqs. (45) and (46) we

get the field components in the bottom region

(153a) $$\overline{E} = i\omega \overline{A}_B - D_1 \frac{D_1 \cdot \overline{A}_B}{i\omega \mu \epsilon \, a^2} - \frac{1}{\epsilon} \overline{\overline{\alpha}}^{-1} \cdot (D_1 \times \overline{F}_B)$$

$$= i\omega \overline{A}_{B1} - (\nabla + i\omega \overline{\Omega}) \frac{[(\nabla + i\omega \overline{\Omega}) \cdot \overline{A}_{B1}]}{i\omega \mu \epsilon \, a^2} - \frac{1}{\epsilon} \overline{\overline{\alpha}}^{-1}$$

$$\cdot [(\nabla + i\omega \overline{\Omega}) \times \overline{F}_{B1}]$$

(153b) $\overline{H} = \dfrac{1}{\mu} \, \overline{\overline{\alpha}}^{-1} \cdot [\, (\nabla + i\omega\overline{\Omega}) \times \overline{A}_{B1} \,] + i\omega \overline{F}_{B1}$

$$- (\nabla + i\omega\overline{\Omega}) \, \dfrac{[\, (\nabla + i\omega\overline{\Omega}) \cdot \overline{F}_{B1} \,]}{i\omega\mu\epsilon \, a^2}$$

or

$$E_x = \dfrac{-\Omega}{\mu\epsilon \, a^2} \, \dfrac{\partial A_{B1}}{\partial x} + \dfrac{1}{\epsilon a} \, \dfrac{\partial F_{B1}}{\partial z} \qquad H_x = \dfrac{-1}{\mu a} \, \dfrac{\partial A_{B1}}{\partial z} - \dfrac{\Omega}{\mu\epsilon \, a^2} \, \dfrac{\partial F_{B1}}{\partial x}$$

$$E_y = i\omega A_{B1} \left(1 - \dfrac{\Omega^2}{\mu\epsilon \, a^2} \right) \qquad H_y = i\omega F_{B1} \left(1 - \dfrac{\Omega^2}{\mu\epsilon \, a^2} \right)$$

$$E_z = \dfrac{-\Omega}{\mu\epsilon \, a^2} \, \dfrac{\partial A_{B1}}{\partial z} - \dfrac{1}{\epsilon a} \, \dfrac{\partial F_{B1}}{\partial x} \qquad H_z = \dfrac{1}{\mu a} \, \dfrac{\partial A_{B1}}{\partial x} - \dfrac{\Omega}{\mu\epsilon \, a^2} \, \dfrac{\partial F_{B1}}{\partial z} \qquad .$$

The tangential components of electric and magnetic field components at the boundary x = 0 are required to be continuous. As a consequence we find

(154a) $\left(\dfrac{-\mu_o I_o}{2i\ell} \right) (1 + R) \, e^{i\ell d} = \left(1 - \dfrac{\Omega^2}{\mu\epsilon \, a^2} \right) f_1$

(154b) $\dfrac{\ell}{\epsilon_o} \, g_2 = \dfrac{\Omega k_z}{\mu\epsilon \, a^2} \, f_1 + \dfrac{k_x}{\epsilon a} \, f_2$

(154c) $g_2 = \left(1 - \dfrac{\Omega^2}{\mu\epsilon \, a^2} \right) f_2$

(154d) $\dfrac{I_o}{2} \, (1 - R) \, e^{i\ell d} = \dfrac{i k_x}{\mu a} \, f_1 - \dfrac{i\Omega k_z}{\mu\epsilon \, a^2} \, f_2 \qquad .$

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Solving these simultaneous equations the results are

(155)
$$R = \frac{[\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell + \mu_o k_x] - \alpha \mu_o \epsilon_o k_z^2}{[\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell - \mu_o k_x] + \alpha \mu_o \epsilon_o k_z^2}$$

(156)
$$g_2 = \frac{i\mu_o \epsilon_o I_o(1-\alpha)\,\Omega\, k_z\, e^{i\ell d}}{[\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell - \mu_o k_x] + \alpha \mu_o \epsilon_o k_z^2}$$

where

$$\alpha = \frac{\Omega^2}{\mu \epsilon a^2} \quad .$$

Hence, the final expressions for the field components in the region

$x \geq d$ are given in the following integral form.

(157a)
$$E_x = -\frac{\mu_o I_o(1-\alpha)\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{k_z^2\, e^{i\ell(x+d)+ik_z z}}{R_d}\, dk_z$$

(157b)
$$E_y = -\frac{\omega_o \mu_o I_o}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell}\left[1 + \frac{R_n}{R_d} e^{i2\ell d}\right] e^{i\ell(x-d)+ik_z z}\, dk_z$$

(157c)
$$E_z = \frac{\mu_o I_o(1-\alpha)\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{\ell\, k_z\, e^{i\ell(x+d)+ik_z z}}{R_d}\, dk_z$$

(157d)
$$H_x = \frac{I_o}{4\pi} \int_{-\infty}^{\infty} \frac{k_z}{\ell}\left[1 + \frac{R_n}{R_d} e^{i2\ell d}\right] e^{i\ell(x-d)+ik_z z}\, dk_z$$

(157e)
$$H_y = -\frac{\omega \mu_o \epsilon_o I_o(1-\alpha)\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{k_z\, e^{i\ell(x+d)+ik_z z}}{R_d}\, dk_z$$

(157f) $\qquad H_z = - \dfrac{I_o}{4\pi} \displaystyle\int_{-\infty}^{\infty} \left[1 + \dfrac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_z z} \, dk_z$

where

$$R_n = [\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell + \mu_o k_x] - \alpha \mu_o \epsilon_o k_z^2$$

$$R_d = [\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell - \mu_o k_x] + \alpha \mu_o \epsilon_o k_z^2 .$$

Two of the branch points are given in (102a) while the other two are

(102c) $\qquad k_{z2} = \pm k_o \sqrt{\dfrac{n^2 - \beta^2}{1 - \beta^2}}$.

One of these two branch points is on the right side of k_o and the other one on the left side of $-k_o$ in the k_z-plane. This is shown in Fig. 12. The location of the poles are found by solving

(158a) $\qquad [\epsilon a(1-\alpha)\ell - \epsilon_o k_x][\mu a(1-\alpha)\ell - \mu_o k_x] + \alpha \mu_o \epsilon_o k_z^2 = 0$

i. e. ,

(158b) $\qquad [\alpha \mu_o \epsilon_o - \mu_o \epsilon_o - \mu \epsilon a^2 (1-\alpha)^2] k_z^2 + \mu_o \epsilon_o k_o^2 \left(n^2 a - \dfrac{b^2}{a} \right)$

$$+ \mu \epsilon a^2 (1-\alpha)^2 k_o^2 .$$

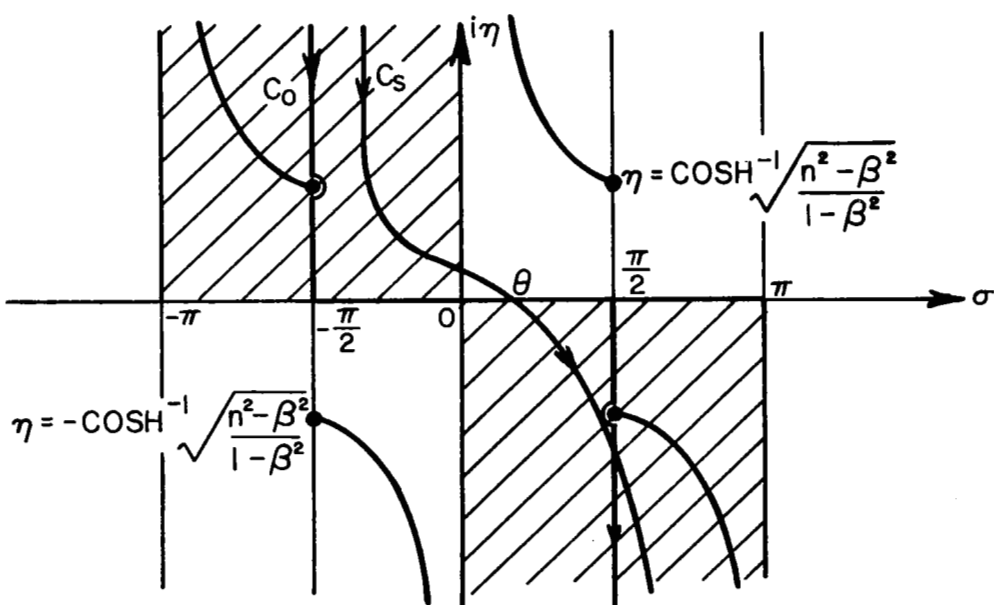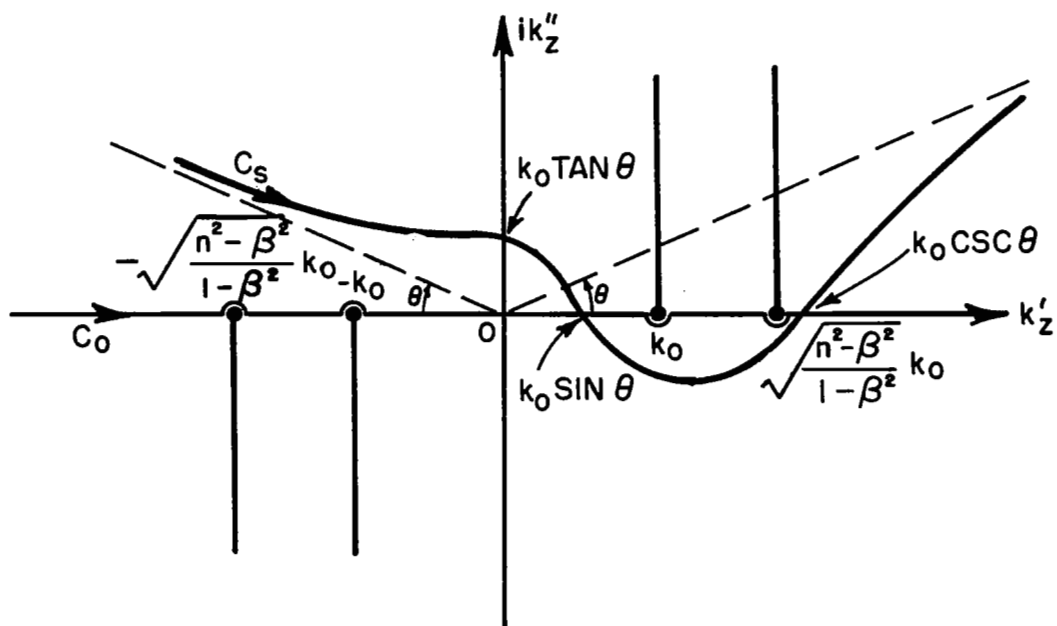$$= a(1-\alpha)(\mu \epsilon_o + \mu_o \epsilon)\ell k_x$$

Fig. 12--Relative positions of branch points, branch cuts
and the path of integration in k_z-plane and ϕ-plane

Squaring both sides of (158b) we get the solutions

(159a) $\qquad k_z = \pm\ k_o[\ (A \pm \sqrt{B})/C\ ]^{\frac{1}{2}}$

where

$$A = -\ [\ 2(n^4 - 1)\ \beta^2 + (\mu_r + \epsilon_r)^2\ (1 + n^2 - 2\beta^2)\ ]$$

$$B = (\mu_r + \epsilon_r)^2\ \{(n^2 - 1)^2\ [\ (\mu_r + \epsilon_r)^2 + 4(n^2 - 2\beta^2)\ ]\ + 16(n^2 - \beta^2)^2\ \}$$

$$C = 2(1 - \beta^2)\ [\ (1 + n^2)^2 - (\mu_r + \epsilon_r)^2\ ]$$

$$\mu_r = \mu/\mu_o \qquad\qquad \epsilon_r = \epsilon/\epsilon_o$$

or

(159b) $\qquad k_z = \pm\ k_o\ (D/E)^{\frac{1}{2}} \qquad$ (if $\mu = \mu_o$ or $\epsilon = \epsilon_o$)

where

$$D = (n^2 + 1)^2\ [(n^2 + 1)\ \beta^2 - n^2\ ] - 4n^2\beta^4$$

$$E = (1 - \beta^2)\ (n^2 + 1)\ (n^2 + 1 - 2\beta)\ (n^2 + 1 + 2\beta)\ .$$

The location of these poles must be examined to see which Riemann sheet they are on.

When $v = 0\ \Omega$ as well as α and g_2 equal zero and the R shown in (155) reduces to that given in (99a) which eventually becomes the reflection coefficient from a semi-infinite stationary medium.[27] When $\mu = \mu_o$ and $\epsilon = \epsilon_o$ both R and g_2 reduce to zero we have the

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same conclusion as the one given in Section IV-1 that the motion of the bottom region have no effect upon the whole radiation field. The motion of the bottom region creates those field components derived from the potential functions $\overline{F}_T$ and $\overline{F}_B$ which do not exist when the bottom region is stationary or moving in the direction perpendicular to the line source.

Applying the transformation given in (110) with k_y replaced by k_z Eq. (157b) becomes

$$(160) \qquad E_y = - \frac{\omega \mu_o I_o}{4\pi} \int_{C_o} P(\phi) \, e^{ik_o r \cos(\phi-\theta)} \, d\phi$$

where

(160a)

$$P(\phi) = 1 + \left\{ \frac{[\epsilon_r a(1-\sigma)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}] \, [\mu_r a(1-\sigma)\cos\phi - (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}] - \sigma \sin^2\phi}{[\epsilon_r a(1-\sigma)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}] \, [\mu_r a(1-\sigma)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}] + \sigma \sin^2\phi} \right\}$$

$$e^{i2d k_o \cos\phi}$$

The transformed path of integration C_o, the path of the steepest descent C_s which is determined by $\cos(\sigma-\theta)\cosh\eta = 1$, the saddle point θ and the branch cuts in the ϕ-plane are shown in Fig. 12. The hatched region corresponding to the proper branch for ℓ. The branch points due to k_x in the ϕ-plane are transformed from (102c) as the solution of

(161a)
$$\sin \phi = \pm \sqrt{\frac{n^2 - \beta^2}{1 - \beta^2}}$$

and the pole from (159) as

(161b)
$$\sin \phi_p = \pm \left[(A + \sqrt{B})/C \right]^{\frac{1}{2}}$$

(161c)
$$= \pm \sqrt{D/E} \quad (\text{if } \mu = \mu_o \text{ or } \epsilon = \epsilon_o).$$

Deforming the path of integration from C_o to C_s the integral given in (160) changes to

(162)
$$E_y = -\frac{\omega \mu_o I_o}{4\pi} \int_{C_s} P(\phi) e^{ik_o r \cos(\phi - \theta)} d\phi$$

$$-\frac{i\omega \mu_o I_o}{2} \sum_p F(\phi_p) e^{ik_o r \cos(\phi_p - \theta)} + K$$

where ϕ_p is given by (161b, c), $F(\phi_p)$ is the residue of $P(\phi)$ at ϕ_p and K is the branch cut integral. If there are no poles of the integrand near the saddle point the first term in (162) can be evaluated asymptotically as

(163a)
$$E_y \approx -\frac{\omega \mu_o I_o}{4\pi} e^{ik_o r - i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{P^m(\theta)}{m!} e^{-im\frac{\pi}{4}}$$

$$\cdot [1 + (-1)^m] \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\left(\frac{k_o r}{2}\right)^{\frac{m+1}{2}}}$$

where $P^m(\theta) = \dfrac{d^m P(\phi)}{d\phi^m}\Bigg|_{\phi=\theta}$ and Γ is gamma function. The leading term is

(163b)
$$E_y \approx - \frac{\omega \mu_o I_o}{2\sqrt{2\pi(k_o r)^{\frac{1}{2}}}} e^{ik_o r - i\frac{\pi}{4}} P(\theta) \ .$$

Since these poles are simple ones the residue of $P(\phi)$ at ϕ_p is

(164)
$$F(\phi_p) = F_n/F_d$$

$$F_n = 2\mu_r a(1-\alpha) \cos\phi_p \left[\epsilon_r a(1-\alpha) \cos\phi_p \pm \left(n^2 a - \frac{b^2}{a} - \sin^2\phi_p \right)^{\frac{1}{2}} \right]$$

$$\cdot \ e^{i2d k_o \cos\phi_p}$$

$$F_d = 2\alpha\sin\phi_p \cos\phi_p - \sin\phi_p \left\{ \left[\epsilon_r a(1-\alpha) \cos\phi_p \right.\right.$$

$$\left. \pm \left(n^2 a - \frac{b^2}{a} - \sin^2\phi_p \right)^{\frac{1}{2}} \right] \left[\mu_r a(1-\alpha) \pm \frac{\cos\phi_p}{\left(n^2 a - \frac{b^2}{a} - \sin^2\phi_p \right)^{\frac{1}{2}}} \right]$$

$$+ \left[\epsilon_r a(1-\alpha) \pm \frac{\cos\phi_p}{\left(n^2 a - \frac{b^2}{a} - \sin^2\phi_p \right)^{\frac{1}{2}}} \right]$$

$$\left. \left[\mu_r a(1-\alpha) \cos\phi_p \pm \left(n^2 a - \frac{b^2}{a} - \sin^2\phi_p \right)^{\frac{1}{2}} \right] \right\} .$$

If the pole happens to be in the near vicinity of the saddlepoint the first

term in (162) can be evaluated asymptotically in the same manner

as the one shown in Section IV-1.

The radiation patterns are symmetric with respect to x-axis.
A branch cut integral must be included if the path C_s is intercepted
by the branch cut when the angle of observation θ extends over the
limit given by

$$(165) \qquad \theta = \sin^{-1}\left[\sqrt{\frac{1 - \beta^2}{n^2 - \beta^2}}\right] \, .$$

The complete contour is constructed in the same way as that shown
in Fig. 11a. The branch cut integral can be performed approxi-
mately as follows.

Making the transform of variable such that

$$(166) \qquad k_z = k_{zo} + it/z$$

where k_{zo} is one of the branch points shown in (102c). In terms of
the new variable t, the following approximations are permissible
for large values of z.

$$\ell^2 = k_o^2 - k_z^2 = k_o^2 - \left(k_{zo} + \frac{it}{z}\right)^2 \approx k_o^2 - k_{zo}^2$$

$$k_x^2 = k_o^2\left(n^2 a - \frac{b^2}{a}\right) - k_z^2 = k_o^2\left(n^2 a - \frac{b^2}{a}\right) - \left(k_{zo} + \frac{it}{z}\right)^2$$

$$= k_o^2\left(n^2 a - \frac{b^2}{a}\right) - k_{zo}^2 - 2ik_{zo}t/z + t^2/z^2$$

$$\approx -2i k_{zo}t/z \, .$$

Equation (157b) reduces to

(167) $\quad E_y = -\dfrac{\omega \mu_o I_o}{4\pi} \displaystyle\int_{\infty}^{(+0)} \dfrac{1}{\ell}$

$$\cdot \left[1 + \frac{\epsilon_r \mu_r a^2 (1-\alpha)^2 \ell^2 - (\mu_r - \epsilon_r) a(1-\alpha)\ell\, k_x - k_x^2 - \alpha k_{zo}^2}{\epsilon_r \mu_r a^2 (1-\alpha)^2 \ell^2 - (\mu_r + \epsilon_r) a(1-\alpha)\ell\, k_x - k_x^2 + \alpha k_{zo}^2}\, e^{i2\ell d} \right]$$

$$\cdot\; e^{ik_{zo}z - t} \left(\frac{i}{z} \right) dt$$

$$\approx \frac{i\omega \mu_o I_o a (1-\alpha)\sqrt{2k_{zo}}}{4\sqrt{\pi}} \left(\frac{i}{z} \right)^{3/2} \left(\frac{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) - \alpha k_{zo}^2}{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) + \alpha k_{zo}^2} \right)$$

$$\cdot \left[\frac{\mu_r + \epsilon_r}{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) + \alpha k_{zo}^2} - \frac{\mu_r - \epsilon_r}{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) - \alpha k_{zo}^2} \right]$$

$$\cdot\; e^{ik_{zo}z - \sqrt{k_{zo}^2 - k_o^2}\,(x+d)}.$$

It is clear that both the branch cut integral shown in (167) and the residue wave term shown in (162) are negligible as compared with the space wave term given in (163) when only the far field is concerned. The same procedures shown above can be used in evaluating other field components given in (157). The leading terms from the saddle point method of integration are,

(163c) $\quad E_x = -\dfrac{I_o(1-\alpha)\Omega k_o}{\epsilon_o \sqrt{2\pi}(k_o r)^{\frac{1}{2}}} \; e^{ik_o r - i\frac{\pi}{4}} \sin\theta\; P_1(\theta)$

(163d) $\quad E_z = \dfrac{I_0(1-\alpha)\Omega\, k_0}{\epsilon_0\sqrt{2\pi(k_0 r)^{\frac{1}{2}}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; \cos\theta\, P_1(1)$

(163e) $\quad H_x = \dfrac{I_0 k_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; \sin\theta\, P(\theta)$

(163f) $\quad H_y = \dfrac{-\omega I_0(1-\alpha)\Omega}{\sqrt{2\pi}\,(k_0 r)^{\frac{1}{2}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; P_1(\theta)$

(163g) $\quad H_z = -\dfrac{I_0 k_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; \cos\theta\, P(\theta)$

where $P(\phi)$ is given in (160a) and

(168) $\quad P_1(\phi) =$

$$\frac{\sin\phi\cos\phi\; e^{i2d\, k_0 \cos\phi}}{\left[\epsilon_r a(1-\alpha)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}\right]\left[\mu_r a(1-\alpha)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}}\right] + \alpha\sin^2\phi}.$$

In cylindrical coordinates the field components and Poynting vector due to potential A_T are

(163h) $\quad H_r \sim 0$

(163i) $\quad H_\theta \sim -\dfrac{I_0 k_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; P(\theta)$

(163b) $\quad E_y \sim -\dfrac{\omega\mu_0 I_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\; e^{ik_0 r - i\frac{\pi}{4}}\; P(\theta)$

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(163j) $\quad \overline{S} = \frac{1}{2} \overline{E} \times \overline{H}^* = \hat{r} \frac{\omega \mu_o I_o^2}{16 \pi r} |P(\theta)|^2$

and those due to potential F_T are

(163k) $\quad E_r \overset{\sim}{\approx} 0$

(163ℓ) $\quad E_\theta \overset{\sim}{\approx} \frac{I_o(1-\alpha) \Omega k_o}{\epsilon_o \sqrt{2\pi(k_o r)^{\frac{1}{2}}}} e^{ik_o r - i\frac{\pi}{4}} P_1(\theta)$

(163f) $\quad H_y \overset{\sim}{\approx} - \frac{\omega I_o(1-\alpha)\Omega}{\sqrt{2\pi(k_o r)^{\frac{1}{2}}}} e^{ik_o r - i\frac{\pi}{4}} P_1(\theta)$

(163m) $\quad \overline{S} \overset{\sim}{\approx} \frac{1}{2} \overline{E} \times \overline{H}^* = \hat{r} \frac{\omega(1-\alpha)^2 I_o^2 \Omega^2}{4\pi \epsilon_o r} |P_1(\theta)|^2 .$

Magnetic Line Source Parallel to Y-Axis

For a magnetic line source located at $x = d$, $z = 0$ and parallel to y-axis we write it as

(169) $\quad \overline{M} = \hat{y} M_o \delta(x-d) \delta(z) .$

The radiated field has no variation with respect to y. The formulation of the problem in this section is similar to the one in the previous section with minor differences. The field components in the upper region are given by (147) but A_T and F_T satisfy respectively the following wave equations.

(170a) $\nabla^2 A_T + k_o^2 A_T = 0$

and

(170b) $\nabla^2 F_T + k_o^2 F_T = - \epsilon_o M_o \, \delta(x-d) \, \delta(z)$.

The solutions are

(171a) $A_T = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} g_1(k_z) \, e^{i\ell x + ik_z z} \, dk_z$

(171b) $F_T = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} g_2(x, k_z) \, e^{ik_z z} \, dk_z$

where $\ell^2 = k_o^2 - k_z^2$ and g_2 is the same as that shown in (132). g_1 and R are determined later on by the boundary conditions at x = 0.

In the bottom region we use the same formulation from (150) through (153). Matching the tangential electric and magnetic field components of both sides at the boundary x = 0 gives the following simultaneous equations.

(172a) $g_1 = \left(1 - \dfrac{\Omega^2}{\mu \epsilon a^2}\right) f_1$

(172b) $\dfrac{M_o}{2} (1-R) e^{i\ell d} = \dfrac{i\Omega k_z}{\mu \epsilon a^2} f_1 + \dfrac{ik_x}{\epsilon a} f_2$

(172c) $\dfrac{-\epsilon_o M_o}{2i\ell} (1+R) e^{i\ell d} = \left(1 - \dfrac{\Omega^2}{\mu \epsilon a^2}\right) f_2$

(172d)
$$\frac{\ell}{\mu_o} g_1 = \frac{k_x}{\mu a} f_1 - \frac{\Omega k_z}{\mu \epsilon a^2} f_2 \; .$$

From (172) we have

(173)
$$R = \frac{\left[\mu a (1-\alpha) \ell - \mu_o k_x\right] \left[\epsilon a (1-\alpha) \ell + \epsilon_o k_x\right] - \alpha \mu_o \epsilon_o k_z^2}{\left[\mu a (1-\alpha) \ell - \mu_o k_x\right] \left[\epsilon a (1-\alpha) \ell - \epsilon_o k_x\right] + \alpha \mu_o \epsilon_o k_z^2}$$

(174)
$$g_1 = \frac{-i\mu_o \epsilon_o (1-\alpha) \Omega M_o k_z e^{i\ell d}}{\left[\mu a (1-\alpha) \ell - \mu_o k_x\right] \left[\epsilon a (1-\alpha) \ell - \epsilon_o k_x\right] + \alpha \mu_o \epsilon_o k_z^2} \; .$$

The expressions in integral form for the field components in the

region $x \geq d$ are

(175a)
$$E_x = \frac{-M_o}{4\pi} \int_{-\infty}^{\infty} \frac{k_z}{\ell} \left[1 + \frac{R_n}{R_d} e^{i2\ell d}\right] e^{i\ell(x-d) + ik_z z} \, dk_z$$

(175b)
$$E_y = \frac{\omega \mu_o \epsilon_o (1-\alpha) \Omega M_o}{2\pi} \int_{-\infty}^{\infty} \frac{k_z}{R_d} e^{i\ell(x+d) + ik_z z} \, dk_z$$

(175c)
$$E_z = \frac{M_o}{4\pi} \int_{-\infty}^{\infty} \left[1 + \frac{R_n}{R_d} e^{i2\ell d}\right] e^{i\ell(x-d) + ik_z z} \, dk_z$$

(175d)
$$H_x = \frac{-\epsilon_o (1-\alpha) \Omega M_o}{2\pi} \int_{-\infty}^{\infty} \frac{k_z^2 e^{i\ell(x+d) + ik_z z}}{R_d} \, dk_z$$

(175e) $\qquad H_y = \dfrac{-\omega \epsilon_o M_o}{4\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{1}{\ell} \left[1 + \dfrac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_z z} \, dk_z$

(175f) $\qquad H_z = \dfrac{\epsilon_o(1-\alpha)\Omega M_o}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{k_z \ell}{R_d} e^{i\ell(x+d)+ik_z z} \, dk_z$

where

$$R_n = \left[\mu a(1-\alpha)\ell - \mu_o k_x \right] \left[\epsilon a(1-\alpha)\ell + \epsilon_o k_x \right] - \alpha \mu_o \epsilon_o k_z^2$$

$$R_d = \left[\mu a(1-\alpha)\ell - \mu_o k_x \right] \left[\epsilon a(1-\alpha)\ell - \epsilon_o k_x \right] + \alpha \mu_o \epsilon_o k_z^2 \quad .$$

Branch points are given in (102a, c) and shown in Fig. 12. Poles can be found from (159). Using the transformation (110) where k_y is replaced by k_z (175e) becomes

(176) $\qquad H_y = - \dfrac{\omega \epsilon_o M_o}{4\pi} \displaystyle\int_{-\infty}^{\infty} Q(\phi) e^{ik_o r \cos(\phi-\theta)} \, d\phi$

where

(176a) $\qquad Q(\phi) = 1 +$

$$\frac{\left[\mu_r a(1-\alpha)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}} \right] \left[\epsilon_r a(1-\alpha)\cos\phi - (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}} \right] - \alpha \sin^2\phi}{\left[\mu_r a(1-\alpha)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}} \right] \left[\epsilon_r a(1-\alpha)\cos\phi + (n^2 a - \frac{b^2}{a} - \sin^2\phi)^{\frac{1}{2}} \right] + \alpha \sin^2\phi}$$

$$\bullet \; e^{i2d k_o \cos\phi} \quad .$$

The transformed path of integration C_O and the steepest descent

path C_S are also shown in Fig. 12. The branch points and poles in

ϕ-plane are given by (161). Shifting the path of integration from

C_O to C_S the integral in (176e) can be written as

(177)
$$H_y = - \frac{\omega \epsilon_o M_o}{4\pi} \int_{C_s} Q(\phi) e^{ik_o r \cos(\phi-\theta)} d\phi$$

$$- \frac{i\omega \epsilon_o M_o}{2} \sum_p G(\phi_p) e^{ik_o r \cos(\phi_p-\theta)} + K$$

where $G(\phi_p)$ is the residue of $Q(\phi)$ at the pole ϕ_p which is given in

(161b, c). If there are no poles located near the saddle point the

first term in (177) can be evaluated approximately for the far field

as

(178a)
$$H_y = - \frac{\omega \epsilon_o M_o}{4\pi} e^{ik_o r - i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{Q^m(\theta)}{m!} e^{-im\frac{\pi}{4}} [1 + (-1)^m]$$

$$\frac{\Gamma\left(\frac{m+1}{2}\right)}{2\left(\frac{k_o r}{2}\right)^{\frac{m+1}{2}}}$$

where the leading term is

(178b)
$$H_y \approx \frac{-\omega \epsilon_o M_o}{2\sqrt{2\pi}(k_o r)^{\frac{1}{2}}} e^{ik_o r - i\frac{\pi}{4}} Q(\theta) .$$

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The residue of $Q(\phi)$ at pole ϕ_p is

(179) $G(\phi_p) = G_n/G_d$

$$G_n = 2\epsilon_r a(1-\alpha)\cos\phi_p\left[\mu_r a(1-\alpha)\cos\phi_p + (n^2 a - \frac{b^2}{a} - \sin^2\phi_p)^{\frac{1}{2}}\right]$$

$$\cdot\, e^{i2d\,k_o\cos\phi_p}$$

$$G_d = 2\alpha\sin\phi_p\cos\phi_p - \sin\phi_p\left\{\left[\epsilon_r a(1-\alpha)\cos\phi_p + (n^2 a - \frac{b^2}{a} - \sin^2\phi_p)^{\frac{1}{2}}\right]\right.$$

$$\cdot\left[\mu_r a(1-\alpha) + \frac{\cos\phi_p}{(n^2 a - \frac{b^2}{a} - \sin\phi_p)^{\frac{1}{2}}}\right]$$

$$+ \left[\epsilon_r a(1-\alpha) + \frac{\cos\phi_p}{(n^2 a - \frac{b^2}{a} - \sin^2\phi_p)^{\frac{1}{2}}}\right]$$

$$\left.\cdot\left[\mu_r a(1-\alpha)\cos\phi_p + (n^2 a - \frac{b^2}{a} - \sin\phi_p)^{\frac{1}{2}}\right]\right\}\quad\cdot$$

A branch cut integral has to be included if the angle of observation θ is beyond the limit given in (165). Its approximate value is

(180) $H_{y_{(branch)}} \underset{\sim}{} \dfrac{i\omega\epsilon_o a\,M_o(1-\alpha)\sqrt{2k_{zo}}}{4\sqrt{\pi}}\left(\dfrac{i}{z}\right)^{3/2}$

$$\cdot\left[\frac{\epsilon_r\mu_r a^2(1-\alpha)^2(k_o^2 - k_{zo}^2) - \alpha k_{zo}^2}{\epsilon_r\mu_r a^2(1-\alpha)^2(k_o^2 - k_{zo}^2) + \alpha k_{zo}^2}\right]$$

$$\cdot \left[\frac{\mu_r + \epsilon_r}{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) + \alpha k_{zo}^2} - \frac{\epsilon_r - \mu_r}{\epsilon_r \mu_r a^2 (1-\alpha)^2 (k_o^2 - k_{zo}^2) - \alpha k_{zo}^2} \right]$$

$$\cdot e^{ik_{zo}z - \sqrt{k_{zo}^2 - k_o^2}\,(x+d)}$$

where k_{zo} is given in (102c). The leading term from the saddle

point method of integration for the other field components in (175)

are

$$(178c) \qquad E_x \stackrel{\sim}{\sim} \frac{-M_o k_o}{2\sqrt{2\pi}\,(k_o r)^{\frac{1}{2}}}\; e^{ik_o r - i\frac{\pi}{4}}\; \sin\theta\; Q(\theta)$$

$$(178d) \qquad E_y \stackrel{\sim}{\sim} \frac{\omega(1-\alpha)\Omega M_o}{\sqrt{2\pi}\,(k_o r)^{\frac{1}{2}}}\; e^{ik_o r - i\frac{\pi}{4}}\; P_1(\theta)$$

$$(178e) \qquad E_z \stackrel{\sim}{\sim} \frac{M_o k_o}{2\sqrt{2\pi}(k_o r)^{\frac{1}{2}}}\; e^{ik_o r - i\frac{\pi}{4}}\; \cos\theta\; Q(\theta)$$

$$(178f) \qquad H_x \stackrel{\sim}{\sim} \frac{-(1-\alpha)\Omega M_o k_o}{\mu_o \sqrt{2\pi}(k_o r)^{\frac{1}{2}}}\; e^{ik_o r - i\frac{\pi}{4}}\; \sin\theta\; P_1(\theta)$$

$$(178g) \qquad H_z \stackrel{\sim}{\sim} \frac{(1-\alpha)\Omega M_o k_o}{\mu_o \sqrt{2\pi}(k_o r)^{\frac{1}{2}}}\; e^{ik_o r - i\frac{\pi}{4}}\; \cos\theta\; P_1(\theta)$$

where $Q(\phi)$ is given in (176a) and $P_1(\phi)$ in (168). In cylindrical

coordinates the field components and Poynting vector due to

potential F_T are

(178j) $E_r \overset{\sim}{\sim} 0$

(178i) $E_\theta \overset{\sim}{\sim} \dfrac{M_0 k_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\, e^{ik_0 r - i\frac{\pi}{4}}\, Q(\theta)$

(178b) $H_y \overset{\sim}{\sim} \dfrac{-\omega \epsilon_0 M_0}{2\sqrt{2\pi}(k_0 r)^{\frac{1}{2}}}\, e^{ik_0 r - i\frac{\pi}{4}}\, Q(\theta)$

(178j) $\overline{S} \overset{\sim}{\sim} \hat{r}\, \dfrac{\omega \epsilon_0 M_0^2}{16\pi r}\, \left| Q(\theta) \right|^2$

and those due to A_T are

(178k) $H_r \overset{\sim}{\sim} 0$

(178ℓ) $H_\theta \overset{\sim}{\sim} \dfrac{M_0(1-\alpha)\Omega\, k_0}{\mu_0 \sqrt{2\pi}\,(k_0 r)^{\frac{1}{2}}}\, e^{ik_0 r - i\frac{\pi}{4}}\, P_1(\theta)$

(178d) $E_y \overset{\sim}{\sim} \dfrac{\omega(1-\alpha)\,\Omega\, M_0}{\sqrt{2\pi}\,(k_0 r)^{\frac{1}{2}}}\, e^{ik_0 r - i\frac{\pi}{4}}\, P_1(\theta)$

(178m) $\overline{S} \overset{\sim}{\sim} \hat{r}\, \dfrac{\omega(1-\alpha)^2 \Omega^2 M_0^2}{4\pi \mu_0 r}\, \left| P_1(\theta) \right|^2 .$

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CHAPTER V
CONCLUSIONS

The definite form of Maxwell's equations for a moving isotropic and lossless medium has been derived by making use of the constitutive relations as found in Minkowski's theory. The wave equations satisfied by the field vectors were given. The integration of these equations is performed by the introduction of the vector and scalar potential functions following a method similar to that used for the stationary medium. The results are then applied to treat two problems. The first is the problem of rectangular and cylindrical waveguides which are filled with isotropic and lossless media moving uniformly in the direction parallel to the axis of the guide. The second problem deals with the effect of uniform motion of a dielectric half-space on the radiation from a line source located above and parallel to the half-space. The sources considered are electric and magnetic line sources, perpendicular or parallel to the direction of motion.

The results thus obtained reduce, as expected, to the known solutions for a medium at rest when the medium becomes stationary or when the constitutive parameters of the medium are set equal to those of free space.

In the waveguide problem it is found that when the velocity of the medium is small such that $n\beta < 1$ there is another frequency limit denoted by f_+ which is larger than the cut off frequency f_c. Two waves with different phase velocity can propagate freely and the faster one travels in the direction opposite to that of the moving medium while the other one travels in this or the reversed direction depending on whether the frequency is below or above f_+. f_+ reduces to f_c when β approaches zero or n approaches one. As the velocity of the medium becomes higher and $n\beta > 1$, the cut off phenomenon disappears but there is still another frequency limit denoted by f_- given by the same formula as that for f_+. It is also possible to have two waves of unequal phase velocity propagating without attenuation. Both will travel in the same direction as that of the medium unless the frequency is lower than f_- in which case the slower one travels in the opposite direction.

In the case of a moving half-space when the line source is oriented perpendicular to the moving direction, the field expressions are similar to those in the stationary case and the far field is linearly polarized. When the line source is oriented parallel to the moving direction another wave component arises and its magnitude varies

mainly with the velocity. Then the radiated field is in general elliptically polarized. However, the direction of power flow for each individual wave component is found to be radial.

The proper location of the branch cuts depends upon the magnitude of the index of refraction and the velocity of the moving medium. Stronger variations are observed when the line source is oriented perpendicular to the moving direction.

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