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ELECTRICAL ANALOGIES AND THE VIBRATION OF LINEAR MECHANICAL SYSTEMS

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ABSTRACT

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This report is tutorial in nature and deals primarily with concepts basic to the vibration of mechanical systems and their equivalent electrical analogies. Although both mechanical impedance and operational amplifier circuits are considered, emphasis is given to passive mobility circuits for the vibration of distributed elastic structures.

The single-degree-of-freedom system is used to illustrate and interrelate many of the concepts and definitions common to mechanical systems and electrical circuits. The mobility concept is then extended to include distributed physical systems described mathematically as partial differential equations. These resulting analogs are electrically equivalent to finite-difference forms of the partial differential equations and can be used directly to synthesize complete electrical models of physical systems. Difference mobility analogs are shown for the vibration of the simple beam, the Timoshenko beam, a curved beam, a rectangular shear panel, and a rectangular plate. Similar analogs are shown for problems typical to heat transfer and fluid flow.

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A	gain factor of an operational amplifier; or, cross-section area of a structural member
a	analog scale factor
B_0	susceptance or imaginary part of a complex admittance
C	a capacitor
$\overline{C}$	capacitor for the generalized mass
C_n	capacitor for the lateral inertia of the nth difference segment
$C_n(\theta)$	capacitor for the rotatory inertia of the nth difference segment
C_θ	capacitor for torsional inertia
c	viscous damping constant; or, specific heat of a material or fluid at constant pressure
D	plate flexural rigidity
dx	differential length measured in the x direction
E	voltage across specified windings of a transformer; or, Young's modulus
e	voltage
e_c	voltage across a capacitor
e_{in}	input voltage of an operational amplifier
e_ℓ	voltage across an inductor
e_o	output voltage of an operational amplifier
e_r	voltage across a resistor

e_θ	voltage denoting an angular velocity
F_{ij}	shear force perpendicular to the ith axis and acting in the j direction
F_r	radial shear force
F_ϕ	circumferential or tangential force
f	external loading applied to a structure
f(t)	external loading for a linear oscillator
f(x, t)	external lateral loading per unit length
f(x, y, t)	external lateral loading per unit area
$f_r(\phi, t)$	external radial loading per unit arc length
$f_\phi(\phi, t)$	external tangential loading per unit arc length
G	shear modulus
G_0	conductance or real part of a complex admittance
$H_0(\omega)$	magnitude or modulus of a complex impedance
h	thickness of a panel or plate
I	current; or, the cross-section area moment of inertia
$\bar{I}$	current flow for a generalized force
I(t)	current for an external force
I_θ	rotatory moment of inertia for a segment of Timoshenko beam
i	current in amps; or, the complex operator $\sqrt{-1}$
i_c	current through a capacitor
i_ℓ	current through an inductor
i_r	current through a resistor

J	rotational mass moment of inertia
K	potentiometer coefficient
K'	geometric correction factor for a Timoshenko beam
K_f	spring constant for a differential segment of an elastic foundation
K_0	equivalent spring constant for a levered system
K_0'	stiffness constant for a viscoelastic spring
K_t	torsional spring constant
k	analog scale factor; or, spring constant; or surface conductivity
L	an arbitrary inductor
$\overline{L}$	inductor for the generalized flexibility
$L_n(K_f)$	inductor for the elastic foundation of the nth difference segment
$L_n(v)$	inductor for the shear deformation of the nth difference segment
$L_n(\theta)$	inductor for the bending of the nth difference segment
L_0	equivalent impedance for a levered system
$L(s)$	shear inductor
L_θ	inductor for torsional flexibility
ℓ	length
M	moment
M_{ii}	bending moment per unit length of a section of plate denoting the bending perpendicular to the ith axis
M_{ij}	twisting moment per unit length of a section of plate denoting the twisting perpendicular to the ith direction

M_n	moment in the nth difference segment
m	mass
$\mathcal{M}$	symbol for mechanical mobility
N	time scale factor; or, number of transformer windings
N^*	notation for auto-transformer windings
n	nth difference segment in finite difference notation
$\bar{n}$	vector normal to a surface
$\rvert_n$	symbol referencing the properties for the nth difference segment
P	electrical power
P_n	primary transformer windings for the nth difference segment
P_θ	analog scale factor
P_ϕ	analog scale factor
ρ	density per unit volume
Q	sharpness of a frequency response curve at the natural frequency of the system
Q_i	shear force per unit length of a section of plate perpendicular to the ith axis
q	generalized coordinate; or, heat flow by conduction
$\bar{q}$	vector for heat flow
q'	heat flow from external sources

R	a resistor; or, radius in cylindrical coordinates
$\overline{R}$	resistor for the generalized damping
$R_n(\theta)$	resistor for the static bending of the nth difference segment
R_0	resistance or real part of a complex impedance
R_θ	resistor corresponding to torsional viscous damping
r	radius of a curved beam segment
$\bar{r}$	vector for an arbitrary spatial location
S	surface area of an arbitrary body
S_n	secondary transformer windings for the nth difference segment
s	the Laplace operator: or, the perimeter of a duct cross-section
T	torque; or, twisting moment; or, temperature of a body
t	time
t_e	electrical time
t_m	mechanical time corresponding to time t
u	longitudinal motion in the x direction
V	shear force; or, volume of a body
V_n	shear in the nth difference segment

v	symbol denoting volts; or, displacement in the radial direction
$\overline{v}$	fluid velocity
w	lateral deflection of a plate from static equilibrium
$\overline{w}$	fluid mass flow rate
X_0	reactance or imaginary part of a complex impedance
x	coordinate for the motion of a mechanical system described as a set of algebraic equations; or, the spatial dimension in the x direction
Y	an electrical admittance
${}_mY$	a mechanical admittance
${}_mY_0$	equivalent mechanical admittance
Y_c	electrical admittance of a capacitor
Y_ℓ	electrical admittance of an inductor
Y_0	equivalent admittance of an electrical network
Y_r	electrical admittance of a resistor
y	displacement from static equilibrium for a linear oscillator; or, the lateral displacement from static equilibrium for a one-dimensional structure; or, the spatial coordinate in the y direction
$\dot{y}$	velocity in the y direction
$\ddot{y}$	acceleration in the y direction
y_b	lateral deflection due to bending
y_s	lateral deflection due to shear deformation

Z	an electrical impedance
$_mZ$	a mechanical impedance
$_mZ_0$	equivalent mechanical impedance
Z_1^*	impedance transferred across the windings of a transformer
Z_c	electrical impedance of a capacitor
Z_ℓ	electrical impedance of an inductor
Z_0	equivalent impedance of an electrical network; or, feedback impedance of an operational amplifier
Z_r	electrical impedance of a resistor
z	spatial dimension in the z direction
$\Delta E(\ )$	symbol denoting the voltage drop across the quantity within the parentheses
$\Delta f(\pm 3db)$	frequency interval denoting the half power bandwidth
Δx	difference length measured in the x direction
ζ	damping ratio
θ	phase angle of an equivalent impedance; or, angular displacement from static equilibrium
$\ddot{\theta}$	angular acceleration
Φ	magnetic flux of a transformer; or, the phase angle for a viscoelastic spring
ω	angular excitation frequency in radians per second
ω_n	undamped natural frequency in radians per second
Ω	symbol denoting ohms

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$\Delta_i(\ )$	finite-difference symbol denoting the difference of ( ) with respect to the ith coordinate
∇	dell operator
$(\ddot{\ })$	second time derivative of the quantity within the parentheses

ELECTRICAL ANALOGIES AND THE VIBRATION OF LINEAR MECHANICAL SYSTEMS

INTRODUCTION

The use of electrical analogies to solve problems common to structures and structural dynamics is not a new concept. Two of the earliest papers dealing with these topics were written in 1933 and 1934 by Firestone (Reference 7) and Bush (Reference 5) respectively. By a concise and rather complete review of the literature, Higgins (Reference 10) provides a historical perspective of electroanalogic methods. Recent authors of mechanical vibration texts as Freberg (Chapter 10 of Reference 8) and Thomson (Chapter 8 of Reference 18) mention electrical analogies although such discussions are introductory in nature. More complete presentations of structural analogies are given by Barnoski (Reference 1) and MacNeal (Reference 13).

Although electrical analogies are discussed here in a general way, prime emphasis is given to passive analogs of elastic structural systems. As contrasted with active electrical analogs consisting of operational amplifier circuits, the passive analog is a circuit consisting of some combination of resistors, inductors, capacitors, and transformers. A direct or one-to-one correspondence exists between the components in the electrical network and the elements of the mechanical system. The exact relationships, however, depend completely on the definition of the analogy.

For dynamic simulation of mechanical structures, two types of passive circuits are defined: 1) force-current, velocity-voltage analog, and 2) force-voltage, velocity-current analog. Using the classic definition of an electrical impedance, the first analogy equates electrical impedance with mechanical mobility whereas the second analogy equates electrical impedance with mechanical impedance.

Thus, the force-current, velocity-voltage analog may be appropriately called a mobility circuit or mobility analog and the force-voltage, velocity-current analog becomes the mechanical impedance circuit or mechanical impedance analog. These two types of circuits are duals of one another and appear topologically distinct. In either case, the same basic circuit analyses and vibration techniques can be applied to efficiently analyze the electrical analogs for both damped and undamped mechanical systems.

In this discussion, the force-current, velocity-voltage passive analog or mobility circuit is emphasized. For this type of circuit, resistors correspond to viscous damping, the inductors to flexibility, the capacitors to mass, and transformers describe geometric relationships. For distributed structures, this analog corresponds mechanically to a lumped parameter model and corresponds mathematically to a finite-difference model. The analog impedances are equivalent to mechanical mobilities and the analog admittances (the reciprocals of the impedances) are equivalent to mechanical impedances.

This report is subdivided into five sections. The first section uses the single-degree-of-freedom system as a means to discuss and illustrate mobility, mechanical impedance, and operational amplifier circuits. In addition, frequency response functions, transformers and current generators are reviewed and placed in context of the electrical analog. The second section cites the use of mobility, mechanical impedance, transformer, and amplifier circuits to simulate a system described as sets of algebraic equations. In addition, mobility circuits are emphasized for two and three-degree-of-freedom systems and an analog equivalence of modal theory is shown.

Section three treats passive mobility analogs of distributed structural systems. Derivation techniques are emphasized although brief discussions of scale factors and boundary conditions are included. Difference analogs are shown for the longitudinal vibration of a rod; the lateral vibration of simple, Timoshenko and curved beams; a beam on an elastic foundation; a rectangular shear panel; and the lateral vibration of a rectangular plate. Section four discusses mobility oriented circuits which are useful for problems typically found in viscoelasticity, heat transfer and fluid flow. Section five presents summary remarks on potential applications of, as well as the distinction between, the analog circuits for structural vibrations and the physical systems typical of section four.

1. THE LINEAR OSCILLATOR

1.1 MOBILITY ANALOG (FORCE-CURRENT, VELOCITY-VOLTAGE ANALOG)

The derivation of passive analog circuits simulating linear mechanical oscillators is discussed in many introductory texts on mechanical vibrations such as in Chapters 9 and 10 of Reference 8 or Chapter 8 of Reference 18. However, this effort is repeated here for completeness of this discussion.

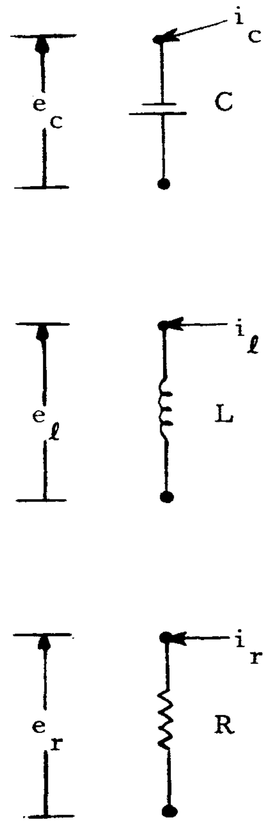
The equation of motion for a mass-excited linear oscillator is

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{1.1}$$

where m is the mass of the system, c the damping constant of the dashpot, k the spring constant, f(t) the external loading applied to the mass, y the displacement of the mass from static equilibrium, $\dot{y}$ the velocity of the mass, and $\ddot{y}$ the acceleration of the mass. Expressed as a function of velocity, the oscillator equation of motion becomes

$$m\frac{d\dot{y}}{dt} + c\dot{y} + k\int \dot{y}\, dt = f(t) \tag{1.2}$$

From basic circuit theory, the currents through a capacitor, inductor, and resistors are shown as Figure 1.1.



$$i_c = C \frac{de_c}{dt}$$

$$i_\ell = \frac{1}{L} \int e_\ell \, dt$$

$$i_r = \frac{e_r}{R}$$

Figure 1.1 Basic Relationships for Current Flow Through a Capacitor, Inductor, and Resistor

Positive currents are shown to flow into the node and from the higher potential to the lower potential where the arrow symbolism denotes a positive voltage drop across the circuit elements. The currents through the components are represented by i and the voltage drops across the components by e where the subscript c refers to a capacitor, ℓ to an inductor, and r to a resistor. The capacitor C has units of farads (fd), the inductor L in henries (h), and the resistor R in ohms (Ω). The physics of current flow associated with a capacitor, inductor and resistor is defined by equations noted as functions of time.

An impedance is a quantity defined as the ratio of the Laplace transforms of the voltage to the current. It is a function of frequency and appears mathematically as a real, imaginary, or complex number. For the components in Figure 1.1, the capacitive impedance Z_c, inductive impedance Z_ℓ, and resistive impedance Z_r appear as

$$Z_c = \frac{L(e_c)}{L(i_c)}$$

$$Z_\ell = \frac{L(e_\ell)}{L(i_\ell)} \qquad (1.3)$$

$$Z_r = \frac{L(e_r)}{L(i_r)}$$

where L( ) denotes the Laplace transform of the quantity ( ). The voltage e has units of volts and the current i has units of amps.

Ignoring initial conditions, the first time derivative and integration with respect to time are represented by Laplace notation as

$$s = \frac{d}{dt}$$

$$\frac{1}{s} = \int dt \qquad (1.4)$$

The capacitive, inductive, and resistive impedances given in Eq. (1.3) become

$$Z_c = \frac{1}{Cs}$$

$$Z_\ell = sL \tag{1.5}$$

$$Z_r = R$$

where the operator s equals $i\omega$ for harmonic steady state assumptions.

By assuming velocity proportional to voltage and force proportional to current and noting the form of the current expressions in Figure 1.1, the terms of Eq. (1.2) are expressed in circuit nomenclature as

$$C\frac{de}{dt} + Re + \frac{1}{L}\int e\,dt = I(t) \tag{1.6}$$

where e is an arbitrary nodal voltage referenced to ground and I(t) corresponds to the external force f(t). Comparing the terms of the above equation with (1.2), capacitance corresponds to mass, inductance to flexibility (the reciprocal of the stiffness), and the resistance to the reciprocal of viscous damping. In terms of mechanical quantities, the analog impedances are

$$Z_c = \frac{1}{sm} = \frac{1}{i\omega m}$$

$$Z_\ell = \frac{s}{k} = \frac{i\omega}{k} \tag{1.7}$$

$$Z_r = \frac{1}{c}$$

Electrically, Eq. (1.6) denotes an algebraic sum of four currents acting at a node which can be easily satisfied by applying Kirchhoff's current law. This law states the summation of currents acting at a node equals zero.

Mechanically, Kirchhoff's current law is noted as equivalent to force equilibrium. In funicular form, Eq. (1.7) is represented as Figure 1.2.

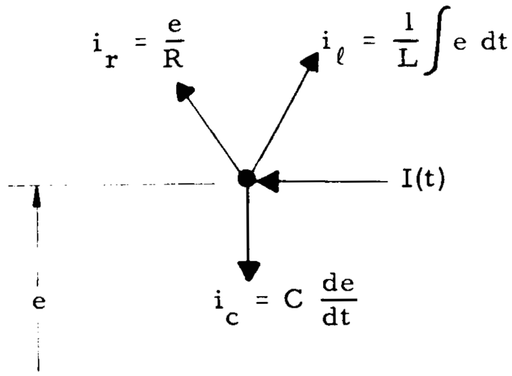
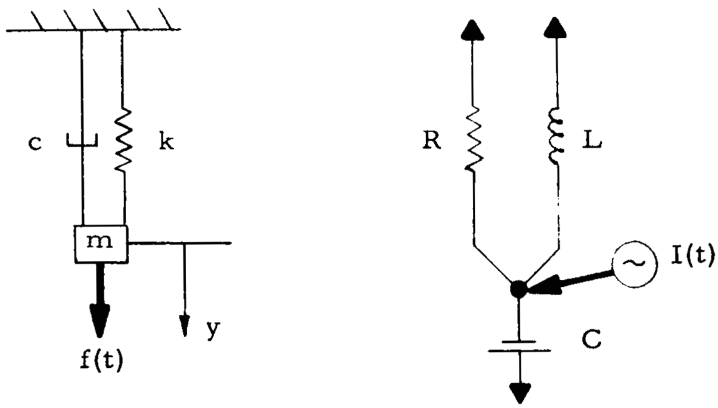


Figure 1.2 Funicular Current Diagram for Mechanical Oscillator

From the relationships provided in Figure 1.1, these currents are simulated by an RLC network where the resistor R, inductor L, and capacitor C are connected in parallel as illustrated in Figure 1.3.



Mechanical Oscillator Analog Circuit

Figure 1.3 Sketch of the Mechanical Oscillator and the Equivalent Mobility Analog Circuit

The topological similarity between the mechanical diagram and the analog circuit is a characteristic typical of this type of analog and readily promotes an intuitive understanding of the electrical circuits. The applied load f(t) is represented as a current generator whose current flow is directed into the node. The fixed boundary conditions at the base of the oscillator are represented as grounded terminals on both the resistor and inductor. The inertial force of the mass is shown as a capacitor referenced to ground. Since Newton's second law of motion requires the inertial force be referenced to an absolute or inertial frame of reference, electrical ground thus corresponds to an inertial reference frame.

1.2 MECHANICAL IMPEDANCE ANALOG (FORCE-VOLTAGE, VELOCITY-CURRENT ANALOG)

Consider the development of the force-voltage, velocity-current or the mechanical impedance analog of the mechanical oscillator. For the circuit elements in Figure 1.1, the voltage drops across the capacitor, inductor, and resistor are

$$e_c = \frac{1}{C}\int i_c \, dt$$

$$e_\ell = L\frac{di_\ell}{dt} \tag{1.8}$$

$$e_r = Ri_r$$

Assuming the current proportional to velocity and the voltage proportional to force, the oscillator equation of motion appears in the form

$$m\frac{di}{dt} + ci + k\int i \, dt = E(t) \tag{1.9}$$

Comparing the terms of the above equation with the voltage expressions of (1.8), mass is noted as an inductance, the dashpot as a resistor, and the flexibility as a capacitor.

Electrically, Eq. (1.9) is an expression of Kirchhoff's voltage law; that is, the sum of the potential drops (or rises) around a closed loop is zero. Applying Kirchhoff's voltage law to Eq. (1.9) results in the circuit shown as Figure 1.4

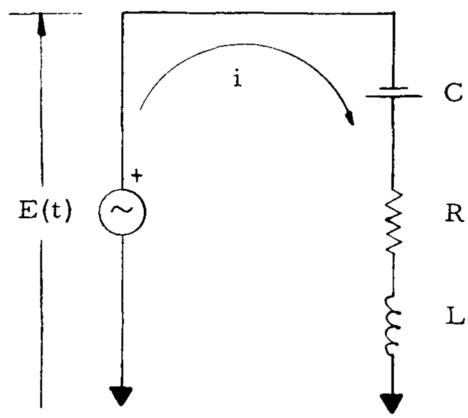


Figure 1.4 Mechanical Impedance Passive Analog Circuit for a Mechanical Oscillator

where the voltage drops across the circuit elements are equivalent to the terms in (1.9). The applied loading is shown as a voltage generator referenced to ground. As contrasted with the parallel arrangement of circuit elements in the mobility analog, the circuit elements are connected in series.

1.3 IMPEDANCE PROPERTIES, MOBILITY AND FREQUENCY RESPONSE FUNCTIONS

In general, analog simulations using mechanical impedance (force-voltage, velocity-current assumptions are associated with concepts common to loop circuits whereas simulations using mobility (force-current, velocity-voltage) assumptions are associated with concepts common to nodal circuits. In appearance, mobility circuits are distinct from mechanical impedance circuits but are related electrically as dual circuits of one another. Irrespective of the particular analog circuit used, the same circuit analysis procedures are applicable. For interpreting the mechanical equivalent of the analog results, however, the analog must be explicitly defined.

In this text, the word impedance implies an electrical impedance of an analog circuit defined as the ratio of the Laplace transforms of the voltage to current both referenced to the same datum at the same instant of time. Any other usage will be specifically noted. If the circuit is a force-current, velocity-voltage analogy, the impedance is mechanically equivalent to mobility and the circuit is called a mobility analog or mobility circuit. If the circuit is a force-voltage, velocity-current analogy, the impedance is mechanically equivalent to mechanical impedance and the circuit is called a mechanical impedance analog or mechanical impedance circuit.

The relationships between the properties of a mechanical system, the mobility analog, and the mechanical impedance analog are summarized in tabular form as Figure 1.5. For example, mechanical mobility (or simply mobility) is equivalent to an impedance in a mobility analog but equivalent to an admittance in a mechanical impedance analog. Thus, mobility and mechanical impedance concepts are reciprocals of one another.

Focusing attention on the mobility analogy, mobility rules are analogous with impedance laws and mechanical impedance rules are analogous with admittance laws. Circuit techniques for adding impedances or admittances in series, in parallel, or in series-parallel combination find direct application. For impedance in series, the resultant impedance is the sum of the individual impedances. For impedances in parallel, the resultant impedance is the reciprocal of the sum of the individual admittances. For a series-parallel

Mechanical System		Mobility or Force-current Velocity-voltage Analog		Mechanical Impedance or Force-voltage Velocity-current Analog	
Mass	m	Capacitance	C	Inductance	L
Viscous Damping	c	Conductance	G(I/R)	Resistance	R
Flexibility	$\frac{1}{k}$	Inductance	L	Capacitance	C
Mobility	$\mathcal{M}$	Impedance	Z	Admittance	Y
Mechanical Impedance	Z_m	Admittance	Y	Impedance	Z

Figure 1.5 Basic Relationships Between Mechanical Systems and Their Equivalent Analog

combination of impedances, the original analog circuit usually can be reduced so that the preceding two rules can be applied to calculate a resultant impedance.

Considering the mobility analog strictly in the sense of an electrical network, the oscillator circuit can be reduced to an equivalent impedance Z_0 to ground as shown in Figure 1.6.

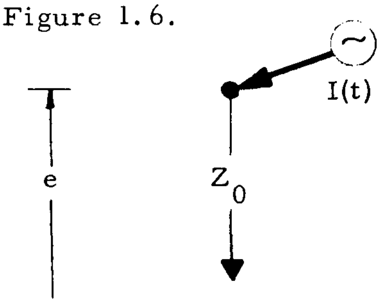


Figure 1.6 Reduced Equivalent Circuit for the Mechanical Oscillator

The equivalent impedance Z_0 is the resultant impedance of the resistor, inductor, and the capacitor connected in parallel. From fundamental circuit theory, the resultant impedance for the RLC network of Figure 1.3 is given as

$$\frac{1}{Z_0} = \frac{1}{Z_c} + \frac{1}{Z_r} + \frac{1}{Z_\ell} \qquad (1.10)$$

where Z_c is the capacitive impedance, Z_r the resistive impedance, and Z_ℓ the inductive impedance. Substituting the impedance relationships given by (1.5) into Eq. (1.10) produces for harmonic motion

$$Z_0 = \frac{1}{\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)} \qquad (1.11)$$

Inserting the mechanical equivalents of the resistor, capacitor, and inductor into the above provides

$$Z_0 = \frac{\frac{1}{c}}{1 + \frac{m}{c} s \left[1 + \frac{k}{ms^2}\right]} = \frac{i\frac{\omega}{\omega_n}}{m\omega_n\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}\right]} \qquad (1.12)$$

where the undamped natural frequency ω_n and damping ratio ζ are related to the oscillator parameters as

$$\omega_n^2 = \frac{k}{m} \quad \text{and} \quad 2\zeta\omega_n = \frac{c}{m} \qquad (1.13)$$

Electrically, the equivalent or resultant impedance defined by Eq. (1.10) may be categorically noted as

$$Z_0 = R_0 + iX_0 \tag{1.14}$$

where R_0 is the resistance of the circuit in ohms and reflects the mechanical damping, while X_0 is the reactance of the circuit in ohms and corresponds to the undamped dynamic properties. In terms of a magnitude and phase angle, the equivalent impedance may be expressed as

$$Z_0 = \left|H_0(\omega)\right| \angle \theta \tag{1.15}$$

where the absolute magnitude is

$$\left|H_0(\omega)\right| = \sqrt{R_0^2 + X_0^2} \tag{1.16}$$

and the associated phase angle (defined as the angle by which the voltage leads the current) appears as

$$\theta = \tan^{-1} \frac{X_0}{R_0} \tag{1.17}$$

The reciprocal of an impedance is an admittance and likewise consists of a real and an imaginary part written as

$$Y_0 = G_0 + iB_0 \tag{1.18}$$

where G_0 is the real component of the admittance noted as the conductance and B_0 is the imaginary component of the admittance noted as the susceptance. The admittance has units of mhos. The relationships between the impedances and admittances of the mobility analog with the mechanical impedance and mechanical admittance are

$$Z_0 = \frac{1}{Y_0} = \frac{1}{{}_mZ_0} = {}_mY_0 \tag{1.19}$$

where ${}_mZ_0$ is the mechanical impedance and ${}_mY_0$ is the mechanical admittance. The mechanical admittance is the ratio of force to velocity and is noted as the reciprocal of the mechanical impedance.

It is sometimes mathematically convenient to use admittances rather than impedances in network analysis. For example, an alternate way to compute the equivalent impedance Z_0 for the oscillator is to add the admittances of the inductor, capacitor, and resistor, then take the reciprocal of the resultant sum. From Eq. (1.11), the equivalent admittance Y_0 of the oscillator is

$$Y_0 = \frac{1}{R} + j(\omega C - \frac{1}{\omega L}) \tag{1.20}$$

and, upon substituting the mechanical equivalents of the resistor, capacitor, and inductor, the admittance Y_0 becomes

$$Y_0 = c + j(m\omega - \frac{k}{\omega}) \tag{1.21}$$

Plotted on log-log paper as curves of magnitude versus the excitation frequency, the mobility and mechanical impedance magnitudes for linear oscillator appear as Figures 1.7 and, 1.8. Also shown are the individual

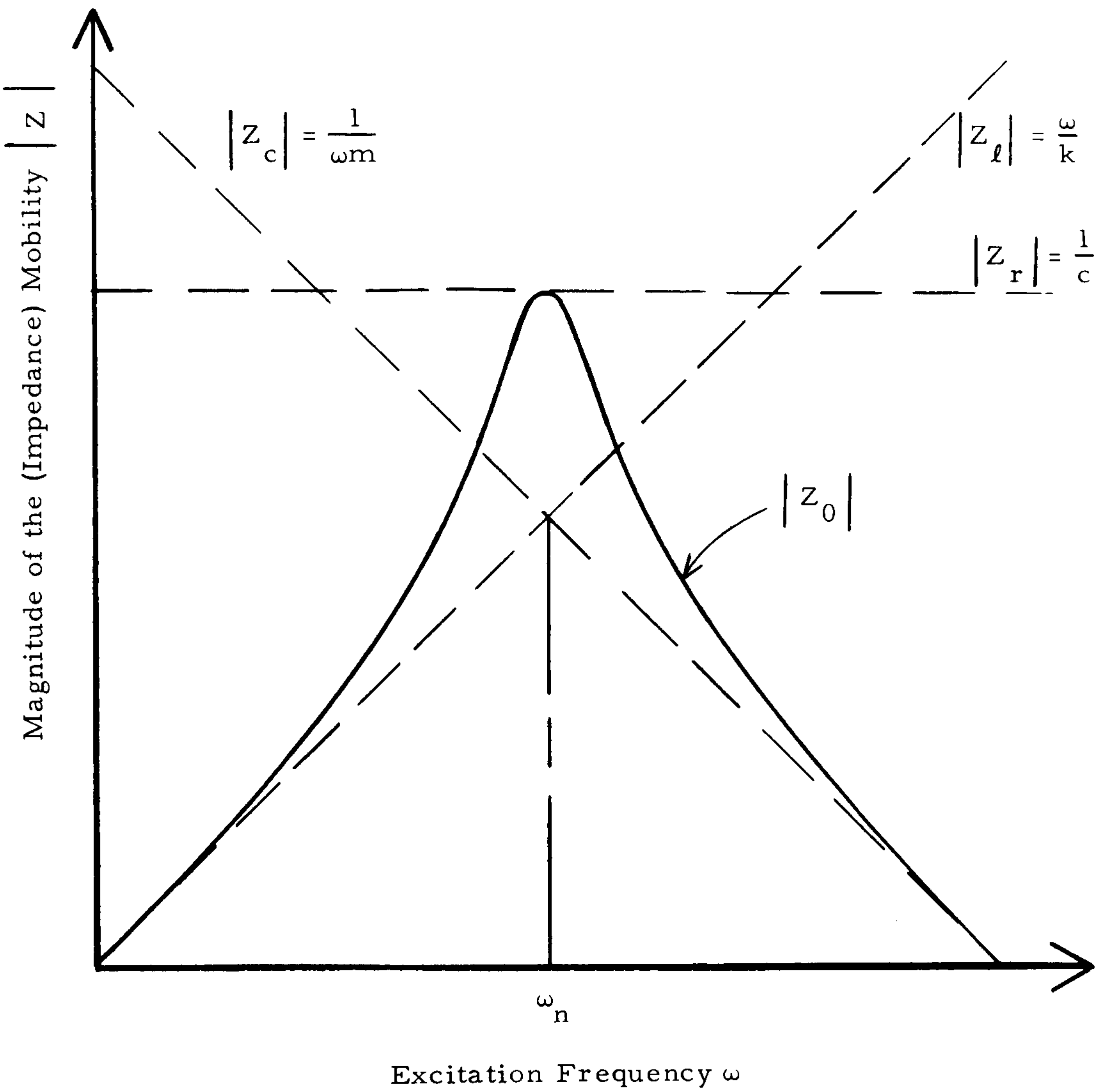


Figure 1.7 Plots of the Magnitudes of (Impedance) Mobility Functions

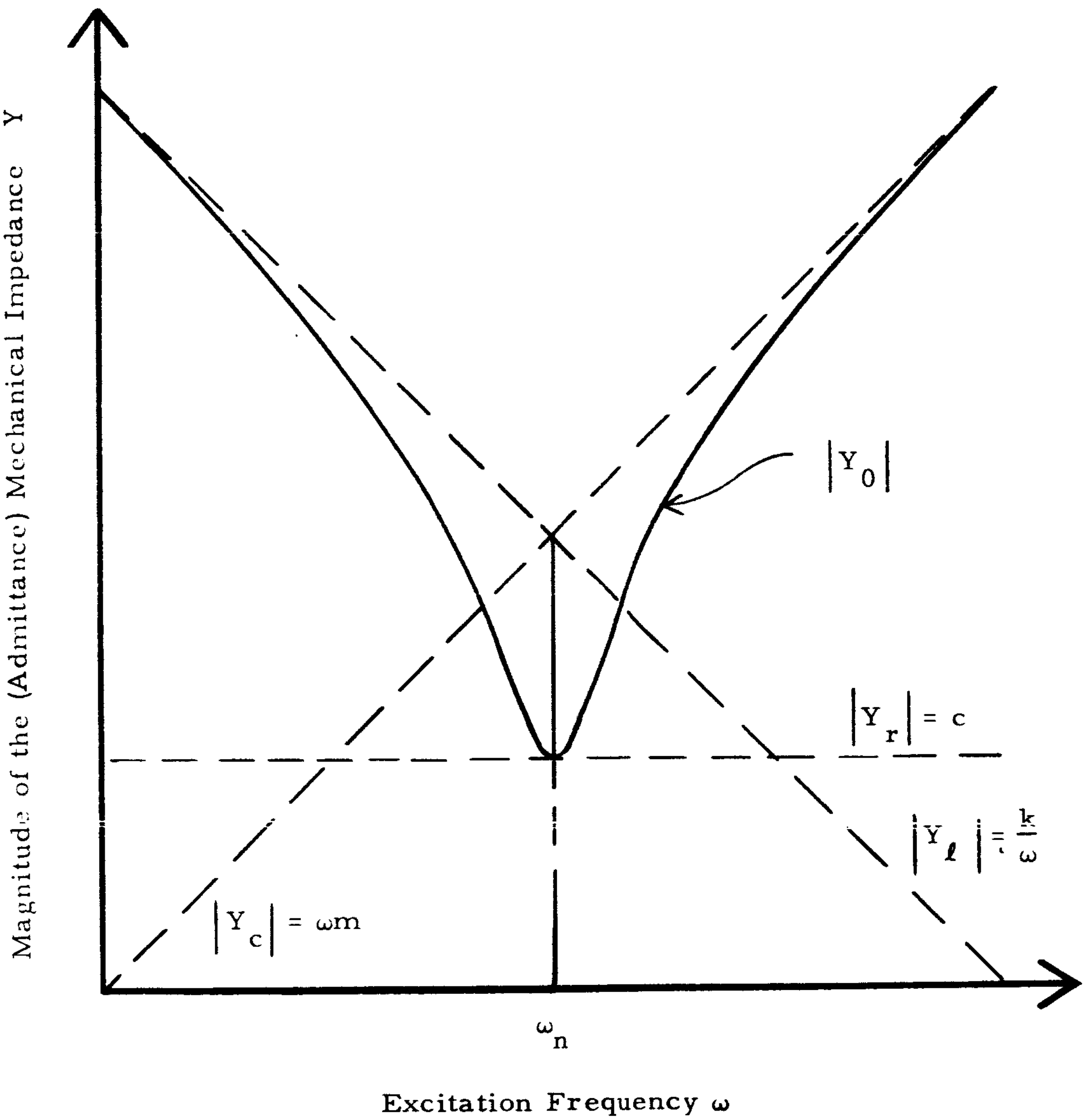


Figure 1.8 Plots of the Magnitudes of (Admittance) Mechanical Impedance Functions

mechanical impedances for the oscillator mass, flexibility, and damping. Since the mobility analog is used, mobility is equivalent to an impedance in the analog circuit whereas mechanical impedance is equivalent to an admittance in the analog circuit. As expressed by Eq. (1.19) the curves in Figures 1.7 and 1.8 are noted as reciprocals of one another.

The plots for the individual components appear as lines with constant slopes. The capacitive impedance plot corresponds to a constant mass line whereas the inductive impedance plot corresponds to a constant stiffness line. The intersection of the capacitive and inductive impedance lines yields the undamped natural frequency ω_n of the oscillator. The response $|Z_0|$ is noted predominantly as a spring at frequencies somewhat below ω_n and as a mass at frequencies $>> \omega_n$. The resistive impedance is independent of the frequency with a magnitude $\frac{1}{c}$ equal to the maximum value of Z_0. Mechanically, this maximum Z_0 value is interpreted as the maximum velocity response of an oscillator excited at the mass by a sinusoidal force, and occurs when the excitation frequency equals the undamped natural frequency.

For a linear system, the ratio of the steady state output response to a simple harmonic input excitation is a frequency response function. This is consistent with the definition of an electrical impedance where $s = i\omega$ for steady state assumptions. Thus, for any arbitrary linear system, many such functions exist and depend only upon the units of both the output response and input excitation. For the mobility analog, the impedance is one such quantity and appears as a complex valued function of frequency in the form given by Eq. (1.15). The absolute magnitude $|H_0(\omega)|$ is commonly called the gain factor and the associated phase angle θ is the phase factor. Expressed in terms of the oscillator parameters, the magnitude and phase angle of the velocity to force frequency response function Z_0 are shown as Eqs. (1.22) and (1.23).

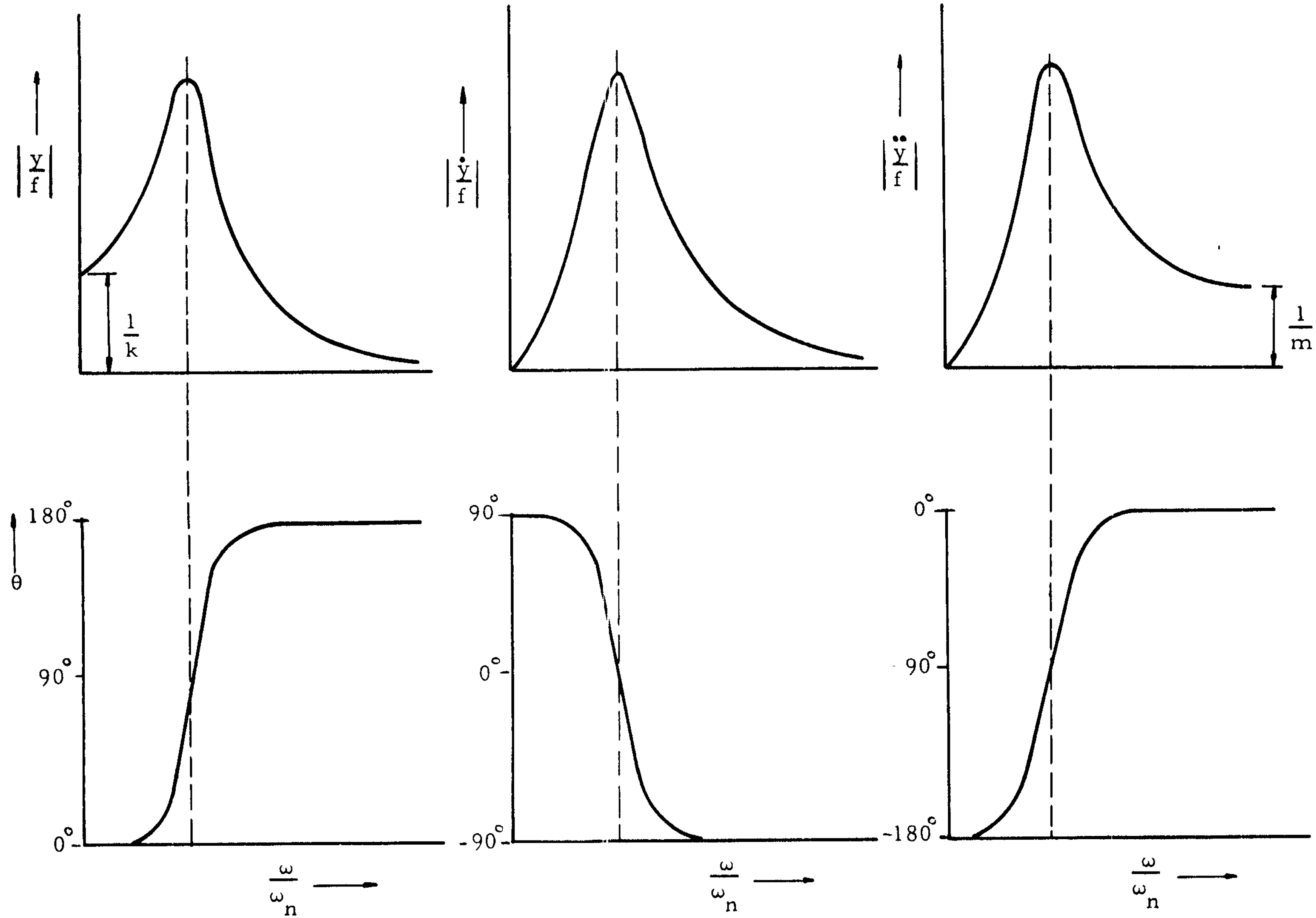


Figure 1. 9 Sketches of Displacement, Velocity, and Acceleration Frequency Response Functions for a Lightly Damped Mechanical Oscillator

$$\left| \frac{\dot{y}}{f} \right| = \left| H_0(\omega) \right| = \frac{1}{m\omega_n} \quad \frac{\frac{\omega}{\omega_n}}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2} \tag{1.22}$$

$$\theta = \tan^{-1} \frac{1 - \left(\frac{\omega}{\omega_n}\right)^2}{2\zeta \frac{\omega}{\omega_n}} \tag{1.23}$$

The $|Z_0|$ plot in Figure 1.7 is a log-log plot of Eq. (1.22) whereas the $|Y_0|$ plot in Figure 1.8 is a log-log plot of the reciprocal of Eq. (1.22).

For steady state conditions, dividing Z_0 by the Laplace operator ($s = i\omega$) yields the displacement to force frequency response function and multiplying Z_0 by s yields the acceleration to force frequency response function. The magnitudes of these response functions in addition to the magnitude of the velocity to force frequency response function $|H_0(\omega)|$ are sketched as Figure 1.9. The top row of curves display the magnitudes of the response functions versus the frequency ratio ω/ω_n and the lower row of curves show the phase angle variation with the frequency ratio ω/ω_n. At the undamped natural frequency ($\omega/\omega_n = 1$), the phase angles are noted as 90 degrees and the magnitudes are approximately maxima for $\left|\frac{y}{f}\right|$ and $\left|\frac{\ddot{y}}{f}\right|$, and is a maximum for $\left|\frac{\dot{y}}{f}\right|$. A more complete discussion of frequency response function is given by Piersol in Section 7 of Reference 15.

The sharpness of the frequency response curves at the natural frequency of the system is usually described by the Q of the system defined as

$$Q = \frac{f_n}{\Delta f\,(\pm 3\text{db})} \tag{1.24}$$

where f_n is the undamped natural frequency of the oscillator or the center frequency (approximately) of the response function and $\Delta f(\pm 3$ db) is the half-power bandwidth. This bandwidth is the frequency interval 3 db down from the response magnitude at the center frequency f_n. The 3 db points correspond to amplitude values which are 0.707 times the magnitude at the center frequency.

1.4 TORSIONAL SYSTEM

This system may be considered as a torsional oscillator and consists of a disk attached to a rigidly mounted shaft as shown in Figure 1.10.

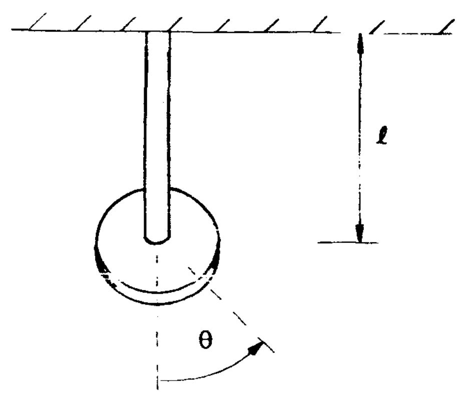


Figure 1.10 Disk-shaft Mechanical System

The equation of motion for this system is

$$J \frac{d^2\theta}{dt^2} + k_t\theta = 0 \tag{1.25}$$

where J is the mass moment of inertia about the axis of rotation, k_t the torsional stiffness (spring constant) of the shaft, θ the angular displacement of the disk from static equilibrium, and t the time. Note that the form of this equation is identical to an undamped linear oscillator. Consequently, the analog circuit for the torsional oscillator appears identical to the analog circuit for an undamped linear oscillator as sketched in Figure 1.11.

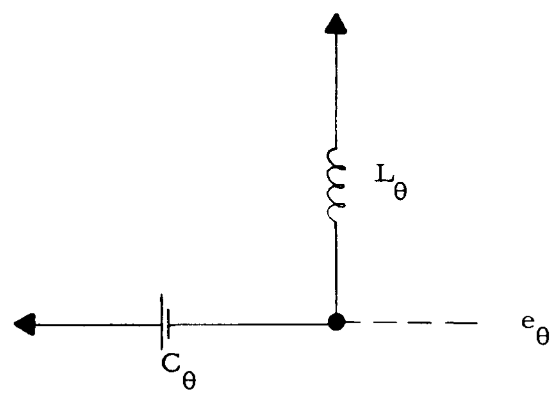


Figure 1.11 Passive Analog Circuit for the Disk-shaft Mechanical System

This simple LC circuit is the mobility (force-current, velocity-voltage) analog for the torsional oscillator. The inductor L_θ corresponds to the torsional flexibility $1/k_t$, the capacitor C_θ to the polar mass moment of inertia of the disk J, the voltage e_θ to the angular velocity, the current through the capacitor as the inertial force, and the current through the inductor as the shaft torque. In terms of impedances, the torsional circuit elements are related as

$$Z_\ell(\theta) = i\omega L_\theta = \frac{i\omega}{k_t}$$

$$Z_c(\theta) = \frac{1}{i\omega C_\theta} = -\frac{i}{\omega J} \qquad (1.26)$$

Damping, if included, is noted as a resistor in parallel with the inductor of magnitude

$$Z_r(\theta) = R_\theta = \frac{1}{c_t}$$

where c_t is the viscous damping coefficient for torsional vibration.

1.5 TRANSFORMERS AND LEVERED SYSTEMS

Typical problems in mechanical vibrations often include systems with rigid, weightless rods or levers. These levers are represented electrically by ideal transformers.

Symbolically, an ideal multi-winding transformer may be sketched as shown in Figure 1.12.

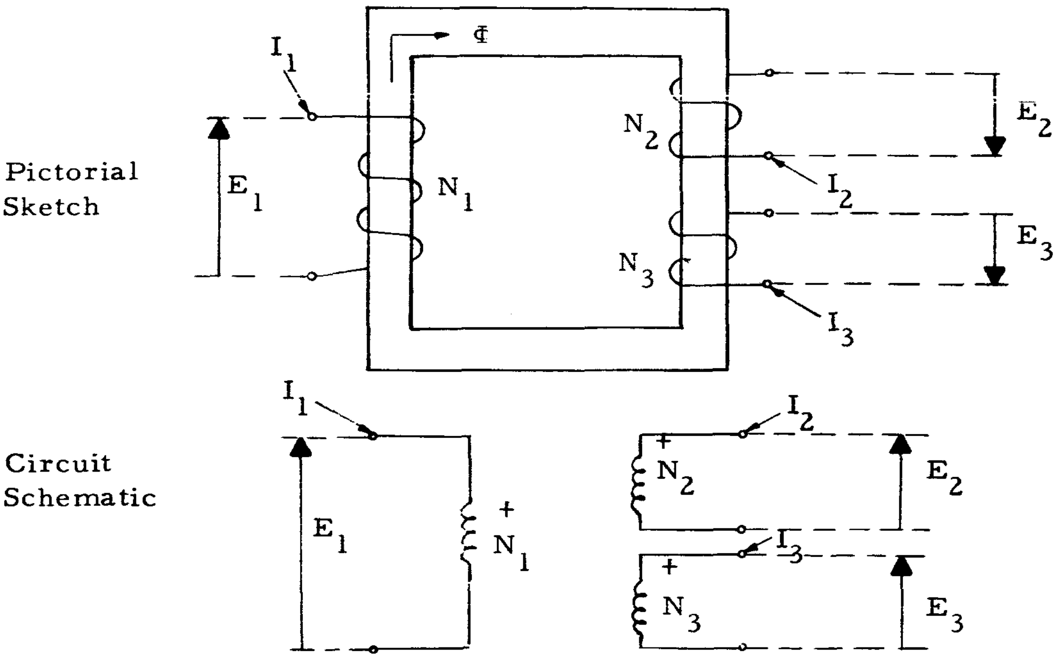


Figure 1.12 Ideal Three-winding Transformer

The pictorial sketch depicts the transformer as discrete windings about a ferromagnetic core where ϕ is the magnetic flux in the core. The direction of the windings about the core dictates the polarity of the transformer and the assumed positive directions of current flow and voltage drop across the individual windings are shown. The transformer ideally is a non-dissipative magnetic circuit where the voltages, currents, and number of turns in the windings are related as

$$\sum_{j=1}^{n} N_j I_j = 0 \qquad (1.27)$$

$$\frac{E_j}{N_j} = \frac{E_{j+1}}{N_{j+1}} \qquad j = 1, 2, 3 \ldots n \qquad (1.28)$$

Eq. (1.27) is the law of Biot and Savart and interrelates the number of turns and current in each individual winding. Eq. (1.28) is Farraday's law of electro-magnetic induction and interrelates the voltage and number of turns in each individual winding. Since an ideal transformer consumes no power, the product of the voltages and currents for all windings is

$$\sum_{j=1}^{n} P_j = \sum_{j=1}^{n} E_j I_j = 0 \qquad (1.29)$$

Applying the current and voltage laws to the three-winding transformer yields

$$I_1N_1 + I_2N_2 + I_3N_3 = 0 \tag{1.30}$$

$$\frac{E_1}{N_1} = \frac{E_2}{N_2} = \frac{E_3}{N_3} \tag{1.31}$$

Substituting (1.31) into (1.30) provides

$$E_1I_1 + E_2I_2 + E_3I_3 = 0 \tag{1.32}$$

For certain applications, it is convenient to convert a conventional two-winding transformer circuit into an equivalent auto-transformer. Schematically, two such circuits are shown as Figure 1.13.

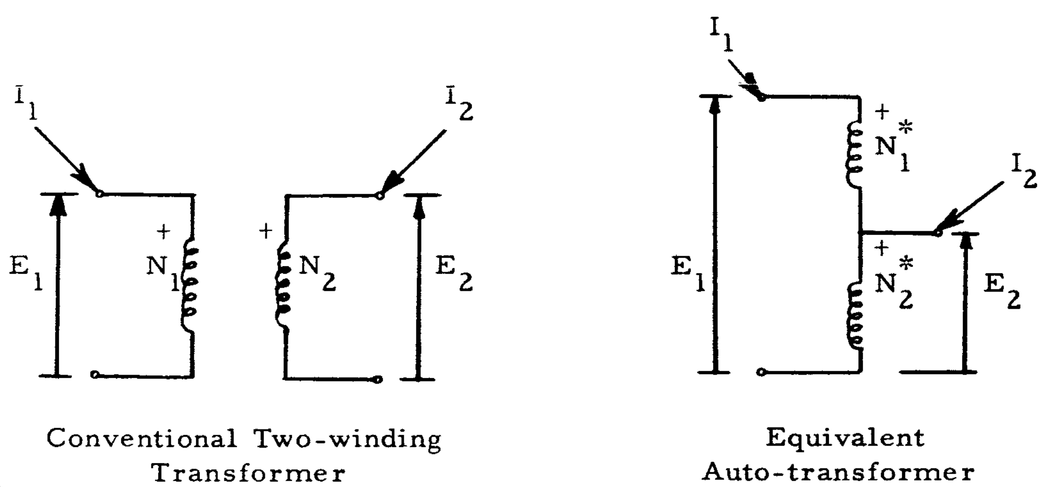


Figure 1.13 Conventional Two-winding Transformer and the Equivalent Auto-transformer

Applied to the conventional two-winding transformer, the current and voltage laws produce

$$I_1 N_1 + I_2 N_2 = 0$$

$$\frac{E_1}{N_1} = \frac{E_2}{N_2} \tag{1.33}$$

For the auto-transformer, the current and voltage laws yield

$$\frac{E_2}{N_2^*} = \frac{E_1}{N_1^* + N_2^*}$$

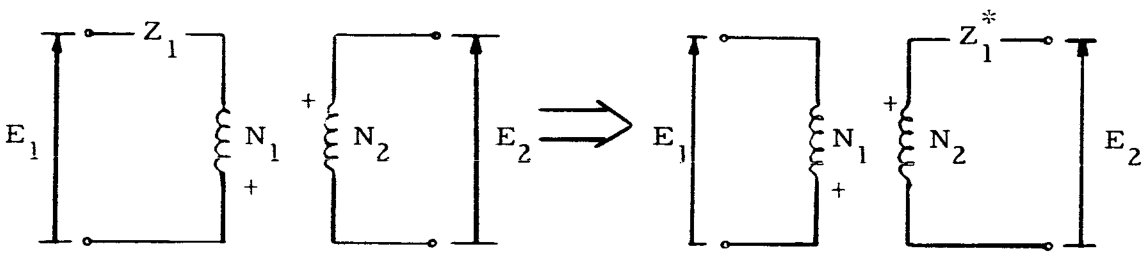
$$I_1 N_1^* + (I_1 + I_2) N_2^* = 0 \tag{1.34}$$

From these equations, the relationships between the turns for the two-winding and auto-transformers are

$$N_1^* = N_1 - N_2$$

$$N_2^* = N_2 \tag{1.35}$$

It is common in transformer usage to transfer a series impedance across the transformer windings. Schematically, this is depicted by Figure 1.14.



Original Circuit

Equivalent Circuit with the Transferred Impedance

Figure 1.14 Impedance Transfer Across a Transformer

Using the transformer voltage laws, the voltages across each of the impedances are

$$\Delta E(Z_1) = E_1 + \frac{N_1}{N_2} E_2$$
$$\Delta E(Z_1^*) = E_2 + \frac{N_2}{N_1} E_1 \tag{1.36}$$

where $\Delta E(\ )$ denotes the voltage difference across the terminals of the element specified within the parentheses. Requiring the power dissipation to be the same in each circuit provides

$$P = \frac{E^2}{Z} = \frac{\left[\Delta E(Z_1)\right]^2}{Z_1} = \frac{\left[\Delta E(Z_1^*)\right]^2}{Z_1^*} \tag{1.37}$$

Substituting the voltage difference equations of (1.36) into the above yields

$$Z_1^* = \left(\frac{N_2}{N_1} \right)^2 Z_1 \tag{1.38}$$

Thus, the transferred impedance Z_1^* is related to the original impedance Z_1 by the square of the turns ratio.

Consider the spring-mass-lever mechanical system depicted as Figure 1.15.

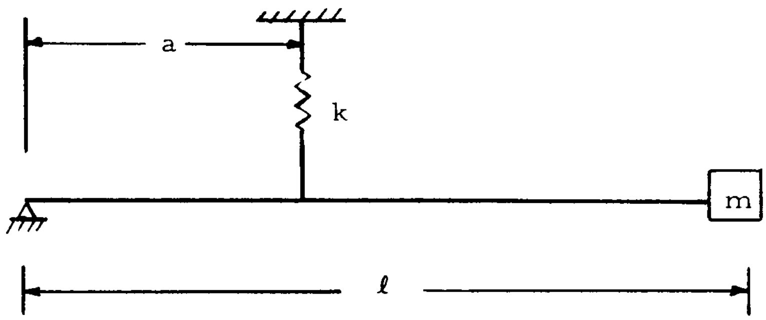


Figure 1.15 Spring-mass-lever Single Degree-of-freedom System

The rod (lever) is assumed rigid, weightless, and of length ℓ. The system is hinged at one end, free at the other end with an attached mass m, and has a restoring torque due to the elastic spring k located at position a.

Assuming small displacements, the lateral deflections of the mass y_m and spring constant y_k are related as

$$\frac{y_m}{\ell} = \frac{y_k}{a} = \theta \tag{1.39}$$

where θ denotes the angular rotation of the rod. Summing torques about the hinged support yields the equation of motion as

$$m\ddot{\theta} + k\left(\frac{a}{\ell}\right)^2 \theta = 0 \qquad (1.40)$$

or in reduced form

$$m\ddot{\theta} + k_0\theta = 0 \qquad (1.41)$$

where k_0 is the equivalent spring constant equal to $k\left(\frac{a}{\ell}\right)^2$.

The mobility circuit or force-current, velocity-voltage analog for the spring-mass-lever system is shown as Figure 1.16.

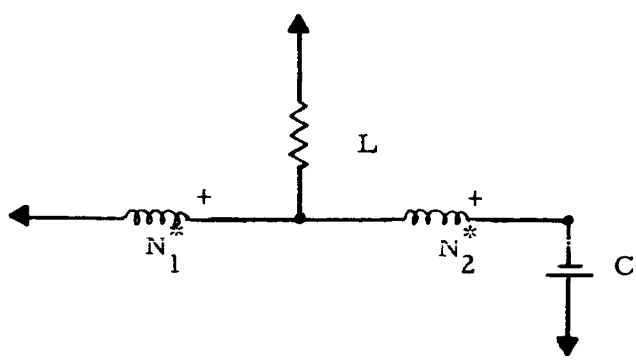


Figure 1.16 Passive Analog for the Spring-mass-lever System of Figure 1.15

The rigid rod is shown as an auto-transformer with $N_1^* = a$ and $N_1^* + N_2^* = \ell$, the spring as the inductor L, and the mass as the capacitor C. The inductor is represented symbolically as a resistor to eliminate confusion between transformer windings and inductors. Although this departure from conventional symbolism provides no particular advantage for this problem, the advantages become apparent when interpreting LC Transformer circuits for more complicated structures such as a beam.

The hinged end condition is simulated by grounding the circuit at the negative end of the N_1^* winding whereas the free end is shown as a node with an attached capacitor. The voltage across the capacitor denotes the angular velocity at the free end of the rod and the voltage across the inductor denotes the angular velocity of the rod at position a. The relationship between the velocity at the end of the rod and the velocity at position a is

$$\frac{\dot{y}_m}{\ell} = \frac{\dot{y}_k}{a} = \dot{\theta} \tag{1.42}$$

which is obtained by taking the first time derivative of (1.39). With voltage proportional to velocity, the above expression is identical in form with the voltage relationships for the auto-transformer given by (1.34).

By making use of the impedance transfer procedure depicted in Figure 1.14, the auto-transformer can be eliminated yielding the L_0C analog circuit of Figure 1.17.

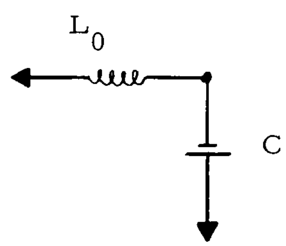


Figure 1.17 Equivalent Mobility Circuit for Figure 1.16

From the impedance transfer expression given as Eq. (1.38), the inductors in the circuits of Figure 1.16 and 1.17 are related as

$$L_0 = \left(\frac{N_1^* + N_2^*}{N_1^*}\right)^2 L = \left(\frac{\ell}{a}\right)^2 L \tag{1.43}$$

Since stiffness is inversely proportional to inductance, Eq. (1.43) yields the relationship between the equivalent spring constant k_0 and the original spring constant k as

$$k_0 = (\frac{a}{\ell})^2 k \quad (1.44)$$

This equivalent spring is noted to be the same as that given by Eq. (1.40).

1.6 ACTIVE ELECTRICAL ANALOGIES

In contrast to modeling an elastic structure using passive components, such a system can be simulated using a differential analyzer. The analyzer is basically an assemblage of passive components and active electrical elements such as multipliers, function generators, and operational amplifiers which are interconnected to reproduce the equations that describe the physical system. These active circuits are specifically intended to analyze sets of ordinary differential equations where time usually is the independent variable and find wide application in control system studies and problems of purely mathematical origin. Other topic areas generally associated with active circuits are nonlinear differential equations, simultaneous solutions to sets of linear or nonlinear algebraic equations and transcendental equations in one variable. Although iterative methods are available to deal with certain types of partial differential equations, these are more easily treated by passive analogs.

The operational amplifier is extremely important to active circuit simulation. Coupled with feedback circuits, this device is used typically to create summing amplifiers, summing integrators, and inverters. Basically, an amplifier is an electronic device where the output-input voltage is related by a negative constant factor as shown in Figure 1.18.

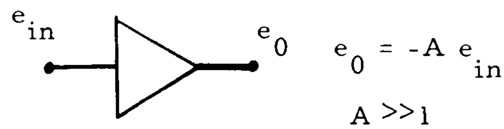


Figure 1.18 Basic Operational Amplifier

The coefficient A is the gain factor and has a magnitude typically ranging from 10^4 to 10^8. A typical feedback circuit is sketched as Figure 1.19

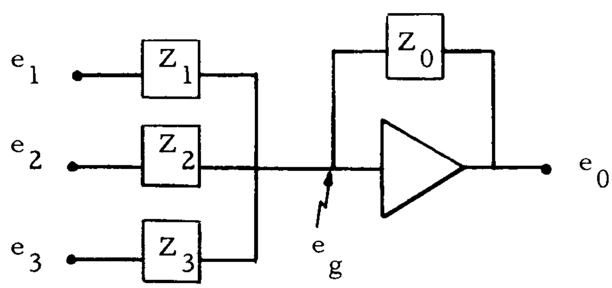


Figure 1.19 General Feedback Circuit

where the input impedances are noted as Z_1, Z_2, and Z_3; the input voltages by e_1, e_2, and e_3; the amplifier input voltage by e_g; the feedback impedance by Z_0; and the output voltage of the amplifier by e_0. It is desired to calculate the relationships between the output voltage and the input voltages assuming no current flow through the amplifier. This assumption corresponds to an infinite impedance for the basic amplifier and guarantees current flow through the feedback impedance. Summing currents at e_g according to Kirchhoff's current law yields

$$\frac{e_1 - e_g}{Z_1} + \frac{e_2 - e_g}{Z_2} + \frac{e_3 - e_g}{Z_3} + \frac{e_0 - e_g}{Z_0} = 0 \qquad (1.45)$$

Noting the amplifier input voltage is approximately zero as $e_g = -\frac{e_0}{A} \simeq 0$, the above equation reduces to

$$-e_0 = \frac{Z_0}{Z_1} e_1 + \frac{Z_0}{Z_2} e_2 + \frac{Z_0}{Z_3} e_3 \tag{1.46}$$

If the same type of components are used for the feedback and input impedances, the impedance ratios reduce to constants and Figure 1.19 appears as a summer circuit. This often is done using precision resistors as these are usually much less expensive than either precision capacitors or inductors. If the feedback and input components are different, the impedance ratios contain the Laplace operators s or 1/s and Figure 1.19 becomes a summer circuit containing differentiators and/or integrators. As an example, consider the feedback circuit shown as Figure 1.20.

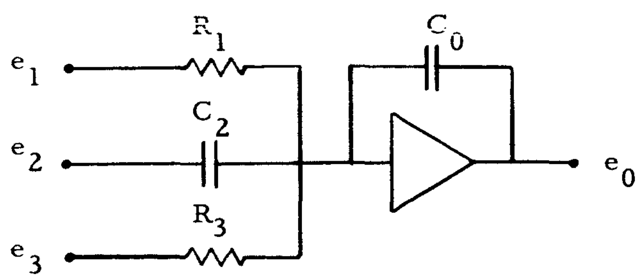


Figure 1.20 Typical Operational Amplifier-feedback Circuit

Substituting the capacitive and resistive impedances of (1.5) into Eq. (1.46) yields

$$-e_0 = \frac{1}{R_1 C_0} \frac{e_1}{s} + \frac{C_2}{C_0} e_2 + \frac{1}{R_3 C_0} \frac{e_3}{s} \tag{1.47}$$

As a function of time, this equation may be expressed as

$$-e_0 = \frac{1}{R_1 C_0} \int e_1 \, dt + \frac{C_2}{C_0} e_2 + \frac{1}{R_3 C_0} \int e_3 \, dt \qquad (1.48)$$

Thus, as shown by Eq. (1.48), this feedback circuit is a combination summer-integrator where the passive components form constant coefficients.

By connecting operational amplifier and feedback circuits in tandem, the linear oscillator is represented by simulating the velocity to force frequency response function. Consider the following network shown as Figure 1.21

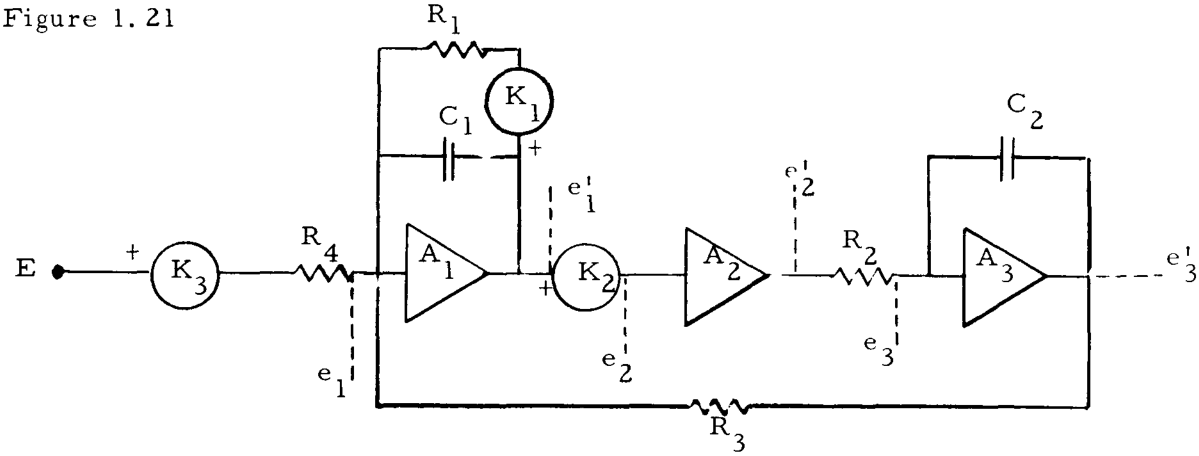


Figure 1.21 Active Circuit for a Linear Oscillator

where K_1, K_2, and K_3 denote variable potentiometers. The input-output relationships for the operational amplifiers are

$$e'_1 = -A_1 e_1$$

$$e'_2 = -A_2 e_2 \qquad (1.49)$$

$$e'_3 = -A_3 e_3$$

where the prime superscripts denote output voltages. Applying Kirchhoff's circuit law at each of the amplifier input nodes provides

$$\text{at } e_1: \quad \frac{e_1 - e_3'}{R_3} + \frac{e_1 - K_3 E}{R_4} + \frac{e_1 - K_1 e_1'}{R_1} + C_1 s\,(e_1 - e_1') = 0 \qquad (1.50)$$

$$\text{at } e_2: \quad e_2' = -A_2 e_2 = -A_2 K_2 e_1' \qquad (1.51)$$

$$\text{at } e_3: \quad \frac{e_3 - e_2'}{R_2} + C_2 s\,(e_3 - e_3') = 0 \qquad (1.52)$$

Substituting (1.51) and (1.52) into Eq. (1.50) and rearranging terms produces the voltage ratio

$$\frac{e_1'}{E} \doteq \frac{-K_3}{\frac{R_4}{R_1} K_1 + R_4 C_1 s + \frac{R_4 A_2 K_2}{R_3 R_2 C_2 s}} \qquad (1.53)$$

which can be expressed in the form

$$\frac{e_1'}{E} = \frac{-\frac{K_3 R_1}{K_1 R_4}}{1 + \frac{R_1 C_1}{K_1} s \left[1 + \frac{A_2 K_2}{R_2 R_3 C_1 C_2 s^2}\right]} \qquad (1.54)$$

The velocity to force frequency response function is given by the impedance of (1.12) and can be restated as

$$Z_0 = \frac{1/c}{1 + \frac{m}{c} s \left[1 + \frac{k}{ms^2}\right]} = \frac{1/c}{1 + \frac{Q}{\omega_n} s \left[1 + \frac{\omega_n^2}{s^2}\right]} \qquad (1.55)$$

where the undamped natural frequency and Q of the oscillator are expressed as

$$\omega_n^2 = \frac{k}{m}$$

$$Q = \frac{k}{c\omega_n} \qquad (1.56)$$

By comparing directly the terms of Eq. (1.54) and (1.55), the relationships between the quantities in the operational amplifier circuit and the quantities in the oscillator frequency response function appear as

$$\frac{1}{c} = \frac{K_3 R_1}{K_1 R_4} = \text{Peak Gain}$$

$$\frac{Q}{\omega_n} = \frac{R_1 C_1}{K_1} \qquad (1.57)$$

$$\omega_n^2 = \frac{A_2 K_2}{R_2 R_3 C_1 C_2}$$

Rather than change the passive components in the amplifier circuits, it becomes clear from (1.57) that the potentiometers are used to conveniently adjust the oscillator parameters.

1.7 CURRENT GENERATORS

In passive element simulation, the elastic structure is represented electrically by some combination of resistors, inductors, capacitors, and/ or transformers. External loadings applied to the mechanical system, however, are usually represented by active circuits connected to form a current generator.

A current or force generator is a device whose output current is independent of its output voltage. One such operational-amplifier circuit called a Type-II current generator (see pg. 190, Reference 13) is represented schematically as Figure 1.22.

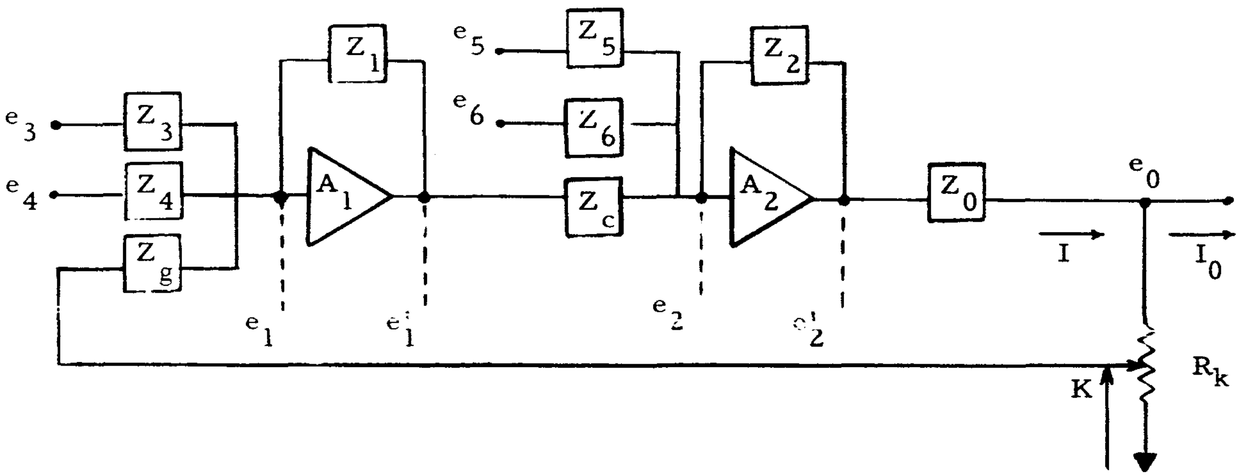


Figure 1.22 Amplifier Circuit Representing a Current Generator

The input-output relationships for the amplifiers are

$$-e_1' = A_1 e_1$$

$$-e_2' = A_2 e_2 \qquad (1.58)$$

and the gains of the amplifiers (A_1 and A_2) are assumed much greater than one.

Applying Kirchhoff's current law to nodes e_1 and e_2 yields

$$\sum I\text{'s at } e_1 \text{:} \; -\frac{e_3}{Z_3} + \frac{e_4}{Z_4} + \frac{K e_0}{Z_g} + \frac{e_1'}{Z_1} = 0 \tag{1.59}$$

$$\sum I\text{'s at } e_2 \text{:} \; -\frac{e_5}{Z_5} + \frac{e_6}{Z_6} + \frac{e_1'}{Z_c} + \frac{e_0 + I Z_0}{Z_2} = 0 \tag{1.60}$$

where

$$I = \frac{e_2' - e_0}{Z_0} = I_0 + \frac{e_0}{R_k} \tag{1.61}$$

Solving the above equations for the output current I_0 produces

$$I_0 = \frac{Z_2}{Z_0 Z_c}\left(\frac{Z_1}{Z_3} e_3 + \frac{Z_1}{Z_4} e_4\right) - \frac{1}{Z_0}\left(\frac{Z_2}{Z_5} e_5 + \frac{Z_2}{Z_6} e_6\right)$$
$$+ \frac{Z_2}{Z_0 Z_c}\left(\frac{Z_1 K}{Z_g} - \frac{Z_0 Z_c}{R_k Z_2} - \frac{Z_c}{Z_2}\right) e_0 \tag{1.62}$$

Setting $Z_2 = Z_c$ and assuming the impedances in the amplifier circuit are resistors, the output current becomes

$$I_0 = \frac{1}{R_c}\left(\frac{R_1}{R_3} e_3 + \frac{R_1}{R_4} e_4\right) - \frac{1}{R_0}\left(\frac{R_2}{R_5} e_5 + \frac{R_2}{R_6} e_6\right)$$

$$+ \frac{1}{R_0}\left(\frac{K R_1}{R_g} - \frac{R_0}{R_k} - 1\right) e_0 \qquad (1.63)$$

Setting the voltage gain of the potentiometer across the output voltage e_0 equal to

$$K = \left(1 + \frac{R_0}{R_k}\right)\frac{R_g}{R_1} \qquad)1.64)$$

reduces the e_0 coefficient in the above equation to zero and provides the output current as

$$I_0 = \frac{1}{R_c}\left(\frac{R_1}{R_3} e_3 + \frac{R_1}{R_4} e_4\right) - \frac{1}{R_0}\left(\frac{R_2}{R_5} e_5 + \frac{R_2}{R_6} e_6\right) \qquad (1.65)$$

This expression is the output current of the current-generator and is noted as independent of the output voltage e_0. Although not shown explicitly in the schematic of Figure 1.22, the common terminals for both amplifiers are connected to ground.

2. MULTI-DEGREE-OF-FREEDOM SYSTEMS

In the previous section, a single degree-of-freedom system was used to introduce concepts common to (1) the mobility or force-current, velocity-voltage analog, (2) the mechanical impedance or force-voltage, velocity-current analog, and (3) operational amplifier or active circuits. In this section, these concepts will be applied to simulating systems with two or more degrees-of-freedom.

2.1 ALGEBRAIC SYSTEMS

Consider, as an example, a mechanical system fully described by the algebraic equations

$$\begin{aligned} 6x_1 - x_2 - 3x_3 &= -1 \\ -x_1 + 4x_2 - x_3 &= 2 \\ -3x_1 - x_2 + 5x_3 &= 3 \end{aligned} \tag{2.1}$$

In matrix form, these equations appear as

$$\begin{bmatrix} 6 & -1 & -3 \\ -1 & 4 & -1 \\ -3 & -1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 2 \\ 3 \end{Bmatrix} \tag{2.2}$$

where the unknowns are expressed as the $\{x\}$ column matrix, the coefficients of the equations are the elements of the square matrix, and the constants on the right-hand side of the equations are given also as a column matrix. The coupling terms of the equations are noted as the off-diagonal elements of the square matrix.

Interpreting the unknowns as nodal voltages, Kirchhoff's current law can be applied to yield the resistive network shown as Figure 2.1.

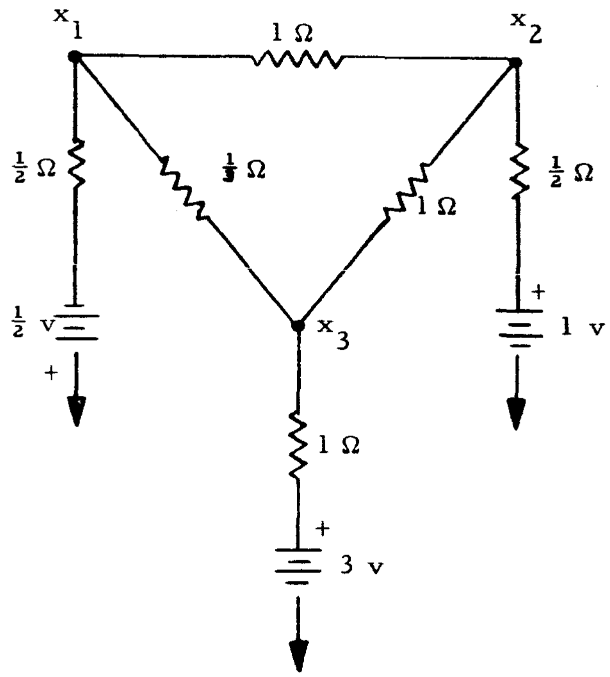


Figure 2.1 Nodal Analogy for the Algebraic System

The coupling terms are noted as resistors interconnecting the appropriate nodes. For this circuit, the coupling resistors form a delta(Δ)circuit whose magnitudes are reciprocals of the off-diagonal matrix elements. The right-hand side of Eq. (2.2) is formed with the use of the batteries as voltage sources and connected as shown. Note that the minus sign (-1) for the first equation of (2.1) is created by grounding the positive terminal of the half-volt battery. This can be easily checked by applying Kirchhoff's current law to the x_1 node.

Interpreting the elements of the x column matrix as currents, Kirchhoff's voltage law can be applied to yield the resistive network of Figure 2.2.

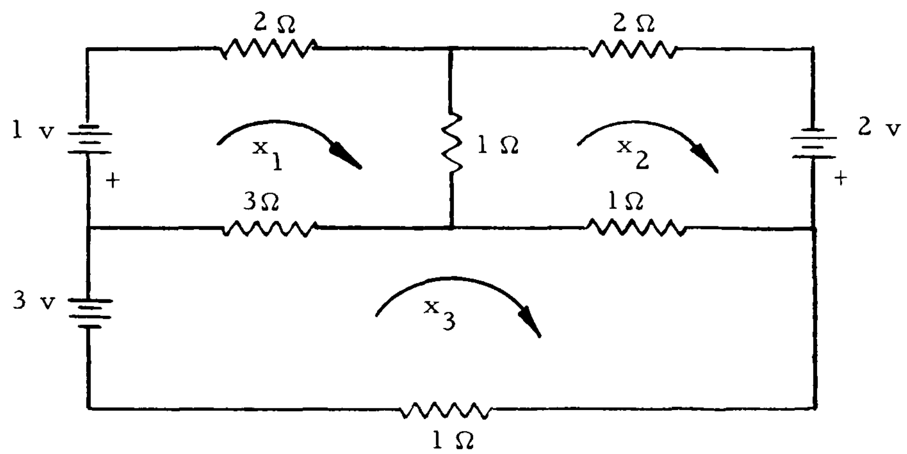


Figure 2. 2 Loop Analogy for the Algebraic System

The coupling terms are shown as resistors common to two circuit loops and have magnitudes equal to the off-diagonal elements in the 3 by 3 matrix of Eq. (2.2). The coupling resistors for these equations form a wye network; and, as before, batteries are used as the voltage sources to represent the righthand column matrix of Eq. (2.2).

To reduce the network of Figure 2.1 into a tractable series-parallel combination, the resistive delta network must be reduced to an equivalent wye network. Conversely, the wye circuit in Figure 2.2 must be reduced to an equivalent delta network before the loop analogy can be analyzed using series-parallel impedance relationships. These delta-wye transformations are found in most standard texts of circuit analysis.

To conveniently simulate Eq. (2.1) using operational amplifiers, the original equations are first restated as

$$x_1 = -\frac{1}{6} + \frac{1}{6}x_2 + \frac{1}{2}x_3$$

$$x_2 = \frac{1}{2} + \frac{1}{4}x_1 + \frac{1}{4}x_3 \qquad (2.3)$$

$$x_3 = \frac{3}{5} + \frac{3}{5}x_1 + \frac{1}{5}x_2$$

Using resistors for the feedback and input elements, amplifier circuits simulating the equations of (2.3) appear as Figure 2.3.

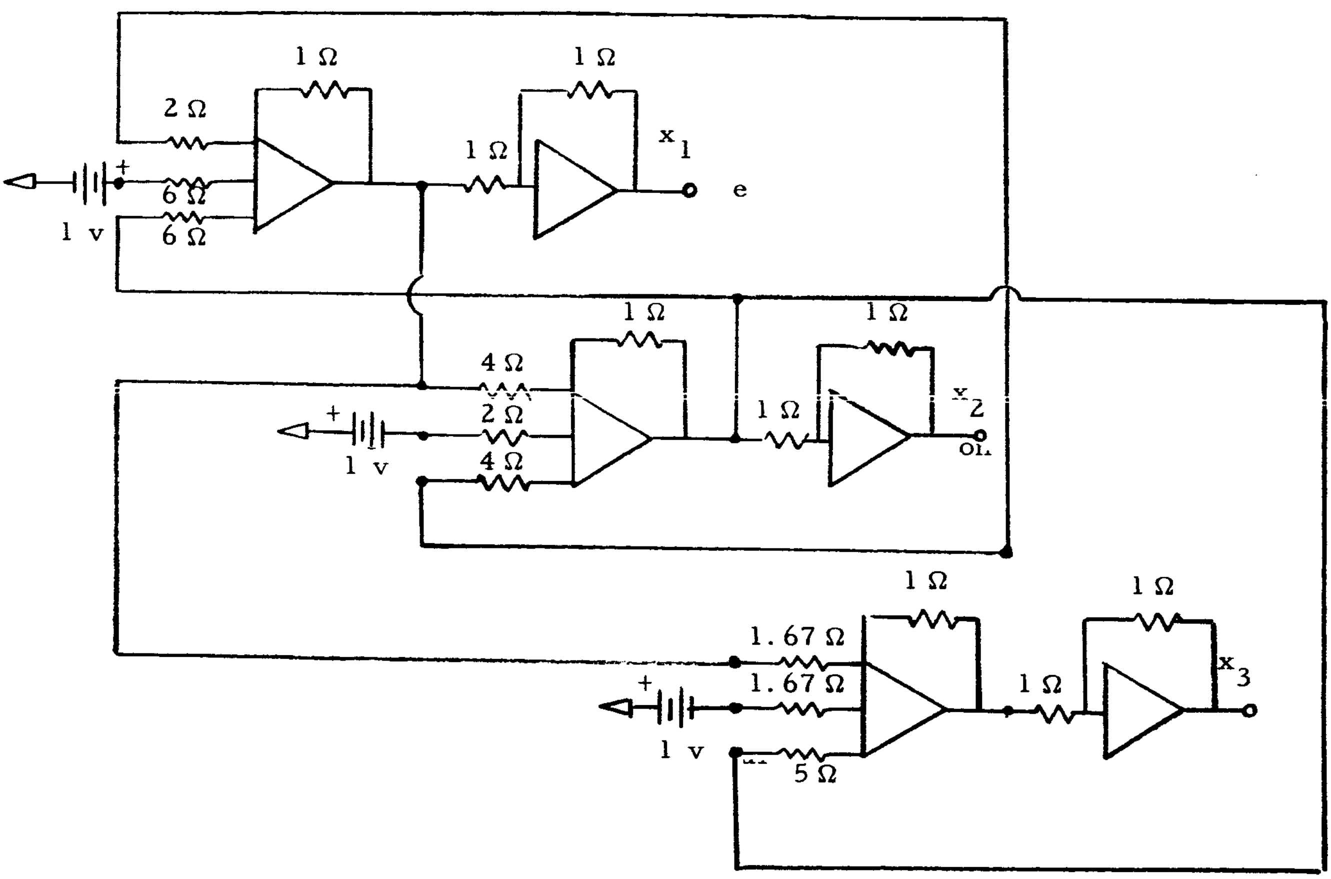


Figure 2.3 Operational Amplifier Circuits for the Algebraic System

The three amplifiers with the input and feedback resistors equal to unity are sign reversers and used to obtain positive values of x_1, x_2, and x_3. The coupling terms are obtained by interconnecting appropriate amplifier output voltages as input voltages to other summing amplifiers. The constants (-1, 2, and 3) are created by using batteries to form biased voltage inputs to the summers.

Defining the unknowns as voltages, the transformer voltage law provides a multi-winding transformer analogy shown as Figure 2. 4.

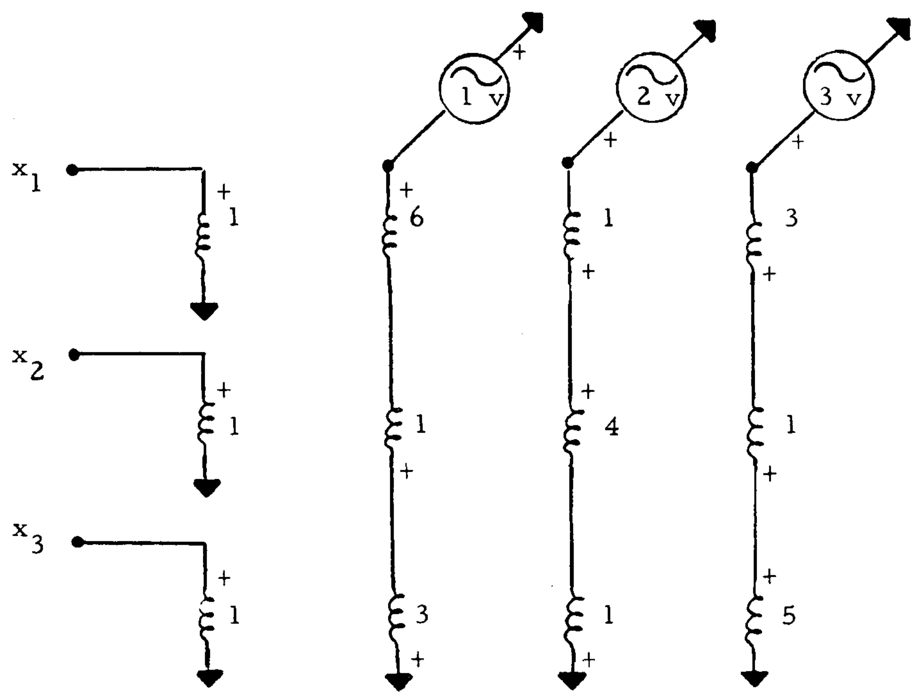


Figure 2. 4 Transformer-voltage Analogy for the Algebraic System

This analogy consists of three interconnected four-winding transformers whose turns ratios are equal numerically to the elements of the 3 by 3 matrix appearing in Eq. (2. 2). The sign changes are accomplished by reversing the polarity of the appropriate transformer windings, and the right-hand column matrix of (2. 2) is simulated by voltage generators.

Defining the unknowns as currents, the transformer current law produces a multi-winding transformer analogy shown as Figure 2. 5.

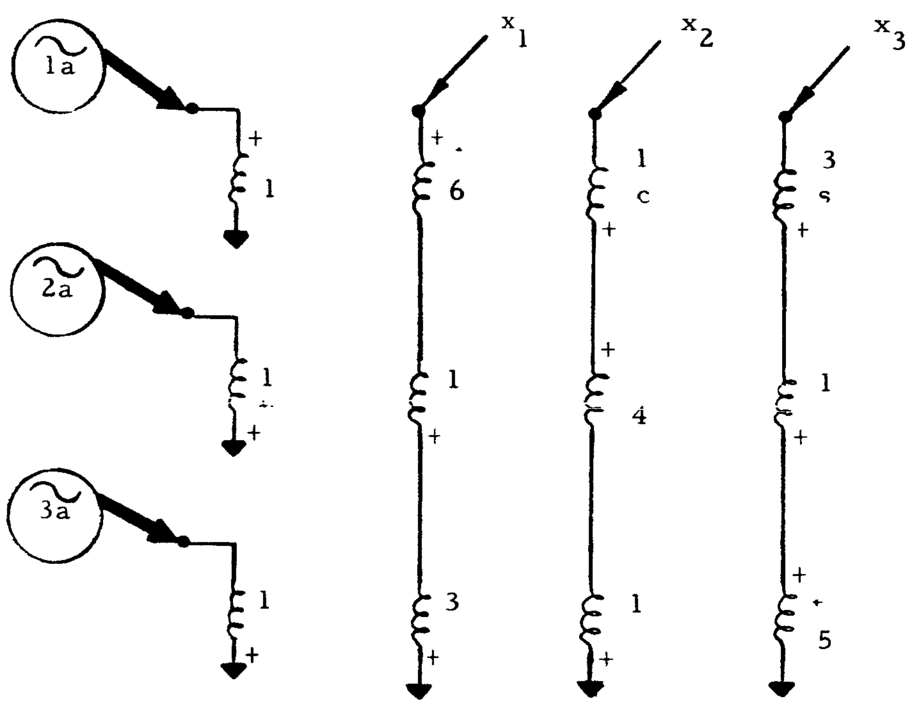


Figure 2. 5 Transformer-current Analogy for the Algebraic System

This circuit is a dual network of Figure 2.4. Except for polarity changes in the windings where the current generators are applied, the transformer circuitry is identical. As contrasted with voltage generators, the right-hand column matrix of (2. 2) is formed by current generators applied at the shown nodal locations. Note that the nodal locations for applying the external generators and measuring the unknowns for the current analogy differ from those used in the transformer circuits of Figure 2. 4.

2. 2 DISCRETE MECHANICAL SYSTEMS

In the previous section, various electrical analogies were illustrated simulating a hypothetical mechanical system depicted as a set of algebraic equations. Although all of the analogies are electrically equivalent to the same three algebraic equations, the form and appearance of each circuit are totally dependent on the defined relationships between quantities in the mathematical and electrical systems.

In many instances, mobility circuits (which is another name for force-current, velocity-voltage analogs) are topologically similar to mechanical systems. Thus, these analogies can be drawn simply by sketching the mechanical system where masses appear as capacitors, springs as inductors, dashpots as resistors, and rigid levers as transformers. Restricting attention to mobility analogs, consider by way of illustration Figures 2.6 through 2.10. depicting mechanical systems and their equivalent mobility circuits.

The nodal voltages represent velocities at the corresponding positions in the mechanical diagram and current generators represent the external loads acting on the mechanical system. Electrical ground is shown symbolically as ↓ and corresponds to the fixed boundaries. For the RLC networks, the spring is sketched as an inductor whereas, whenever transformers are used, the spring is sketched as a resistor.

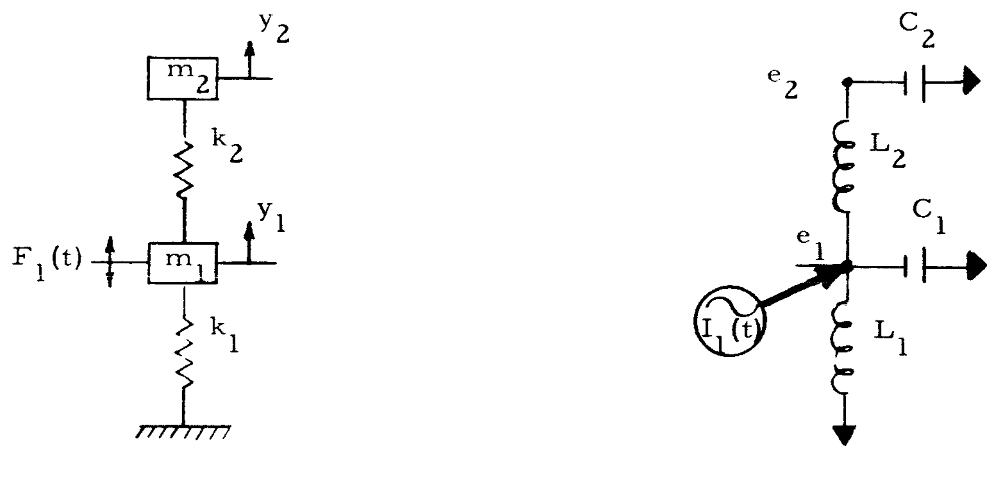


Figure 2. 6 The Vibration Isolator System and its Equivalent Mobility Analog

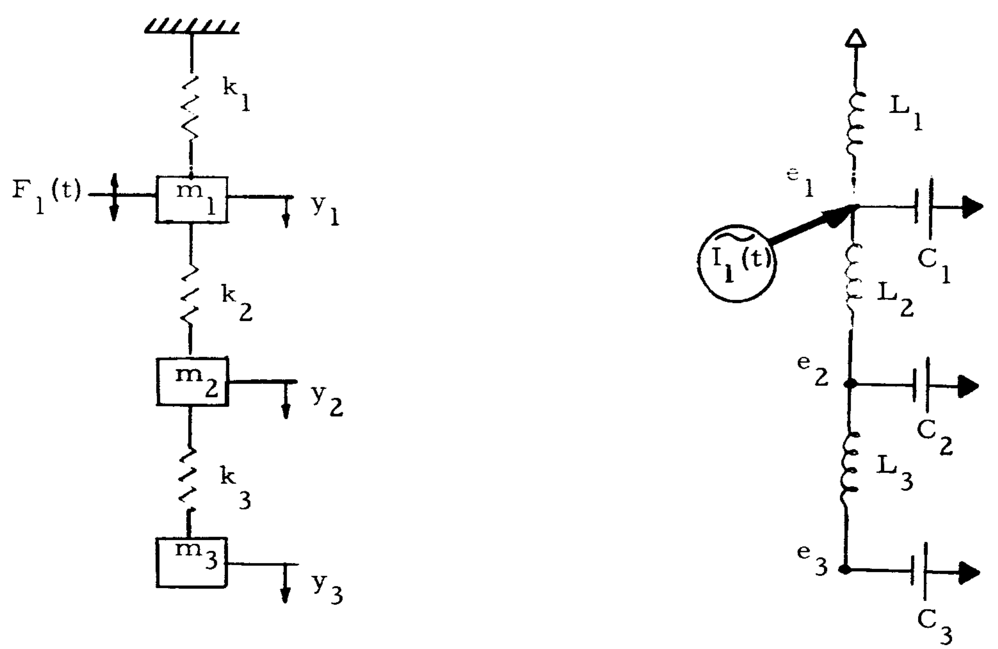


Figure 2. 7 Three-Degree-of-Freedom System and its Equivalent Mobility Analog

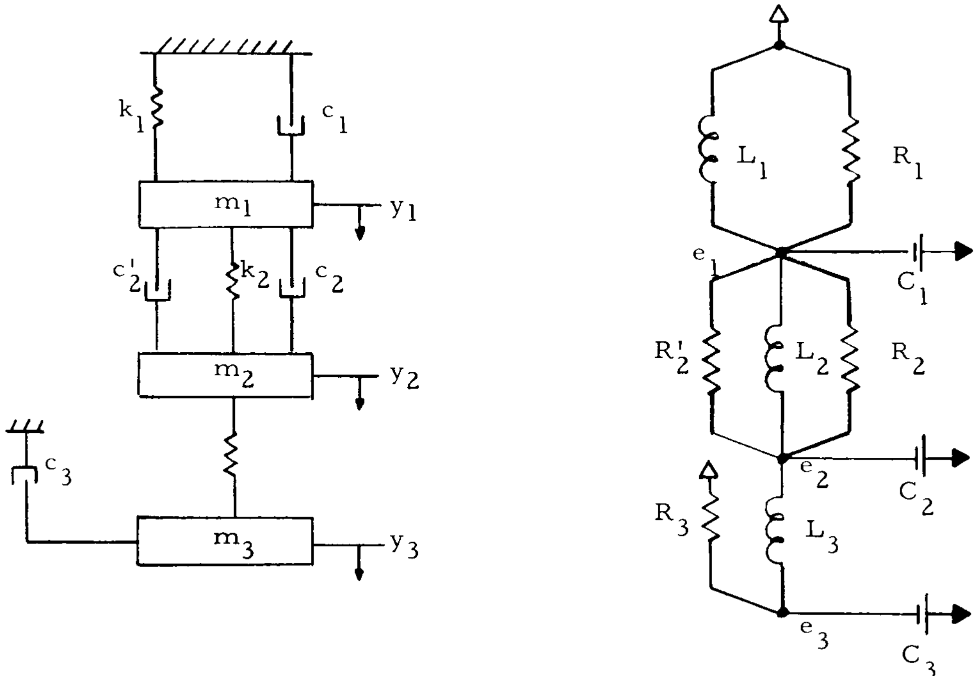


Figure 2. 8 Composite Three-Degree-of-Freedom System and its Equivalent Mobility Analog

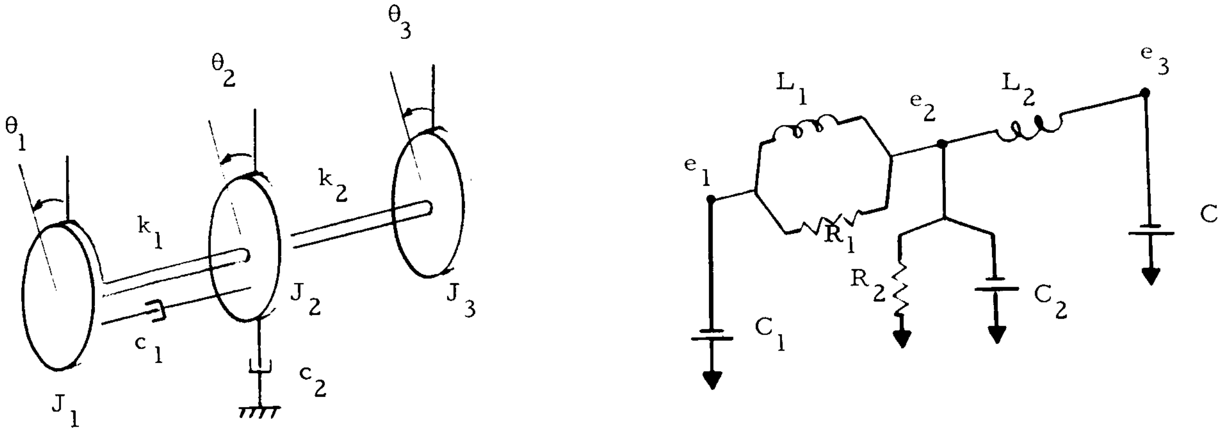


Figure 2. 9 Composite Torsional System and its Equivalent Mobility Analog

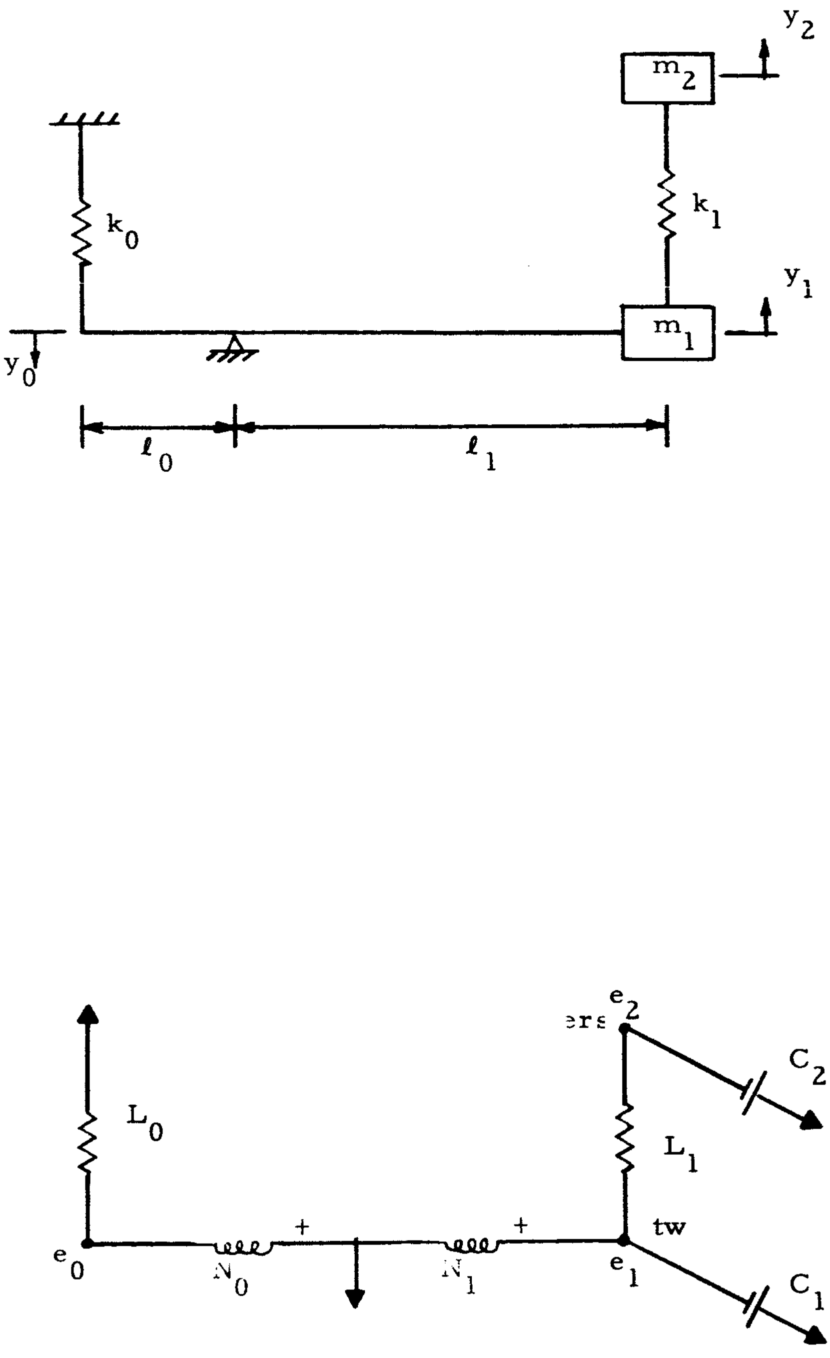


Figure 2.10 Levered Mechanical System and its Equivalent Mobility Analog

2.3 MODAL THEORY AND ITS ANALOG EQUIVALENCE

A typical and classical approach for calculating the dynamic response of multi-degree-of-freedom systems is the use of modal theory. This theory is a separation of variables technique and represents the displacements at various positions on the structure as

$$\{y\} = [\phi]\{q\} \tag{2.4}$$

where

$\{y\}$ is a column matrix denoting the displacement-time histories at spatial locations on the structure

$[\phi]$ is a square matrix consisting of the mode shapes of the structure

$\{q\}$ is a column matrix denoting displacement-time histories of modal oscillators in generalized coordinates

The $\{y\}$ column matrix is a function both of space and time whereas the modal matrix is a function only of the space coordinates and the generalized coordinate column matrix is a function only of time. Mathematically, Eq. (2.4) can be thought of as a transformation relating the physical coordinates of the mechanical system with the generalized coordinates defined by modal theory. By using the orthogonality properties of normal modes, it is found that the generalized coordinates represent the output response from linear oscillators whose coefficients are described in terms of generalized mass, generalized stiffness, generalized damping, and the generalized force. Each mode shape has an associated linear oscillator in generalized coordinates which is appropriately called the modal oscillator.

Using the transformer voltage laws, an electrical representation of Eq. (2.4) is sketched as Figure 2.11.

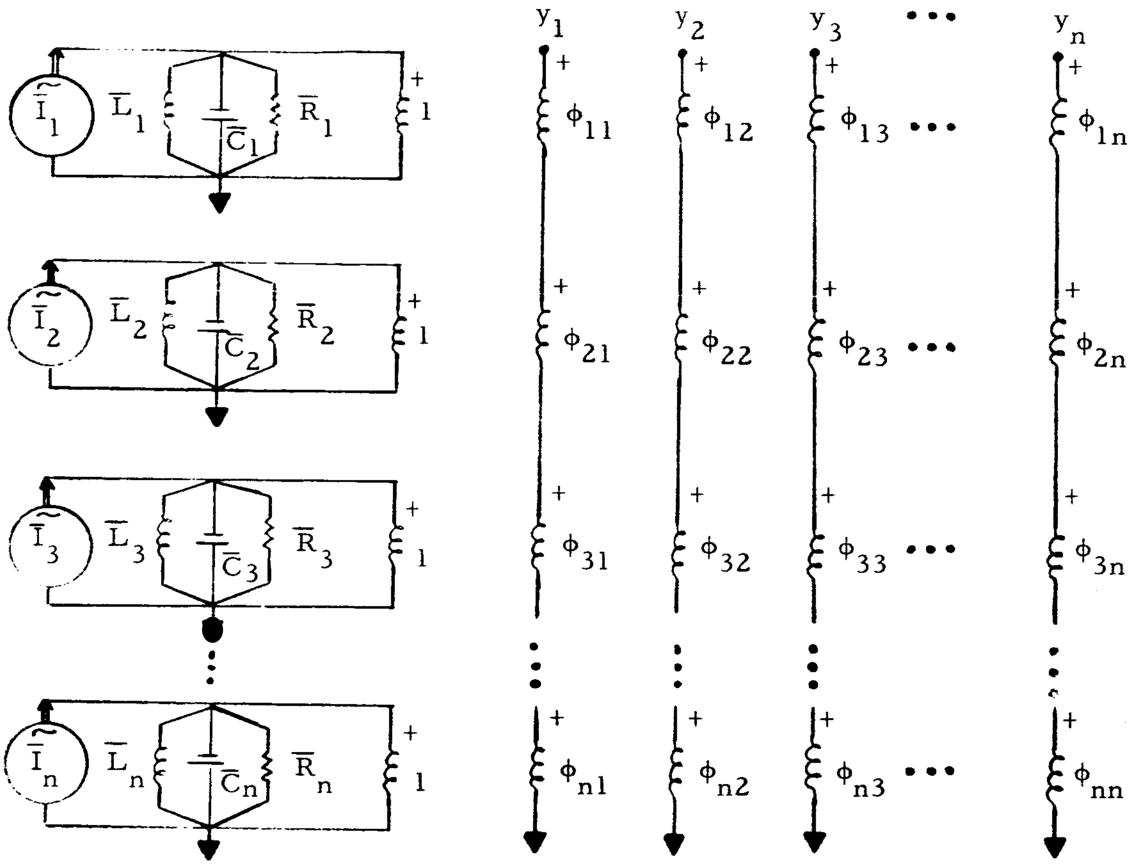


Figure 2.11 Modal Analog for an N-degree-of-freedom Mechanical System

The transformer windings denote the mode shapes of the structure and the RLC circuits denote the modal oscillators. The overbar in the modal circuit refer to the generalized force (I), the generalized flexibility (L), the generalized mass (C), and the generalized damping (R). The transformers shown are noted as n-winding transformers which, depending on the magnitude of n, may be impractical to obtain electrically. However, in concept, any linear n-degree-of-freedom system may be simulated by the modal analog shown above.

3. DISTRIBUTED STRUCTURAL SYSTEMS

Contrasted with discrete multi-degree-of-freedom systems, the vibration of a distributed or continuous elastic structure appears mathematically as a partial differential equation instead of sets of coupled ordinary differential equations. This equation is functionally dependent upon both the spatial coordinates and time. Except for simple structural problems, explicit analytical solutions to such equations are nontrivial mathematical tasks and, in many cases, an understanding of the physical problem is masked by the shear preponderance of mathematical details.

Consequently, in engineering practice, it is common to study the dynamic behavior of a distributed structure by examining a lumped model of the original structure. Mathematically, matrix techniques are used to analyze the multi-degree-of-freedom or lumped model. Electrically, passive mobility circuits can be used to simulate the lumped model, and the analysis can proceed using either a passive analog computer or digital computer.

The circuits to be considered here are mobility analogs for differential segments of a variety of elastic structures. These analogs represent the differential structural segments as difference segments and are used to synthesize complete electrical models of structural systems. In addition to elastic structures, the derivation procedures can be applied to most physical systems described by partial differential equations.

3.1 LONGITUDINAL VIBRATION OF A ROD

The equation of motion for the longitudinal oscillation of a thin, uniform, homogeneous rod appears as

$$AE \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2} \tag{3.1}$$

where A is the cross-section area, E the Young's modulus, m the mass per unit length, t the time, u the longitudinal displacement and a function of x and t, and x the spatial position along the length of the rod. Expressing the spatial derivative as a finite-difference expression yields the original partial differential equation as

$$\frac{AE}{\Delta x}\left[(u_{n+1} - u_n) - (u_n - u_{n-1})\right] = m\Delta x\, \ddot{u}_n \qquad (3.2)$$

where the double dot notation $(\ddot{\ })$ denotes the second time derivative of ( ). The second partial derivative with respect to x is reduced to difference form as

$$\frac{\partial^2 u_n}{\partial x^2} = \frac{\Delta_x[\Delta_x(u_n)]}{(\Delta x)^2} = \frac{\Delta_x[u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}}]}{(\Delta x)^2} = \frac{(u_{n+1} - u_n) - (u_n - u_{n-1})}{(\Delta x)^2} \qquad (3.3)$$

while the first partial derivative is noted as

$$\frac{\partial(\ )}{\partial x} = \frac{\Delta_x(\ )}{\Delta x} \qquad (3.4)$$

where Δx is the difference length of the rod segment and $\Delta_x(\ )$ denotes the change of the quantity ( ) with respect to the x coordinate. For more complete discussions of finite-difference techniques applied to structures, the reader is directed to Section 2.9 of Reference 4.

Interpreting the longitudinal displacements of (3.2) as voltages at the nodal position n + 1, n, and n - 1, Ohm's law and Kirchhoff's current law are applied to yield the funicular current diagram shown as Figure 3.1.

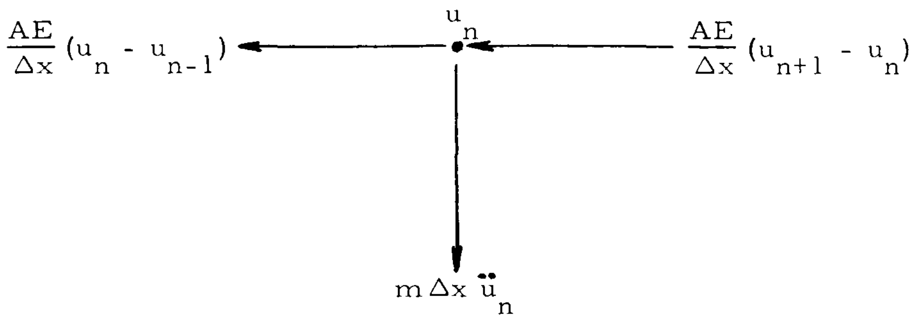


Figure 3.1 Funicular Diagram for a Difference Segment of a Rod

In a form consistent with mobility procedures, the above currents can be expressed as functions of velocities as

$$\frac{AE}{\Delta x}(u_{n+1} - u_n) = \frac{AE}{\Delta x}\int(\dot{u}_{n+1} - \dot{u}_n)\, dt \tag{3.5}$$

$$\frac{AE}{\Delta x}(u_n - u_{n-1}) = \frac{AE}{\Delta x}\int(\dot{u}_n - \dot{u}_{n-1})\, dt \tag{3.6}$$

$$m \Delta x \ddot{u}_n = m \Delta x \frac{d}{dt}(\dot{u}_n) \tag{3.7}$$

The first two equations denote current flow through inductors whereas the last equation describes the current flow of a capacitor. Therefore, the mobility circuit for the longitudinal vibration of a difference segment of rod becomes the LC network shown as Figure 3.2.

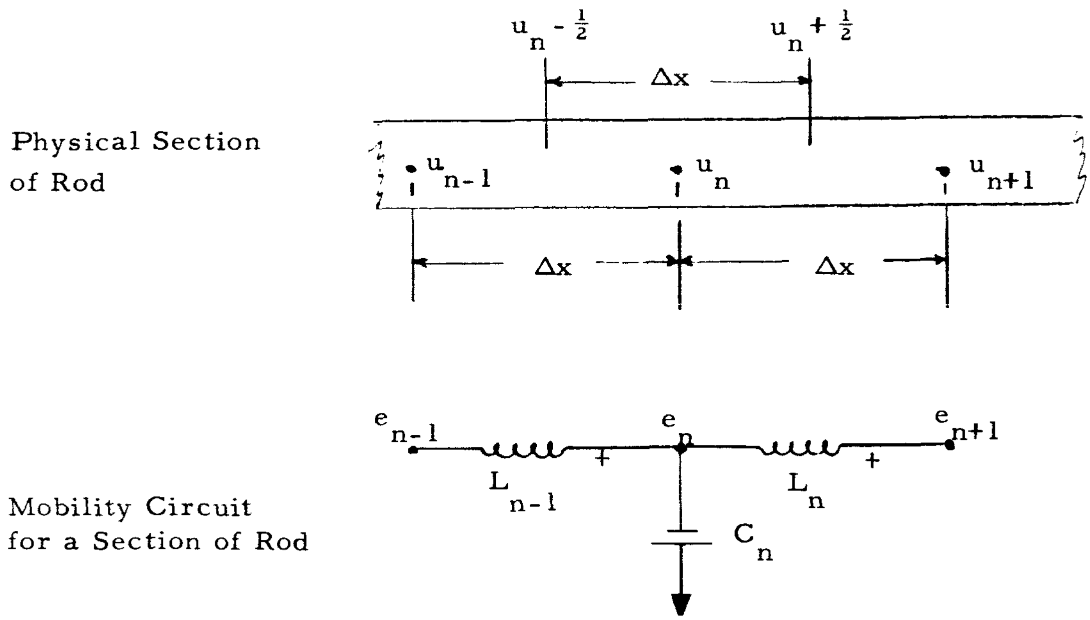


Figure 3. 2 Mobility Circuit for the Longitudinal Vibration of a Rod

Interpreted as a lumped model, the mobility circuit describes a system of undamped linear oscillators connected in tandem. The nodal voltages denote the velocities at the spatial locations and the current flows denote extensional forces and inertial forces. The circuit components are related to the mechanical parameters as

$$L_n = \left.\frac{\Delta x}{AE}\right|_n$$

$$L_{n-1} = \left.\frac{\Delta x}{AE}\right|_{n-1}$$

$$C_n = m\Delta x\Big|_n$$

where the subscripts below the vertical line refer to the difference segment over which each of the mechanical quantities are to be evaluated. For the nth inductor (L_n), the flexibility distribution of the rod is integrated from spatial position n to n + 1. For the nth capacitor (C_n), however, the mass distribution of the rod is integrated from spatial position $n - \frac{1}{2}$ to $n + \frac{1}{2}$.

Although sketched as equal difference lengths, the difference segments can vary in length while the form of the circuit remains unchanged. For a rod with nonuniform mass and flexibility properties, the form of the mobility circuit likewise does not change. The circuit component values are calculated by integrating the nonuniform properties over the appropriate segmental lengths thus representing the nonuniform rod by "weighted" uniform sections.

3.2 LATERAL VIBRATION OF A SIMPLE BEAM

The section of beam to be treated here is assumed uniform, homogeneous, and to obey small deflection theory. Although uniform beam properties are assumed for the analog derivation, the form of the analog circuit will be the same for nonuniform properties

Consider a differential length of a simple beam in bending (also commonly called the Bernoulli-Euler beam) shown as Figure 3.3.

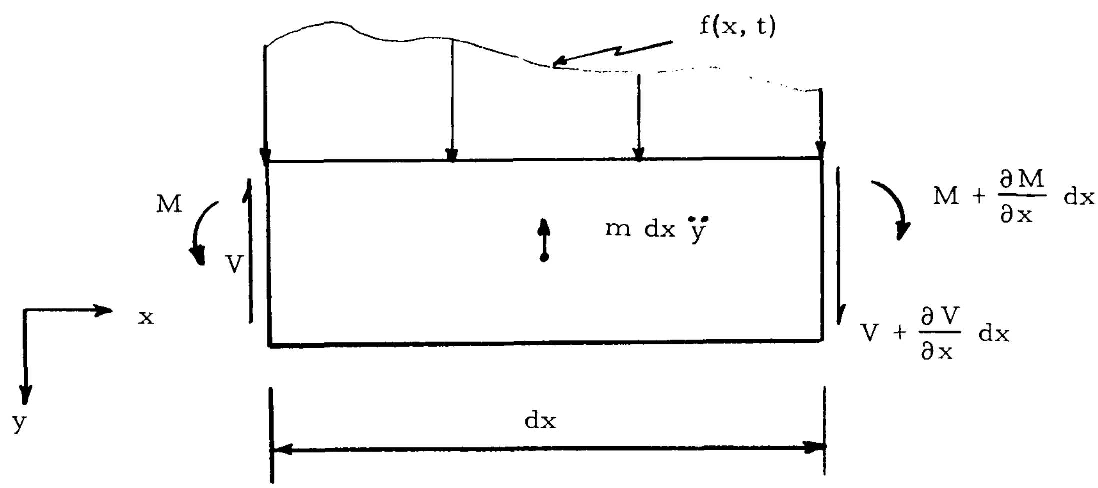


Figure 3.3 Differential Segment of a Simple Beam

where M denotes the bending moment, V the shear, f(x, t) the external loading per unit segmental length, and $m \ddot{y}\, dx$ the inertial loading acting at the center of gravity of the differential segment. The equation of motion for this beam is given by

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = f(x, t) \tag{3.11}$$

where E is Young's modulus, I the cross-section area moment of inertia about the bending axis, f(x, t) the external lateral loading per unit length, m the mass per unit length, t the time, x the position along the length of the beam, and y the lateral deflection from static equilibrium noted as a function of both the spatial coordinate x and time t.

The derivation of this fourth-order partial differential equation is based upon four, first-order differential expressions which may be written as

$$\begin{aligned} &(a) \qquad \theta = \frac{dy}{dx} \\ &(b) \qquad M = EI \frac{d\theta}{dx} \\ &(c) \qquad -V = \frac{dM}{dx} \\ &(d) \qquad m \frac{d^2 y}{dt^2} = m\ddot{y} = \frac{\partial V}{\partial x} + f(x, t) \end{aligned} \tag{3.12}$$

where θ is noted as the slope in bending. As first-order difference expressions, these equations become

$$(a) \qquad \theta_{n+\frac{1}{2}} = \frac{\Delta_x(y_n)}{\Delta x} = \frac{y_{n+1} - y_n}{\Delta x}$$

$$(b) \qquad M_n = M_{n+\frac{1}{2}, n-\frac{1}{2}} = \frac{EI}{\Delta x} \Delta_x(\theta_n) = \frac{EI}{\Delta x} (\theta_{n+\frac{1}{2}} - \theta_{n-\frac{1}{2}}) \tag{3.13}$$

$$\text{(c)} \quad -V_n = -V_{n+1,n} = \frac{\Delta_x(M)_{n+\frac{1}{2}}}{\Delta x} = \frac{1}{\Delta x}\left(M_{n+\frac{3}{2},\, n+\frac{1}{2}} - M_{n+\frac{1}{2},\, n-\frac{1}{2}}\right)$$

$$\text{(d)} \quad \Delta_x(V_n) = V_{n+1,n} - V_{n,n-1} = m_n \Delta x \ddot{y}_n - f_n \Delta x$$

where M_n is the moment in the beam segment at position n and corresponds to moment flow from position $n+\frac{1}{2}$ to $n-\frac{1}{2}$, V_n the shear force in the nth beam segment and corresponds to shear flow from position n+1 to n, f_n the external loading per unit length acting on the nth beam segment, m_n the mass per unit length for the difference segment centered at position n, $\theta_{n+\frac{1}{2}}$ the slope of the beam segment at position $n+\frac{1}{2}$, Δx the difference length of a beam segment, $\Delta_x(\ )$ the change of the quantity ( ) with respect to the x coordinate. Electrically, these equations are simulated by the difference circuits of Figure 3. 4.

Defining a composite difference gridwork, the circuits in Figure 3. 4 are combined to yield the general symbolic circuit shown as Figure 3. 5. Note that the slope coordinates are shown at positions intermediate to the deflection coordinates. The bending moment M_n is defined as the moment in the beam from spatial position $n+\frac{1}{2}$ to $n-\frac{1}{2}$ while the shear force V_n is defined as the shear flow in the beam from spatial position n+1 to n-1. The bending moment corresponds to the current measured in the θ slope circuit and the shear flow corresponds to the current measured in the y deflection circuit. The transformer windings couple the slope and deflection circuits and are shown using conventional primary P_n and secondary S_n notation. The primary winding is located in the deflection circuit and, with the secondary winding set to unity, corresponds numerically to the beam segmental length defined from spatial position n to n+1. The inertial loading is shown as the impedance Z_n while the bending flexibility is noted as the impedance $Z_n(\theta)$. Consistent with the positive sign convention, the external load on the nth difference segment $f_n \Delta x$ is shown as a current generator directed into the nth node of the deflection circuit.

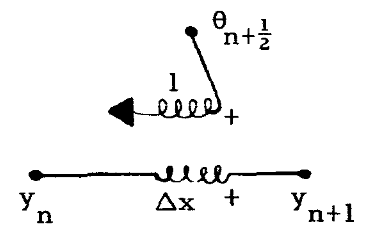
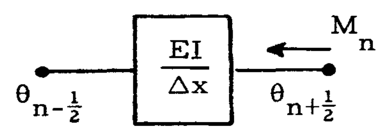
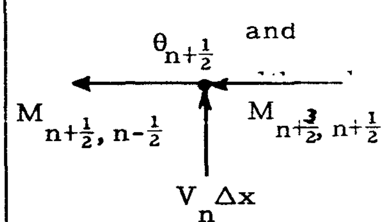
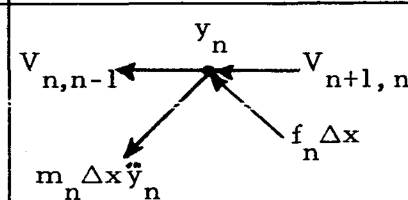
<table>
<tr><td>$$e_{n+\frac{1}{2}} = \frac{y_{n+1} - y_n}{\Delta x}$$</td><td>Voltage law for the Ideal Transformer</td><td>
</td></tr>
<tr><td>$$M_n = \frac{EI}{\Delta x}(\theta_{n+\frac{1}{2}} - \theta_{n-\frac{1}{2}})$$</td><td>Ohm's Law</td><td>
</td></tr>
<tr><td>$$V_n \Delta x + M_{n+\frac{3}{2}, n+\frac{1}{2}} - M_{n+\frac{1}{2}, n-\frac{1}{2}} = 0$$</td><td>Kirchhoff's current law applied to the slope circuit at location $n+\frac{1}{2}$</td><td>
</td></tr>
<tr><td>$$-V_{n,n-1} - m_n \Delta x \ddot{y}_n + f_n \Delta x + V_{n+1,n} = 0$$</td><td>Kirchhoff's current law applied to the deflection circuit at location n</td><td>
</td></tr>
<tr><td>Difference Equation</td><td>⟺ Electrical Circuit Concept ⟺</td><td>Equivalent Difference Circuit</td></tr>
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Figure 3.4 Circuit Representations for the Difference Equations of a Simple Beam

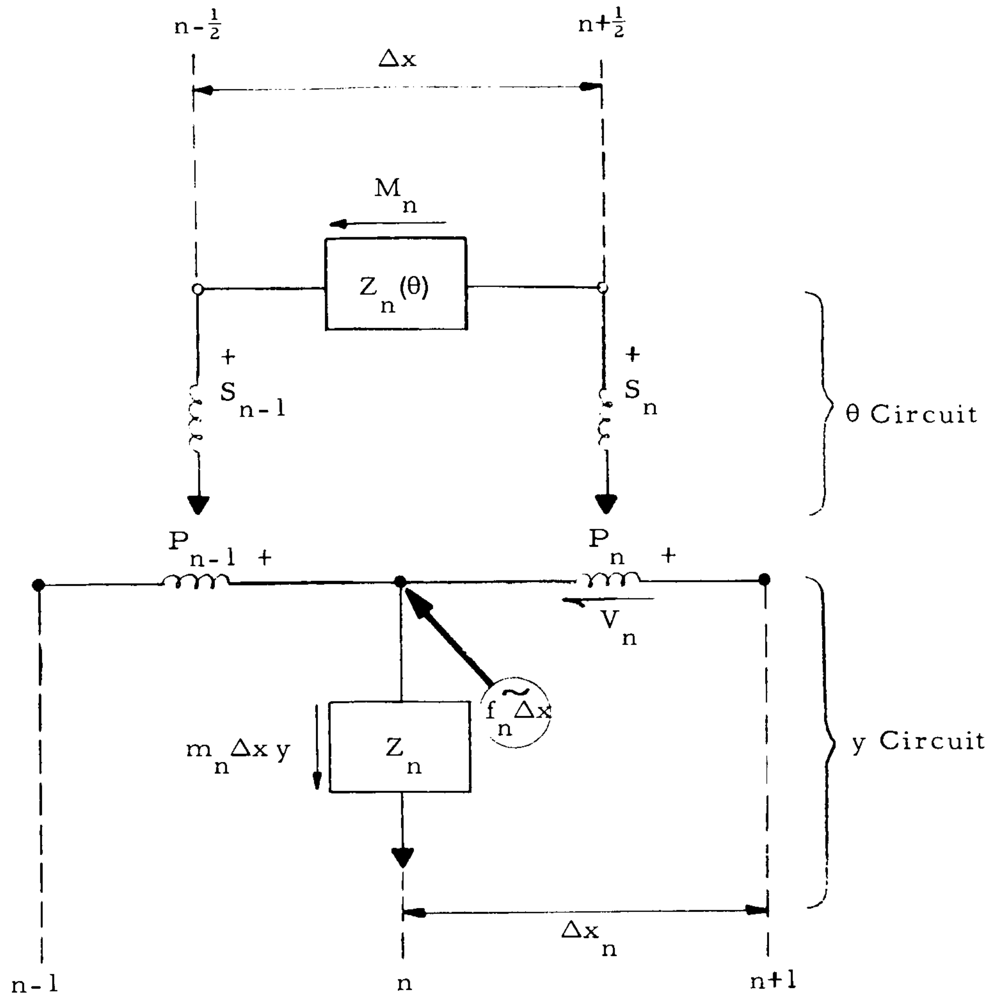
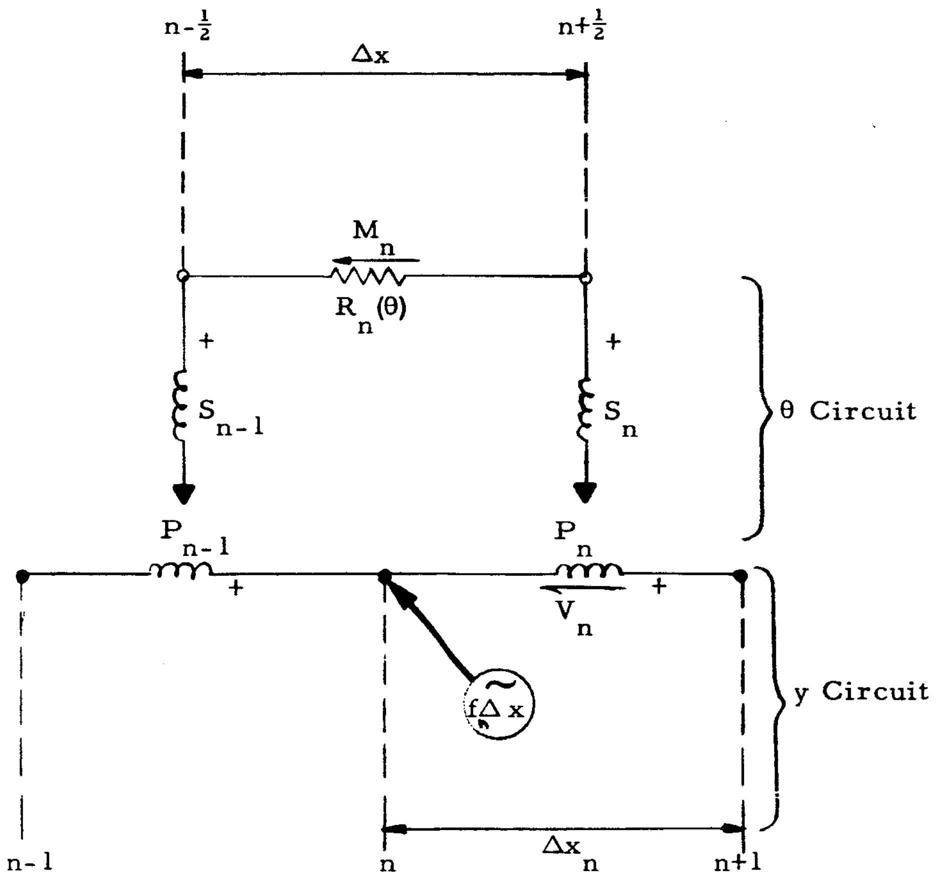


Figure 3.5 General Mobility Circuit for the Lateral Vibration of a Simple Beam

From the circuits of Figure 3.5, consider the development of an analog depicting the bending of a simple beam due to an external static load. Consistent with the mobility approach, the required analog is a force-current, displacement-voltage circuit and differs from a mobility analog only in the definition of the mechanical equivalence of voltage. Such a circuit for the static bending of a beam is appropriately called a static analog and appears as Figure 3.6.

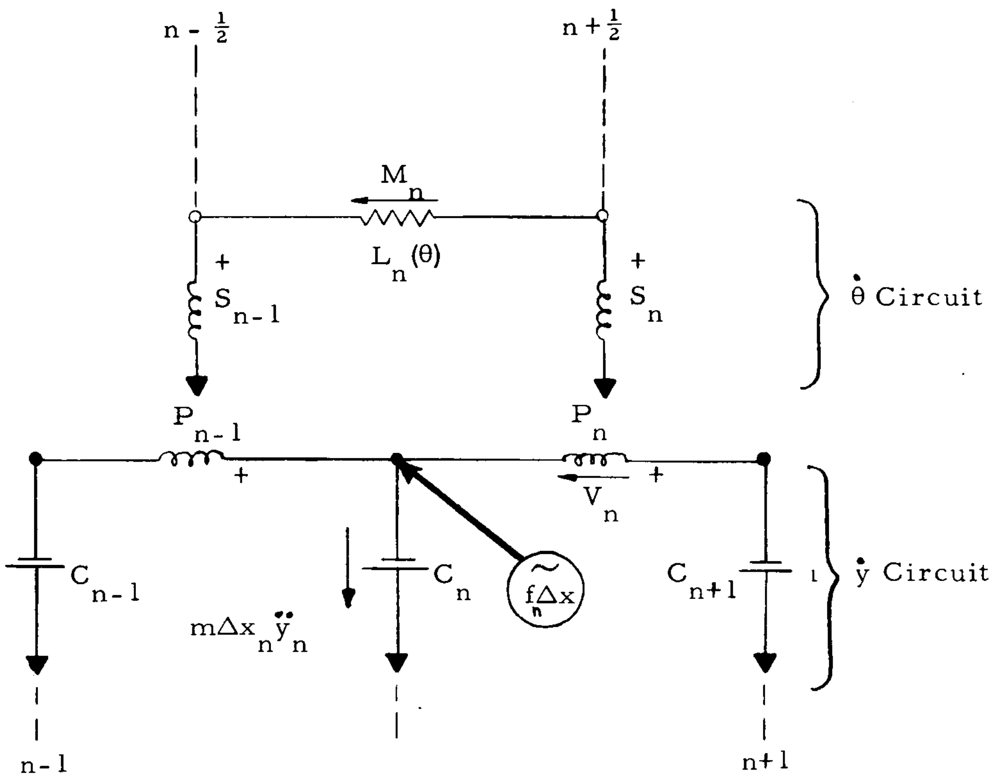


$$R_n(\theta) = \left.\frac{\Delta x}{EI}\right|_n \qquad \frac{P_n}{S_n} = \Delta x_n$$

Figure 3.6 Static Passive Analog for a Simple Beam in Bending

Since there is no inertial load, the inertial capacitor Z_n does not appear. The bending impedance $Z_n(\theta)$ is noted as a resistor and all other parts of the circuit remains unchanged. The magnitude of the nth bending resistor is obtained by integrating the beam flexibility distribution from spatial position $n-\frac{1}{2}$ to $n+\frac{1}{2}$ whereas the nth transformer winding ratio is the length of difference segment from position n to n+1. The current generator corresponds to the applied static loading lumped at the deflection coordinates. For example, the magnitude of the loading at the nth position is found by integrating the distribution of the external load from $n-\frac{1}{2}$ to $n+\frac{1}{2}$.

From the general circuit of Figure 3.5, a mobility circuit for the vibration of a simple beam is shown as Figure 3.7.



$$L_n(\theta) = \left.\frac{\Delta x}{EI}\right|_n$$

$$C_n = m\Delta x\Big|_n$$

$$\frac{P_n}{S_n} = \Delta x_n$$

Figure 3.7 Mobility Analog for the Lateral Vibration of a Simple Beam

Consistent with mobility relationships, the currents correspond to shear force and bending moments whereas the nodal voltages correspond to slope and lateral velocities. This analog is equivalent to a finite-difference mathematical model of the simple beam and is sometimes called a difference mobility circuit. The circuit components are expressed in terms of their mechanical equivalents where the flexibility appears as an inductor (shown symbolically as a resistor), the inertial loading as a capacitor, and the slope geometry as a transformer coupling the deflection circuit with the slope circuit. The magnitude of the bending inductor is obtained by integrating the flexibility distribution from position $n-\frac{1}{2}$ to $n+\frac{1}{2}$. Likewise, the capacitor and current generator magnitudes are calculated by integrating the mass and external loading distributions over the segmental length between position $n-\frac{1}{2}$ and $n+\frac{1}{2}$.

Constructed as a mechanical model, the mobility difference circuit shown in Figure 3.7 (without the external loading) is depicted as Figure 3.8.

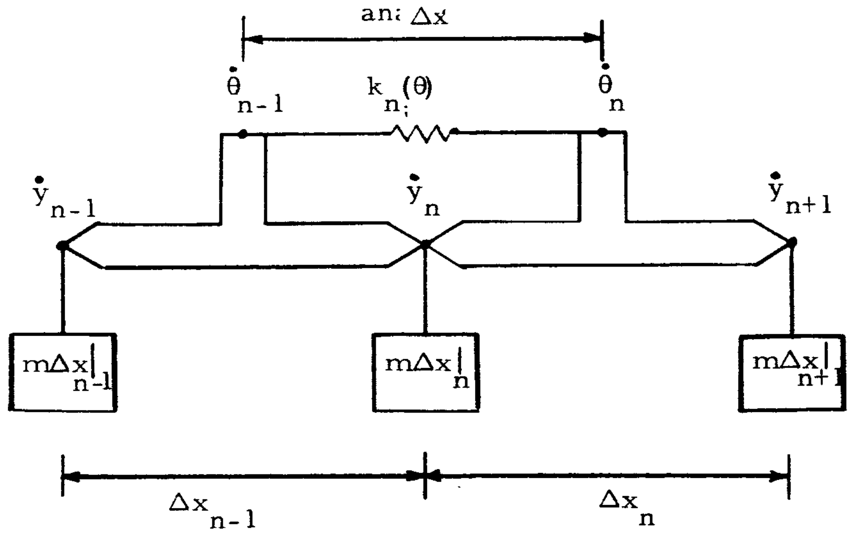


Figure 3.8 Equivalent Mechanical Model for the Mobility Analog of a Simple Beam Segment

Note the topological similarity with the circuit. The transformers are sketched as the levers while the bending inductor $L_n(\theta)$ and inertial capacitor C_n appear as the spring $k_n(\theta)$ and concentrated mass $m\Delta x|_n$ respectively.

By comparing the static circuits of Figure 3.6 with the mobility circuit of Figure 3.7, the form of the two circuits are identical. To convert the static circuit to the mobility circuit, the nodal voltages are first assumed as slope and lateral velocities. Then, an inertial capacitor C_n is added to the nth mode in the deflection circuit and the bending resistor $R_n(\theta)$ is converted to a bending inductor $L_n(\theta)$. Since the transformers describe the slope geometry, they remain unchanged. In a similar manner, most dynamic analogs for structures can be conveniently derived from their static circuits. This conversion requires that the resistors be changed to inductors and capacitors added to the appropriate nodes to account for inertial loadings.

The mobility analog shown here has the accuracy limitations of finite-difference equations. For accuracy improvement, the use of higher order difference methods can be used as is discussed for beam analysis by Greenwood in Reference 9. As contrasted to passive analogs, however, Greenwood uses active or operational amplifier circuits.

Other beam circuits also are available in the literature. One very useful mobility circuit is the Russell beam analogy discussed by Russell and MacNeal in Reference 17 and again by MacNeal in Reference 13. In contrast with the mobility circuits emphasized in this discussion, a mechanical impedance circuit (force-voltage, velocity-current analogy) for the beam is discussed by Molloy in Reference 14 as a four-pole-parameter network.

3.3 LATERAL VIBRATION OF A TIMOSHENKO BEAM

This beam may be described as the simple beam in bending plus the effects of both rotary inertia and shear deformation. If the shear deformation is ignored in a Timoshenko beam, the resultant structure is called the Rayleigh beam. Pictorially, a differential segment of the Timoshenko beam is shown as Figure 3.9

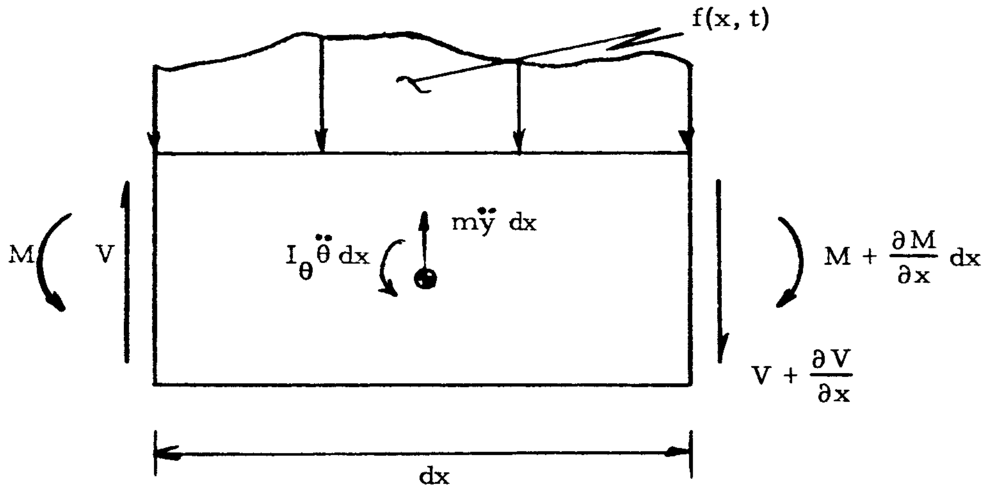


Figure 3.9 Differential Segment of Timoshenko Beam

where M denotes the bending moment, V the shear force, f(x, t) the external loading per unit length, $m\ddot{y}\, dx$ the lateral inertial force of the differential segment, and $I_\theta \ddot{\theta}\, dx$ the rotatory inertial force of the differential segment. The equations of motion for this beam may be written as

$$\gamma = \frac{\partial y_s}{\partial x} = \frac{1}{GK'}\left(I_\theta \ddot{\theta} - EI\frac{\partial^3 y}{\partial x^3}\right)$$

$$m\ddot{y} + EI\frac{\partial^4 y}{\partial x^4} - \frac{\partial}{\partial x}\left(I_\theta \ddot{\theta}\right) = f(x, t) \qquad (3.14)$$

where A is the cross-section area, E the Young's modulus, I the cross-section area moment of inertia about the bending axis, I_θ the mass moment of inertia about the bending axis, G the shear modulus, K' a geometric correction factor noted as the ratio of average shear stress to maximum shear stress over a cross section, m the mass per unit length, y the total beam deflection from the static equilibrium position, y_b the beam deflection due to bending, y_s the beam deflection due to shear deformation, θ the slope due to bending, and $(\ddot{\ })$ the second time derivative of the function within the parentheses.

The derivation of the equations in (3. 14) is based upon the following first-order differential equations.

$$
\begin{aligned}
&\text{(a)} \qquad m\ddot{y} = \frac{\partial V}{\partial x} + f(x,t) \\
&\text{(b)} \qquad I_\theta \ddot{\theta} = V + \frac{\partial M}{\partial x} \\
&\text{(c)} \qquad \theta = \frac{d}{dx}(y_b) \\
&\text{(d)} \qquad M = EI\frac{d\theta}{dx} \qquad\qquad (3.15) \\
&\text{(e)} \qquad -V = \frac{dM}{dx} \\
&\text{(f)} \qquad \gamma = \frac{\partial}{\partial x}(y_s) = \frac{V}{K'AG} \\
&\text{(g)} \qquad y = y_s + y_b
\end{aligned}
$$

The first two equations (a, b) are derived from force equilibrium conditions, the next three equations (c, d, e) are applicable statements from simple beam theory, the sixth equation (f) is obtained from elementary strength of materials, and the last equation (g) is a geometric definition.

As finite-difference expressions, the above equations appear as

$$\text{(a)} \qquad m_n \Delta x \ddot{y}_n = \Delta_x (V)_n + f_n \Delta x$$

$$\text{(b)} \qquad I_\theta \Delta x \ddot{\theta}_{n+\frac{1}{2}} = V_n \Delta x + \Delta_x (M)_{n+\frac{1}{2}}$$

$$\text{(c)} \qquad \theta_{n+\frac{1}{2}} = \frac{1}{\Delta x} \Delta_x (y_b)$$

$$\text{(d)} \qquad M_n = \frac{EI}{\Delta x} \Delta_x (\theta_n) \qquad (3.16)$$

$$\text{(e)} \qquad -V_n = \frac{1}{\Delta x} \Delta_x (M)_{n+\frac{1}{2}}$$

$$\text{(f)} \qquad \Delta_x (y_s) = \frac{\Delta x}{K'AG} V_n$$

$$\text{(g)} \qquad \Delta_x (y_n) = \Delta_x (y_s) + \Delta_x (y_b)$$

The first two equations are electrically satisfied by Kirchhoff's current law, equations c, d, and 3 are shown in Figure 3. 4, and equation f is electrically satisfied by Ohm's law. Both the f and g equations are simulated by the circuit shown as Figure 3. 10. CR-510

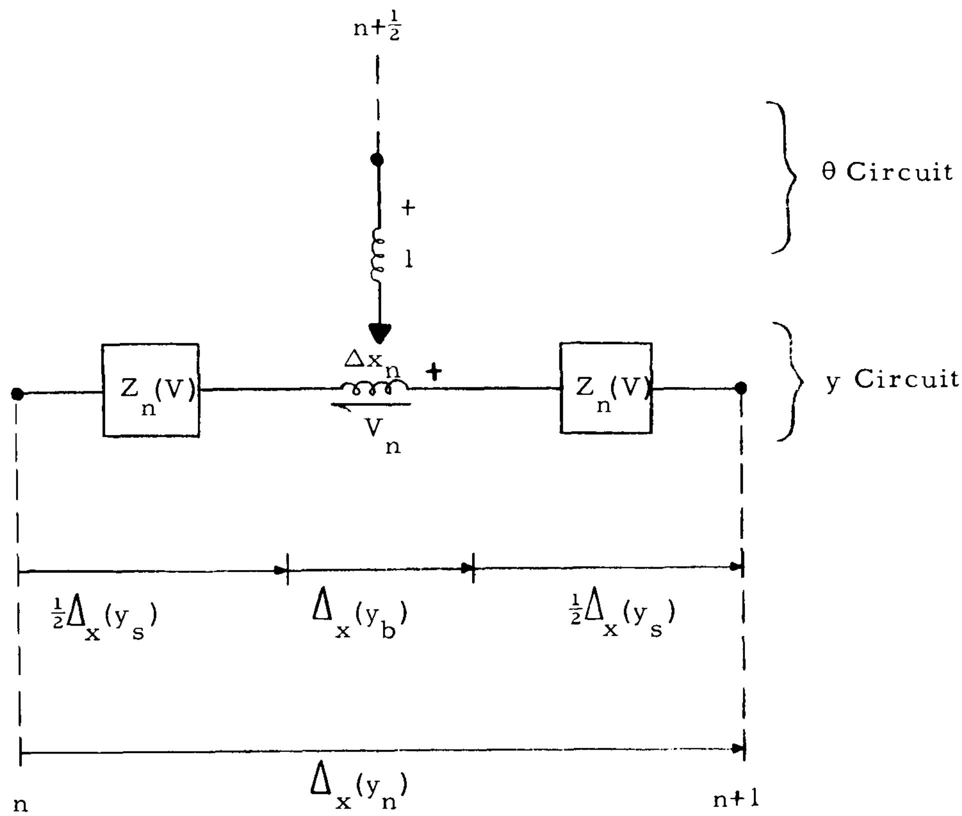
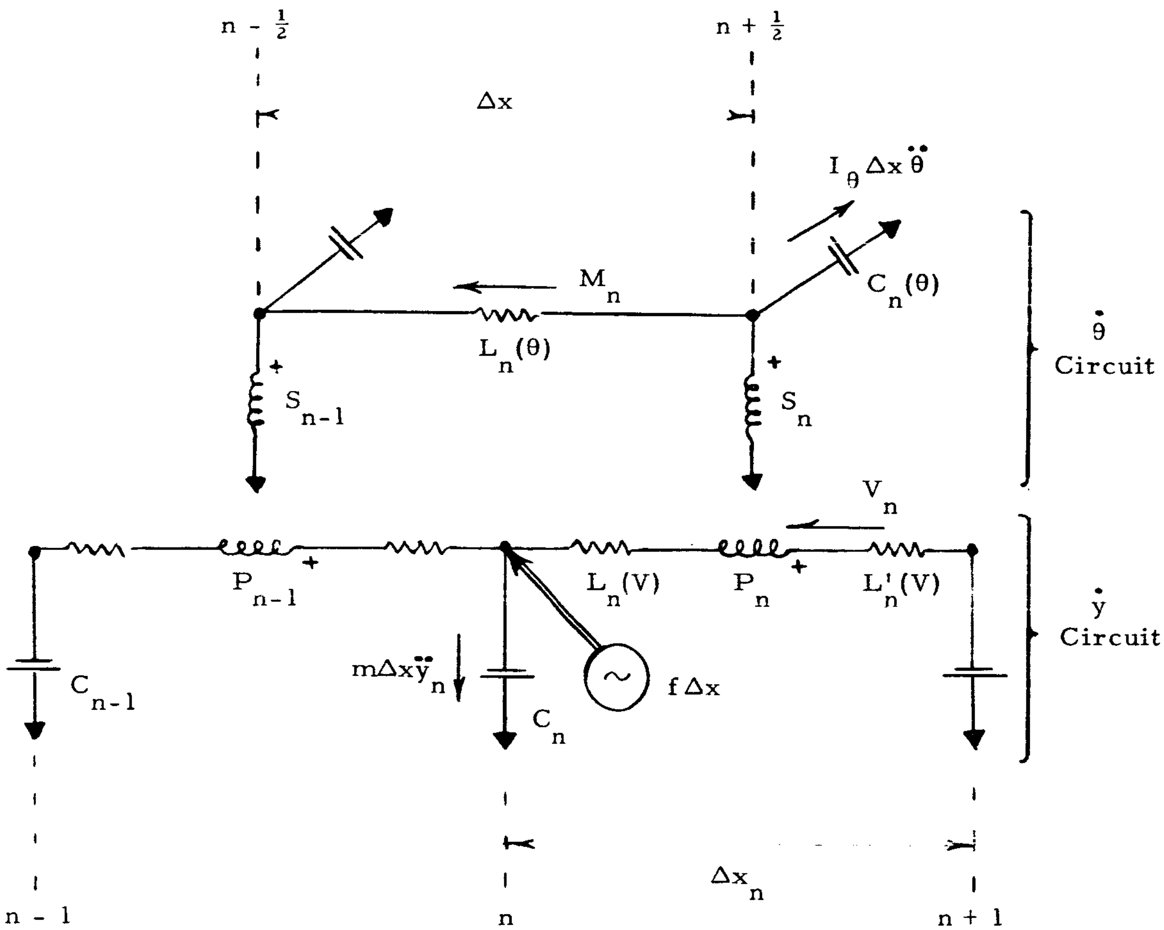


Figure 3. 10 General Difference Circuit for Shear Deformation and Total Deflection

By applying Ohm's law where $\Delta_x(y_s)$ denotes the voltage drop and V_n the current, the shear impedances $Z_n(V)$ acount for the deflection due to shear deformation given by Eq. (3.16-f). By adding the positive voltage drops $\Delta_x(y_s)$ and $\Delta_x(y_b)$, the total voltage drop $\Delta_x(y_n)$ is obtained as specified by Eq. (3.16-g).

By expressing the lateral and rotational accelerations as well as the slope and displacement coordinates in terms of angular and lateral velocities, a passive mobility circuit for the lateral vibration of a segment of Timoshenko beam appears in Figure 3. 11. The difference between this circuit and that of a simple beam is the addition of (1) the rotatory inertial capacitors $C(\theta)$ in the slope circuit and (2) the shear deformation inductors $L(V)$ and $L'(V)$ in the lateral velocity circuit. Shorting the shear deformation inductors from the above circuit yields the mobility circuit for a Rayleigh beam. The magnitude



$$L_n(\theta) = \frac{\Delta x}{EI}\Big|_n \qquad C_n = m\Delta x\Big|_n$$

$$L_n(V) = \frac{\Delta x}{K'AG}\Big|_n \qquad C_n(\theta) = I_\theta \Delta x\Big|_n$$

$$L_n'(V) = \frac{\Delta x'}{K'AG}\Big|_n \qquad \frac{P_n}{S_n} = \Delta x_n$$

Figure 3.11. Mobility Circuit for the Lateral Vibration of Timoshenko Beam

of the rotatory inertial capacitor for the nth beam segment $C_n(\theta)$ is obtained by integrating the distribution of the mass moment of inertia from spatial position n to n+1. The magnitudes of $L_n(V)$ and $L'_n(V)$ are calculated by integrating the distribution of $\frac{1}{K'AG}$ from spatial position n to $n+\frac{1}{2}$ for $L_n(V)$ and from $n+\frac{1}{2}$ to n+1 for $L'_n(V)$.

3.4 LATERAL VIBRATION OF A SIMPLE BEAM ON AN ELASTIC FOUNDATION

The structure considered here is the simple or Bernoulli-Euler beam on a distributed, elastic foundation. The remarks of this discussion, however, are equally appropriate to a beam defined by Timoshenko beam theory.

For a differential segment of a beam, the elastic foundation is assumed equivalent to an elastic spring attached below the center of gravity of the beam segment. The equation of motion is expressed as

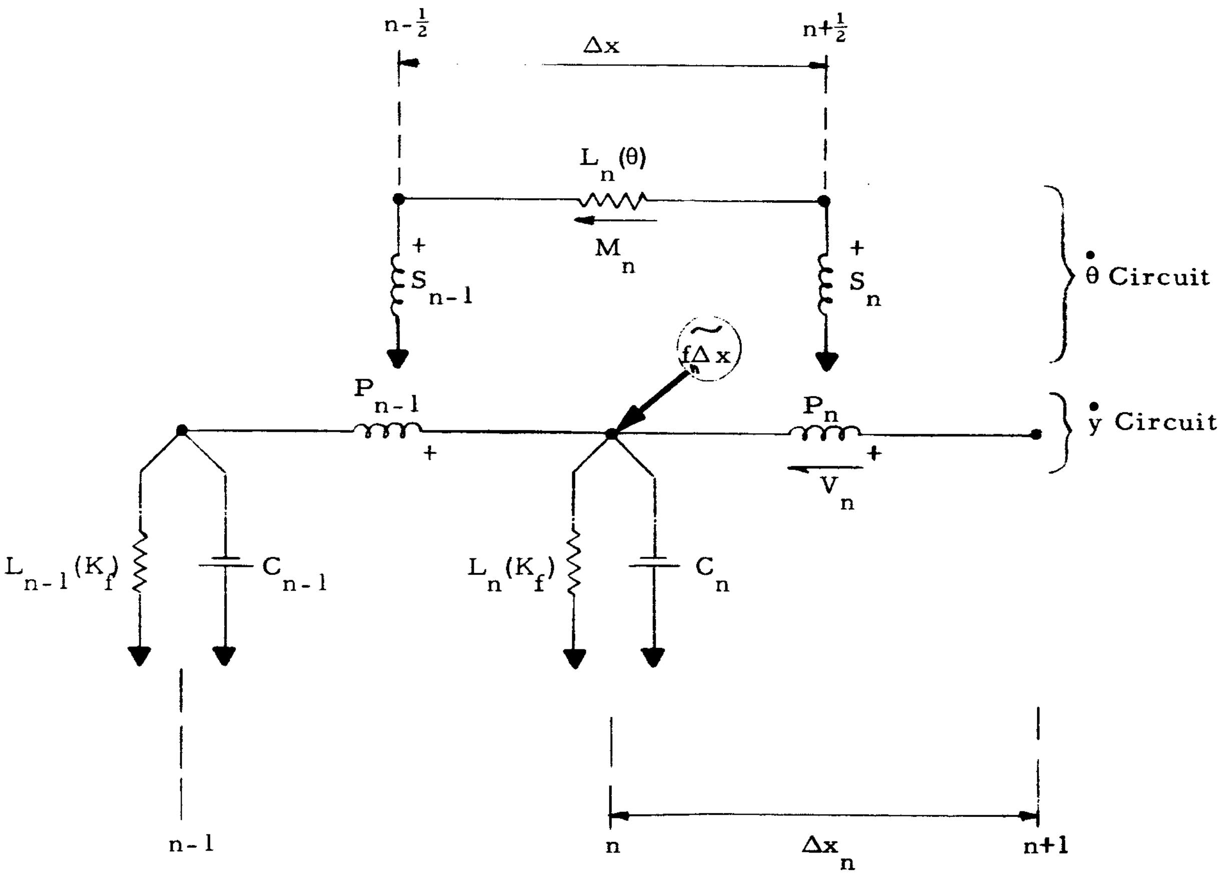
$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} + K_f y = f(x, t) \tag{3.17}$$

where the symbols are those for the simple beam with the additional term K_f denoting the foundation stiffness per unit length. This fourth-order equation is based on four first-order differential equations which are expressed in finite-difference form as

$$\begin{aligned}
&\text{(a)} \quad \theta_{n+\frac{1}{2}} = \frac{\Delta_x(y_n)}{\Delta x} \\
&\text{(b)} \quad M_n = \frac{EI}{\Delta x}\Delta_x(\theta_n) \\
&\text{(c)} \quad -V_{n+1,\,n} = V_n = \frac{\Delta_x(M)_{n+\frac{1}{2}}}{\Delta x_n} \\
&\text{(d)} \quad \Delta_x(V)_n = m\Delta x\ddot{y}_n + K_f\Delta x y_n - f_n\Delta x
\end{aligned} \tag{3.18}$$

Except for the term in the d equation denoting the force due to the elastic foundation, these equations are the same as those for the simple beam.

By applying Kirchhoff's current law at the nth node, the foundation force is incorporated into the simple beam analogy and yields the mobility analog of Figure 3. 12.



$$L_n(\theta) = \left.\frac{\Delta x}{EI}\right|_n \qquad C_n = m\Delta x\big|_n$$

$$L_n(K_f) = K_f\Delta x\big|_n \qquad \frac{P_n}{S_n} = \Delta x_n$$

Figure 3. 12 Mobility Circuit for the Lateral Vibration of a Simple Beam on an Elastic Foundation

The elastic foundation is shown electrically as an inductor $L_n(K_f)$ to ground and is inserted in parallel with the lateral inertial capacitor C_n. The magnitude of the foundation inductor for the nth segment is obtained by integrating the distribution of the foundation flexibility from spatial position $n-\frac{1}{2}$ to $n+\frac{1}{2}$.

3.5 IN-PLANE VIBRATION OF A CURVED BEAM

The structure considered here is a curved beam loaded externally in the radial direction by the force $f_r(\phi, t)$. The beam theory is essentially that for a curved simple beam. A differential section of a curved beam with the external loading, internal force, and displacement coordinates is shown as Figure 3.13.

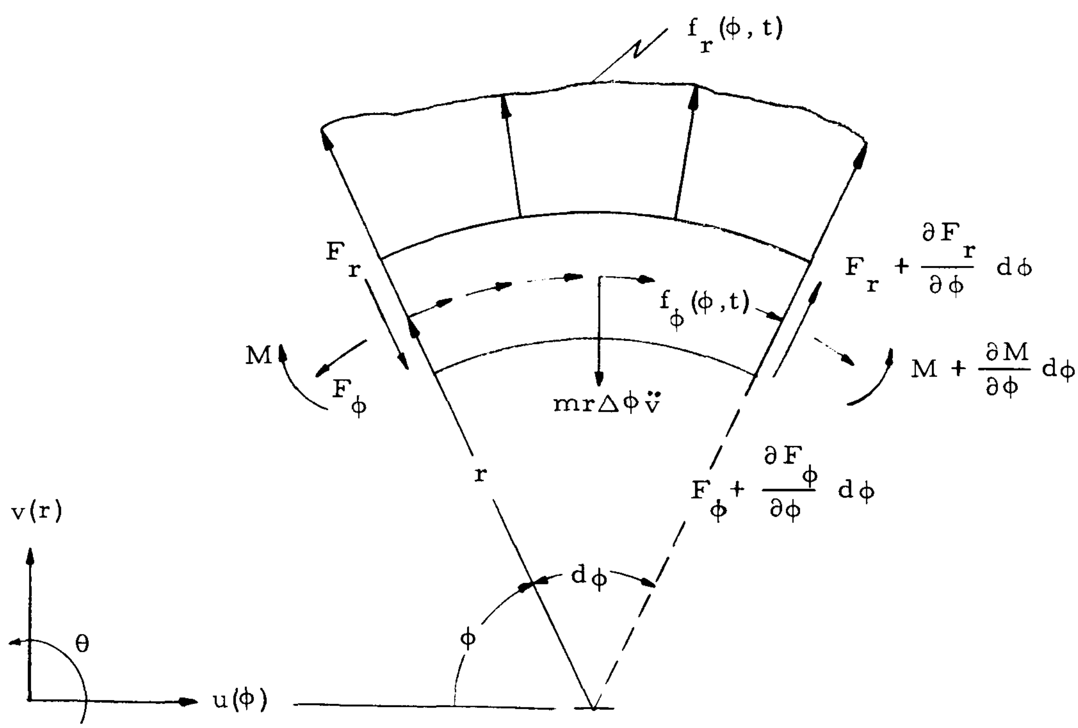


Figure 3.13 Curved Beam Segment with the External Loading, Internal Forces and Displacement Coordinates

where F_r is the shear force acting in the radial direction, F_ϕ the extensional or tangential force acting in the angular coordinate ϕ direction, $f_r(\phi, t)$ the external radial loading per unit arc length, M the bending moment, $f_\phi(\phi, t)$ the external tangential loading per unit arc length, m the mass per unit arc length, r the radius of the curved segment, u the axial displacement measured in the ϕ direction, v the radial displacement measured in the r direction, θ the slope of the beam, ϕ the angular coordinate, and $(\ddot{\ })$ the second time derivative of the quantity ( ).

Consistent with the format of the previous derivations, the first-order differential equations for this beam segment are

$$\begin{aligned}
&\text{(a)} \quad \frac{dF_r}{d\phi} = F_\phi - f_r r + m r \ddot{v} \\
&\text{(b)} \quad \frac{dF_\phi}{d\phi} = -F_r - f_\phi r \\
&\text{(c)} \quad \frac{dM}{d\phi} = -F_r r \\
&\text{(d)} \quad \frac{d\theta}{d\phi} = \frac{r}{EI} M \\
&\text{(e)} \quad \frac{dv}{d\phi} = u + \theta r \\
&\text{(f)} \quad \frac{du}{d\phi} = -v
\end{aligned} \qquad (3.19)$$

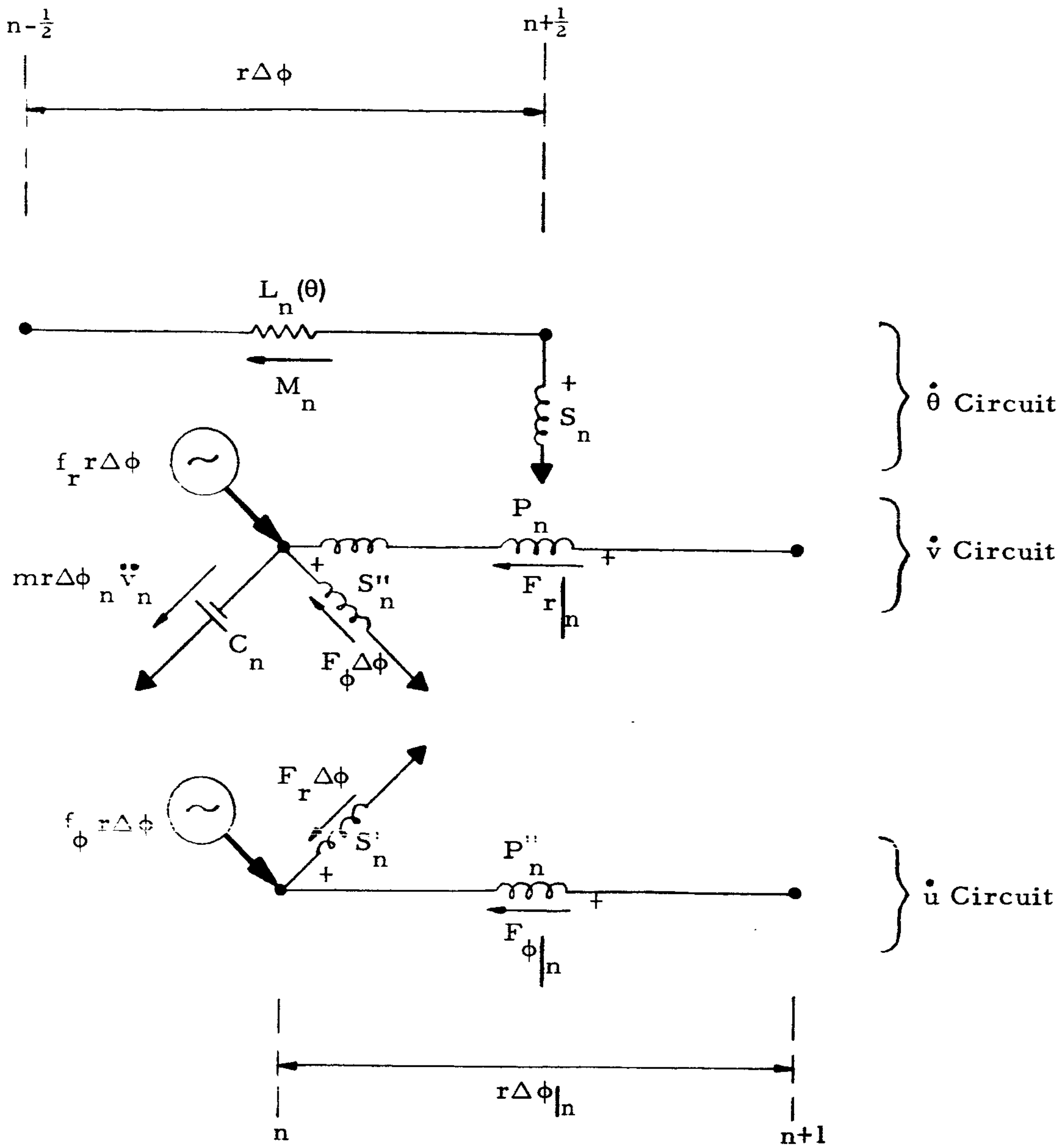
Except for notational changes and the radial inertial force, these equations are found in Reference 12.

Expressed as first-order difference equations, the equations of (3.19) appear as

$$
\begin{aligned}
&\text{(a)} \quad \Delta_\phi (F_r)_n = (F_\phi - f_r + m r \ddot{v}) \, \Delta\phi \\
&\text{(b)} \quad \Delta_\phi (F_\phi)_n = -(F_r + f_\phi r) \, \Delta\phi \\
&\text{(c)} \quad \Delta_\phi (M)_{n+\frac{1}{2}} = -F_r \, r \, \Delta\phi \\
&\text{(d)} \quad \Delta_\phi (\theta_n) = \frac{r\Delta\phi}{EI} M_n \\
&\text{(e)} \quad \Delta_\phi (v_n) = (u_n + \theta r) \, \Delta\phi \\
&\text{(f)} \quad \Delta_\phi (u_n) = -v_n \Delta\phi
\end{aligned}
\tag{3.20}
$$

The first three equations are obtained from force equilibrium conditions and are simulated electrically by applying Kirchhoff's current law at appropriate nodes. The fourth equation is determined from stress-strain conditions and is simulated electrically by applying Ohm's law in the slope circuit. The last two equations are obtained from strain-displacement conditions and are simulated by ideal transformers.

Defining a conventional difference gridwork, a mobility circuit for the in-plane vibration of a curved beam is shown as Figure 3.14. The magnitude of the bending inductor is calculated by integrating the bending flexibility distribution over the arc length from angular position $\phi_{n-\frac{1}{2}}$ to $\phi_{n+\frac{1}{2}}$. Similarly, the magnitude of the radial inertial capacitor is obtained by integrating the mass distribution over the same angular positions. Setting the S_n winding equal to unity, the P_n winding corresponds to the difference arc length for the nth beam segment, whereas the remaining transformers denote the magnitude of the ϕ angle.



$$C_n = mr\Delta\phi\Big|_n \qquad \frac{P_n}{S_n} = r\Delta\phi\Big|_n$$

$$L_n(\theta) = \frac{r\Delta\phi}{EI}\Big|_n \qquad \frac{P'_n}{S'_n} = \frac{P''_n}{S''_n} = \Delta\phi\Big|_n$$

Figure 3. 14 Mobility Analog for the In-plane Vibration of a Curved Beam

Deleting the primed transformers reduces the curved beam circuits to those for the lateral vibration of a simple beam. Thus, these transformers account for the geometry change necessary to convert a straight beam segment into a curved beam segment and is shown to couple the radial (v) and tangential (u) motions.

By making use of the previous derivations, the basic circuit of Figure 3.14 can be intuitively altered to include additional effects. As an example, rotatory inertia and shear deformation are accounted for in the Timoshenko beam circuit by the inertial capacitor $C_n(\theta)$ and the shear inductors $L_n(V)$ and $L_n'(V)$. These effects are included in the curved beam by adding a rotatory inertial capacitor $C_n(\theta)$ to the slope circuit at $n + \frac{1}{2}$ and adding (in series) the shear inductors $L_n(V)$ and $L_n'(V)$ to either side of the P_n winding in the v circuit.

3.6 RECTANGULAR SHEAR PANEL

A shear panel is a two-dimensional structure capable of resisting shear forces applied to its edges. A differential segment of a rectangular shear panel with the shear forces, the accompanying displacements, and coordinate directions is shown as Figure 3.15. This sketch depicts u and u' as the deflection components in the x direction, v and v' the deflection components in the y direction, F_{yx} the shear force perpendicular to the y axis and in the x direction, and F_{xy} the shear force perpendicular to the x axis and in the y direction. From force equilibrium conditions

$$F_{xy}\,dx = F_{yx}\,dy \tag{3.21}$$

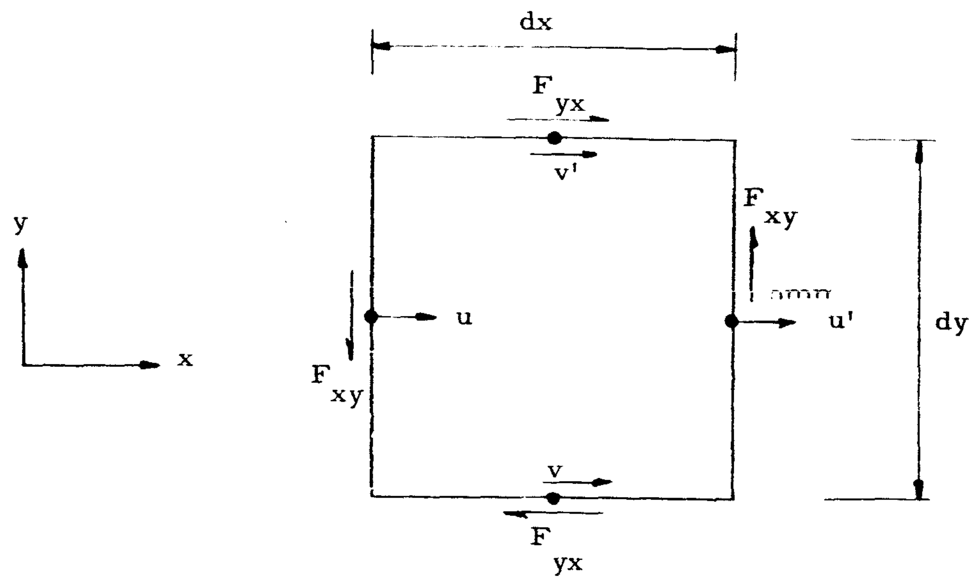


Figure 3. 15 Differential Segment of a Rectangular Shear Panel

From stress-strain considerations, the shear flow appears as

$$\frac{F_{yx}}{dx} = Gh\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3.22}$$

where G is the shear modulus and h the panel thickness. As first-order difference equations, the above differential equations become

$$\begin{aligned} F_{xy}\Delta x &= F_{yx}\Delta y \\ \frac{F_{yx}}{\Delta x} &= Gh\left\{\frac{\Delta_y(u)}{\Delta y} + \frac{\Delta_x(v)}{\Delta x}\right\} \end{aligned} \tag{3.23}$$

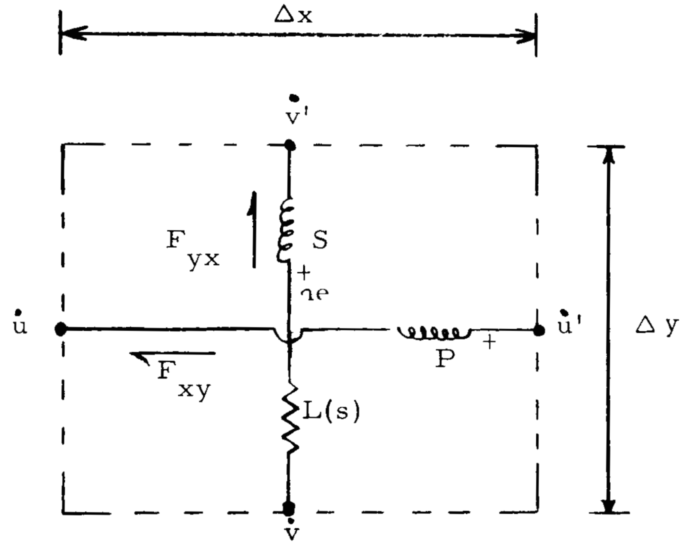
where the lower equation describes a coupling of mutually perpendicular motions.

Consistent with the mobility approach, the equations of (3. 23) are expressed in terms of force and velocity as

$$F_{xy}\,\Delta x = F_{yx}\,\Delta x$$

$$F_{yx} = \frac{G h \Delta x}{\Delta y} \int \left\{ \Delta_y(\dot{u}) + \frac{\Delta y}{\Delta x}\, \Delta_x(\dot{v}) \right\} dt \qquad (3.24)$$

Defining a two-dimensional rectangular difference grid, these equations are simulated in the circuit shown as Figure 3. 16.



$$L(s) = \frac{\Delta y}{Gt\Delta x} \qquad \frac{P}{S} = \frac{\Delta x}{\Delta y}$$

Figure 3. 16 Mobility Circuit for a Rectangular Shear Panel

The transformer serves to couple the $\dot{u}$ and $\dot{v}$ motions and the shear inductor L(s) accounts for the shear strain energy of the panel. If the inertial forces of the panel were included, capacitors would be added to the $\dot{u}$ and $\dot{v}$ nodes.

3.7 LATERAL VIBRATION OF A FLAT, RECTANGULAR PLATE

As contrasted to a one-dimensional structure such as a beam, the plate considered here is a two-dimensional elastic structure assumed as homogeneous, uniform, and to obey small deflection theory. Although the plate physical properties are assumed uniform, the circuits to be derived are equally appropriate for nonuniform properties.

A differential section of a laterally loaded plate with the accompanying moments and shear forces is sketched as Figure 3. 17. The force notation shows M as the bending moments per unit length, Q the shear forces per unit length, and f(x, y, t) the external loading per unit area. The differential segment is of thickness h and has the differential dimensions dx and dy. From the content of Chapter 4 in Reference 19, the equation of motion for the lateral vibration of a flat rectangular plate can be expressed as

$$D\nabla^4(w) + m\ddot{w} = f(x, y, t) = f \tag{3.25}$$

where m is the mass per unit area, w the lateral deflection from the static equilibrium position, ν the Poisson's ratio, and $(\ddot{\ })$ the second time derivative of the quantity ( )

$$D = \text{plate flexural rigidity} = \frac{Eh^3}{12(1 - \nu^2)}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

The loading is assumed normal to the surface and the deflections are small in comparison with the plate thickness. The effect on bending due to the shearing forces Q_x and Q_y and the compressive stress in the lateral direction due to the external loading are both neglected.

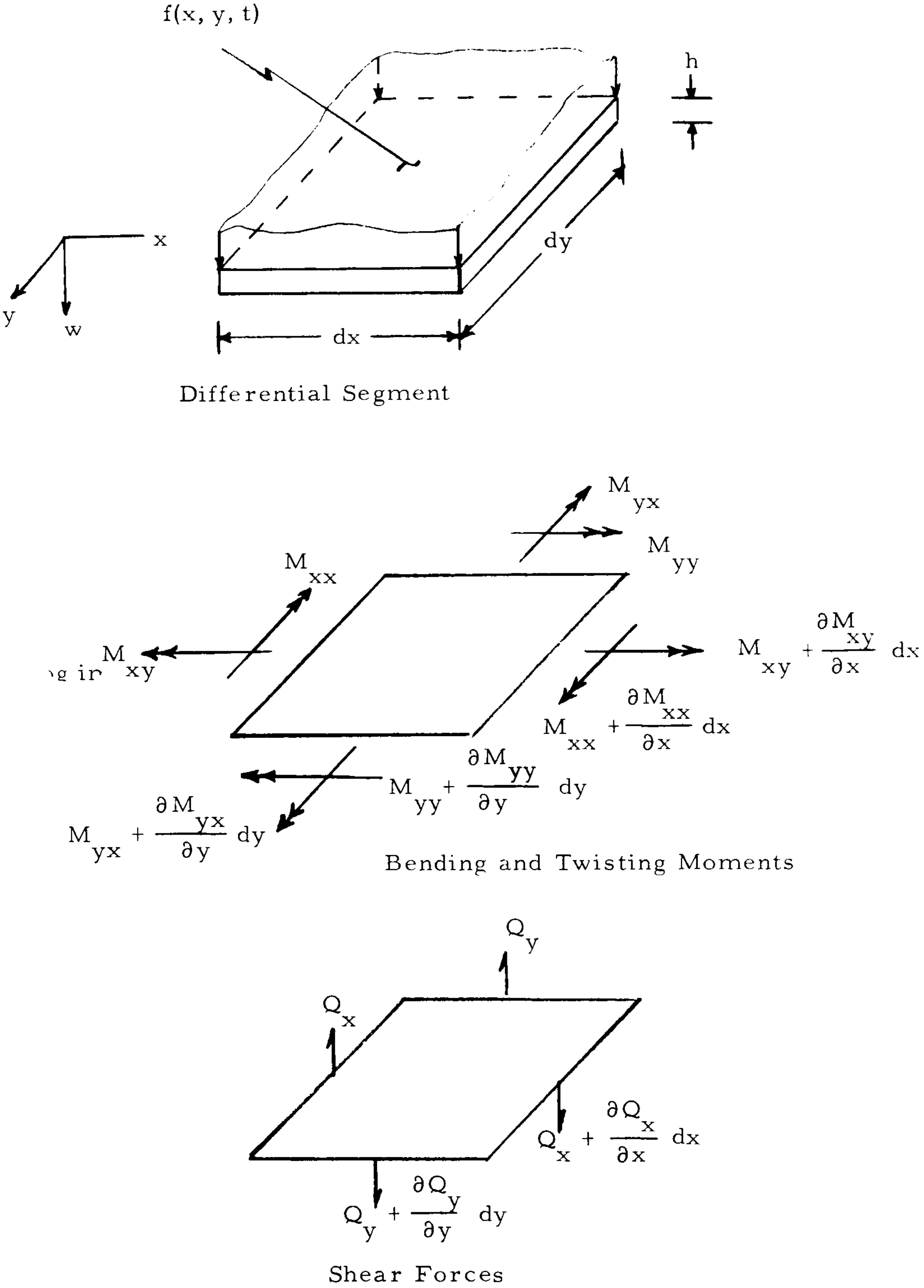


Figure 3.17 Differential Segment of Rectangular Plate and the Associated Forces

The equation of motion for the plate is derived from the following differential equations

$$
\begin{aligned}
&(a) \qquad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + f - m\ddot{w} = 0 \\
&(b) \qquad \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_{xx}}{\partial x} - Q_x = 0 \\
&(c) \qquad \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{yy}}{\partial y} + Q_y = 0 \\
&(d) \qquad M_{xx} = -D\left\{\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right\} \\
&(e) \qquad M_{yy} = -D\left\{\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right\} \\
&(f) \qquad -M_{xy} = M_{yx} = D(1 - \nu)\frac{\partial^2 w}{\partial x \, \partial y} \\
&(g) \qquad \theta_x = \frac{dw}{dx} \\
&(h) \qquad \theta_y = \frac{dw}{dy}
\end{aligned}
\tag{3.26}
$$

The first three equations (a, b, and c) are obtained from equilibrium conditions, the next three moment equations (d, e, and f) from stress-strain relationships, and the last two equations (g and h) from strain-displacement definitions.

In finite-difference form without the spatial position notation, the above equations appear as

$$
\begin{aligned}
&(a) && \frac{\Delta_x(Q_x)}{\Delta x} + \frac{\Delta_y(Q_y)}{\Delta y} + f - m\ddot{w} = 0 \\
&(b) && \frac{\Delta_y(M_{yx})}{\Delta y} + \frac{\Delta_x(M_{xx})}{\Delta x} - Q_x = 0 \\
&(c) && \frac{\Delta_x(M_{xy})}{\Delta x} - \frac{\Delta_y(M_{yy})}{\Delta y} + Q_y = 0 \\
&(d) && M_{xx} = -D\left\{\frac{\Delta_x(\theta_x)}{\Delta x} - \nu \frac{\Delta_y(\theta_y)}{\Delta y}\right\} \\
&(e) && M_{yy} = -D\left\{\frac{\Delta_y(\theta_y)}{\Delta y} + \nu \frac{\Delta_x(\theta_x)}{\Delta x}\right\} \\
&(f) && -M_{xy} = M_{yx} = D(1 - \nu) \frac{\Delta_x(\theta_y)}{\Delta x} \\
&(g) && \theta_x = \frac{\Delta_x(w)}{\Delta x} \\
&(h) && \theta_y = \frac{\Delta_y(w)}{\Delta y}
\end{aligned}
\qquad (3.27)
$$

Expressing the coordinate motions as velocities, the mobility analog for the flat rectangular plate is obtained by electrically simulating the above equations and appears as Figures 3.18.

These circuits are shown as three distinct sets: (1) the lateral velocity circuit $\dot{w}$, (2) the slope circuit θ_x for bending of the x axis, and (3) the slope circuit θ_y for bending of the y axis. The gridwork displays four difference rectangular plate segments where the incremental distance between consecutive numbers is one-half of a difference length. Although the nodes in the $\dot{w}$ circuit are defined at odd-odd coordinate locations, the nodes in the slope circuits occur at odd-even coordinate positions.

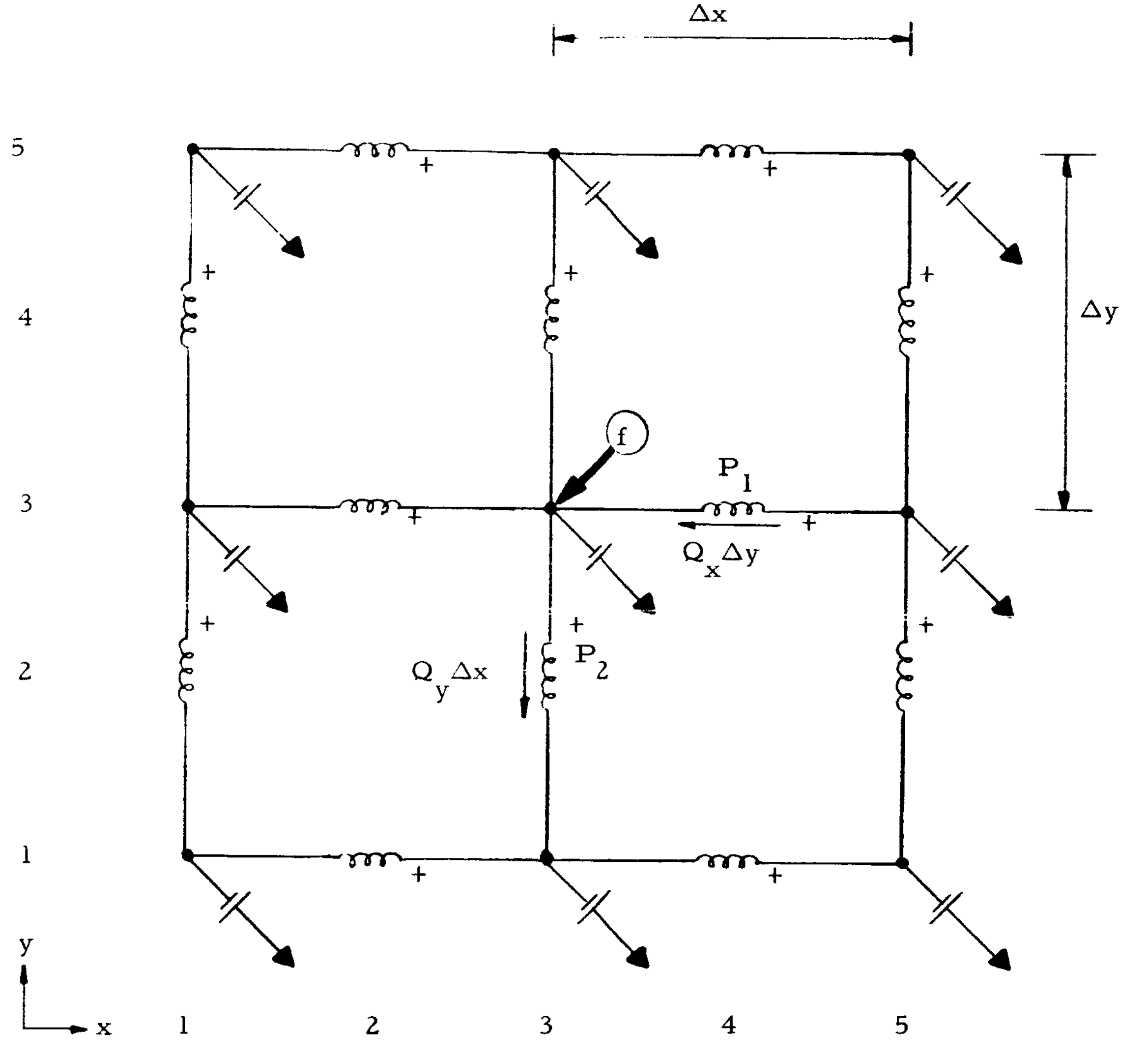


Figure 3. 18-a $\dot{w}$ Circuit for the Lateral Vibration of a Flat Rectangular Plate

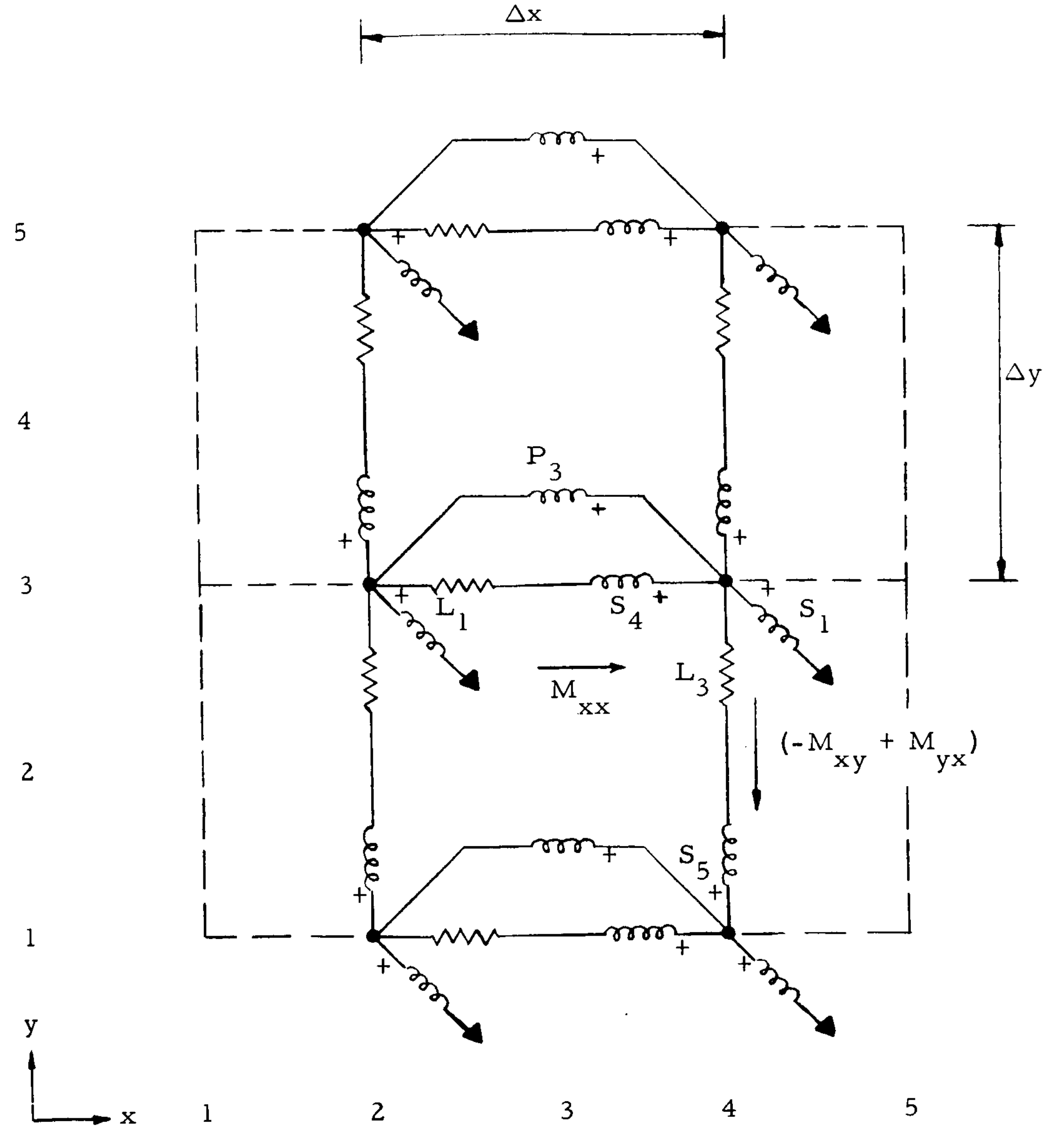


Figure 3. 18-b $\dot{\theta}_x$ Circuit for the Lateral Vibration of a Flat Rectangular Plate

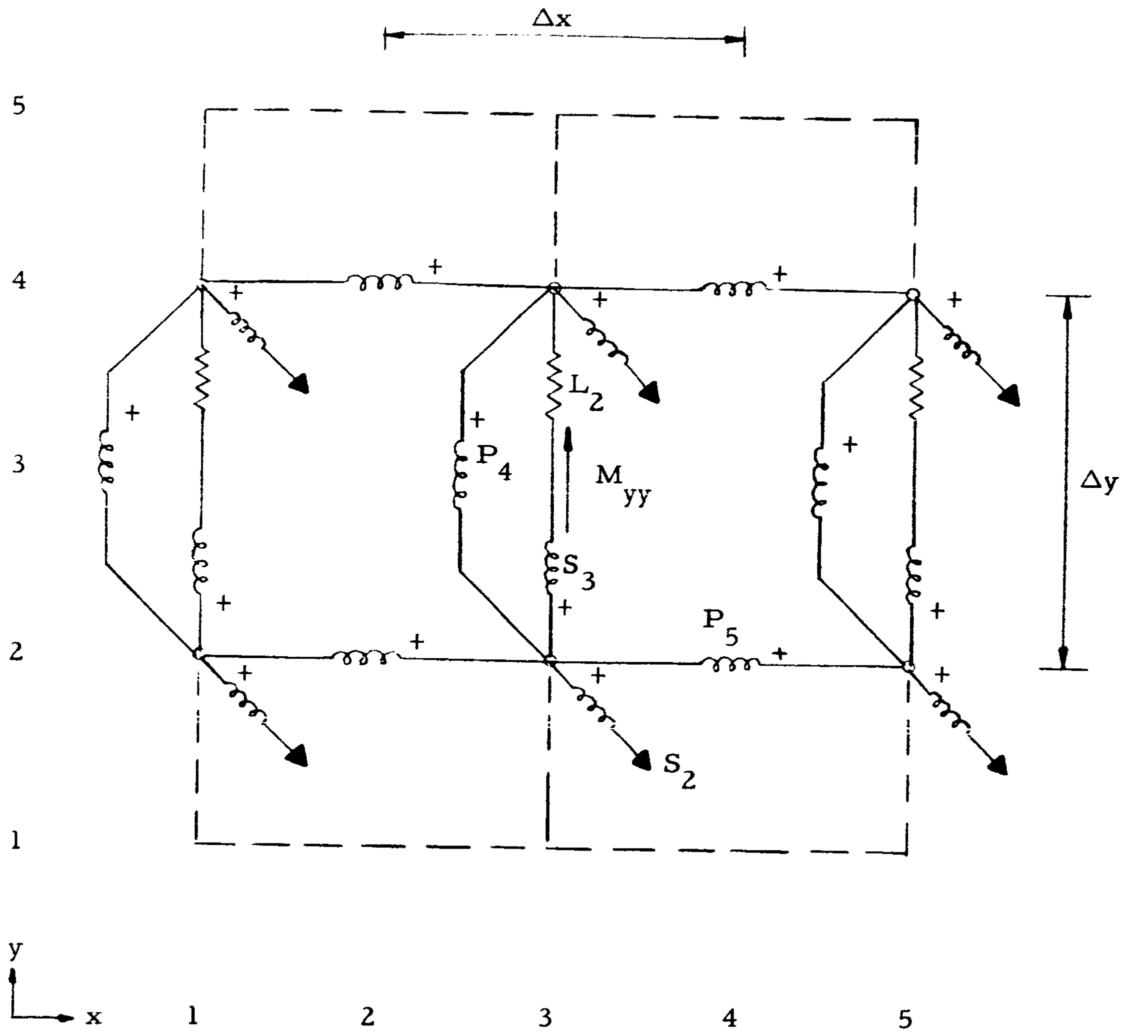
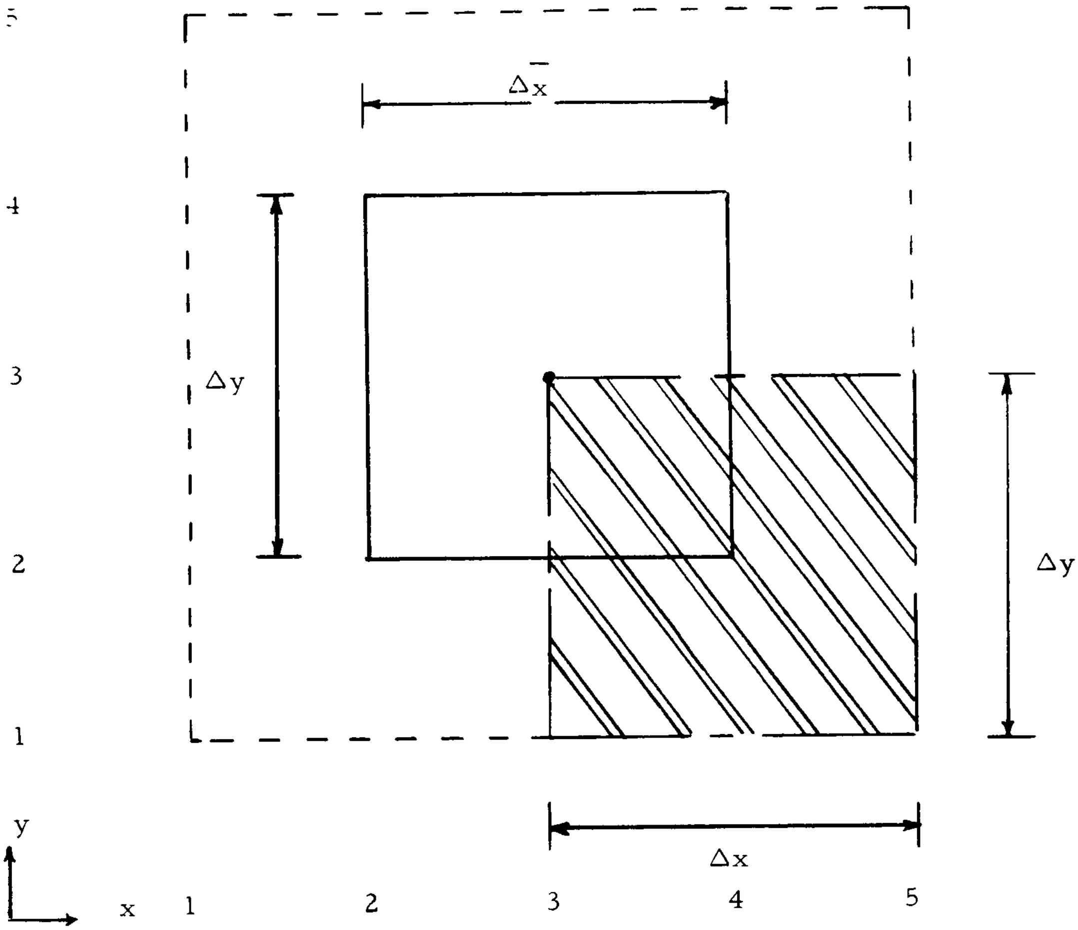


Figure 3.18-c θ_y Circuit for the Lateral Vibration of a Flat Rectangular Plate



$$C = m\Delta y\,\Delta x \qquad \frac{P_1}{S_1} = \Delta x \qquad \frac{P_3}{S_3} = \frac{\Delta x}{\nu \Delta y}$$

$$L_1 = \frac{\Delta x}{D} \qquad \frac{P_2}{S_2} = \Delta y \qquad \frac{P_4}{S_4} = \frac{\Delta y}{\nu \Delta x}$$

$$L_2 = \frac{\Delta y}{D}$$

$$L_3 = \frac{\Delta y}{D(1-\nu)} \qquad D = \frac{Eh^3}{12(1-\nu^2)} \qquad \frac{P_5}{S_5} = \frac{\Delta x}{\Delta y}$$

Figure 3. 18-d Component Values for the Rectangular Plate Circuits

The currents are shown in terms of their mechanical equivalents and correspond to the shear forces and moments in the difference segments. The L_1 and L_2 inductors account for the strain energy in bending, the L_3 inductor for the torsional strain energy, the capacitor for the kinetic energy of a plate segment, and the transformers for the plate geometry.

Figures (3.18) represent the flat plate as a rectangular array of simple beams constrained in bending by interlocking torsion members. The $\dot{w}$ circuits are those for a two-dimensional rectangular gridwork of simple beams in bending. The slope circuits, however, differ from those corresponding to simple beams in bending due to (1) Poisson coupling terms relating the M_{xx} and M_{yy} bending moments, and (2) the torsional restraint due to the M_{xy} and M_{yx} twisting moments. The Poisson coupling is represented by the transformers number three and four and the torsional restraint by transformer number five.

By way of review, consider the simulation of the torsional restraint. From the $\dot{\theta}_x$ circuit in Figure 3.18-b, the voltage across the L_3 inductor is

$$\Delta E\Big|_{L_3} = \Delta_y(\dot{\theta}_x) + \frac{S_5}{P_5}\,\Delta_x(\dot{\theta}_y) \tag{3.28}$$

Substituting the mechanical equivalent of the transformer ratio yields

$$\Delta E\Big|_{L_3} = \Delta_y(\dot{\theta}_x) + \frac{\Delta y}{\Delta x}\,\Delta_x(\dot{\theta}_y) \tag{3.29}$$

By adding the equations given as (3.27-f), the sum of the twisting moments becomes

$$-M_{xy} + M_{yx} = \frac{D(1-\nu)}{\Delta y}\int\left\{\Delta_y(\dot{\theta}_x) + \frac{\Delta y}{\Delta x}\,\Delta_x(\dot{\theta}_y)\right\}dt \tag{3.30}$$

Consistent with mobility procedures, the twisting moments are assumed as currents and the voltage is shown as the equation of (3.29). By imposing Ohm's law, the form of Eq. (3.30) is that for current flow through the inductor

$$L_3 = \frac{\Delta y}{D(1 - \nu)} \tag{3.31}$$

where L_3 is noted as the torsional inductor.

The mechanical equivalents of the circuit components for an arbitrary plate segment are shown as Figure 3.18-d. The inertial capacitor C, the bending inductors L_1 and L_2, and the bending transformers (numbers one, two, three, and four) are referenced to the plate segment centered at the grid position 3, 3. The torsional coupling inductor L_3 and the coupling transformer (number 5) are referenced to the cross-hatched plate segment.

3.8 SCALE FACTORS

To use the analog circuits shown here on a passive element computer, scale factors often must be considered to obtain electrical values compatible with available setting values for the circuit components. As shown, the circuit components for the analog circuits are given only in terms of their mechanical equivalents or, in other words, the scale factors are assumed as unity. Scale factors, therefore, are introduced so that the mechanical quantities can be conveniently adjusted to the available setting values on the computer. These factors are constants which relate the magnitude of the quantities in the mechanical system with the corresponding quantity in the electrical system.

Requiring the power in the analog circuits to be equal to the energy of the mechanical system yields the scale factor relationships shown as Figure 3.19. These factors are derived in a general way by MacNeal in Section 3.3 of Reference 13. The scaling constants are normally selected

FORCES	
$F = \frac{k}{a} I$	F = mechanical force
$M_\theta = \frac{kP_\theta}{a} I_\theta$	M_θ = bending moment
$T_\phi = \frac{kP_\phi}{a} I_\phi$	T_θ = torque or twisting moment
	I = current corresponding to F
	I_θ = current corresponding to M_θ
	I_ϕ = current corresponding to T_ϕ

COORDINATE MOTION	
$\dot{w} = \frac{ka}{N} e_{\dot{w}}$	$\dot{w}$ = lateral or linear velocity
$\dot{\theta} = \frac{ka}{NP_\theta} e_{\dot{\theta}}$	$\dot{\theta}$ = slope velocity
$\dot{\phi} = \frac{ka}{NP_\phi} e_{\dot{\phi}}$	$\dot{\phi}$ = angular (twisting) velocity
$t_m = N t_e$	t_m = real or mechanical time
	$e_{\dot{w}}$ = voltage corresponding to $\dot{w}$
	$e_{\dot{\theta}}$ = voltage corresponding to $\dot{\theta}$
	$e_{\dot{\phi}}$ = voltage corresponding to $\dot{\phi}$
	$k, a, P_\theta, P_\phi, N$ = scaling constants to be selected
	t_e = electrical time

Figure 3.19 General Scale Factor Relationships for Mobility Analogs

such that the capacitor setting range from .050 microfarads (μfd) to .900 microfarads, inductor settings from .050 henries (h) to 1 henry, resistor settings from 100 ohms (Ω) to 10,000,000 ohms. The time scale factor N is selected such that the electrical frequency range for the specific problem extends from 80 cycles per second (cps) to approximately 800 cycles per second.

Consider an application of these scale factors to determine the scaled magnitude of the circuit components for the lateral vibration of a simple beam. The lateral inertial force of a beam segment may be expressed as

$$F_{\dot{y}} = m_n \Delta x \frac{d\dot{y}}{dt_m} \tag{3.32}$$

where t_m is noted as mechanical time. Substituting the force, lateral velocity, and time scale factors from Figure 3.19 into the above expression

$$I_f = \frac{a^2}{N^2} m_n \Delta x \frac{de_{\dot{y}}}{dt_e} \tag{3.33}$$

where I_f denotes the current corresponding to the inertial force $F_{\dot{y}}$. This equation corresponds in form to current flow through a capacitor of magnitude

$$C_n = \frac{a^2}{N^2} m_n \Delta x \tag{3.34}$$

In a similar manner, the bending inductor and transformer ratio are calculated and shown to equal

$$L_n(\theta) = \left(\frac{P_\theta}{a}\right)^2 \left.\frac{\Delta x}{EI}\right|_n \tag{3.35}$$

$$\frac{P_n}{S_n} = \frac{\Delta x_n}{P_\theta} \tag{3.36}$$

As an additional example, consider the calculation of the shear deformation inductors $L_n(V)$ in the Timoshenko beam circuits. The relationship between the shear force and the deformation due to shear is given by Eq. (3. 16-f) as

$$V_n \frac{\Delta x}{K'AG} = \Delta_x(y_s) = \Delta_x\left(\int \dot{y}_s \, dt_m\right) \tag{3. 37}$$

Substituting the shear, lateral velocity, and time scale factors into the above difference expression provides

$$I_v = a^2 \frac{K'AG}{\Delta x} \Delta_x \left(\int e_y t_e\right) \tag{3. 38}$$

where I_v denotes the current corresponding to the shear force V_n. This equation corresponds in form to current flow through an inductor of magnitude

$$L_n(V) = \frac{1}{a^2} \frac{\Delta x}{K'AG} \tag{3.39}$$

Note that the Δx in the above expression is twice the difference length of the shear inductors shown in Figure 3.11. Thus, the sum of the inductors in the Timoshenko beam circuit equals the magnitude given by Eq. (3.39).

3.9 BOUNDARY CONDITIONS

The mobility circuits derived in this discussion are electrical models of difference segments for specific distributed elastic systems. These circuits are elemental building blocks to be used in synthesizing complete structural systems. In constructing a complete electrical model, the boundary conditions of the original system must be accounted for electrically. Several beam problems will be considered to illustrate the handling of typical boundary conditions.

Consider a six-segment representation of a cantilever beam described by simple beam theory. The basic circuit for a segment of simple beam is shown as Figure 3.7. Interconnecting six such circuits (without the lateral loading) in tandem yields the analog circuit shown as Figure 3.20.

The electrical network is a mobility analog, thus, the voltages and currents in the slope circuit $\dot{\theta}$ correspond to the slope velocities and bending moments respectively; and, the voltages and currents in the lateral motion circuit $\dot{y}$ correspond to lateral velocities and shear forces respectively. The circuit components and their mechanical equivalents for the nth segment are shown: the capacitor C_n corresponds to the lumped mass m_n, the inductor $L_n(\theta)$ to the spring $k_n(\theta)$, and the nth transformer to the rigid lever labeled as Δx_n. The beam grid defines the lateral motion nodes as even digits and the slope nodes as odd digits. Therefore, the difference between any two odd or any two even positions is the difference segment Δx.

The boundary conditions for a cantilever beam are expressed as

$$y(0, t) = 0$$

$$\frac{\partial y(0, t)}{\partial x} = \theta(0, t) = 0$$

$$EI \frac{\partial^2 y(\ell,t)}{\partial x^2} = M(\ell, t) = 0 \qquad (3.40)$$

$$EI \frac{\partial^3 y(\ell,t)y}{\partial x^3} = V(\ell, t) = 0$$

where the fixed or built-in end condition (x=0) allows no lateral deflection nor slope and the free end ($x=\ell$) can carry no shear force nor bending moment. No lateral nor slope motion at the built-in end is equivalent to

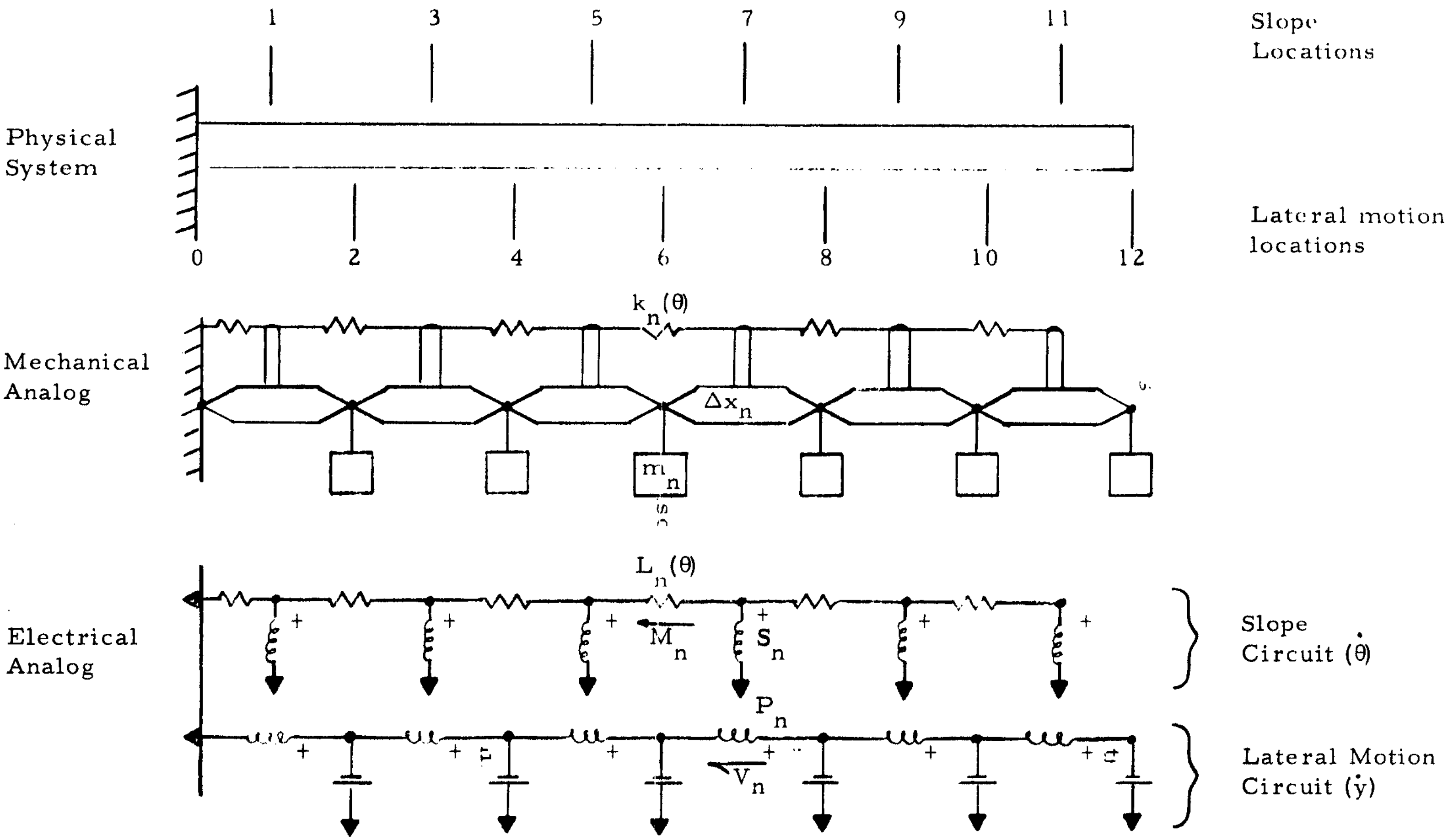


Figure 3. 20 Mechanical and Electrical Lumped Parameter Analogs for a Cantilevered Beam

zero voltage at the appropriate nodes in the lateral motion and slope circuits. This is accomplished electrically by grounding the nodes at the spatial location corresponding to $x = 0$. No bending moment nor shear at the free end is equivalent to zero current flow out of the appropriate nodes in the $\dot{y}$ and $\dot{\theta}$ circuits. This is done electrically by opening the circuit at the spatial positions shown as 11 and 12.

Instead of the fixed boundary conditions at $x = 0$, suppose the cantilever beam was attached to a flexible boundary allowing both lateral and bending motions at the root. This is accounted for electrically by inserting inductors L_θ and L_y to ground in both the $\dot{y}$ and $\dot{\theta}$ circuits at the $x = 0$ spatial location as depicted in Figure 3.21. The magnitudes of these inductors depend upon the amount of flexibility in the boundaries. Setting the inductors to zero (shorting them from the circuit) reduces the flexible boundaries to an infinite stiffness or the original rigid boundary conditions for the fixed end of a cantilever beam.

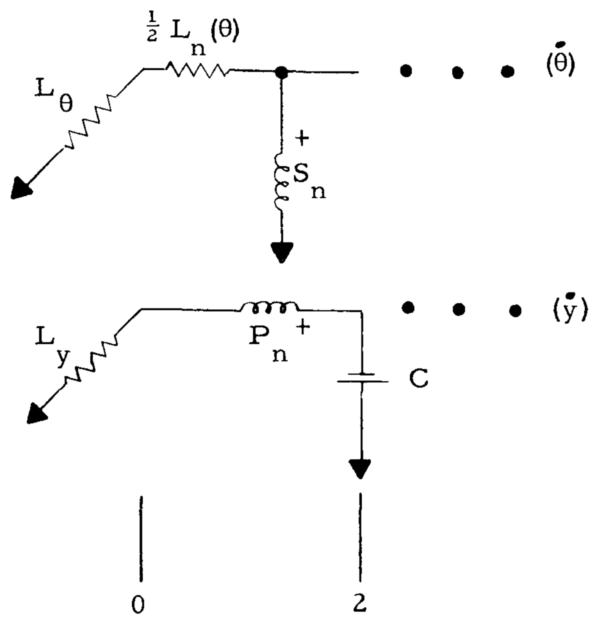


Figure 3.21 Flexible Boundary for the Built-in End Condition of a Cantilever Beam

Consider the six-segment simulation of the partially loaded beam structure shown as Figure 3.22. Simple beam theory is assumed adequate to describe the structure. The boundary conditions are those of a cantilever beam with an intermediate support at the center span. Thus, the analog simulation of the end conditions are identical with those shown in Figure 3.20. The intermediate support allows no deflection and requires continuity in the bending moment and slope at the mid span. These are simulated electrically by grounding only the $\dot{y}$ circuit at the node corresponding to spatial position 6. The inertial capacitor is shorted from the circuit at this position and is not shown. The current generators correspond to the external loading lumped at nodal positions 8, 10, and 12.

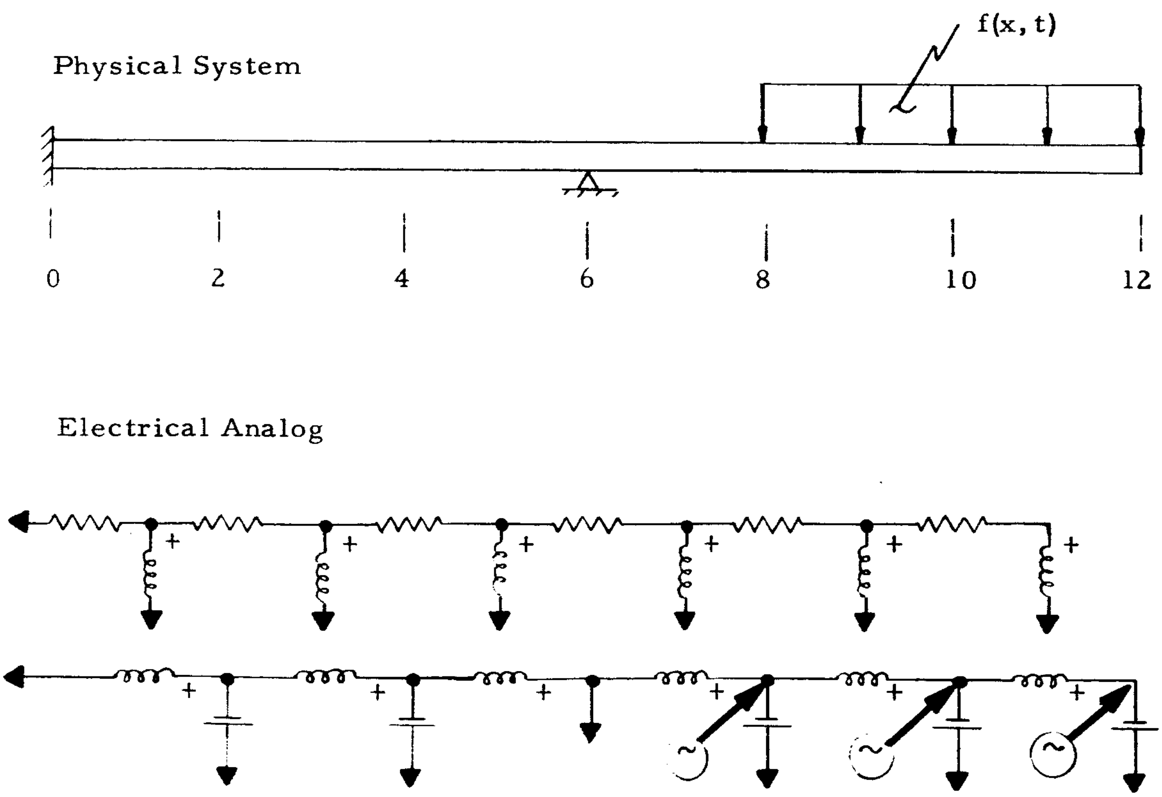


Figure 3.22 Mobility Analog of a Partially Loaded Distributed Beam

4. OTHER PHYSICAL SYSTEMS

Passive analog concepts are generally applicable for most any physical system described as a function of both space and time, i.e., a partial differential equation. Although various electrical analogs were considered in the previous sections, mobility circuits were emphasized for simulating the vibration of linear elastic structures. In this section, mobility-oriented circuits are considered for physical systems other than elastic structures.

4.1 VISCOELASTIC MODEL

For some problems, the properties associated with a viscoelastic material such as solid propellant can be described as a complex stiffness where the force-displacement relationship is

$$F = K_0'(1 + i\Phi)x \tag{4.1}$$

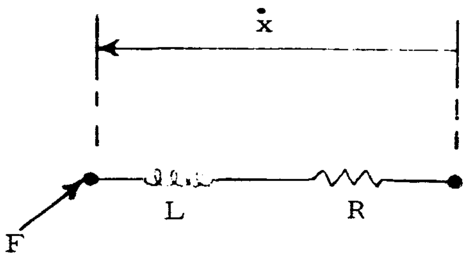
Expressed in terms of the velocity, the above equation becomes

$$F = K_0'(1 + i\Phi)\int \dot{x}\, dt = \frac{K'(1 + i\Phi)}{i\omega}\,\dot{x} \tag{4.2}$$

Consistent with mobility procedure, the impedance appears as

$$Z = \frac{e}{i} = \frac{i\omega}{K_0'(1 + i\Phi)} = \frac{i\omega}{K_0'(1 + \Phi^2)} + \frac{\omega\Phi}{K_0'(1 + \Phi^2)} \tag{4.3}$$

where force is assumed analogous to current and velocity analogous to voltage. By Ohm's law, Equation (4.3) is electrically equivalent to the combined impedance of an inductor in series with a resistor shown as Figure 4.1. The mechanical equivalents of the inductor and resistor are found by writing the impedance for the RL series combination and comparing the resulting terms with Eq. (4.3).



$$L = \frac{1}{K'(1 + \Phi^2)}$$

$$R = \frac{\Phi}{K'(1 + \Phi^2)}$$

Figure 4.1 Mobility Circuit for a Viscoelastic Spring

This simple mobility circuit can be used in conjunction with known analogs of elastic structures to create models of viscoelastic structures. For example, to create an analog depicting the lateral motion of a viscoelastic beam, the RL series combination is substituted for the bending inductor $L_n(\theta)$ in Figure 3.7. For an analog model depicting the vibration of an elastic beam on a viscoelastic foundation, the RL series combination is substituted for the elastic foundation inductor $L_n(K_f)$ in Figure 3.12. Other viscoelastic models can be created by similar substitutions.

4.2 HEAT TRANSFER OF A CYLINDRICAL ROD

Consider the derivation of an analog circuit depicting the temperature characteristics of a homogeneous cylindrical section of rod. As discussed by Pipes in Section 14 of Reference 16, equating the loss of heat energy in a volume of material to the heat flow from the volumetric surface yields

$$-\frac{d}{dt}\iiint c\rho\, T\, dV = \iint \bar{q} \cdot \bar{n}\, dS \qquad (4.4)$$

where S is the surface of the body, T the temperature of the body, V the volume of the body, c the specific heat of the material, $\bar{n}$ the vector normal to the cylindrical surface, $\bar{q}$ the vector denoting the heat flow, ρ the density of the body per unit volume, and t the time.

Expressing this equation as a volumetric integral provides

$$\iiint \left(c\rho \frac{dT}{dt} + \nabla \bar{q} \right) dV = 0 \qquad (4.5)$$

where Gauss' theorem is used to transform the surface integral and ∇ is the dell operator common to vector analysis.

Since the volume is finite and the integral is continuous, the integrand of Eq. (4.5) can vanish only if

$$c\rho \frac{dT}{dt} + \nabla \bar{q} = 0 \qquad (4.6)$$

In expanded form, Eq. (4.6) appears as the diffusion equation for heat flow

$$c\rho \frac{dT}{dt} = k \nabla^2 T \qquad (4.7)$$

where the heat flow is given by

$$\bar{q} = -k \nabla T \qquad (4.8)$$

and k is the thermal conductivity. For cylindrical coordinates, the square of the dell operator is expressed as

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial T}{\partial R} \right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (4.9)$$

Substituting the above equation into (4.7) yields the heat flow equation for a cylindrical section

$$\frac{c\rho}{k}\frac{dT}{dt} = \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial T}{\partial R} \right) + \frac{1}{R^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \qquad (4.10)$$

Assuming the temperature to be independent of the angular coordinate θ, Eq. (4.10) reduces to the two-dimensional heat flow equation

$$c\rho \frac{dT}{dt} = k\left\{\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) + \frac{\partial^2 T}{\partial z^2}\right\} \qquad (4.11)$$

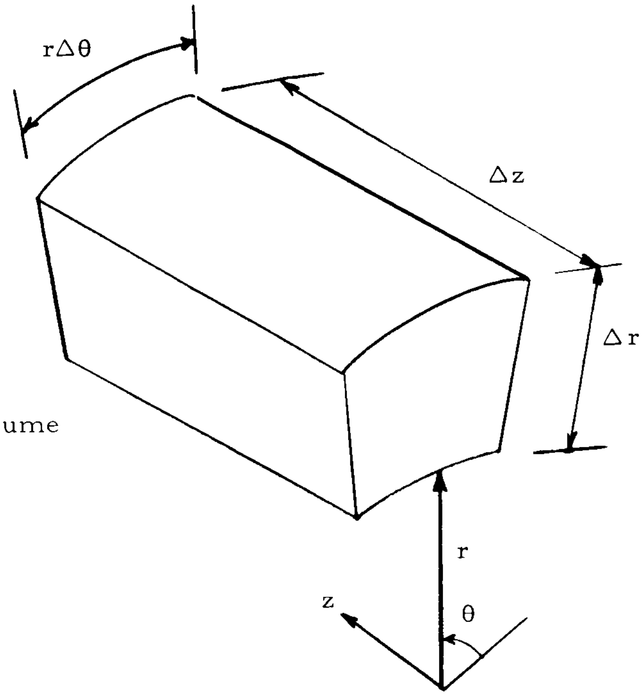
Expressed as a finite-difference equation for a difference volume of the material, the two-dimensional heat flow equation appears as

$$c\rho r\,\Delta r\,\Delta z\,\frac{dT_n}{dt} = \frac{k\,r\Delta z}{\Delta r}\left\{(T_{n+1} - T_n) - (T_n - T_{n-1})\right\}_r + \frac{kr\,\Delta r}{\Delta z}\left\{(T_{n+1} - T_n) - (T_n - T_{n-1})\right\}_s \qquad (4.12)$$

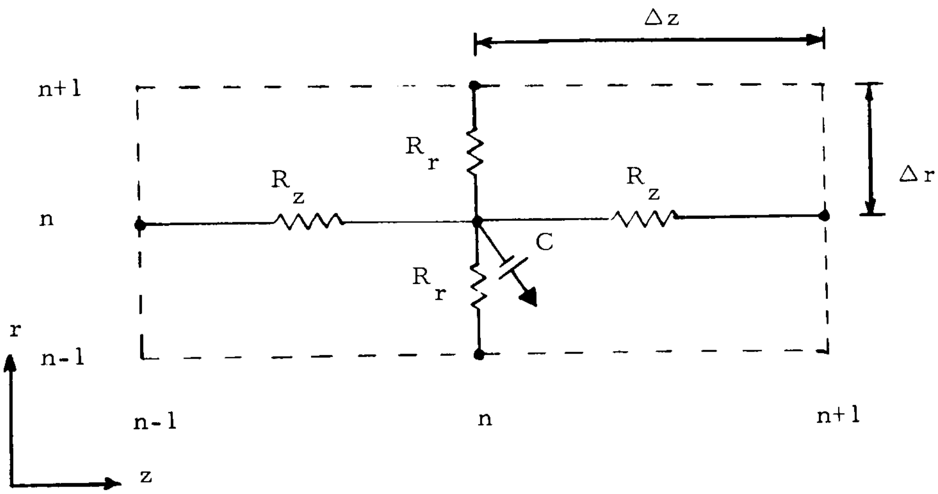
where subscripts below the braces denote the coordinate directions and $\Delta\theta$ is assumed as one radian.

Defining a r-z rectangular grid, a difference RC network simulating Eq. (4.12) is shown as Figure 4.2. Consistent with the mobility approach, the temperature is proportional to voltage and the heat flow proportional to the current. The capacitor denotes the heat capacity of the material and the resistor the thermal resistivity in the radial and axial directions. A heat or cooling source would be represented as a current generator connected to the appropriate temperature node. Similar analogs for heat transfer are discussed in greater depth by Karplus and Soroka in Chapter 10 of Reference 11.

Difference Volume in Cylindrical Coordinates



Difference Network



$$R_r = \frac{\Delta r}{kr\Delta z}$$

$$R_z = \frac{\Delta z}{kr\Delta r}$$

$$C = c\rho r\Delta r\Delta z$$

Figure 4.2 Analog Difference Network for Two-dimensional Heat Flow in Cylindrical Coordinates

4.3 THERMAL CHARACTERISTICS OF FLUID FLOW IN A DUCT

Consider the derivation of an electrical analog depicting the thermal characteristics of fluid flow through a duct. The quantities to be considered in this representation are shown in Figure 4.3.

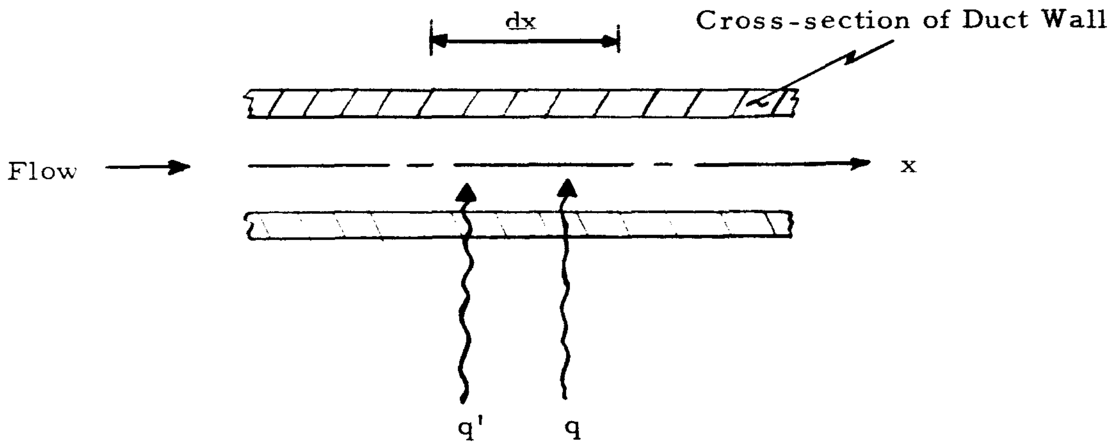


Figure 4.3 Sketch Depicting Thermal and Fluid Flow Through a Section of Duct

In this sketch, q denotes the heat flow into the fluid by conduction through the duct wall, q' the heat flow into the fluid from sources other than the surrounding body (such as internal friction or radiation), and x a spatial location along the flow path.

The differential equation describing the fluid temperature as a function of time and spatial position may be expressed as

$$\overline{w}\, c \left(\frac{1}{\overline{v}_1} \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \right) = q + q' . \tag{4.13}$$

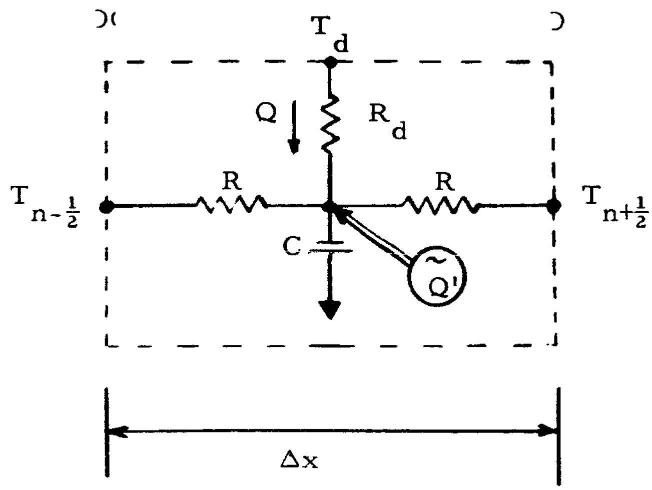
where T is the temperature of the fluid, c the specific heat of the fluid at constant pressure, t the time, $\overline{v}$ the velocity of the fluid, and $\overline{w}$ the mass flow rate of the fluid. Quantities such as the temperature and velocity are assumed constant over the duct cross-section at any instant. Although such quantities may vary over the cross-section, mean values are multiplied by the cross-section area to yield correct values of mass flow and heat flow of the fluid. The heat flow q into the fluid due to a temperature difference between the fluid and the duct wall appears as

$$q = sk(T_d - T) \tag{4.14}$$

where T_d is the temperature of the ambient surrounding such as the duct wall, k the surface conductivity coefficient between the fluid and the ambient surroundings, and s the perimeter of the duct cross-section. Substituting (4.14) into (4.13) and expressing the resultant equation in finite-difference form yields

$$\frac{\Delta x \overline{w} c}{\overline{v}} \frac{dT_n}{dt} + \overline{w} c \left(T_{n+\frac{1}{2}} - T_{n-\frac{1}{2}}\right) - Q'_n = sk\Delta x (T_d - T_n) \tag{4.15}$$

An analog circuit electrically simulating this difference equation is shown as Figure 4.4.



$$C = \frac{\Delta x \bar{w} c}{\bar{v}}$$

$$R = \frac{1}{2\bar{w}c}$$

$$R_d = \frac{1}{sk\Delta x}$$

Figure 4.4 Difference Circuit Depicting Heat Flow and Fluid Flow Through a Duct

The current is analogous to heat flow, the nodal voltages to the duct and fluid temperatures, the capacitor to the heat capacity of the fluid, the resistors R and R_d to the thermal resistivity of the fluid and duct wall respectively. The capital Q and Q' symbols denote the heat flows by conduction and external sources for the difference length Δx. A thorough discussion of this topic area is given by Dixon in Reference 6.

5. CONCLUDING REMARKS

The distinction between structural vibration problems and the topics of Section 4 should be clearly understood. Vibration of the simpler elastic structures is categorized mathematically as a type of wave equation whereas problems common to heat transfer and fluid flow are categorized mathematically as diffusion equations. The difference between wave and diffusion equations is the order of the time derivative; the wave equation has a second time derivative while the diffusion equation has a first time derivative.

Although the same general procedure can be applied to derive passive analog circuits for both types of equations, the resultant analogs are distinct. Analogs for the vibration of structures are force-current, velocity-voltage finite-difference circuits of the wave equation. These types of circuits are called mobility analogs. Electrically these circuits consist of capacitors, inductors and transformers. If viscous damping is included, resistors then are added in the mobility circuits. The electrical impedance of mobility analogs corresponds to mobility whereas the electrical admittance of mobility analogs corresponds to mechanical impedance.

Analogs for heat transfer and fluid flow are 'flow'-current 'dependent-variable'-voltage, finite-difference circuits of the diffusion equation. These circuits are referred to as quasi-mobility analogs; and differ from mobility analogs in the type of electrical components needed and in the mechanical interpretation of the current and voltage. Current is assumed analogous to heat flow and temperature is the dependent variable for heat transfer simulation. Electrically, quasi-mobility analogs consist chiefly of resistors and capacitors, although transformers may sometimes be required.

Another important analogy common to static analyses of structures is a force-current, displacement-voltage, finite-difference circuit. This type of circuit is referred to as a static-mobility analog and consists of resistors and transformers. Such analogs electrically describe, in finite-difference form, the static behavior of linear eleastic structures as defined by elasticity theory where the body forces are neglected. Since the spatial derivatives remain unchanged, the equations provided by the elasticity theory differ in form from that of a wave equation only in the absence of the time derivative term. Thus, since elasticity theory contains all of the spatial derivatives for a given structure, an important use of static-mobility analogs is to derive mobility analogs. By simply changing the resistors to inductors and adding capacitors to the nodes (to account for the inertial or body forces), static analogs are routinely converted to mobility analogs. This procedure is illustrated only for the simple beam.

In contrast with the RLC (resistor-inductor-capacitor) or RC circuits common to the simpler, discrete physical systems, transformers are introduced to describe the spatial characteristics of distributed systems. Mathematically, these spatial characteristics are defined as the differential operators of the partial differential equations and become increasingly complex in progressing from rectangular to polar coordinates. Synthesizing circuits for higher order spatial derivatives is one of the most difficult tasks in analog derivation.

The analog derivation procedure emphasized in this discussion consists of recognizing circuit laws electrically equivalent to first-order finite difference equations representing the dynamic behavior of the structures. Force equilibrium corresponds to Kirchhoff's current laws, stress-strain relationships to Ohm's law, and compatibility expressions to voltage drop relationships. In contrast to this procedure, Barnoski (Reference 1) and Barnoski and Freberg (Reference 3) equate strain energy with electrical power to obtain static analogs of complex structures.

The mobility analogs of this discussion are considered as elemental structural building blocks and can be used to synthesize electrical models of complete structural systems. In general, these electrical models are set up on a passive element analog computer, and a planned experimental program is performed electrically. An alternate application of the analog circuits is to use circuit analysis procedures to analyze complex structures by digital computers. Such approaches are suggested by Barnoski (Reference 2) and Walker (Reference 20) and, although still undeveloped, should result in analysis procedures for complex structural systems incorporating the advantages of both analog and digital computers.

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