X-642-66-273

NASA TM X-55502

INVERSE L TO I TRANSFORMATION

BY

E. G. STASSINOPOULOS

APRIL 1966

NASA

GODDARD SPACE FLIGHT CENTER

GREENBELT, MD.

INVERSE L TO I TRANSFORMATION

SUMMARY: It is shown that the second adiabatic invariant I may be defined as a function of the magnitude of the field B and the shell parameter L. A method is developed to obtain solutions for I in terms of arbitrary B and L. Results are tested and discussed.

INTRODUCTION

The wide acceptance of McIlwain's magnetic coordinates B and L (McIlwain 1961) and their almost exclusive use in the mapping of geomagnetically trapped particles, has resulted in the gradual disuse of the second adiabatic or longitudinal invariant I. Many studies pertaining to the motion or the spatial distribution of trapped particles, employ predominantly the parameters B and L in developing their theories (Galbraith 1965), analysing phenomena (Hess 1964), and presenting experimental data (McIlwain 1963).

Already a great amount of data have been given exclusively in L without any reference to I. This trend will probably prevail even more, as new and faster methods of obtaining L are being devised and used (Stern 1965), making its calculation entirely independent of I.

When considering however topics like the distribution, the shell configuration or the drift of these particles, it might be desirable, if not advantageous, to use I in addition to B and L or instead of L. In most such instances the integral invariant will not be readily available and will have to be obtained by some computational method.

The conventional computation of I may not be of help, because it can only be performed, if the appropriate geographic coordinates are known, in which case one must resort to the relatively slow process of integrating numerically along a field line an expression equivalent to that defining I:

$$I = \int_{B_m}^{B_m'} \left(1 - \frac{B(s)}{B_m}\right)^{1/2} ds \qquad (1)$$

where B(s) is the field strength along a line of force, B_m and B_m' are the magnitudes of the field at the conjugate mirror points, and ds is the differential path length on a field line from B_m to B_m'. Also it is not unlikely, that sometimes a line integration to determine L has already been performed and we just wish to save another integration when deriving 'I'.

This paper will present a simple way, by which 'I' may be obtained from the known quantities B and L only, without the need of integration.

METHOD

Let us define the integral invariant as a function of the magnetic parameters B and L:

$$I = f(B, L) \tag{2}$$

and let us stipulate that an acceptable solution to this function should be positive, real and single valued.

Recalling from McIlwain (1961) that:

$$\frac{L^3B}{M} = F\left(\frac{I^3B}{M}\right) \tag{3}$$

where:

$$F\left(\frac{I^3B}{M}\right)^{*} = 1 + \exp\left(\Sigma a_m \alpha^m\right) \qquad m = 0, 1 \ldots 9 \tag{4}$$

and:

$$\alpha = \ln\left(\frac{I^3B}{M}\right) \tag{5}$$

---

*Equation (4) is an approximation to a high order of accuracy, where the coefficients a_m have been evaluated for different ranges of α (Table 1).

we require that a solution to (2) must be derivable from (3). Writing out equation (3) and rearranging, we get:

$$\ln\left(\frac{L^3B}{M} - 1\right) = \Sigma a_m \alpha^m \equiv \beta \tag{6}$$

By defining β as a function of $\alpha = g(I,B)$ we have reduced the problem to the level of plain matrix manipulation, where:

$$\beta_i = \Sigma a_m \alpha_i^m \qquad \begin{array}{l} m = 0, 1 \ldots n \\ i = 1, ---, n + 1 \\ n = 9 \end{array} \tag{7}$$

will give us a system of $i = n + 1$ equations, for which the values of α may be autonomously generated, that is, arbitrarily assigned in such a way, as to be evenly distributed over the range of the given set of coefficients a_m. Since equation (7) is monotonic within the fixed range, the β_i's are nonsingular. Hence it can be shown (appendix) that for all pairs of (β_i, α_i) of a domain, some new set of coefficients b_m exists, that will make the inverse relation hold true:

$$\alpha_i = \Sigma b_m \beta_i^m \qquad \begin{array}{l} m = 0, 1 \ldots n \\ i = 1, --- , n + 1 \\ n = 9 \end{array} \tag{8}$$

Combining (5) and (8) and writing the equation in a form corresponding to (3) we have:

$$\frac{I^3B}{M} = \exp\left(\Sigma b_m \beta^m\right) \tag{9}$$

We shall call this expression the "Inverse Dipole Function" and define it as:

$$\frac{I^3B}{M} = F^{-1}\left(\frac{L^3B}{M}\right) \tag{10}$$

where:

$$F^{-1}\left(\frac{L^3B}{M}\right) = \exp\left(\Sigma b_m \beta^m\right) \qquad m = 0, 1 -- 9 \tag{11}$$

and:

$$\beta = \ln\left(\frac{L^3M}{B} - 1\right) \tag{12}$$

Expression (2) is now sufficiently determined from (10) by:

$$f\ (B,L) = \left[\frac{M}{B}\ F^{-1}\ \left(\frac{L^3B}{M}\right)\right]^{1/3} \tag{13}$$

To obtain the new coefficients b_m, the corresponding β's were computed through equation (7) for a selected set of (n + 1) α's, belonging to a given range of a_m. Using these values of α and β, the resulting linear system of equation (8) was solved for b_m. Table 2 gives the values of b_m calculated by this method for the indicated domains of a_m.

CONCLUSIONS AND RESULTS

We have shown that a solution to I = f(B, L), equation (2), exists in the form of the Inverse Dipole Function, which does satisfy the imposed requirement of derivability. When conforming with the expressed stipulations, the second adiabatic invariant may unambiguously be obtained for any arbituary B and L with great accuracy and speed and without integration, from:

$$I = \left[\frac{M}{B}\ F^{-1}\ \left(\frac{L^3B}{M}\right)\right]^{1/3} \tag{14}$$

where:

$$F^{-1}\ \left(\frac{L^3B}{M}\right) = \exp\ \left(\Sigma b_m \beta^m\right) \qquad m = 0,\ 1\ \text{---}\ 9 \tag{11}$$

and:

$$\beta = \ln\left(\frac{L^3B}{M} - 1\right) \tag{12}$$

In similarity to (4), Equation (11) is also an approximation, but with a higher degree of accuracy. Tests and recomputations of α and L from the calculated 'I' have shown the average error to be less than 10^{-6} with a maximum around 10^{-4}.

Sample computations are given in Table 4. They were handled by a program written in Fortran IV for the IBM 7094, Moonlight System. Actual running time per 'I' computation accounted to a very small fraction of a second in all domains.

APPENDIX (I)

Given a polynomial of the nth degree in α that is monotonic in the interval $[\alpha^{*}, \alpha^{**}]$, for which the coefficients a_m exist:

$$(A) \qquad \beta = g(\alpha) = \sum_{m=0}^{n} a_m \, \alpha^m \qquad n = 9$$

we select (n+1) points $\{\alpha_1, \alpha_2, \cdots \alpha_{n+1}\}$ such that:

$$\{\alpha_i\} \subset [\alpha^{*}, \alpha^{**}] \qquad i = 1, n+1$$

and calculate from (A) the pairs:

$$(B) \qquad (\alpha_1, \beta_1 = g(\alpha_1)), (\alpha_2, \beta_2 = g(\alpha_2)), \cdots (\alpha_{n+1}, \beta_{n+1} = g(\alpha_{n+1}))$$

This gives (n+1) points lying on $g(\alpha)$.

Further, from the monotonicity of g, it can be expected that, if the α_i's are distinct, so are the β_i's.

Now let us write (B) as:

$$(C) \qquad (\beta_1, \alpha_1), (\beta_2, \alpha_2), \cdots (\beta_{n+1}, \alpha_{n+1})$$

and regard (C) as giving us (n+1) points on the function:

$$(D) \qquad \alpha = g^{-1}(\beta)$$

Accordingly, this is a unique polynomial of nth degree passing through these points, i.e.:

$$(E) \qquad \alpha = \sum_{m=0}^{n} b_m \, \beta^m = g^{-1}(\beta) \qquad n = 9$$

Now if we change our selection of the $\{\alpha_1, \alpha_2, \cdots \alpha_{n+1}\}$, we could expect very nearly the same polynomial in (E) again, provided the $\{\alpha_i\}$'s were well dispersed over the interval $[\alpha^{*}, \alpha^{**}]$.

Of course, if one selected the α_i's all clustered together, at one end of the interval $[\alpha^*, \alpha^{**}]$ for example, the β_i's would be clustered together also, and some difficulty might be experienced in solving for the β_m's, which in that case might be different. But with a reasonable selection of the $\{\alpha_i$'s$\}$, the values for $\{\beta_m\}$ will be approximately the same, their difference remaining insignificantly small.

ACKNOWLEDGEMENTS

I am indebted to Dr. A. Stokes, consultant to the Mathematics and Computing Branch, for his assistance in formulating the contents of the Appendix, and to Dr. D.P. Stern of the Theoretical Studies Branch for his valuable criticism and his useful suggestions. Thanks are also due to Dr. J. Roederer, Senior Post Doctoral Research Associate of the Theoretical Studies Branch, for many helpful comments.

REFERENCES

1. STERN, David P., "The Classification of Magnetic Shells," J. Geophys. Res., 70, 3629, 1965.

2. MCILWAIN, Carl E., "Coordinates Mapping the Distribution of Magnetically Trapped Particles," J. Geophys. Res., 66, 3681, 1961.

3. MCILWAIN, Carl E., "The Radiation Belts, Natural and Artificial," Science, 142, 355, 1963

4. GALBRAITH, T. L., "A Model for the Protons Trapped in the Magnetic Field of the Earth," paper presented at the A.G.U. meeting in Dallas, Texas, 1-3 Sept. 1965

5. HESS, W. N., "Lifetime and Time Histories of Trapped Radiation Belt Particles," Space Research IV, p. 60, 1964.

TABLE 1

McIlwain's Polynomial Expansion Coefficients for the Dipole Function Used in the Calculation of L (Source: McIlwain's INVAR Subroutine)

	$\alpha\le\alpha_1^*$	$\alpha_1^*<\alpha\le\alpha_2^*$	$\alpha_2^*<\alpha\le\alpha_3^*$	$\alpha_3^*<\alpha\le\alpha_4^*$	$\alpha_4^*<\alpha\le\alpha_5^*$	$\alpha>\alpha_5^*$
a_0	3.0062102	0.62337691	0.62286440	0.62223550	2.00718690	-3.04608090
a_1	0.3333380	0.43432642	0.43352788	0.43510529	-0.18461796	1.0
a_2	0	0.15017245×10^{-1}	0.14492441×10^{-1}	0.12817956×10^{-1}	0.12038223	0
a_3	0	0.13714667×10^{-2}	0.11784234×10^{-2}	0.21680398×10^{-2}	-0.67310338×10^{-2}	0
a_4	0	0.82711095×10^{-4}	0.38379916×10^{-4}	-0.32077032×10^{-3}	0.21702240×10^{-3}	0
a_5	0	0.32916354×10^{-5}	-0.33408821×10^{-5}	0.79451312×10^{-4}	-0.38049275×10^{-5}	0
a_6	0	0.81048661×10^{-7}	-0.53977641×10^{-6}	-0.12531932×10^{-4}	0.28212094×10^{-7}	0
a_7	0	0.10066362×10^{-8}	-0.21997982×10^{-7}	0.99766147×10^{-6}	0	0
a_8	0	0.83232531×10^{-12}	0.23208676×10^{-8}	-0.39583059×10^{-7}	0	0
a_9	0	$-0.81537734\times10^{-13}$	0.26047023×10^{-9}	0.63271665×10^{-9}	0	0

TABLE 2

NEW POLYNOMIAL EXPANSION COEFFICIENTS FOR THE INVERSE DIPOLE FUNCTION USED IN THE CALCULATION OF I

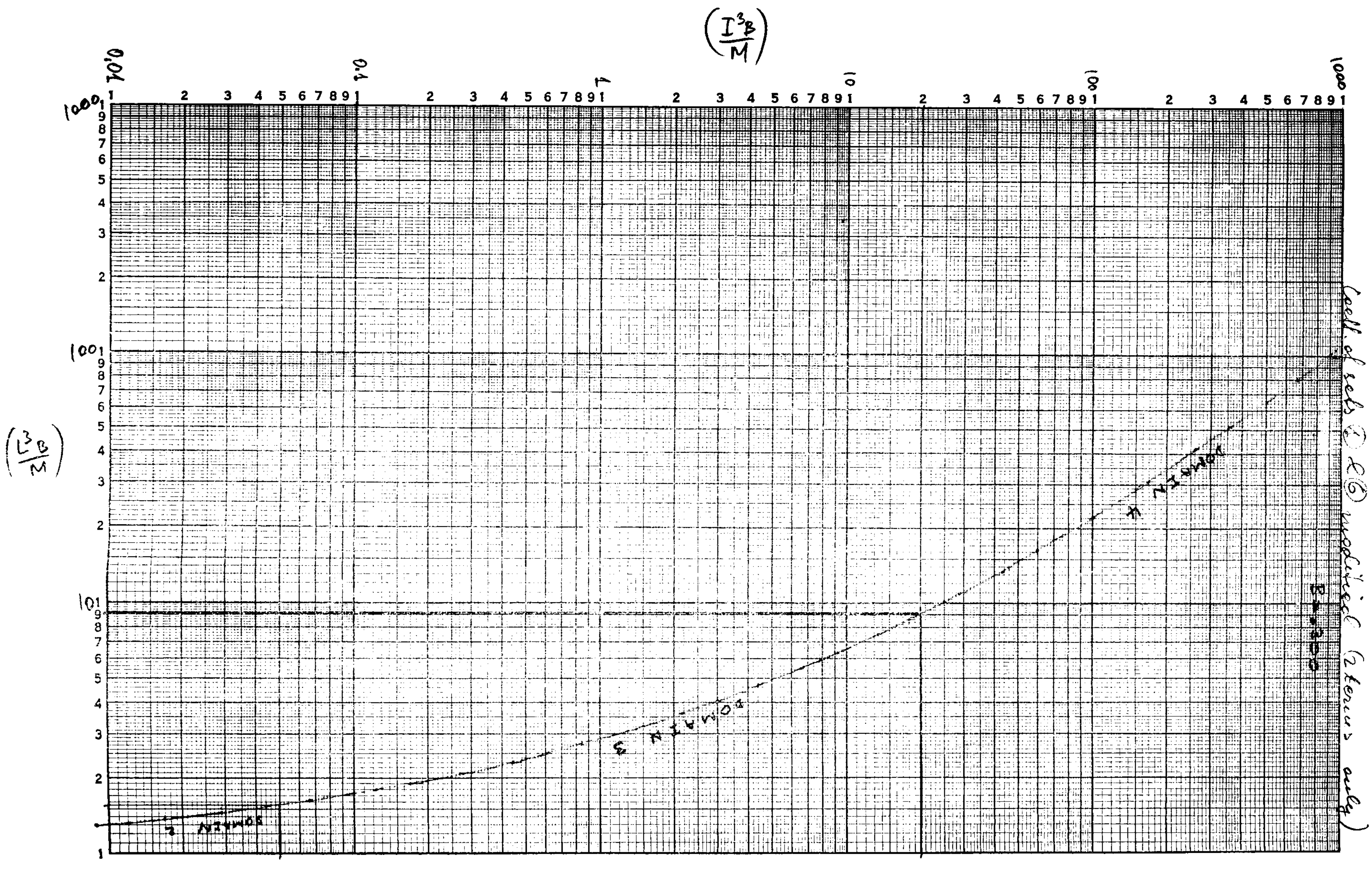
	$\beta \le \beta_1^*$	$\beta_1^* < \beta \le \beta_2^*$	$\beta_2^* < \beta \le \beta_3^*$	$\beta_2^* < \beta \le \beta_4^*$	$\beta_4 < \beta \le \beta_5^*$	$\beta > \beta_5^*$
b_0	-9.0185043	-1.50332138	-1.50356655	-1.48829786	-1.15852794	3.04608090
b_1	2.9999580	2.51792982	2.51664081	2.47107351	2.47335759	1.0
b_2	0	$-1.52832614 \times 10^{-1}$	$-1.55786011 \times 10^{-1}$	$-9.62546301 \times 10^{-2}$	$-2.53042388 \times 10^{-1}$	0
b_3	0	$-1.30130862 \times 10^{-2}$	$-1.69741516 \times 10^{-2}$	$-6.11302775 \times 10^{-2}$	$2.75046200 \times 10^{-2}$	0
b_4	0	$6.78193533 \times 10^{-3}$	$3.34384981 \times 10^{-3}$	$2.32827449 \times 10^{-2}$	$-2.04813427 \times 10^{-3}$	0
b_5	0	$2.86709833 \times 10^{-3}$	$9.28348470 \times 10^{-4}$	$-4.35763061 \times 10^{-3}$	$1.06449099 \times 10^{-4}$	0
b_6	0	$5.45117462 \times 10^{-4}$	$-1.07266191 \times 10^{-4}$	$5.04112720 \times 10^{-4}$	$-3.80249637 \times 10^{-6}$	0
b_7	0	$5.91509795 \times 10^{-5}$	$-9.61658788 \times 10^{-5}$	$-3.64886370 \times 10^{-5}$	$8.89227426 \times 10^{-8}$	0
b_8	0	$3.54120123 \times 10^{-6}$	$2.84261262 \times 10^{-5}$	$1.52088347 \times 10^{-6}$	$-1.22426523 \times 10^{-9}$	0
b_9	0	$9.11333585 \times 10^{-8}$	$-2.44924125 \times 10^{-6}$	$-2.79306844 \times 10^{-8}$	$7.51524144 \times 10^{-12}$	0

TABLE 3

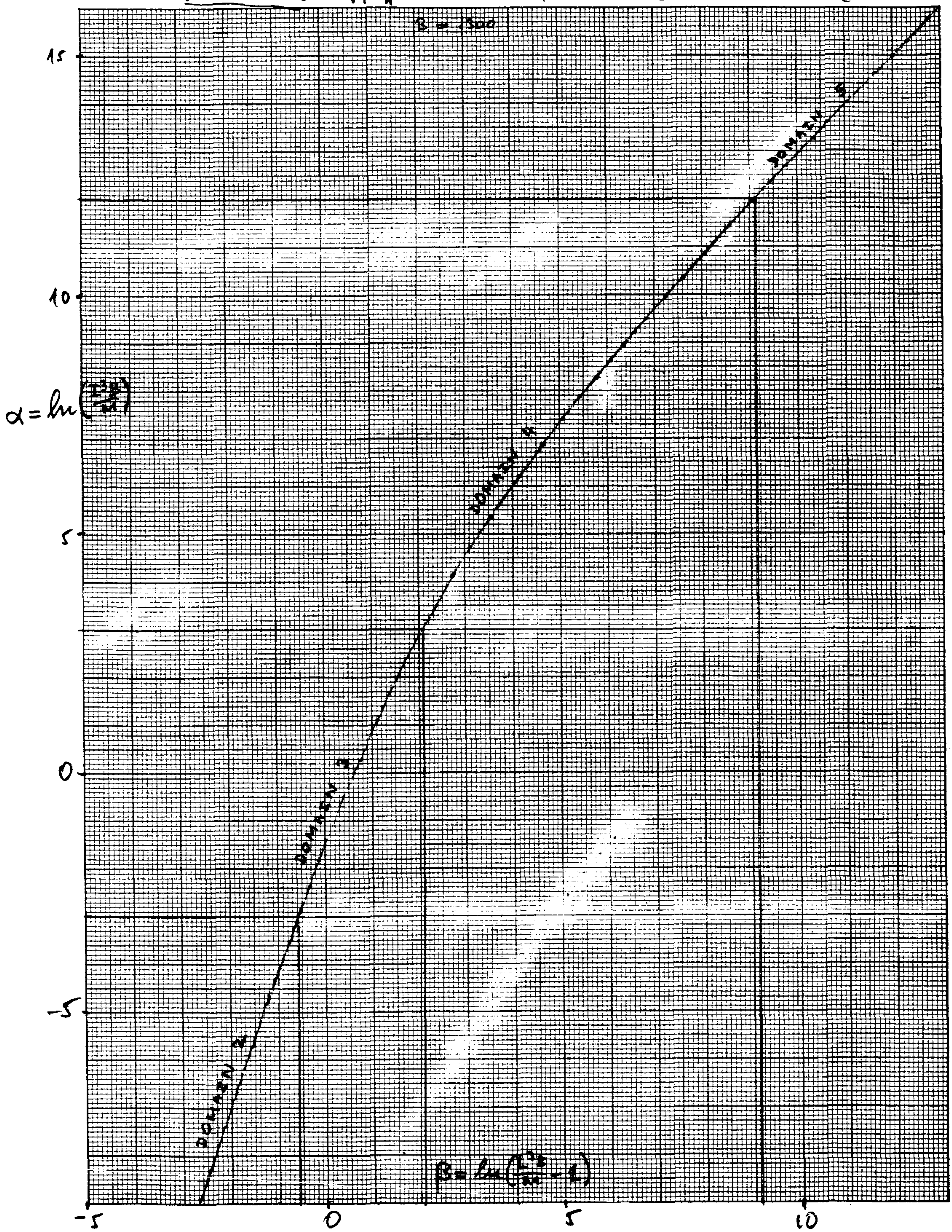
BOUNDARIES OF α AND β RANGES

$\alpha_1^* = -22.$	$\beta_1^* = -7.0328651$
$\alpha_2^* = -3.$	$\beta_2^* = -0.57551527$
$\alpha_3^* = +3.$	$\beta_3^* = 2.0875796$
$\alpha_4^* = +12.$	$\beta_4^* = 9.1267546$
$\alpha_5^* = +23.$	$\beta_5^* = 19.958566$

$$\alpha = \ln\left(\frac{I^3 B}{M}\right) \qquad \beta = \left(\frac{L^3 M}{M} - 1\right)$$


Coeff. of sets 1 & 6 modified (2 terms only)
B = 300
DOMAIN 4
DOMAIN 3
DOMAIN 2
(I³B/M)
0.01
0.1
1
10
100
1000

version 2: coeffts with modified sets ① & ⑥ (2 terms only)



ACKNOWLEDGEMENTS

I am indebted to Dr. A. Stokes, consultant to the Mathematics and Computing Branch, for his assistance in formulating the contents of the Appendix, and to Dr. D. P. Stern of the Theoretical Studies Branch for his valuable criticism and his useful suggestions. Thanks are also due to Dr. J. Roederer, Senior Post Doctoral Research Associate of the Theoretical Studies Branch, for many helpful comments.

REFERENCES

1. STERN, David P., "The Classification of Magnetic Shells," J. Geophys. Res., 70, 3629, 1965.

2. MCILWAIN, Carl E., "Coordinates Mapping the Distribution of Magnetically Trapped Particles," J. Geophys. Res., 66, 3681, 1961.

3. MCILWAIN, Carl E., "The Radiation Belts, Natural and Artificial," Science, 142, 355, 1963.

4. GALBRAITH, T. L., "A Model for the Protons Trapped in the Magnetic Field of the Earth," paper presented at the A.G.U. meeting in Dallas, Texas, 1-3 Sept. 1965.

5. HESS, W. N., "Lifetime and Time Histories of Trapped Radiation Belt Particles," Space Research IV, p. 60, 1964.