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A WORST DISTURBANCE DESIGN CRITERION IN THE THEORY OF ANALYTICAL CONTROL SYSTEMS SYNTHESIS

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ABSTRACT

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Design techniques of linear optimal control are found to apply to a minimax problem. With a bounded energy constraint on the class of admissible disturbances, the minimax value of an integral quadratic form of state variables and control can be obtained by finding a positive definite steady-state solution of a matrix Riccati equation. The optimal strategies for control and disturbance are linear functions of state which depend on the numerical bound of the class of disturbances, the set of initial conditions, and relative weighting of the state variables in the cost functional. Analytical design procedures, such as the root square locus of Chang, which appear in optimal linear control problems are also valid for this problem. An equivalent optimal multivariable control problem has been found whose steady-state solution is obtained by solving the same matrix Riccati equation as was obtained from the minimax problem. Sufficient conditions for existence of solutions to the minimax problem are thus obtained from the properties of the equivalent optimal multivariable control problem. The results are illustrated by solving a second order example.

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SUMMARY

Design techniques of linear optimal control are found to apply to a minimax problem. With a bounded energy constraint on the class of admissible disturbances, the minimax value of an integral quadratic form of state variables and control can be obtained by finding a positive definite steady-state solution of a matrix Riccati equation. The optimal strategies for control and disturbance are linear functions of state which depend on the numerical bound of the class of disturbances, the set of initial conditions, and relative weighting of the state variables in the cost functional. Analytical design procedures, such as the root square locus of Chang, which appear in optimal linear control problems are also valid for this problem. An equivalent optimal multivariable control problem has been found whose steady-state solution is obtained by solving the same matrix Riccati equation as was obtained from the minimax problem. Sufficient conditions for existence of solutions to the minimax problem are thus obtained from the properties of the equivalent optimal multivariable control problem. The results are illustrated by solving a second order example.

I. INTRODUCTION

LINEAR CONTROL, DIFFERENTIAL GAMES, AND A MINIMAX PROBLEM

In recent years the mathematical conditions for which linear control is optimal have been well defined.[1] In 1961, S. S. L. Chang[2] published a book on optimal control synthesis which included a root square locus technique permitting mathematically optimum linear systems to be analyzed by the well-known root locus diagram. This technique has recently been applied to multivariable[3,4] control problems to provide an analytical approach to control synthesis of high order dynamical systems.

In the above analytical synthesis studies, the problem of disturbances was not considered in the initial design phase. It is shown in this work that the same analytical synthesis techniques can be applied to a problem with a single control and a bounded energy type disturbance. The control and disturbance are scalars for this problem, but no additional difficulty is encountered, in principle, if both are vectors. The problem is formulated as a minimax problem and is solved by the theory of differential games. For this problem, the control and the disturbance are viewed as opposing players in a differential game. The solution leads to linear feedback as the optimum strategy for both players. Linear control can thus be interpreted as the best control for the worst of a well defined class of bounded energy disturbances.

The study of differential games was begun by Isaacs[8]. His book[5] is used as a basic reference for much of the present work and some of his terminology will be used herein. Unfortunately, his terminology is not in agreement with much of the usage in optimal control theory. These differences are pointed out in Reference 14. A more rigorous foundation for the theory of differential games can be found in the work of Berkowitz[9]. Application of differential games to problems in optimal control[13] and also to pursuit and evasion problems can be found[10,11,12]. In Reference 12 the authors solve a pursuit-evasion problem which leads to linear feedback strategies for both the pursuer and evader, and the results are closely related to those for the linear optimal control problem. In the present work a similar relationship with the linear optimal control problem exists, but in this case, linear "analytical design concepts" are extended to include systems subject to worst disturbances. The design philosophy presented herein can be used to define a set of linear controls, and a corresponding class of worst disturbances for which these controls are optimal.

II. FORMULATION

The problem is formulated as follows: Consider the dynamical system

$$\dot{x} = Ax + bu + cw \qquad x(t_o) = x_o \tag{1.1}$$

in which x is a real n vector denoting the state of the system, b and c are real constant n vectors, A is a real constant $n \times n$ matrix, u is a scalar control belonging to the class C_p of piecewise continuous functions of time t, w is a scalar disturbance belonging to some class W of functions of time. The dot denotes differentiation with respect to time, and t_o is the initial time. The output of the dynamical system is

$$y = Hx \tag{1.2}$$

where y is a p-dimensional vector and H is a constant $p \times n$ matrix. The cost functional for this problem is

$$J[u,w] = \int_{t_o}^{T} [y^T Q y + u^2] dt + x^T(T) P x(T) \tag{1.3}$$

where Q is a constant positive definite $p \times p$ matrix, P is a positive definite constant $n \times n$ matrix, and T is the time required to attain a terminal surface of dimension n-1 which divides the state space into two disjoint n-dimensional subsets. (See Reference 5, Chapter 2.) For each admissible $u(t)$ and $w(t)$, the solution of (1.1) uniquely determines (1.3). The problem consists of finding a control $\overline{u}$ in C_p and a disturbance $\overline{w}$ in W so that

$$J[\overline{u},\overline{w}] = \min_{u \epsilon C_p} \max_{w \epsilon W} J[u,w]. \tag{1.4}$$

The above problem belongs to a class of problems known as minimax problems. V. P. Grishin[6] has solved a similar problem in which the set W of disturbances is the class of piecewise continuous functions of time whose magnitude is bounded by a positive constant. We shall consider here the class W of admissible disturbances to be the set of piecewise continuous functions of time which satisfy the relation

$$\int_{t_o}^{T} w^2(t) dt \leq \rho^2 \tag{1.5}$$

where ρ is a positive constant.

In aerospace vehicle control problems, for example, the class W of admissible disturbances is a class of winds. The relation (1.5) allows a wind in this class to take on arbitrarily large magnitudes, either positive or negative, but an average square magnitude is bounded. Admitted in this class are large disturbances of short duration (gusts) or small disturbances of long duration (breezes). Not admitted are large disturbances of long duration (gales). If T is infinite, it can be seen that since ρ is finite, any wind in this class must eventually die out.

The number ρ^2 can be, for a particular application, determined empirically. An estimate can be obtained by computing

$$\int_{t_0}^{T} w^2(t)dt$$

for a large number of recorded disturbances and taking the largest value of this integral as ρ^2. It should be pointed out, however, that the worst disturbance which arises from a solution of the problem may not resemble any of the reocrded disturbances and should be examined for reasonableness. What a solution of the problem does provide is the shape of the worst disturbance associated with a given dynamical system.

In order to solve the minimax problem in which the disturbance is subject to a bounded integral constraint, we shall introduce a positive Lagrange multiplier λ and consider the new cost functional

$$G[u,w] = \int_{t_0}^{T} [y^TQy + u^2 - \lambda w^2]dt + x^T(T)Px(T) \qquad (1.6)$$

in which the disturbance w(t) now belongs to the class C_ρ and the value of λ depends on ρ and x_0. We shall require that the players w and u employ strategies (functions of state) rather than time functions. For this reason the function w(t) and u(t) will be given by

$$w(t) = w[x(t)]$$

$$u(t) = u[x(t)] \qquad (1.7)$$

where x(t) is the solution of (1.1). The class of strategies w(x) and u(x) with the property that w[x(t)] and u[x(t)] belong to C_ρ will be denoted Ω. In terms of game theory our problem is: given x_0 and ρ, find strategies $\overline{u}$ and $\overline{w}$ in Ω so that

$$G[\overline{u},\overline{w}] = \min_{u\epsilon\Omega} \max_{w\epsilon\Omega} G[u,w]. \tag{1.8}$$

With the specification of a terminal surface, the problem is in a form for which the theory of differential games can be applied. We shall restrict our attention to terminal surfaces for which we can find a solution having stable motion with respect to the origin of the state space. This can be accomplished in two ways.

One way is to choose time as another state variable x_{n+1} and define the terminal surface as the set of points for which $x_{n+1} = T$ where T is a fixed terminal time. Under dertain controllability and observability conditions, the limit of the minimax value for this problem exists as T approaches infinity. It is then shown that this limit is a Liapunov function, thus insuring stability. Since this approach follows the usual development found in linear optimal control[7], it will not be presented here.

The other way does not require the concept of a Liapunov function. Instead, we choose as a terminal surface an appropriate n-1 dimensional ellipsoid which is always between the initial state and the origin. Stability of motion is assured by shrinking the target ellipsoid so that the terminal state becomes arbitrarily near the origin. Using this approach, the problem to be solved by application of differential game theory is as follows:

Given the dynamical system (1.1), and the terminal surface

$$x^T Px = \delta \tag{1.9}$$

which is defined by the positive number δ, and the cost function (1.6) which depends on δ and will be denoted $G_\delta[u,w]$, find strategies $\overline{u}(x)$ and $\overline{w}(x)$ and a positive number λ so that

$$G_\delta[\overline{u}(x),\overline{w}(x)] = \min_{u\epsilon\Omega} \max_{w\epsilon\Omega} G_\delta[u(x),w(x)] \tag{1.10}$$

and the constraint (1.5) is not violated.

The quantity $G_\delta[\overline{u}(x),\overline{w}(x)]$ is, for a particular ρ and δ, a function of x_o. It will be denoted $V_\delta(x_o)$ and will be called the "value" of the game at x_o. We shall be interested in the limit of the value of the game as δ approaches zero.

III. SOLUTION OF THE PROBLEM

There are two computational methods for analyzing the problem. The first method is to solve what Isaacs refers to in Reference 5 as the "main equation." This leads to the problem of finding a steady-state solution of a matrix-Riccati equation. The second method is to solve the "path equations" associated with the problem. For our problem the path equations lead to a root square locus technique similar to that developed by Chang.

A. SOLUTION VIA THE MAIN EQUATION

According to the differential games approach, we form the $\mathcal{H}$ function defined as follows:

$$\mathcal{H}(x,V_X,u,w) = x^T H^T Q H x + u^2 - \lambda w^2 + V_X^T(Ax+bu+cw) \tag{2.1}$$

where V_X denotes the gradient of a scalar valued function of x. For this $\mathcal{H}$ function, the "minimax assumption"

$$\min_u \max_w \mathcal{H}(x,V_X,u,w) = \max_w \min_u \mathcal{H}(x,V_X,uw) \tag{2.2}$$

is valid since $\mathcal{H}$ is the sum of functions of u and w. Since $\lambda > 0$, a necessary and sufficient condition that $\mathcal{H}$ be minimax is that

$$u = -\tfrac{1}{2}V_X^T b$$
$$w = \frac{1}{2\lambda}V_X^T c, \tag{2.3}$$

and the main equation is given by

$$\min_u \max_w \mathcal{H}(x,V_X,u,w) = 0. \tag{2.4}$$

For this problem the main equation becomes

$$x^T H^T Q H x + V_X^T A x - \tfrac{1}{4}V_X^T S V_X = 0 \tag{2.5}$$

where $S = bb^T - \frac{1}{\lambda}cc^T$. (2.6)

It is known from the theory of differential games that the value of the game must satisfy the main equation and the boundary condition given by the terminal surface. If the solution of the main equation satisfying the boundary condition is unique, then that solution is the value of the game. Since condition (2.3) is necessary and sufficient for $\mathcal{H}$ to be minimax, the problem has only one main equation. The boundary condition is given by the surface $x^TPx = \delta$. We then choose P so that $V_\delta(x) = x^TPx$ is a solution of the main equation, if possible, because such a choice of P will provide a terminal value given by

$$V_\delta(x(T)) = \delta. \tag{2.7}$$

Assuming a solution of this form gives

$$x^TH^TQHx + 2x^TP^TAx - x^TP^TSPx = 0. \tag{2.8}$$

If equation (2.8) has a solution which is valid for each x in the state space exterior to the region enclosed by the terminal set and which satisfies the boundary condition (1.9), P must satisfy the matricial equation

$$P^TA + A^TP - P^TSP = -H^TQH. \tag{2.9}$$

Let us assume for the moment that A, S, H, and Q satisfy whatever conditions are needed in order that a unique symmetric positive definite matrix P exists satisfying (2.9). It then follows that $V_\delta(x) = x^TPx$ is the value of the game in which (1.9) is the terminal surface. In this case $V_\delta(x)$ represents the value at x of a game whose terminal surface is an n-dimensional ellipsoid containing the origin. We are primarily interested in this game because taking the limit as δ approaches zero, the terminal state approaches the origin. Recalling that $V_\delta(x)$ is the value associated with (1.6) and the terminal surface (1.9) and that $V_\delta(x) = \delta$ on this surface, we observe that

$$\lim_{\delta \to 0} V_\delta(x) = V(x) \tag{2.10}$$

where $V(x)$ is defined as the value associated with (1.6) and the terminal state $x = 0$.

A result of the above arguments is the following theorem: If P is a unique positive definite solution of (2.9), then the minimax value of (1.6) for the terminal surface (1.9) is given by $V(x) = x^T Px$, and the strategies (2.3) are optimal for any $\delta > 0$; the motion of (1.1) for these strategies is stable with respect to the origin.

Since $V_x = 2Px$, we can now write explicitly the expressions for the "best" control and the "worst" wind. These are

$$\overline{u} = - b^T Px \tag{2.11}$$

$$\overline{w} = \frac{1}{\lambda} c^T Px. \tag{2.12}$$

The optimal strategy for both players is linear. Given A, the optimal play depends on Q, H, b, c, and λ. These characteristics uniquely define the optimal feedback strategy for control. In previous linear theory in which the disturbance was assumed zero, the optimal feedback was of the same form, but depended only on Q, H, and b. Note that the more general minimax control reduces to the zero disturbance optimal linear control as λ becomes infinite. This is seen from equations (2.6), (2.9), and (2.12).

B. CALCULATION OF THE ELLIPSOID OF INITIAL STATES

The optimal strategies (2.11) and (2.12) depend on λ, which, in turn, depends on ρ and the initial conditions by (1.5) and (2.12). As the problem has been formulated x_0 and ρ are assumed fixed, and we solve for λ and strategies $\overline{u}(x)$ and $\overline{w}(x)$ for which the minimax value of (1.3) is attained. From the standpoint of design, however, it may be advantageous to specify λ and work backward to find the set of all initial states for which the strategies are optimal. (Note also that λ is a convenient parameter for representing the combined effects of ρ and x_0, and it will be used in this manner in the work to follow.)

If λ is specified so that there exists a unique positive definite solution of (2.9), then by using the strategies (2.11) and (2.12), the differential system (1.1) becomes, in closed loop form,

$$\dot{x} = \tilde{A}x \qquad x(t_o) = x_o \tag{2.13}$$

where $$\tilde{A} = A - SP \tag{2.14}$$

and the initial state x_o belongs to a set for which (1.5) is satisfied where the disturbance is given by (2.12).

Let us define a function W of initial state by

$$W(x_o) = \int_{t_o}^{T} \overline{w}^2(t)dt. \tag{2.15}$$

Since $\overline{w}^2(t)$ is nonnegative, $W(x_o)$ is at least positive semi-definite. Its time derivative along solutions of (2.13) is given by

$$\dot{W}(x(t)) = -\frac{x^T(t)P^Tcc^TPx(t)}{\lambda^2} \tag{2.16}$$

For this reason, it is apparent that $W(x_o)$ is a quadratic form

$$W(x_o) = x_o^T B x_o, \tag{2.17}$$

if the n×n matrix B satisfies the linear equation

$$B\tilde{A} + \tilde{A}^T B = -\frac{P^Tcc^TP}{\lambda^2}. \tag{2.18}$$

The matrix B is the entity that relates λ, ρ, and x_o. It is obtained from λ and P by solving (2.18). From (1.5), (2.15), and (2.17), the initial states for which strategies (2.11) and (2.12) are optimal for a particular λ are contained in the region

$$x_o^T B x_o \leq \rho^2. \tag{2.19}$$

If B is positive definite, the geometric representation of (2.19) is the set of points on and interior to the n-1 dimensional ellipsoid

$$x^T B x = \rho^2 . \tag{2.20}$$

The value of λ from which the region (2.19) is determined defines the best control given by (2.11) for the worst disturbance in the class W and for the worst initial state in the region (2.19). For initial states exterior to this region, another value of λ is needed in order to define the optimal strategies.

C. SOLUTION VIA THE PATH EQUATIONS

Equations (2.11 - 2.14) provide a means of computing the optimal strategies and resulting motion of the differential system. In order to perform these computations, however, a positive definite solution of the matricial equation (2.9) must be found. A root locus analysis, for example, would require finding a steady-state solution of a matrix Ricatti equation for each new value of λ or of the elements of Q in order to find the eigenvalues of $\tilde{A}$ from (2.14). It is possible, however, to find eigenvalues of $\tilde{A}$ without directly solving (2.9). This is done by a root square locus method which can be developed from the path equations presented in Chapter 4 of Reference 5.

The path equations can be written in the form

$$\dot{V}_x(x) = -\mathcal{H}_x(x, V_x, u, w) \tag{2.21}$$

where $\mathcal{H}(x, V_x, u, w)$ is given by (2.1). The subscripts denote gradients with respect to x; u and w are given by (2.3), and V(x) solves the main equation.

Equation (2.21) presents an expression for the time rate of change of the gradient of the value of the game along solution trajectories of the differential system. The differential system (1.1) for the optimal strategies (2.3) becomes

$$\dot{x} = Ax - \tfrac{1}{2} S V_x . \tag{2.22}$$

From equations (2.1) and (2.21) the path equations become

$$\dot{V}_x = -2H^T Q H x - V_x{}^T A . \tag{2.23}$$

Equations (2.22) and (2.23) can be thought of as a system of 2n first order differential equations in terms of unknowns V_{x_j} and x_j $(j = 1,\dots,n)$. Written in this manner the path equations and differential system can be expressed as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{V}_x \end{bmatrix} = \begin{bmatrix} A & -\tfrac{1}{2}S \\ -2H^TQH & -A^T \end{bmatrix} \begin{bmatrix} x \\ V_x \end{bmatrix}. \qquad (2.24)$$

The characteristic equation of this system is given by

$$\left| \begin{array}{c|c} A - sI & -\tfrac{1}{2}S \\ \hline -2H^TQH & -A^T - sI \end{array} \right| = 0. \qquad (2.25)$$

where I denotes the n×n identity matrix. The root square locus is obtained from the roots of this equation and will be shown in the next section to be equivalent under certain conditions to a multivariable root square locus of an undisturbed optimal multichannel control system[3,4].

If a positive definite solution of the matricial equation (2.9) exists, then n of the 2n roots of (2.25) are in the left half plane, and these stable roots are the eigenvalues of A, which is given by (2.13). Sufficient conditions for existence of a positive definite solution of (2.9) are presented in the section following.

D. AN EQUIVALENT OPTIMAL MULTIVARIABLE CONTROL PROBLEM

Minimax and optimal control problems are solved by similar methods. It will be demonstrated that the minimax problem considered here can be reduced to an optimal control problem with several control variables. The two problems are equivalent in the sense that both have the same terminal conditions and both lead to the same Riccati equation. A solution of one problem leads immediately to a solution of the other.

The only condition we shall need in order to demonstrate the existence of an equivalent optimal control problem is the requirement that the matrix S given by (2.6) be positive semi-definite. This condition places a restriction on λ which confines the class of admissible disturbances so that controllability is possible. We use a generalized definition of controllability here which is relative to the class of admissible disturbances.* The plant is said to be completely controllable relative to the class W if for each disturbance in W there exists a finite control capable of bringing the motion from any initial state to any desired state in finite time. This generalized controllability for a system with disturbances as (1.1) can be shown to be the same as controllability in the usual sense for a certain multivariable control system with no disturbance if a certain matrix exists. This is seen as follows.

Suppose there exists an $n \times r$ matrix F where $r \leq n$ which has the property that for each admissible u(t) and w(t) there is a finite piecewise continuous r-vector function v(t) satisfying the equation

$$Fv(t) = bu(t) + cw(t) \tag{2.26}$$

and also b has the property that for each such v(t) and each w(t) in W there is an admissible u(t) such that equation (2.26) is satisfied. Under these conditions for every motion of (1.1) under a given disturbance and control, there is an identical motion of the multi-channel system

$$\dot{x} = Ax + Fv \tag{2.27}$$

for some v(t) and conversely. It is seen then that the disturbed plant is completely controllable in the generalized sense relative to W if and only if the multi-channel controlled plant is completely controllable in the usual sense. With this motivation we shall analyze the differential game by analysis of an equivalent optimal control problem whose plant equations are in the form of (2.27).

A well known property of a positive semi-definite matrix is that it can be written as the product of a rectangular matrix and its transpose. Since S is positive semi-definite there exists an $n \times r$ matrix F having the property that

$$FF^T = S \tag{2.28}$$

* Compare with the relative controllability concept in the "Linear Pursuit-Evasion Game" of Ho, Bryson, and Baron [12].

where r is the rank of S. This matrix F is used to define the plant equations (2.27). The performance index of the multi-channel optimal control problem is given by

$$G[v] \doteq \int_{t_o}^{T} [y^T Qy + v^T Iv]dt + x^T(T)Px(T) \tag{2.29}$$

where I is the r×r identity matrix, y is given by (1.2), and the terminal set is exactly the same as for the minimax problem.

Solving the Hamilton-Jacobi equation for this problem, a minimum of G[v] is given by $V = x^T Px$ where P is a solution of the steady state matrix Riccati equation

$$P^T A + A^T P - PFI^{-1}F^T P + H^T QH = 0 \tag{2.30}$$

and the optimal control law is given by

$$\overline{v} = -\tfrac{1}{2}F^T Px. \tag{2.31}$$

We see from (2.28) that (2.30) is exactly the same equation as (2.9); hence, the minimum cost of (2.29) is exactly the same as the minimax value of (1.6) since the terminal set is the same in both problems. Furthermore, sufficient conditions for existence of a unique positive definite solution of (2.30) have been developed by Kalman[7]. These are that the pair [A,F] is completely controllable and the pair [A,H] is completely observable. A unique positive definite solution of (2.9) is sufficient to insure existence and uniqueness of the solution of the optimal control problem and asymptotic stability of the motion. Since the same can be said of the minimax problem, the function $V(x) = x^T Px$ may be interpreted either as the minimum cost of (2.29) or as the minimax value of (1.6).

IV. A SECOND ORDER EXAMPLE

The application of the preceding theory to a worst disturbance design problem is illustrated by solving a second order example. Higher order problems can be solved by this theory but the complexity of the numerical analysis will, in most cases, require a computer. The sample problem is as follows.

Given the second order dynamical system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \end{bmatrix} u + \begin{bmatrix} 0 \\ c \end{bmatrix} w \tag{3.1}$$

and the cost functional

$$G[u,w] = \int_{t_0}^{T} [q_1 x_1^2 + q_2 x_2^2 + u^2 - \lambda w^2]dt + x^T(T)Px(T) \tag{3.2}$$

where x_1, x_2, a, b, c, q_1, q_2, and λ are scalars the last three being non-negative, and the terminal set is

$$\lim_{\delta \to 0} \left\{ x \mid x^T P x = \delta \right\}, \tag{3.3}$$

find the optimal strategies so that (3.2) is minimax.

Before solving this problem we shall examine it for existence of unique stable solutions. For this problem expression (2.6) becomes

$$S = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} \tag{3.4}$$

where

$$\Delta = b^2 - \frac{c^2}{\lambda} . \tag{3.5}$$

---

* A physical model leading to this equation is the rotational equation of a typical rocket in which the lateral drift is neglected in the angle-of-attack expression. The rotational equation is $\ddot{\varphi} + c_1 \alpha + c_2 \beta = 0$ where $\alpha = \varphi + \alpha_W$. In this model φ represents the vehicle attitude angle, α the angle of attack, β the engine thrust angle, α_W the angle of attack due to wind disturbance, and c_1 and c_2 are the normalized rotational acceleration coefficients associated with aerodynamic and thrust forces.

Necessary and sufficient for S to be positive semi-definite is the condition

$$b^2 - \frac{c^2}{\lambda} \geq 0. \tag{3.6}$$

Noting that the rank of S is at most one, we find that the two solutions of (2.28) are given by

$$F_1 = \begin{bmatrix} 0 \\ +\sqrt{\Delta} \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ -\sqrt{\Delta} \end{bmatrix}. \tag{3.7}$$

Each of these matrices defines an equivalent optimal control problem of the form given by (2.27). One of these equivalent optimal control problems has its plant equations given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{\Delta} \end{bmatrix} v, \tag{3.8}$$

and its cost function is

$$G[v] = \int_{t_0}^{T} [q_1 x_1^2 + q_2 x_2^2 + v^2]dt + x^T(T)Px(T) \tag{3.9}$$

where for this case the control v is a scalar.

Forming the controllability matrix [F,AF] we see that for either of the two solutions of (2.28), its rank is two if and only if $\Delta \neq 0$; hence, [A,F] is controllable if and only if

$$\lambda > \frac{c^2}{b^2} \text{ and } b \neq 0. \tag{3.10}$$

Let us choose Q = I which is clearly positive definite, and

$$
H = \begin{bmatrix} \pm\sqrt{q_1} & 0 \\ 0 & +\sqrt{q_2} \end{bmatrix} . \tag{3.11}
$$

Forming the observability matrix $[H^T, A^T H^T]$ we find its rank is two if $a = 0$ and q_1 and q_2 are non-negative, but not both zero. If $a=0$, the rank is two only if q_1 is positive. The observability condition that the rank is two, along with the controllability condition (3.10) is sufficient for the existence of a unique stable solution of the problem.

After this examination, we shall solve the problem. Due to the simplicity of this problem, we shall solve the main equation directly rather than taking the alternate route via the path equations. By forming either the main equation of the differential game or the Hamilton-Jacobi equation of an equivalent optimal control problem and assuming a solution in quadratic form we obtain the matricial equation (2.9). Of the two solutions of this equation the requirement of positive definiteness excludes one, and the remaining one is given by

$$
P = \begin{bmatrix} \dfrac{r\sqrt{2(a+r)+\Delta q_2}}{\Delta} & \dfrac{a+r}{\Delta} \\[2ex] \dfrac{a+r}{\Delta} & \dfrac{\sqrt{2(a+r)+\Delta q_2}}{\Delta} \end{bmatrix} \tag{3.12}
$$

where

$$
r = \sqrt{a^2 + \Delta q_1} . \tag{3.13}
$$

It is seen from (3.12) and (3.13) that a positive definite solution exists if $\Delta>0$ and the non-negative numbers q_1 and q_2 are not both zero. If, however, $a=0$, it is required that $q_1>0$. These conditions for a solution given by (3.12) to be positive definite are exactly the same as those determined by the controllability and observability examination before the solutions of the matricial equation were found.

Knowing P in terms of λ from equations (3.5), (3.12) and (3.13) the optimal strategies are given as functions of λ by (2.11) and (2.12), and the motion is given by (2.13) and (2.14). For this problem, equation (2.14) becomes

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ & \\ -r & -\sqrt{2(a+r)+\Delta q_2} \end{bmatrix}. \tag{3.14}$$

Because of positive definiteness of P, $\tilde{A}$ is a stable matrix.

The eigenvalues of $\tilde{A}$ are presented in Figures 1 and 2 with q_1, q_2, and Δ as variable gains. Figure 1 shows the root locus pattern of the disturbed system with Δq_1 increasing where the ratio

$$\frac{\Delta q_1}{\Delta q_2}$$

assumes fixed values. These loci are, in actuality, the stable half of the root square locus diagram we would have obtained if the problem had been solved via the path equations.

We can now specify a desired natural frequency and damping ratio of the disturbed system and determine from these corresponding values of Δq_1 and Δq_2. Since we have a second order system, the expressions for Δq_1 and Δq_2 are

$$\Delta q_1 = \omega_n^4 - a^2, \ \Delta q_2 = 4\zeta^2\omega_n^2 - 2(a+\omega_n^2). \tag{3.15}$$

These expressions show the relationship between q_1, q_2, and Δ which provides the desired point on the root locus. Since Δ depends on λ, which in turn depends on x_0 and ρ, we have a way of picking the weighting factors q_1 and q_2 in terms of desired frequency and damping, the initial state, and the class of admissible disturbances which is defined by ρ. Figure 2 shows a locus of roots as Δ varies for fixed q_1 and q_2.

For this second order system, we can compute an "ellipse of initial states" in terms of the natural frequency and damping ratio. Equation (2.20) for the second order system becomes

$$b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2 = \rho^2 \tag{3.16}$$

where b_{11}, b_{12}, and b_{22} are the elements of the matrix B which is obtained by solving equation (2.18). For this problem those elements are given in terms of natural frequency ω_n, damping ratio ζ, and λ as follows:

$$\begin{aligned} b_{11} &= \mu\left[\frac{(a+\omega_n^2)^2 + 4\zeta^2 a^2}{4\zeta\omega_n}\right] \\ b_{12} &= \mu\left[\frac{(a+\omega_n^2)^2}{2\omega_n^2}\right] \\ b_{13} &= \mu\left[\frac{(a+\omega_n^2)^2 + 4\zeta^2\omega_n^4}{4\zeta\omega_n^3}\right] \end{aligned} \tag{3.17}$$

where

$$\mu = \frac{c^2}{\lambda b^2 - c^2} \; . \tag{3.18}$$

The equations (3.16), (3.17), and (3.18) provide a relationship between the initial state, the class of disturbances, and the multiplier λ. A sketch of equation (3.16) for $\zeta = .866$ and $\omega_n = 4\sqrt{a}$ is presented in Figure 3.

For this problem the worst disturbance is given by

$$\overline{w}(t) = \mu[(a+\omega_n^2)\, x_1(t) + 2\zeta\omega_n\, x_2(t)]. \tag{3.19}$$

The shape of the worst disturbance is presented in Figure 4 for the above values of ζ and ω_n and the initial condition $x_1(t_o) = x_{1o}, x_2(t_o) = 0$.

IV. CONCLUSION

It has been found that the analytical design techniques of linear optimal control can be extended to apply to a minimax problem in which the energy of disturbance is bounded. The main equation for this problem corresponds to the Hamilton-Jacobi equation of a class of equivalent optimal multivariable control problems. A certain quadratic form was found to provide a relationship between the class of admissible disturbances and the set of initial conditions. Solution of the problem yields a linear feedback control law which minimizes a performance index subject to the disturbance which maximizes it. For a given plant, therefore, the class of admissible disturbances for which linear control is optimal, and the shape of the worst disturbance in the class, are well defined.

Knowing the worst disturbance for which linear control is optimal, the following observation becomes apparent. The shape of the worst disturbance depends on the plant dynamics in a very simple manner. Since the disturbance is a linear combination of state variables, the frequencies of the oscillation of the worst disturbance are the same as the closed loop oscillations of the plant. A structural bending mode, for example, in a boost vehicle would cause the worst disturbance for this particular model to have an oscillatory component at exactly the same frequency as the bending vibration.

For this reason, a worst disturbance design criterion might be expected to be over-conservative. The philosophy that nature is totally against you is, hopefully, unrealistic. It might, however, provide better control than a theory which ignores disturbance effects altogether, as much linear optimum control theory does. The question, from a practical standpoint, is as follows.

Does a worst disturbance design criterion provide a good control system for a less "hostile," more reasonable class of disturbances?

This question should be investigated.

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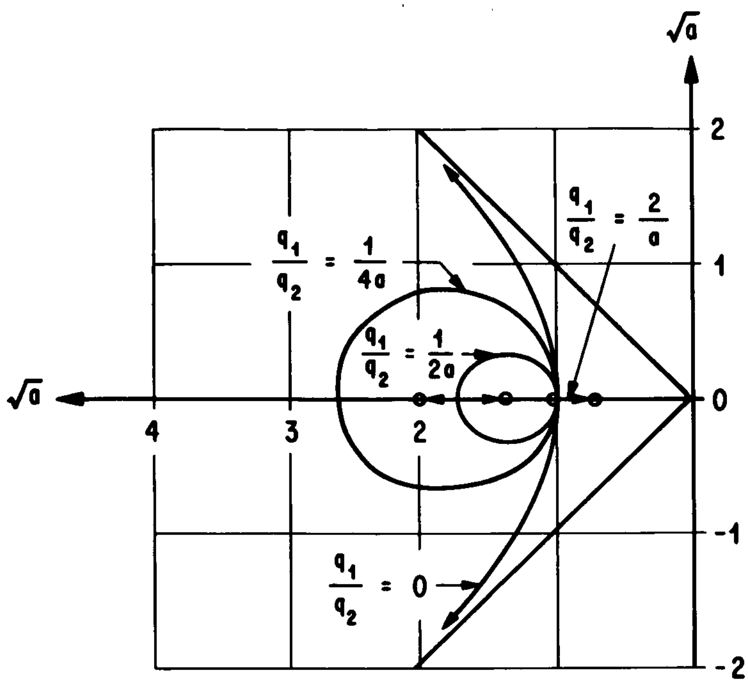

FIG. 1. ROOT LOCUS OF SECOND ORDER MINIMAX PROBLEM: Δq_1 INCREASING

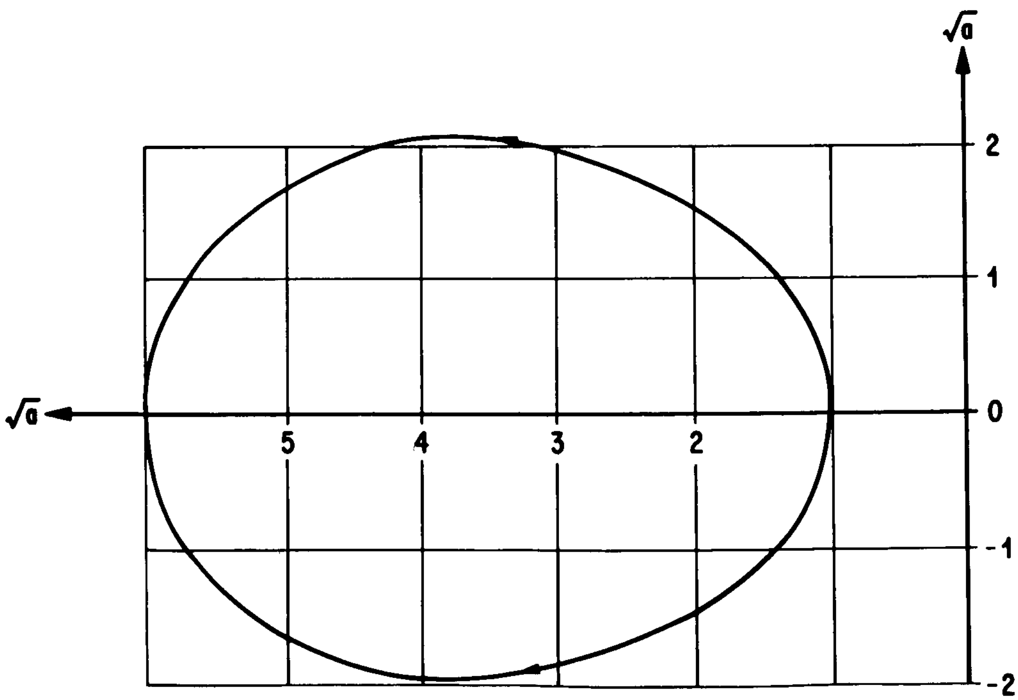

FIG. 2. ROOT LOCUS OF SECOND ORDER MINIMAX PROBLEM: Δ INCREASING $q_1 = 1{,}224\,a^2$, $q_2 = 67.7a$

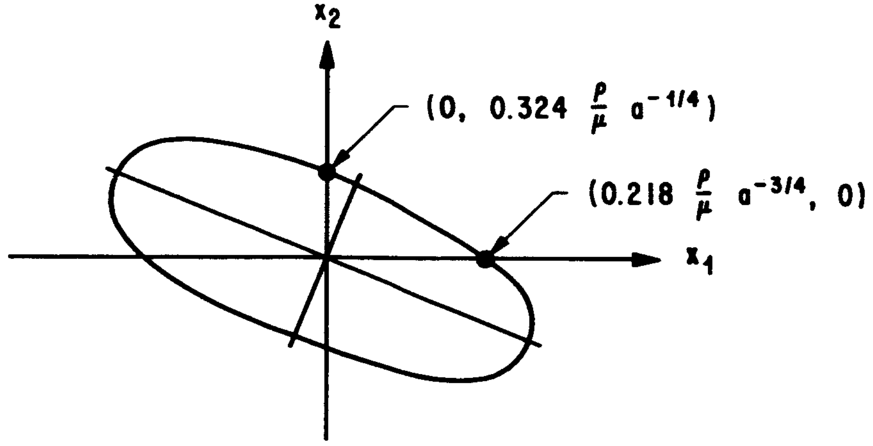

FIG. 3. ELLIPSE OF INITIAL STATES:

$\zeta = .866,\ \omega_n = 4\sqrt{a}$

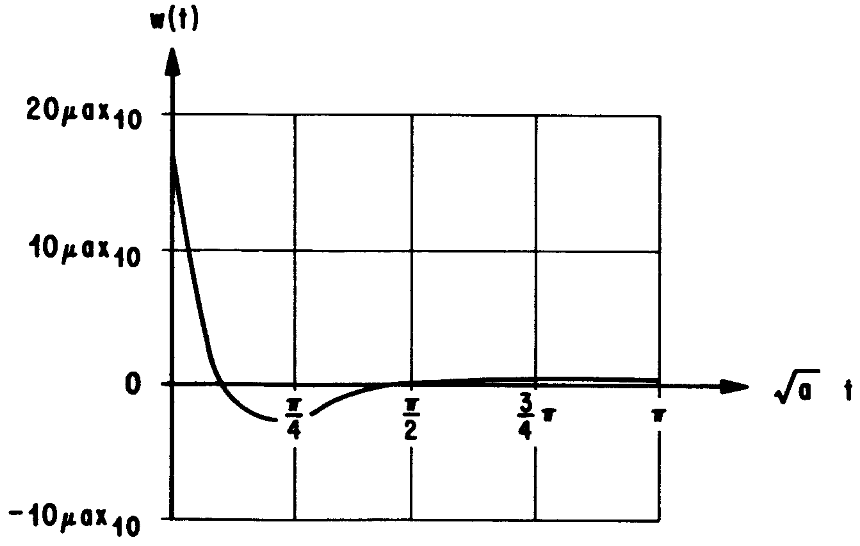

FIG. 4. WORST DISTURBANCE:

$\zeta = .866,\ \omega_n = 4\sqrt{a},\ x_{20} = 0$

APPROVAL

A WORST DISTURBANCE DESIGN CRITERION IN THE THEORY OF ANALYTICAL CONTROL SYSTEMS SYNTHESIS

By

Thomas E. Carter

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This report has also been reviewed and approved for technical accuracy.

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