STABILITY OF CYLINDRICAL SHELLS UNDER AXISYMMETRIC MOVING LOADS

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NOMENCLATURE

E, ν	elastic constants
h	shell thickness
a	shell radius
D	$Eh^3/12(1 - \nu^2)$
β^4	$(h/a)^2/12(1 - \nu^2)$
P	surface load
ρ	mass density
V_L	load velocity
V_{co}	$(E/\rho)^{1/2}\{(h/a)/[3(1 - \nu^2)]^{1/2} + N_X^o/Eh\}^{1/2}$
U, V, W	midsurface displacements
X, Y	axial and circumferential coordinates
$-N_X^o$	initial axial compression of cylinder
F	stress function
w	W/a
w_s	axisymmetric response
x	X/a
θ	Y/a
t	$(T/a)(E/\rho)^{1/2}$
T	time
q	Pa/Eh
f	F/a^2Eh
N_x^*	N_X^o/Eh

M	$V_L/(E/\rho)^{1/2}$
P_c, P_n, P_n^*, Ω_n, Ω_n^*	load parameters
ξ	$x - Mt$
τ	t
C_j, C_j^*, α_j, α_j^*	displacement constants
ζ, η	perturbation quantities
n	number of circumferential half waves
ζ_n, η_n	coefficients of Fourier series (24)
Z_n	two-dimensional vector, $\begin{Bmatrix} \zeta_n \\ \hline \eta_n \end{Bmatrix}$
$\bar{Z}_n$	Laplace transform of Z_n with respect to τ.
$\bar{\bar{Z}}_n$	Laplace transform of $\bar{Z}_n$ with respect to ξ
p	first transform parameter
s	second transform parameter
L_j	2 x 2 matrices defined by (33)
A_j	$L_o^{-1} L_j$
ψ	two-dimensional vector, equation (34)
Λ	see page 17
ϕ	$L_o^{-1}(\psi + \Lambda)$
ρ_j	roots of Det $L_o(s) = 0$

$R_j(k)$	See (41a)
$D(i)$	defined by (41b)
$\underline{\lambda}$	defined by (42)
S_i	defined by (43)
G_i, G_i^*	integration constants
$\left.\begin{matrix} b_1(r,i) \\ b_2(r,i) \end{matrix}\right\}$	defined by (46b)
$C_{r,i;j_1 \ldots j_N}$	defined by (46c)
A	matrix defined by (53a)
λ	load parameter

1. INTRODUCTION

Questions concerning the stability of thin shells subjected to moving loads occur frequently and may be of considerable importance in the area of aerospace boosters where thin shell structures in a shock wave environment are common. This paper will focus on a small portion of the general problem namely, on the stability of elastic thin-walled cylindrical shells subjected to a class of axisymmetric pressure distributions moving with constant velocity in the axial direction.

For a geometrically perfect cylindrical shell, the response to a moving axisymmetric load will be axisymmetric. A number of investigators [1-8] have examined this response in the light of linear shell theory. To date this work covers transient and steady-state responses of infinite and finite length shells in vacuo, and steady-state responses of infinite length shells in an acoustic medium. Similar work on the closely related moving load on the beam-on-elastic-foundation problem is discussed in [9-13].

Under certain circumstances, and when a geometrically non-linear shell theory is considered, the foregoing axisymmetric motion can be unstable with respect to nonaxisymmetric disturbances. Since such instabilities imply the initiation of a new nonaxisymmetric motion, which may result in either shell buckling (in the dynamic sense) or a state of nonaxisymmetric oscillation, they are of considerable interest. Unfortunately, and in contrast to the linear axisymmetric investigations, the present literature is void on the associated stability problem. This report establishes a method of solution of a number of idealized problems in this area concerning infinite length shells in vacuo.

In the analysis the shell is modeled by a system of nonlinear partial differential equations and the primary[1] response to the assumed class of loads is sought as a static (steady-state) solution of the non-linear system in a coordinate system moving with the load. Such motions can, in the load velocity range to be considered, be visualized as the limiting case of a transient problem in which the load is applied and brought up to speed from rest. The stability of this response is defined according to the classical concept of Poincaré, i. e., on the basis of the behavior of a nonaxisymmetric perturbation about the primary motion. Under this concept the analysis consists of a study of a set of perturbation or variational equations. These equations are linearized, assuming infinitesimal disturbances, and stability in the small is considered. Difficulties regarding the existence of variable coefficients in the variational equations are overcome by use of a multiple Laplace transform-functional difference method of solution. As applications of the theory the moving uniform line load (ring load) and the decayed step (shock wave type) loads are treated.

2. FORMULATION OF THE PROBLEM

2.1 The Shell

Consider an infinitely long, elastic, isotropic and homogeneous cylindrical shell with a uniform thickness, h, and a middle surface of radius a. The shell will be assumed thin so that $h/a \ll 1$.

2.2 Equations of Motion

All motions of the shell will be referred to a Lagrangian or fixed coordinate system as illustrated in Fig. 1. Employing a Donnell-type

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[1] "Primary" refers to the axisymmetric response prior to the introduction of nonaxisymmetric disturbances.

theory [14] the equations of motion can be written as a set of two equations, one governing the radial equilibrium of the shell and the other being the compatibility condition. In terms of the radial displacement W, of the midsurface, and a stress function F, these are respectively

$$D\nabla^4 W = P + \frac{\partial^2 F}{\partial Y^2}\frac{\partial^2 W}{\partial X^2} - 2\,\frac{\partial^2 F}{\partial X \partial Y}\,\frac{\partial^2 W}{\partial X \partial Y} + \frac{\partial^2 F}{\partial X^2}\,\frac{\partial^2 W}{\partial Y^2} + \frac{1}{a}\frac{\partial^2 F}{\partial X^2} - \rho h\,\frac{\partial^2 W}{\partial T^2} \tag{1}$$

$$\nabla^4 F = Eh\left[\left(\frac{\partial^2 W}{\partial X \partial Y}\right)^2 - \frac{\partial^2 W}{\partial X^2}\,\frac{\partial^2 W}{\partial Y^2} - \frac{1}{a}\,\frac{\partial^2 W}{\partial X^2}\right]$$

where F is related to the stress resultants by

$$N_X = \frac{\partial^2 F}{\partial Y^2}, \qquad N_Y = \frac{\partial^2 F}{\partial X^2}, \qquad N_{XY} = -\frac{\partial^2 F}{\partial X \partial Y}$$

Here D denotes flexure rigidity, P the normal surface loading, ρ mass density, h shell thickness, a shell midsurface radius, and ∇^4 the biharmonic operator.

The above theory assumes that strains and rotations about a normal to the midsurface are small compared to unity, while the square of rotations about an axis lying in the middle surface are small compared to unity. (See Novozhilov [15] for amplification of these statements). A further assumption, usually associated with Donnell [16] is that $V << \partial W/a\partial Y$. This latter approximation is valid if the displacements of the middle surface are such that the square of the number of circumferential waves, n, is large compared to unity. For thin shells $n > 3$ is usually sufficiently large. For the special case $n = 0$ (axisymmetric motion)

Donnell's approximation is not involved since both V and $\partial W/\partial Y$ vanish. Finally, only the effects of radial inertia are included, i.e., the effects of (1) longitudinal inertia, (2) circumferential inertia, (3) rotary inertia, and (4) transverse shear deformation have been neglected. (Since (3) and (4) are neglected, equations (1) have a diffusive character in place of a correct hyperbolic form, and energy transfer will take place at infinite velocity.) The neglect of these quantities will necessitate a restriction on the magnitude of the load velocity.

2.3 Loading Condition

The shell will be assumed loaded by an axial stress resultant, $N_X{}^o$ (positive in tension) and an axisymmetric lateral pressure distribution P, moving with a velocity V_L and defined by

$$\begin{aligned} P(\xi_1) &= P_c\ \delta(\xi_1) + H(\xi_1)\left[\sum_{n=o}^{N} P_n\ e^{-\Omega_n \xi_1}\right] \\ &\quad + H(-\xi_1)\left[\sum_{k=o}^{k} P_k^*\ e^{\Omega_k^* \xi_1}\right] \end{aligned} \tag{2}$$

Here $\xi_1 = X - V_L T$ represents a coordinate system moving with the load, N and K are assumed to be finite, P_c, P_o, P_o^* are real valued constants and P_n, P_k^*, Ω_n, Ω_k^* $(n, k > 0)$ are complex valued constants with Re Ω_n, $\Omega_k^* > 0$ and $\Omega_o = \Omega_o^* = 0$. The quantities $\delta(\xi_1)$ and $H(\xi_1)$ are, respectively, the Dirac delta function and the Heaviside step function. The load velocity will be restricted, because of the approximations mentioned previously, to

$$V_L < V_{co} = (E/\rho)^{1/2} \left\{ \frac{h/a}{[3(1-\nu^2)]^{1/2}} + \frac{N_X^o}{Eh} \right\}^{1/2} \qquad (3)$$

where N_X^o will be considered only in compression, i. e., $N_X^o \leq 0$. V_{co} in (3) represents the minimum propagation velocity (cut-off velocity) of undistorted axisymmetric sinusoidal wave trains in the shell. For steel shells, with $N_X^o = 0$, V_{co} lies between 400 - 2000 fps for a/h = 1000 - 40 respectively. The effect of $N_X^o < 0$ is to lower these values. The value of N_X^o yielding $V_{co} = 0$ is the classical buckling load due to axial compression. For all compressive loads less than this value, $V_{co} > 0$. Physically, V_{co} marks a basic change in the character of the axisymmetric response. The necessity of (3) will be discussed later.

Several examples of the type of loads that can be constructed from equation (2) are illustrated in Figs. 2(a), (b), (c), and (d). They include the moving ring load, step load, decayed steps, and general pulse (including internal pressure and axial compression), respectively. Many load distributions not falling directly into the foregoing class can be approximated closely by (3) in which the coefficients can be determined by suitable means. The question of the completeness of the exponential portion of (3) as N or $k \to \infty$ has been discussed by Erdélyi [17].

2.4 Primary Response

The primary response of the shell to the load (3) will be obtained as a bounded solution of the nonlinear system (1) in the steady state form

$$W = W_s(X - V_L T)$$

$$\frac{\partial^2 F}{\partial X^2}, \frac{\partial^2 F}{\partial Y^2} = \frac{\partial^2 F_s}{\partial X^2}, \frac{\partial^2 F_s}{\partial Y^2} (X - V_L T) \tag{4}$$

$$\frac{\partial^2 F}{\partial X \partial Y} = \frac{\partial^2 F_s}{\partial X \partial Y} = 0$$

2.5 Definition of Stability

The problem is to determine the conditions under which the response (4) is stable or unstable. For this purpose we perturb the primary response W_s, F_s by, respectively, the quantities $W^*(X, Y, T)$ and $F^*(X, Y, T)$. If W_p and F_p denote the perturbed motion, we have

$$W_p = W_s + W^*, \qquad F_p = F_s + F^* \tag{5}$$

Inserting W_p and F_p into the nonlinear equations (1) and neglecting powers of W^*, F^* above the first, one obtains linear variational equations for W^*, F^*. Based on the physics of the problem only those solutions of the variational equations which are bounded as $|X| \rightarrow \infty$ for fixed Y and T will be considered. If all such solutions, corresponding to prescribed initial conditions over an arbitrary but finite X interval, are bounded as $T \rightarrow \infty$, the shell will be said to be stable, otherwise unstable.

3. GENERAL ANALYSIS

3.1 System in Nondimensional Form

It will be convenient to introduce the nondimensional quantities

$$w = W/a, \quad f = F^2/a^2Eh, \quad x = X/a, \quad \theta = Y/a$$

$$t = (T/a)\,(E/\rho)^{1/2}, \quad M = V_L \Big/ (E/\rho)^{1/2} \tag{6}$$

Substitution of (6) into (1) and (2) yields

$$\beta^4 \nabla^4 w = q\left[a(x - Mt)\right] + f_{\theta\theta}\, w_{xx} - 2f_{x\theta}\, w_{x\theta} + f_{xx}\,(1 + w_{\theta\theta}) - w_{tt}$$

$$\nabla^4 f = (w_{x\theta})^2 - w_{xx}\,(1 + w_{\theta\theta}) \tag{7a}$$

Here

$$\beta^4 = (h/a)^2/12(1 - \nu^2)$$

$$\nabla^4 (\;) \equiv (\;)_{xxxx} + 2(\;)_{xx\theta\theta} + (\;)_{\theta\theta\theta\theta}$$

Where $(\;)x$ denotes $\partial(\;)/\partial x$, etc. The function q in (7) is given by

$$q(a\xi) = q_c\, \delta(\xi) + H(\xi)\left[\sum_{n=o}^{N} q_n\, e^{-\gamma_n \xi}\right]$$

$$+ H(-\xi)\left[\sum_{k=o}^{k} q_k^*\, e^{\gamma_k^* \xi}\right] \tag{7b}$$

where

$$q_c = P_c/Eh, \quad (q_n, q_k^*) = (P_n, P_k^*)\, a/Eh, \quad (\gamma_n, \gamma_k^*) = (\Omega_n, \Omega_k^*)a$$

and

$$\xi = x - Mt$$

3.2 Primary Response

The governing equations for the axisymmetric motion of the shell are obtained by requiring that $w_\theta = f_{x\theta} = 0$, and that $f_{\theta\theta}$ and f_{xx} be independent of θ. Under these restrictions (7a) takes the form

$$\beta^4 w_{s_{xxxx}} = q\left[a(x-Mt)\right] + f_{s_{\theta\theta}} w_{s_{xx}} + f_{s_{xx}} - w_{s_{tt}}$$

$$(f_{s_{xx}} + f_{s_{\theta\theta}})_{xx} = - w_{s_{xx}} \tag{8}$$

It is advantageous at this point to note that the second of equations (8) can be written

$$\left(\frac{N_X}{Eh} + \frac{N_Y}{Eh}\right)_{xx} = - w_{s_{xx}} \tag{9}$$

The equilibrium equation in the axial (X) direction under the present theory has the form

$$\partial N_X/\partial X + \partial N_{XY}/\partial Y = 0$$

Whereby $\partial N_X/\partial X = 0$ in the axisymmetric state and therefore $N_X = N_X^o(T)$ only. It will be assumed that N_X^o is maintained as constant at the "ends" $X = \pm \infty$ and we define

$$N_x^* = \overset{o}{N_X}/Eh = \text{const.} \tag{10}$$

(The quantity $-\overset{o}{N_X}$ represents an initial compression of the cylinder).

Upon use of (10), integration of (9), application of a bounded requirement on N_Y as $X \rightarrow \pm \infty$, and noting that under a uniform compression $N_Y = 0$, $w_s = -\nu N_X^*$, one obtains

$$N_Y/Eh = -w_s + \nu N_x^* \tag{11}$$

and therefore the second of (8) can be replaced by

$$f_{s_{xx}} = -w_s + N_x^*, \quad f_{s_{\theta\theta}} = N_x^* \tag{12}$$

Substitution of (12) into the first of (8) yields the following linear partial differential equation for the radial displacement w_s:

$$\beta^4 w_{s_{xxxx}} - N_x^* w_{s_{xx}} + w_s + w_{s_{tt}} = q\left[a(x - Mt)\right] + \nu N_x^* \tag{13}$$

The primary response is now obtained by solution of (13) under the condition $w_s = w_s(x - Mt) = w_s(\xi)$. This leads to the following total differential equation for w_s:

$$\beta^4 w_s^{IV} + (M^2 - N_x^*) w_s'' + w_s = q(a\xi) + \nu N_x^* \tag{14}$$

where $(\ )' = d/d\xi$. Requiring only that the solution of (14) be bounded as $\xi \longrightarrow \infty$, one obtains $w_s(\xi)$ for $V_L < V_{co}$ ($M < \sqrt{2}\,\beta$) as:

$$w_s(\xi) = \int_{-\infty}^{\infty} g(\xi, \lambda)\left[q(a\lambda) + \nu N_x^*\right] d\lambda \tag{15}$$

where $g(\xi, \lambda)$ represents the Green's function of (14) and has the form

$$g(\xi, \lambda) = \sum_{i=1}^{2} g_i\, e^{-\alpha_i(\xi - \lambda)}, \qquad \xi - \lambda > 0$$

$$= \sum_{i=1}^{2} g_i\, e^{\alpha_i(\xi - \lambda)}, \qquad \xi - \lambda < 0 \tag{16a}$$

Here

$$g_{1,2} = \frac{1}{4(\bar{\beta}^4 - \bar{M}^4)^{1/2}} \left[(\bar{\beta}^2 + \bar{M}^2)^{1/2} \pm i(\bar{\beta}^2 - \bar{M}^2)^{1/2}\right]$$

$$\alpha_{1,2} = \frac{1}{\bar{\beta}^2} \left[(\bar{\beta}^2 - \bar{M}^2)^{1/2} \pm i\, (\bar{\beta}^2 + \bar{M}^2)^{1/2}\right] \tag{16b}$$

$$\bar{\beta} = \sqrt{2}\, \beta, \; \bar{M}^2 = M^2 - N_x^*$$

Evaluation of the integral (15) under the assumption that the arguments of the exponentials of the loading function 7(b) are not roots of the characteristic equation of (14) yields $w_s(\xi)$ formally as

$$w_s(\xi) = C_o + \sum_{j=1}^{\ell} C_j\, e^{-\alpha_j \xi}, \qquad \ell = N + 2;\; \xi > 0$$

$$= C_o^* + \sum_{j=1}^{\ell} C_j^*\, e^{\alpha_j \xi}, \qquad \ell^* = K + 2;\; \xi < 0 \tag{17}$$

where C_j, C_j^*, α_j, α_j^* are in general complex valued, C_o, C_o^* are real valued, and Re α_j, $\alpha_j^* > 0$.

Recalling that the effects of longitudinal inertia, rotary inertia, and shear deformation were neglected in (1) and hence in (13) and (14), an estimate of the validity of these approximations can be made by referring to the investigations of Tang [8] and Jones and Bhuta [1]. Consider first the phase (or load velocity) spectrum of (13), obtained by the assumption of steady-state wave trains of the form

$$w_s(x, t) = A\, e^{iK(x - ct)} \tag{18}$$

Substitution of (18) into the homogeneous portion of (13) yields the following frequency equation of (13) or characteristic root (K) equation of (14) (when c is replaced by M):

$$\beta^4 K^4 + K^2 (N_x^* - c^2) + 1 = 0 \tag{19}$$

A typical comparison of (19) with the first mode spectrum of a more exact (linear) theory corresponding to Timoshenko's theory of beam vibration is reproduced from Tang's work in Figs. 3(a) and 3(b) for the case $h/a = .06$, $N_x^* = 0$. It can be observed that when $c < \sqrt{2}\,\beta$ ($V_L < V_{co}$) the spectrums of both theories possess complex wave numbers (indicating an attenuated primary response) and are found to be practically identical (indicating shear and rotational effects do not appreciably modify the characteristic roots of (14) for $V_L < V_{co}$). When $c > \sqrt{2}\,\beta$ ($V_L > V_{co}$) however, only real wave numbers exist (indicating a pure oscillatory primary response for the first mode) and the two spectrums are seen to agree for only small wave numbers (long wave lengths). In terms of the characteristic roots (K) of the primary response, this implies half the roots will be in error for $V_L > V_{co}$, the magnitude of the error increasing with increasing V_L (a second (shear) mode not discussed above is also known to play an important role for $V_L > V_{co}$. This mode is of course not embodied in the elementary theory). It is evident therefore, that, as far as steady-state solutions are concerned, the effects of rotary inertia and shear deformation can be neglected only if $V_L < V_{co}$. In view of this development, the restriction (3) was placed on V_L.

In the analysis of Jones and Bhuta [1] the axisymmetric response to moving loads on infinite length cylindrical shells was studied. Their calculations indicate the effect of longitudinal inertia on the steady-state solution is negligible for load velocities which are considerably less than the plate velocity, which is given by

$$V_p^2 = E/\rho(1 - \nu^2) \tag{20}$$

The ratio of $(V_{co})_{max} = V_{co}(N_x^* = 0)$ to the plate speed V_L is

$$\left(\frac{V_{co}}{V_p}\right)^2 = \frac{h}{a} \frac{(1 - \nu^2)^{1/2}}{3} \tag{21}$$

Thus, since it is assumed $h/a \ll 1$, the effect of longitudinal inertia is apparently not important if $V_L < V_{co}$.

Finally, the basic character of the primary response for $V_L < V_{co}$ can be summarized as follows: 1.) the motion is attenuated on either side of $\xi = 0$ if $C_o = C_o^* = 0$, or attenuates to a constant value at $x = \pm \infty$ if $C_o, C_o^* \neq 0$, 2.) for loads symmetric about $\xi = 0$ the motion is also symmetric in ξ, 3.) the amplitude of the primary response increases without bound (approaches a resonance condition) as $V_L \rightarrow V_{co}$. These statements follow directly from the character of the Green's function (16). In the following section the stability of the primary response is discussed.

3.3 Variational Equations

It will be convenient at this point to introduce the transformation

$$\xi = x - Mt, \quad \theta = \theta, \quad \tau = t \tag{22}$$

Application of (22) to (7) and substitution of the perturbed motions (5) into the resulting differential equations yields the following variational equations:

$$\begin{aligned} \beta^4 \nabla^4 \zeta &= (N_x^* - M^2)\zeta_{\xi\xi} + w_s''(\xi)\eta_{\theta\theta} + (\nu N_x^* - w_s(\xi))\zeta_{\theta\theta} \\ &\quad + \eta_{\xi\xi} - \zeta_{\tau\tau} + 2M\zeta_{\xi\tau} \end{aligned} \tag{23}$$

$$\nabla^4 \eta = - w_s''(\xi)\zeta_{\theta\theta} - \zeta_{\xi\xi}$$

where

$$\nabla^4(\ ) = (\ )_{\xi\xi\xi\xi} + 2(\ )_{\xi\xi\theta\theta} + (\ )_{\theta\theta\theta\theta}; \quad \zeta = \frac{W^*}{a}; \quad \eta = \frac{F^*}{a^2 Eh}$$

3.4 Solution of the Variational Equations

3.4.1 Series Representation

To construct that class of solutions of (23) pertinent to the stability analysis we begin by representing the functions ζ and η by the following Fourier series:

$$\zeta(\xi,\theta\ \ ) = \sum_{n=o}^{\infty} \zeta_n\ (\xi,\tau)\cos n\,\theta$$

$$\eta(\xi,\theta,\ \ ) = \sum_{n=o}^{\infty} \eta_n(\xi,\tau)\ \cos n\,\theta \tag{24}$$

By use of (24) one obtains from (23) the following set of coupled partial differential equations governing ζ_n and η_n for each integer $n = 0, 1, 2,$ ...

$$\beta^4 \zeta_{n_{\xi\xi\xi\xi}} + (M^2 - N_x^* - 2n^2\beta^4)\zeta_{n_{\xi\xi}} + (\beta^4 n^4 - n^2 w_s(\xi) + n^2 \nu N_x^*)\zeta_n$$

$$+ \zeta_{n_{\tau\tau}} - 2M\zeta_{n_{\xi\tau}} = \eta_{n_{\xi\xi}} - n^2 w_s''(\xi)\eta_n \tag{25}$$

$$\eta_{n_{\xi\xi\xi\xi}} - 2n^2\eta_{n_{\xi\xi}} + n^4\eta_n = n^2 w_s''(\xi)\zeta_n - \zeta_{n_{\xi\xi}}$$

3.4.2 First Laplace Transform

Next a Laplace transform of (25) with respect to τ is performed. This yields in matrix form

$$\begin{bmatrix} \beta^4 & 0 \\ 0 & 1 \end{bmatrix} \bar{Z}_n^{IV} + \begin{bmatrix} M^2-2n^2\beta^2-N_x^* & -1 \\ 1 & -2n^2 \end{bmatrix} \bar{Z}_n'' + \begin{bmatrix} -2Mp & 0 \\ 0 & 0 \end{bmatrix} \bar{Z}_n'$$

(26)

$$+\begin{bmatrix} \beta^4 n^4 - n^2(w_s(\xi) - \nu N_x^*) + p^2 & n^2 w_s''(\xi) \\ -n^2 w_s''(\xi) & n^4 \end{bmatrix} \bar{Z}_n = \begin{Bmatrix} p\zeta_n(\xi,0) - 2M\zeta_{n_\xi}(\xi,0) + \zeta_{n_\tau}(\xi,0) \\ 0 \end{Bmatrix}$$

where Z_n is the two-dimensional vector

$$Z_n = \begin{Bmatrix} \zeta_n \\ \eta_n \end{Bmatrix} \qquad (27)$$

and $\bar{Z}_n$ is defined by

$$\bar{Z}_n(\xi,p) = \int_0^\infty e^{-p\tau} Z_n(\xi,\tau)d\tau \;, \qquad \tau > 0, \; \mathrm{Re}\,p > c \qquad (28)$$

From the regularity condition on ζ and η at $\xi = \pm\infty$ we have in addition the requirement

$$\bar{Z}_n(\xi,p) \text{ remain bounded as } \xi \rightarrow \infty, \; \mathrm{Re}\,p > c \qquad (29)$$

The terms on the right side of (26) represent the initial conditions of the problem or the form of the initial disturbance. It will be assumed that the initial disturbance is bounded in the arbitrary, but finite interval $|\xi| < x_o = \text{const.}$, and is zero for $|\xi| \geqslant x_o$.

3.4.3 Second Laplace Transform

The solution of the set of total differential equations (26) will now be constructed. Since the variable coefficients of (26) consist of a sum of exponentials it is possible to perform a Laplace transform of (26) with respect to ξ. Considering the interval $0 < \xi < \infty$, a unilateral transform will be applied. Inversion will yield a solution for $\xi > 0$ from which the solution for $\xi < 0$ is easily deduced.

Denoting the transform of $\vec{Z}_n$ with respect to ξ by

$$\bar{\bar{Z}}_n(s,p) = \int_0^\infty \bar{Z}_n(\xi,p)\, e^{-s\xi} d\xi, \quad \xi > 0, \ \mathrm{Re}\ s > b \tag{30}$$

and noting the shift property of the Laplace transform

$$\int_0^\infty e^{-s\xi} \left[e^{-\alpha_j \xi} \bar{Z}_n(\xi,p) \right] d\xi = \bar{\bar{Z}}_n(s+\alpha_j,p) \tag{31}$$

one obtains the transformed version of (26) in the form

$$L_o(s,p)\, \bar{\bar{Z}}_n(s,p) = \sum_{j=1}^{\ell} L_j(s,p)\, \bar{\bar{Z}}_n(s+\alpha_j,p) + \psi(s,p) + \Lambda(s,p) \tag{32}$$

where L_o, L_1, ..., L_ℓ represent the following 2 x 2 matrices:

$$L_o = \begin{bmatrix} \beta^4 s^4 + s^2(M^2 - N_x^* - 2n^2\beta^4) - 2Mps & -s^2 \\ \quad + \beta^4 n^4 - n^2(C_o - \nu N_x^*) + p^2 & \\ s^2 & (s^2-n^2)^2 \end{bmatrix} \tag{33}$$

$$L_j = n^2 C_j \begin{bmatrix} 1 & -\alpha_j^2 \\ \alpha_j^2 & 0 \end{bmatrix} \quad ; \; j = 1, 2, \cdots, \ell$$

and where $\Lambda(s,p)$ is the transform of the right hand side of (26) with respect to ξ, and $\psi(s,p)$ is a two-dimensional vector containing initial data at $\xi = 0^+$ of the form

$$\psi(s,p) = \sum_{i=o}^{3} s^{(3-i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{d^i \bar{Z}_n(0^+, p)}{d\xi^i} + $$

$$+ \sum_{i=o}^{1} s^{(1-i)} \begin{bmatrix} (M^2 - N_x^* - 2n^2\beta^4) & -1 \\ 1 & -2n^2 \end{bmatrix} \frac{d^i \bar{Z}_n(0^+, p)}{d\xi^i} \tag{34}$$

$$+ \begin{bmatrix} -2Mp & 0 \\ 0 & 0 \end{bmatrix} \bar{Z}_n(0^+, p)$$

Premultiplication of (32) by L_o^{-1}, the inverse of L_o, yields

$$\bar{\bar{Z}}_n(s,p) = \sum_{j=1}^{\ell} A_j(s,p)\, \bar{\bar{Z}}_n(s+\alpha_j, p) + \Phi(s,p) \qquad (35)$$

where

$$A_j = L_o^{-1} L_j, \qquad \Phi = L_o^{-1}(\Psi + \Lambda) \qquad (36)$$

Equation (36) represents a system of linear functional-difference equations with variable coefficients[3]. Our next task is to obtain a suitable solution of these equations. Note first that the variable coefficients of (26) possess the property

$$\lim_{\xi \to \infty} w_s(\xi) = C_o, \qquad \lim_{\xi \to \infty} w_s''(\xi) = 0$$

Thus it is not surprising that the solutions of (26) are of exponential order, i. e.,

$$|\bar{Z}_n(\xi)| \leqslant a e^{b\xi}$$

where a and b are constants, and $|\bar{Z}_n|$ denotes the norm of $\bar{Z}_n$, defined by

$$|\bar{Z}_n| = |\zeta_n| + |\eta_n|$$

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3 For a discussion of the relationship between the Laplace transform and difference equations, and the solution of difference equations, see [21], [22], and [23].

However, this implies

$$|\bar{\bar{Z}}_n| \leq \int_0^{\infty} e^{-(\mathrm{Re}\ s)\xi}\ |\bar{Z}_n|\ d\xi \leq \frac{a}{\mathrm{Re}\ s - b}\ , \quad \mathrm{Re}\ s > b$$

Therefore the following quiescent condition on the second transform must be satisfied:

$$\lim_{\mathrm{Re}\ s \to \infty} \bar{\bar{Z}}_n(s) = 0 \tag{37}$$

This is sufficient to render the second transform unique, or more specifically, the solution of the difference equation unique. This follows from the fact that all solutions of the homogeneous counterpart of (27):

$$\bar{\bar{Z}}_n(s,p) = \sum_{j=1}^{\ell} A_j(s,p)\ \bar{\bar{Z}}_n(s+\alpha_j, p)$$

which represents the difference between any two particular solutions, are unbounded as $\mathrm{Re}\ s \to \infty$. Thus, on the basis of (37), only the trivial solution of the homogeneous equation can be accepted. There is thus a unique particular solution of (37) to be found.

The desired particular solution can be constructed by the method of ascending continued fractions [22], and has the form

$$\bar{\bar{Z}}_n(s) = \Phi(s) + \sum_{j=1}^{\ell} A_{j_1}(s)\Phi(s+\alpha_{j_1}) + \sum_{j_1,j_2=1}^{\ell} A_{j_1}(s)A_{j_2}(s+\alpha_{j_1})\Phi(s+\alpha_{j_1}+\alpha_{j_2})$$

$$+ \sum_{j_1,j_2,j_3=1}^{\ell} A_{j_1}(s)A_{j_2}(s+\alpha_{j_1})A_{j_3}(s+\alpha_{j_1}+\alpha_{j_2})\Phi(s+\alpha_{j_1}+\alpha_{j_2}+\alpha_{j_3})$$

$$+ \ldots \qquad (38a)$$

which can be written in closed form as

$$\bar{\bar{Z}}_n(s) = \Phi(s) + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{\ell} A_{j_k}(s + \sum_{q=o}^{k-1} \alpha_{j_q})\Phi(s + \sum_{r=1}^{N} \alpha_{j_r}) \qquad (38b)$$

where $\alpha_{j_o} \equiv 0$. The parameter p has been suppressed for simplicity.

The vector function (38) formally satisfies the difference equation (35) and the quiescent requirement (37). Further, the component series for the vector $\bar{\bar{Z}}_n$ are absolutely and uniformly convergent with respect to s and represent analytic functions of s when $s \epsilon R$ where the region R of the complex s-plane is defined by

$$|s - (\rho_i - m_1\alpha_1 - m_2\alpha_2 - \ldots - m_\ell \alpha_\ell)| \geq \epsilon > 0$$

$$m_j = 0, 1, 2, \ldots ; \; j = 1, 2, \ldots, \ell$$

Here ρ_i are the roots of the 8th order polynomial in s defined by Det $L_o(s) = 0$. The singularities of $\bar{\bar{Z}}_n(s)$ are isolated poles, located at

$$s = \rho_i - m_1\alpha_1 - m_2\alpha_2 - \dots - m_\ell \alpha_\ell$$

$$m_j = 0, 1, 2, \dots; \qquad j = 1, 2, \dots, \ell$$

If all problem parameters are held fixed, including p, these poles lie a finite distance to the right of Re $s = 0$, and $\bar{\bar{Z}}_n(s)$ is regular for Re $s > C_1$ = constant.

3.4.4 Inversion of the s-Transform

The series (38) will now be inverted term by term under the assumption that the roots ρ_i are nonrepeated (repeated roots will be discussed later). Consider first the definitions

$$B_j(s) = \frac{\mathrm{Det}\ L_o(s)}{\beta^4} A_j(s) = \prod_{q=1}^{8} (s-\rho_q)\, A_j(s)$$

$$\Phi(s) = \prod_{q=1}^{8} (s - \rho_q)\, \phi(s) \tag{39}$$

Now, the Nth term of (38b) is composed of the product

$$\frac{B_{j_1}(s)}{\Pi(s-\rho_q)} \cdot \frac{B_{j_2}(s+\alpha_{j_1})}{\Pi(s-\rho_q+\alpha_{j_1})} \cdots \frac{B_{j_N}(s+\alpha_{j_1}+\dots+\alpha_{j_{N-1}})}{\Pi(s-\rho_q+\alpha_{j_1}+\dots+\alpha_{j_{N-1}})} \cdot \frac{(s+\alpha_{j_1}+\dots+\alpha_{j_N})}{\Pi(s-\rho_q+\alpha_{j_1}+\dots+\alpha_{j_N})}$$

where Π denotes $\prod_{q=1}^{8}$. Each factor of the resulting matrix is the ratio of polynomials, the denominator being of higher order than numerator. In the Nth term, each factor above is inverted separately by the residue theorem and the inversion of the entire term is then obtained by repeated use of the convolution integral. After some manipulation the following series is obtained for $\overline{Z}_n(\xi, p)$:

$$\overline{Z}_n(\xi, p) = \overline{Z}_{n_1}(\xi, p) + \overline{Z}_{n_2}(\xi, p) \tag{40a}$$

where

$$\overline{Z}_{n_1}(\xi, p) = \sum_{i=1}^{8} \Bigg\{ I \exp(\rho_i \xi) + \sum_{j_1=1}^{\ell} \sum_{k_1=1}^{8} R_{j_1}(k_1) \int_0^{\xi} \exp\left[(\rho_i - \alpha_{j_1})(\xi - \xi_1) + \right.$$

$$\left. + \rho_{k_1} \xi_1\right] d\xi_1 + \sum_{j_1, j_2=1}^{\ell} \sum_{k_1, k_2=1}^{8} R_{j_1}(k_1) R_{j_2}(k_2) \int_0^{\xi}\int_0^{\xi_1} \exp\Big[$$

$$(\rho_i - \alpha_{j_1} - \alpha_{j_2})(\xi - \xi_1) + (\rho_{k_1} - \alpha_{j_1})(\xi_1 - \xi_2) + \rho_{k_2} \xi_2\Big] d\xi_2 d\xi_1$$

$$+ \ldots \Bigg\} Q_i ; \quad \xi \geq 0$$

which can be written in closed form as

$$\overline{Z}_{n_1}(\xi,p)=\sum_{i=1}^{8}\left\{I\exp(\rho_i\xi)+\sum_{N=1}^{\infty}\left(\prod_{q=1}^{N}\sum_{j_q=1}^{\ell}\right)\left(\prod_{r=1}^{N}\sum_{k_r=1}^{8}\right)\left(\prod_{s=1}^{N}R_{j_s}(k_s)\right)\int_0^{\xi}\left(\prod_{t=1}^{N-1}\int_0^{\xi_t}\right)\right. \tag{40b}$$

$$\left.\exp\left[(\rho_i-\sum_{u=1}^{N}\alpha_{j_u})(\xi-\xi_1)+\rho_{k_N}\xi_N+\sum_{V=1}^{N-1}(\rho_{k_v}-\sum_{w=1}^{N-v}\alpha_{j_w})(\xi_v-\xi_{v+1})\right]d\xi_N\cdots d\xi_1\right\}\cdot$$

$$\cdot Q_i;\quad \xi>0$$

and

$$\overline{Z}_{n_2}(\xi,p)=\sum_{i=1}^{8}\left\{D(i)\int_0^{\xi}\underline{\lambda}(\xi_1)\exp\left[\rho_i(\xi-\xi_1)\right]d\xi_1+\right.$$

$$+\sum_{j_i=1}^{\ell}\sum_{k_1=1}^{8}R_{j_1}(k_1)D(i)\int_0^{\xi}\int_0^{\xi_1}\underline{\lambda}(\xi_2)\exp\left[\rho_{k_1}(\xi-\xi_1)+(\rho_i-\alpha_{j_1})(\xi_1-\xi_2)-\alpha_{j_1}\xi_2\right]d\xi_2 d\xi_1$$

$$+\sum_{j_1,j_2=1}^{\ell}\sum_{k_1,k_2=1}^{8}R_{j_1}(k_1)R_{j_2}(k_2)D(i)\int_0^{\xi}\int_0^{\xi_1}\int_0^{\xi_2}\underline{\lambda}(\xi_3)\exp\left[\rho_{k_1}(\xi-\xi_1)+(\rho_{k_2}-\alpha_{j_1})(\xi_1-\xi_2)\right.$$

$$\left.+(\rho_i-\alpha_{j_1}-\alpha_{j_2})(\xi_2-\xi_3)-(\alpha_{j_1}+\alpha_{j_2})\xi_3\right]d\xi_3 d\xi_2 d\xi_1$$

$$+\ldots,\qquad \xi>0$$

which can be written in closed form as

$$\bar{Z}_{n_2}(\xi,p) = \sum_{i=1}^{8} \Bigg\{ D(i) \int_0^{\xi} \underline{\lambda}(\xi_1) \exp\,[\rho_i(\xi-\xi_1)]\, d\xi_1$$

$$+ \sum_{N=1}^{\infty} \left(\prod_{q=1}^{N} \sum_{j_q=1}^{\ell} \right) \left(\prod_{r=1}^{N} \sum_{k_r=1}^{8} \right) \left(\prod_{s=1}^{N} R_{j_s}(k_s) \right) \cdot \tag{40c}$$

$$D(i) \int_0^{\xi} \left(\prod_{t=1}^{N} \int_0^{\xi_t} \right) \underline{\lambda}(\xi_{N+1}) \exp\Big[(\rho_i - \sum_{u=1}^{N} \alpha_{j_u})(\xi_N - \xi_{N+1}) - (\sum_{v=1}^{N} \alpha_{j_v})\xi_{N+1}$$

$$+ \rho_{k_1}(\xi-\xi_1) + \sum_{w=2}^{N} (\rho_{k_w} - \sum_{m=1}^{w=1} \alpha_{j_m})(\xi_{w-1} - \xi_w)\Big]\, d\xi_{N+1} \ldots d\xi_1 \Bigg\}$$

$$, \quad \xi > 0$$

Here $R_j(k)$ and $D(i)$ are 2 x 2 matrices of the form

$$R_j(k) = \frac{n^2 C_j}{\prod\limits_{\substack{r=1 \\ r \neq k}}^{8} (\rho_k - \rho_r)} \begin{bmatrix} (\rho_k^2 - n^2)^2 + \alpha_j^2 \rho_k^2 & -\alpha_j^2 (\rho_k^2 - n^2)^2 \\ \begin{aligned} &-\rho_k^2 + \alpha_j^2 [\beta^4 \rho_k^4 - 2Mp\rho_k \\ &+ \rho_k^2 (M^2 - N_x^* - 2n^2\beta^4) \\ &+ (\beta^4 n^4 - n^2 C_o + n^2 \nu N_x^* + p^2] \end{aligned} & s^2 \alpha_j^2 \end{bmatrix} \tag{41a}$$

$$D(i) = \frac{1}{\prod_{\substack{r=1\\ r\neq i}}^{8}(\rho_i-\rho_r)} \begin{bmatrix} (\rho_i^{\,2}-n^2)^2 & \rho_i^{\,2} \\ & \\ -\rho_i^{\,2} & \begin{array}{c} \beta^4\rho_i^{\,4}+\rho_i^{\,2}\left[M^2-N_x^* -2n^2\beta^4\right] \\ -2Mp\rho_i+\left[\beta^4 n^4-n^2C_o+n^2\nu N_x^*+p^2\right] \end{array} \end{bmatrix} \tag{41b}$$

and $\underline{\lambda}$ is given by

$$\underline{\lambda} = \left\{ \begin{array}{c} p\zeta_n(\xi,0) - 2M\,\zeta_{n_\xi}(\xi,0) + \zeta_{n_\tau}(\xi,0) \\ 0 \end{array} \right\} \tag{42}$$

The vectors Q_i in (40b) contain the unknown initial data at $\xi = 0^+$. They can be regarded as unknown constants, to be evaluated later from the boundary conditions. The elements of each Q_i vector, however, are not independent, but are related through the differential equations (26). By direct substitution of the homogeneous solution (40b) into (27) one finds the differential equations are formally satisfied for each $i = 1, 2, 3, \ldots, 8$, if the elements of Q_i are related by

$$Q_i = \left\{ \begin{array}{c} S_i \\ 1 \end{array} \right\} G_i \tag{43a}$$

where

$$S_i = -(\rho_i^2 - n^2)^2/\rho_i^2 \tag{43b}$$

and the G_i are arbitrary scalar constants.

For each value of $i = 1, 2, 3, \ldots, 8$, equation (40b) represents a linearly independent solution of the homogeneous part of (27) when the ρ_i are nonrepeated. Equation (40c) is a particular solution of (27). Therefore (40a) represents the general solution of (27) for $\xi > 0$ when the ρ_i are nonrepeated.

The function $\overline{Z}_n(\xi, p)$ for $\xi < 0$ is easily obtained by inspection from equations (40). One need only replace ℓ, C_j α_j, ρ_i in (40), (41) and (42) by ℓ^*, C_j^*, $-\alpha_j^*$, ρ_i^* respectively. The quantities ρ_i^* are the roots of Det $L_o(s) = 0$ when C_o is replaced by C_o^*. These changes will be denoted by "starring" all quantities where changes occur. Thus, for $\xi < 0$ $\overline{Z}_n$ has the form (40) with the changes

$$\rho_i \rightarrow \rho_i^*, \quad \ell \rightarrow \ell^*, \quad R(k) \rightarrow R^*(k), \quad \alpha_j \rightarrow -\alpha_j^*, \quad G_i \rightarrow G_i^*, \quad D(i) \rightarrow D^*(i)$$

The solution (40) contains eight unknown constants of integration G_i ($i = 1$ to 8). Similarly, for $\xi < 0$ there exist eight constants, G_i^*. These constants are evaluated by requiring that 1) $\overline{Z}_n$ and its first three derivatives with respect to ξ be continuous across $\xi = 0$[4] and 2) $\overline{Z}_n$ remain bounded[5] as $\xi \rightarrow \infty$, Re $p > c$. Consider first the matching condition at $\xi = 0$. Using the relation[6]:

4. This follows from the assumed boundedness of $\lambda(\xi)$
5. This follows from the regularity condition imposed on Z_n as $x \rightarrow \infty$ for fixed t.
6. This is easily verified by expanding the leading term in partial fractions.

$$\sum_{k=1}^{N} \rho_k{}^m / \prod_{\substack{q=1\\q\neq k}}^{N} (\rho_k - \rho_q) = 0 \left\{ \begin{array}{ll} N > m+1 & \\ \rho_q & \text{nonrepeated} \end{array} \right.$$

one can show that the series (40) and its counterpart for $\xi < 0$ possess the following property at $\xi = 0^+$ and $\xi = 0^-$:

$$\begin{aligned} \frac{d^m}{d\xi^m} \bar{Z}_n(0^+, p) &= \rho_i{}^m \begin{Bmatrix} S_i \\ 1 \end{Bmatrix} G_i \,, \qquad m < 7 \\ \frac{d^m}{d\xi^m} \bar{Z}_n(0^-, p) &= \rho_i{}^{*m} \begin{Bmatrix} S_i^* \\ 1 \end{Bmatrix} G_i^* \,, \qquad m < 7 \end{aligned} \tag{44}$$

Application of the foregoing matching conditions, 1), with use of (44) yields

$$\begin{aligned} &\sum_{i=1}^{8} \rho_i{}^m G_i - \rho_i{}^{*m} G_i^* = 0, \qquad m = 0, 1, 2, 3 \\ &\sum_{i=1}^{8} \rho_i^m S_i G_i - \rho_i{}^{*m} S_i^* G_i^* = 0, \qquad m = 0, 1, 2, 3. \end{aligned} \tag{45}$$

To apply the quiescent condition 2) above it is necessary to obtain from (40) the limiting form of $\bar{Z}_n$ for large ξ. One finds

$$\bar{Z}_n \underset{\xi\to\infty}{\backsim} \sum_{r=1}^{8} e^{\rho_r \xi} \left[\sum_{i=1}^{8} \begin{Bmatrix} b_1(r,i) \\ b_2(r,i) \end{Bmatrix} G_i + \begin{Bmatrix} d_1(r) \\ d_2(r) \end{Bmatrix} \right] \tag{46a}$$

where

$$\begin{Bmatrix} b_1(r_1 i) \\ b_2(r, i) \end{Bmatrix} = \left[I \ \delta ri + \sum_{N=1}^{\infty} (\prod_{q=1}^{N} \sum_{j_q=1}^{\ell})(\prod_{r=1}^{N} \sum_{k_r=1}^{8})R_{j_1}(r). \right. \tag{46b}$$

$$\cdot (\prod_{s=2}^{N} R_{j_s}(k_{s-1}))(\prod_{t=1}^{N-1} C_{r,k_t;j_1,\ldots,j_t})$$

$$\left. \cdot C_{r,i;j_1,\ldots,j_N} \right] \begin{Bmatrix} S_i \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} d_1(r) \\ d_2(r) \end{Bmatrix} = D(r)\ \gamma_1(r) + \sum_{N=1}^{\infty} (\prod_{q=1}^{N} \sum_{j_q=1}^{\ell})(\prod_{r=1}^{N} \sum_{k_r=1}^{8})R_{j_1}(r)(\prod_{s=2}^{N} R_{j_s}(k_{s-1})). \tag{46c}$$

$$\cdot\ D(k_N)(\prod_{t=1}^{N} C_{r,k_t;j_1,\ldots,j_t})\gamma_{N+1}(r;j_1,\ldots,j_N)$$

In equation (46) δ_{ri} represents the Kronecker delta, the quantities $C_{r,i;j_1,\ldots,j_N}$ are defined by

$$C_{r,i;j_1,j_2,\ldots,j_N} = (\rho_r - \rho_i + \alpha_{j_1} + \alpha_{j_2} + \ldots + \alpha_{j_N})^{-1}$$

$$\text{if} \qquad (\rho_r - \rho_i + \alpha_{j_1} + \alpha_{j_2} + \ldots + \alpha_{j_N}) \neq 0 \tag{46d}$$

$$= 0 \ \text{ if } \quad (\rho_r - \rho_i + \alpha_{j_1} + \ldots + \alpha_{j_N}) = 0$$

and $\gamma_1(r)$, $\gamma_N(r;j_1,\dots,j_{N-1})$ are defined by

$$\gamma_1(r) = \int_0^{x_o} \underline{\lambda}(y)\exp(-\rho_r y)dy$$

$$\gamma_N(r;j_1,\dots,j_{N-1}) = \int_0^{x_o} \underline{\lambda}(y)\exp[-(\rho_r + \sum_{q=o}^{N-1} \alpha j_q)y]dy \tag{46e}$$

Equation (46a) indicates that the growth or decay of $\bar{Z}_n$ as $\xi \to \infty$ depends entirely on the sign of Re ρ_r, $r = 1, 2, \dots, 8$. To ascertain if (29) can be satisfied it is necessary to consider c large in (30) and determine the large p behavior of the roots $\rho_i(p)$. This is accomplished by noting that Det $L_o(s) = 0$ is satisfied by the asymptotic series

$$s = p\left[S_o + \frac{s_1}{\sqrt{p}} + \frac{s_2}{(\sqrt{p})^2} + \frac{s_3}{(\sqrt{p})^3} + \dots\right] \tag{47}$$

Equations governing the coefficients, s_n, are obtained by substitution of (47) into Det $L_o(s) = 0$ and equating terms of the same p-order. Solution of those equations for the leading terms of (47) yield the following asymptotic values

$$\rho_1, \rho_1^* \sim \frac{\sqrt{p}(1+i)}{\sqrt{2}\,\beta};\ \rho_2, \rho_2^* \sim \frac{\sqrt{p}(1-i)}{\sqrt{2}\,\beta};\ \rho_3, \rho_3^*, \rho_4, \rho_4^* \sim n$$

$$\rho_5, \rho_5^* \sim -\frac{\sqrt{p}(1+i)}{\sqrt{2}\,\beta};\ \rho_6, \rho_6^* \sim -\frac{\sqrt{p}(1-i)}{\sqrt{2}\,\beta};\ \rho_7, \rho_7^*, \rho_8, \rho_8^* \sim -n \tag{48}$$

where the designations as to root number were purely arbitrary.

Identifying the roots ρ_i, ρ_i^* by their asymptotic values above and selecting the positive branch of $\sqrt{p}$ in the p-plane, it is evident that (29) can be satisfied as $\xi \to \infty$ only if

$$\sum_{i=1}^{8} \begin{Bmatrix} b_1(r,i) \\ b_2(r,i) \end{Bmatrix} G_i + \begin{Bmatrix} d_1(r) \\ d_2(r) \end{Bmatrix} = 0, \quad r = 1 \text{ to } 4 \tag{49}$$

Equations (49) represent eight equations in the eight unknowns, G_i. However, it can be shown that only four (one row) equations are linearly independent. With the vector components $b_1(r,i)$, $b_2(r,i)$, $d_1(r)$, $d_2(r)$ as defined by (46b and c), these four equations can be written (the choice of the second row over the first was purely arbitrary)

$$\sum_{i=1}^{8} b_2(r,i)G_i + d_2(r) = 0, \qquad r = 1 \text{ to } 4 \tag{50}$$

In a similar manner, (29) can be satisfied for $\xi \to -\infty$ only if

$$\sum_{i=1}^{8} b_2^*(r,i)G_i^* + d_2^*(r) = 0, \qquad r = 5 \text{ to } 8 \tag{51}$$

where b_2^*, d_2^* are obtained from b_2, d_2 by the parametric exchanges described previously.

The constants G_i and G_i^* can now be determined from equations (45), (50) and (51). In matrix form we have

$$Ag = e \tag{52}$$

where A is a 16 x 16 matrix, the elements of which are given by

$$A = \left[\begin{array}{c|c} A_{r,i} = b_2(r,i) & A_{r,i+8} = 0 \\ r = 1 \text{ to } 4,\ i = 1 \text{ to } 8 & r = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \\ \hline A_{m+4,i} = \rho_i^{m-1} S_i & A_{m+4,i+8} = -\rho_i^{*m-1} S_i^* \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 & m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \\ \hline A_{m+8,i} = \rho_i^{m-1} & A_{m+8,i+8} = -\rho_i^{*m-1} \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 & m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \\ \hline A_{r+8,i} = 0 & A_{r+8,i+8} = b^*_2(r,i) \\ r = 5 \text{ to } 8,\ i = 1 \text{ to } 8 & r = 5 \text{ to } 8,\ i = 1 \text{ to } 8 \end{array}\right] \tag{53a}$$

where, as usual, the first subscript refers to the row and the second to the column. The quantities g and e represent the following 16 dimensional vectors:

$$g \doteq \begin{Bmatrix} G_1 \\ \cdots \\ G_8 \\ G_1^* \\ \cdots \\ G_8^* \end{Bmatrix}, \qquad e = \begin{Bmatrix} -d_2(1) \\ \cdots \\ -d_2(4) \\ \left.\begin{matrix} 0 \\ \cdots \\ 0 \end{matrix}\right\} 8 \\ -d_2^* \\ \cdots \\ -d_2^*(8) \end{Bmatrix} \tag{53b}$$

Premultiplying (52) by A^{-1}, g is obtained as

$$g = A^{-1} e \tag{53c}$$

The solution in the p-plane is now complete. Next we discuss a few of the properties of the series for $\overline{Z}_n$.

3.4.5 Properties of $\overline{Z}_n(\xi, p)$ and Remarks Related to s-Inversion

The assumption was made, upon inverting $\overline{\overline{Z}}_n$, that the roots ρ_i and ρ_i^* were not repeated. For all points in the p-plane such that

$$\frac{\partial \operatorname{Det} L_o(s,p)}{\partial s} \neq 0$$

the roots of $\operatorname{Det} L_o(s,p) = 0$ are nonrepeated. It can be shown that the condition

$$\operatorname{Det} L_o = \frac{\partial \operatorname{Det} L_o}{\partial s} = 0 \tag{54}$$

defines the location of branch points of the roots in the p-plane.

Let us define the region R_1 of the complex p-plane by 1) $|p - p_b| \geq \epsilon_1 > 0$, where p_b are root branch points in the p-plane, defined by (54), ii) $|p - p_\Delta| \geq \epsilon_2 > 0$, where p_Δ are zeros of the determinant, Det A. If $p \epsilon R_1$, and $-\infty < -B \leq \xi \leq B < \infty$, B = arbitrary constant, the series (40), and its equivalent for $\xi < 0$, can be shown, by standard techniques, to be absolutely and uniformly convergent with respect to both ξ and p. Since the series obtained by an nth term-by-term ξ-derivative possesses the same property of uniform convergence, and each term is a continuous function of ξ, our differentiation of the series was justified. If appropriate branch cuts are made within the region R_1, rendering the roots analytic functions of p, then each term of the series will be an analytic function of p in the new region, call it R_2, defined by R_1 and the cuts. The uniform convergence with respect to p then indicates $\overline{Z}_n$ is an analytic function of p when $p \in R_2$. The points p for which the determinant of A vanish represent poles of $\overline{Z}_n$. The points p_b are possible branch points of $\overline{Z}_n$.

3.5 Determination of Stability

3.5.1 General Remarks

Stability is determined by examining the behavior of $Z_n(x,t)$ as $t \to \infty$. However, care must be exercised in the manner in which the limit in t is taken. If the length of the shell were finite it would be sufficient to investigate stability by determining the boundedness of $Z_n(x,t)$ along lines of finite constant x as $t \to \infty$. Since an infinite x interval is under consideration, however, this is not sufficient; perturbed motions may exist that are bounded in any finite x interval, but are unbounded in the infinite x interval (an example is a pulse, traveling with constant velocity with amplitude growing with distance).

One method of analyzing stability is to determine the boundedness of $Z_n(x,t)$ along arbitrary rays in the x,t plane as $t \to \infty$. The entire upper half of this plane can be "covered" by such rays. One particular ray of interest is defined by $\xi = x - Mt$ = constant (Fig. 4). This ray plays a fundamental role. In what follows $Z_n(x,t)$ will first be investigated along ξ = constant and then it will be shown, by considering an arbitrary ray, that the ξ = constant examination is sufficient for stability purposes.

3.5.2 The Ray ξ = Constant

The boundedness of $Z_n(x,t)$ along a line of constant ξ as $= t \rightarrow \infty$ is governed entirely by the location and type of singularities of $\overline{Z}_n$ in the region Re $p \geqslant 0$ of the p-plane. From the theory of the Laplace transform[6] one can state (with only minor restrictions that do not concern the present transform):

1) If $\overline{Z}_n(\xi, p)$ possesses singularities in Re $p > 0$, then $Z_n(\xi, \tau)$ is unbounded as $\tau = t \rightarrow \infty$ and the system is unstable.

2) If $\overline{Z}_n(\xi, p)$ possesses no singularities in Re $p \geq 0$, then $Z_n(\xi, \tau)$ vanishes as $\tau \rightarrow \infty$.

3) Assume $\overline{Z}_n(\xi, p)$ is regular in Re $p > 0$ with isolated singularities on the imaginary axis. In the neighborhood of such a singularity (call it p_s), $\overline{Z}_n(\xi, p) \sim f(\xi, p_s)/(p - p_s)^k$, where $f(\xi, p_s)$ is finite. If $k \leq 1$, then $Z_n(\xi, \tau)$ is bounded as $\tau \rightarrow \infty$ whereas if $k > 1$, $Z_n(\xi, \tau)$ is unbounded.

The singularities that must be dealt with in Re $p \geq 0$ are branch points and poles. Consider first the branch points. $\overline{Z}_n$ can possess no more branch points in Re $p > 0$ than those of the roots ρ_i, ρ_i^*. However, not all branch points of the roots may belong to $\overline{Z}_n$ due to squaring, cancellation, etc. The equation Det $L_o(s, p) = 0$, which governs the roots ρ_i, ρ_i^*, depends in general on M, N_x^*, C_o, n and β, but not on C_j, C_j^*, α_j, α_j^* $(j > 0)$. Thus the location of the branch points is independent of the magnitude and distribution of $w_s(\xi)$, except for the uniform term. This implies that we may determine the location of the

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6 See, for example, Erdelyi's theorems in [25].

branch points of $\overline{Z}_n$ by setting $C_j = C_j^* = 0$. The problem is then one of a cylinder loaded in axial compression and by a uniform pressure distribution. We can, for convenience, investigate the stability of such a shell in the ξ, Υ coordinates; the transformation clearly does nothing to alter the stability of the shell. If the axial and pressure loads are restricted to be less than those required for static buckling, one therefore concludes, by virtue of items 1) and 3) above, that those branch points belonging to the roots ρ_i, ρ_i^* in Re $p > 0$ cannot be branch points of $\overline{Z}_n$ and those on Re $p = 0$ must be such that $k \leq 1$ in 3) above. One therefore concludes that it is the poles of the transform that govern the stability of the system, and the analysis reduces to the determination of the location and type of zeros of Det A. Because of the complex nature of the matrix A the investigation of Det A must be primarily numerical. This problem is discussed next.

The simplest case to treat is that class of static problems obtained by setting $M = 0$. In this case the variational equations represent a conservative system. Therefore the energy method of analyzing stability and the present dynamic method are equivalent [24]. If, under the same approximations associated with equations (1) the potential energy of the shell and loading system are formulated, one finds that equations (23) with $\partial/\partial\Upsilon = 0$ are the result of requiring that the second variation of the potential energy vanish (a necessary condition for the transition from stability to instability). The function (40b), with $p = 0$, represents (following a θ separation) a solution to these equations. The solution is completed by requiring that $\mathcal{T}_n(\pm\infty) = \eta_n(\pm\infty) = 0$ and continuity of and η_n and their derivatives with respect to ξ, up to an including the third, at $\xi = 0$. Applications of these conditions leads to:

$$Ag = 0$$

whence a solution exists if and only if $\text{Det } A = 0$. This, however, implies the transition from stability to instability takes place at $p = 0$ in the p-plane. The stability transition in the static case can therefore be determined by simply plotting Det A versus a characteristic load parameter, determining that value of the parameter for which Det A first vanishes. A minimization with respect to M will yield the desired result.

If $M \neq 0$, the situation is more complex since the variational equations do not represent a conservative system. If $p = 0$, the parameter M occurs everywhere in the combination $M^2 - N_x^*$. It therefore has the same effect as an axial compression of the cylinder. Since the determinant, $\text{Det } A(p = 0, M \neq 0)$ possesses the same form and properties as $\text{Det } A(p = 0, M = 0)$, save an effective change in N_x^*, one can say, by analogy to the static problem, that a zero of Det A will appear at $p = 0$ for some set of load parameters. In all probability this again represents the transition from stability to instability. This supposition can be verified numerically. One method of accomplishing this task is as follows:

Zeros of $\text{Det } A(p)$ in $\text{Re } p \geq 0$ can be detected with the aid of a numerical mapping and a theorem from the theory of complex variables, frequently called the Principle of the Argument. This theorem [25] states:

If $f(p)$ is analytic in a region R bounded by a contour C, and does not vanish on C, then the number of zeros[10] minus the number of poles of $f(p)$ within C is $1/2\pi$ times the increase of arg $f(p)$ as p goes once around C in the positive direction. (The positive direction is defined such that the enclosed region R appears to the left of an observer moving along C).

Let us consider as the function $f(p)$ the determinant Det $A(p)$, and as the contour C the contour illustrated in Fig. 5 which covers the right half plane as $R \to \infty$. The function Det $A(p)$ can be made analytic in R by use of branch cuts (recall the root branch points are defined by (54)) which render the roots ρ_i, ρ_i^* analytic. Choosing the principal branches (i. e., the cuts run from the branch points to $-\infty$, parallel to the real ρ -axis) the contour C can be deformed to a new contour C' (e. g., see Figs. 9 of the numerical example) so that Det $A(p)$ is analytic on the new contour and within the enclosed region. To determine if poles exist in Re $p \geq 0$ one can map the function Det A as p goes once around the contour C and apply the theorem. Since the elements of A are finite for $p < \infty$, Det A possesses no poles within and on the contour, and hence the number of zeros is directly proportional to the number of times arg Det $A(p)$ increases by 2π.

If zeros on C are encountered during the mapping we have:

$$\text{Det } A(p) \backsim f(p)\,(p - p_\Delta)^n$$

in a neighborhood of the zero, where $f(p)$ is finite. Let $p - p_\Delta = \epsilon e^{i\theta}$ then

$$\text{Det } A(p) = \epsilon^n e^{in\theta} f(p_\Delta) + 0(\epsilon)$$

[10] A zero or pole of order 2, 3, ..., is counted as 2, 3, ... zeros or poles.

Therefore, as p goes once around a semi-circular arc, from $\theta = \pi/2$ to $\theta = -\pi/2$, of radius ε, and as $\varepsilon \to 0$, the change in phase of Det A(p) is $n\pi$. Thus, such zeros can be circumvented by a circular arc, of small radius and their order determined from the mapping. Conversely their order also can be determined by observing the decay of Det A(p) in the neighborhood of the zero and the corresponding change in phase of arg Det A(p) can be included in the mapping without numerically circumventing that point. If a pole is found to coalesce with a branch point on Re $p = 0$ for some load magnitude or velocity, the constant in item 3) of the above discussion can be determined in a similar manner.

During the course of the mapping one must guarantee that the correct branches of the roots in the p-plane are selected and retained. This selection is governed by the asymptotic values given by equation (48). Since the deformed contour C circumvents branch points, the roots are analytic on C and are non-repeated. A root can therefore be identified at any point on the contour during numerical calculations by requiring that it be a smooth function of the path traced from its asymptotic values. One must, of course, begin with a point on the semi-circular portion of the contour and R must be selected large enough to allow the roots to approach sufficiently close to their asymptotic values to be properly identified. There is a certain degree of freedom in identifying those roots with first term asymptotic values $\pm$ n. Although two roots have similar asymptotic values, one need only make an initial choice of the branch and then be consistent. A procedure suitable for root tracing is discussed in [26].

For the static case $(M = 0)$ it suffices to associate those roots ρ_i with positive real parts with the indices $i = 1 - 4$ in any order and those ρ_i^* with negative real parts with $i = 5$ to 8 in any order. Explicit formulas for the roots for $M = 0$ can be found in [27].

3. 5. 3 Arbitrary Rays

If $Z_n(x,t)$ is found to be unbounded as $t \to \infty$ along ξ = constant in the x,t plane, then by our definition of stability the shell is unstable. The question that remains is: if Z_n is bounded along ξ = constant does this imply stability? To answer this question it is necessary to show that boundedness along ξ = constant implies boundedness along all other rays. With this in mind, consider $\overline{Z}_n(\xi,p)$ again. It will suffice to consider the large ξ, τ domain of the ξ, τ plane. From the theory of the Laplace transform the behavior of $Z_n(\xi,\tau)$ for large τ is governed by those singularities of $\overline{Z}_n$ farthest to the right of the p-plane. Consider the contribution from one such singularity. In the neighborhood of a singularity at, say $p = p_s$, and considering ξ large, the function $\overline{Z}_n$ has the asympotic form (see 46).

$$\overline{Z}_n(\xi,p) \underset{\substack{p \to p_s \\ \xi \to \infty}}{\sim} \sum_{i=5}^{8} a_i(p_s)(p-p_s)^{-k} \exp\left[\rho_i(p_s)\xi\right],$$

$$Z_n(\xi,p) \underset{\substack{p \to p_s \\ \xi \to -\infty}}{\sim} \sum_{i=1}^{4} a_i^*(p_s)(p-p_s)^{-k} \exp\left[\rho_i^*(p_s)\xi\right], \tag{55}$$

where a_i and a_i^* are finite. By use of a theorem by Erdelyi [25] the contribution from the singularity is:

$$Z_n \underset{\substack{\xi \to \infty \\ \tau \to \infty}}{\sim} \sum_{i=5}^{8} \left[a_i(p_s)/\tau^{1-k}\Gamma(k)\right] \exp\left[\rho_i(p_s)\xi + p_s\tau\right],$$

(56)

$$Z_n \underset{\substack{\xi \to -\infty \\ \tau \to \infty}}{\sim} \sum_{i=1}^{4} \left[a_i^*(p_s)/\tau^{1-k}\Gamma(k)\right] \exp\left[\rho_i^*(p_s)\xi + p_s\tau\right],$$

Now, let an arbitrary ray in the x, t plane be defined by $x = t/b$ (see Fig. 4). Then

$$\xi = x - Mt = (1/b - M)t$$

and in the large (x, t) domain we have

$$Z_n \underset{\substack{t \to \infty \\ x = t/b}}{\sim} \sum_{i=5}^{8} \left[a_i(p_s)/t^{1-k}\Gamma(k)\right] \exp\left[\rho_i(p_s)(1/b-M)t + p_s t\right], \qquad (1/b-M) \geq 0$$

(57)

$$Z_n \underset{\substack{t \to \infty \\ x = t/b}}{\sim} \sum_{i=1}^{4} \left[a_i^*(p_s)/t^{1-k}\Gamma(k)\right] \exp\left[\rho_i^*(p_s)(1/b-M)t + p_s t\right], \qquad (1/b - M) < 0$$

To satisfy the quiescent condition at x or $\xi = \pm\infty$ for t or τ fixed, the roots must be selected such that

$$\text{Re}\ \rho_i(p_s) < 0 \quad (i = 5 \text{ to } 8)\ , \qquad \text{Re}\ \rho_i^*(p_s) > 0 \quad (i = 1 \text{ to } 4) \tag{58}$$

It is therefore clear that the boundedness of Z_n along the given ray is determined entirely by the sign of Re p_s, or by k if Re $p_s = (1/b\text{-}M) = 0$.

Let us now assume that Z_n was found to be bounded along ξ = constant. This implies the singularity p_s must lie either on the imaginary axis or in the left half of the p-plane and thus Re $p_s \leq 0$. Further, if Re $p = 0$, boundedness along ξ = constant implies $k \leq 1$. Thus boundedness along ξ = constant implies boundedness along all other rays in the x, t plane as $t \rightarrow \infty$. (It should be noted at this point that the converse is not necessarily true, i. e, unboundedness along ξ = constant does not imply unboundedness along all other rays. From (56) it can be observed that the motion at any finite value of x is unbounded as $t \rightarrow \infty$ only if Re $[p_s - \rho_i^*(p_s)M] > 0$ for some ρ_i^*.)

3.6 Summary of the Procedure

The original rather complex stability problem has, at this point, been reduced to a simpler one of locating and classifying the zeros of Det A(p). It has been pointed out that this latter task can be accomplished with the aid of a numerical mapping and the Principle of the Argument. It is appropriate now to summarize the basic steps involved:

(1) Select the parameters n, M and β as well as the load distribution (6). By virtue of (15) one can obtain the parameters C_j, C_j^*, α_j, α_j^*, ℓ and ℓ^* as indicated in (17). Choose a parameter λ which characterizes the "magnitude" of the load such that λ is a function only of C_j and C_j^* for $j > 0$.

Case (a): M = 0

(2) If $M = 0$, set $p = 0$ and calculate the roots ρ_i from the polynomial in s obtained by setting $\text{Det } L_o = 0$. The roots ρ_i^* are obtained by replacing C_o by C_o^*. Select the four roots ρ_i with negative real parts and denote them as $\rho_i (i = 5 \text{ to } 8)$, in any order, and the remaining ρ_i as $i = 1$ to 4, in any order. Select the four roots ρ_i^* with positive real parts and denote them as $\rho_i^* (k = 1 \text{ to } 4)$ in any order, and the remaining ρ_i^* as $i = 5$ to 8 in any order.

(3) Calculate Det A from (53) by truncating the series (46b) after N terms. Plot Det A against λ beginning with $\lambda = 0$. By the above choice of λ, ρ_i and ρ_i^* do not depend on λ and thus they need not be recalculated for each value of λ. Increase λ until a zero of Det A is first obtained. This represents the transition from stability to instability for the particular value of n considered.

(4) Repreat the above procedure for various n values until a minimum λ has been found for a wide range of n's. This represents the static buckling load of the shell.

Case (b): M = 0

(5) Repeat the above process for each value of M. Denote the critical value of λ by λ_{CRM}.

(6) Determine the branch point locations of the roots $\rho_i(p)$ and $\rho_i^*(p)$ by numerical solution of equations (54). (The parameter p can be eliminated and a single polynomial in s can be obtained.)

(7) Select a contour in the p-plane as illustrated in Fig. 5(b), circumventing the branch points in $\text{Re } p \geq 0$. Choose R large enough to allow proper identification of ρ_i, ρ_i^* from their asymptotic values

(48). Select appropriate increments of p on the contour and calculate the roots ρ_i and ρ_i^* for each value of p. Order these roots at each station by requiring that they be smooth functions of the path traced from the asymptotic values.

(8) Verify that $\lambda > \lambda_{CRM}$ is unstable by mapping $\text{Det } A(p)$ as p goes once around the contour. Use the Principle of the Argument to show that a zero exists in $\text{Re } p > 0$ or a zero of order greater than unity exists on $\text{Re } p = 0$. Verify that $\lambda < \lambda_{CRM}$ is stable by again mapping $\text{Det } A(p)$, showing that no zeros exist in $\text{Re } p > 0$ and all those on $\text{Re } p = 0$ are of order less than or equal to unity.

4. NUMERICAL EXAMPLES

4.1 Ring Load

As the first example we shall take the case of the ring load moving with constant velocity. Here the load is defined by

$$P = P_c \, \delta(X - V_L T) , \qquad \overset{o}{N}_X = 0 \tag{59}$$

and is illustrated in Fig. 2(a). The primary response for this load has the form (20) if the constants g_j are multiplied by P_c/Eh, i.e.,

$$C_j = C_j^* = \frac{P_c}{Eh} g_j , \quad j = 1,2; \quad C_j = C_j^* = 0, \; j = 0, \; j > 2$$

$$\alpha_j = \alpha_j^* \text{ given by (20b) for } j = 1,2 \tag{60}$$

If we select as the parameter λ the quantity P_c/Eh, the matrix A (53) can be written

$$A = K_o + \lambda K_1 + \lambda^2 K_2 + \lambda^3 K_3 + \cdots$$

where the matrices K_j do not depend on λ. They are obtained from (53) by grouping terms of like λ-powers.

Truncating the series (61) and setting $p = 0$, Det A was numerically evaluated by use of a digital computer. A Reguli Falsi method was employed to determine the minimum λ for which a zero of Det A occurred at $p = 0$ (Det A was found to be real-valued for $p = 0$). In the neighborhood of this value of λ, (61) was found to converge quite rapidly for a wide range of shell parameters ($a/h = 100$ to 1000). For all cases where $M < 0.95$, the correction due to the retention of more than three terms of (61) was apparently negligible.

The behavior of the minimum eigenvalue as a function of the number of circumferential waves, n, and velocity, V_L is illustrated in Fig. 6(a) for the case $a/h = 100$ and $\nu = 0.3$. For each value of n, a curve similar to that of Fig. 7(a) can be constructed. Fig. 6(b) is the minimum envelope of all such curves and represents, as will be discussed momentarily, the transition from stability to instability. For all a/h values in the range of 100-1000, the form of the curve in Fig. 6(b) was found to remain essentially invariant. (Compare Figs. 7(a) and 7(b)).

For M or $V_L = 0$, the results obtained were the buckling load of an infinite shell subject to a uniform radial line load. Below, a comparison is made with existing analyses on the subject for $a/h = 100$ and $\nu = 0.3$.

Present theory:	$P_c/Eh = 3.91 \times 10^{-4}$
Brush [18], long finite shell:	$P_c/Eh = 4.20 \times 10^{-4}$
Hahne [19], long finite shell:	$P_c/Eh = 4.61 \times 10^{-4}$

The agreement is quite good. No comparison can be made for the dynamic case since analyses on the subject apparently do not exist. Fig. 8 indicates the behavior of the above static buckling load as a function of a/h.

It should be noted that the effect of axial compression can easily be incorporated into these results by replacing V_L/V_{co} in Figs. 6 by

$$\left\{ (V_L/V_{co})^2 - \left(\frac{N_x^o}{Eh}\right)\left(\frac{a}{h}\right)\left[3(1-\nu^2)\right]^{1/2} \right\}^{1/2}$$

where $V_{co} = V_{co}$ at $N_x^o = 0$. This latter result is a consequence of the occurrence of M and N_x^* in the combination $M^2 - N_x^*$ everywhere when $p = 0$. That is, the effect of velocity squared is seen to create the same effect on the critical load as an axial compression of the cylinder.

It now remains to verify that the above λ values represent the transition between stability and instability. Let us take as typical points of Fig. 7(a) $V_L/V_{co} = 0$, $V_L/V_{co} = .2$ on the $n = 6$ curve, and attempt a mapping for λ values slightly greater and less than the point of the curve. The branch point locations, the contour C', and the final mapping for these cases are shown in Figs. 9 and 10. (Numerical investigations indicated three terms of the series give sufficient accuracy for all selected points on C'.) Corresponding points in the p and Det A planes are noted by similar numbers. Since the mapping of Det A(p) on the dotted portion of C' was found to be a mirror image (about the real axis) of the solid line; only one half of the mappings are shown. The solid lines in the Det A(p) plane represent a mapping at the λ value given by Fig. 7(a), while the dashed lines represent values slightly above and below this value. The three curves, in each case, were found to merge for points some distance away from the origin of the p-plane, as shown.

The results of the mapping were quite similar. First, a mapping along $\operatorname{Re} p = 0$ for $\lambda < \lambda_{CRM}$ indicated first order poles at mirror images above and below $\operatorname{Re} p = 0$. At $\lambda = \lambda_{CRM}$ these poles merged to form a second order pole at $p = 0$. Then as $\lambda > \lambda_{CRM}$, the second order pole split into two first order poles at mirror images about $\operatorname{Im} p = 0$ on $\operatorname{Re} p = 0$. (The mappings shown circumvents the poles on $\operatorname{Re} p = 0$.)

A rather extensive investigation of other similar points indicated Fig. 7(a) to be actual transition points for $n = 2, 3, 4, \ldots$ It was concluded, therefore, that the minimum envelope, Fig. 6(b), represents the transition from stability to instability.

4.2 Decayed Step Loads

Another case of interest is the decayed step wave moving with constant velocity (see Fig. 2(b)). Here the load is defined by

$$P(X - V_L T) = P_1^* \; e^{\Omega_1^*(X-V_L T)} \; H\left[-(X-V_L T)\right] \tag{62}$$

where Ω_1^* is assumed to be real valued and positive. The load constants, as obtained from (19) including axial compression are:

$$C_j = \left(\frac{P_1^* a}{Eh}\right)\frac{g_j}{b+\alpha_j}, \quad j = 1,2; \quad C_3 = 0; \quad C_o = -\nu\frac{N_x^o}{Eh}$$

$$C_j^* = \left(\frac{P_1^* a}{Eh}\right)\frac{g_j}{b-\alpha_j}, \quad j = 1,2; \quad C_3^* = 2\left(\sum_{i=1}^{2}\frac{g_i\alpha_i}{\alpha_i^2-b^2}\right); \; C_o^* = -\nu\frac{N_x^o}{Eh}$$

$$\alpha_j = -\alpha_j^*, \; j = 1,2 \text{ given by (20b)}; \; \alpha_3 = 0; \; \alpha_3^* = -\Omega_1^* a \equiv -b \tag{63}$$

The results of a numerical investigation of this case are illustrated in Figs. 11-15. Again it was found that the procedure described previously under the case $M = 0$ predicted the transition curve itself. Figs. 11-13 represent the critical nondimensional load amplitude (λ was selected as $P_1^* a/Eh$) versus the decay parameter b for various shell radius to thickness ratios. Fig. 14 indicates a typical interaction curve for $a/h = 100$, $\nu = 0.3$. The case $b = 0.1$ approaches a step load while $b = 100$ approaches the ring load. Note again, the effect of axial compression and velocity can be grouped as one parameter. (The abscissa of Fig. 14 can also be written $\left[(V_L/V_{co})^2 + \sigma/\sigma_{c\ell}\right]^{1/2}$ where $\sigma_{c\ell}$ represents the classical buckling stress of a cylinder in axial compression only.)

Fig. 15 indicates the number of circumferential waves, n, corresponding to the stability transition curve, Fig. 14. This number was found to increase with both an increase in shell radius to thickness ratio and load velocity. Because of the Donnell type shell theory employed here, those results for $n < 4$ will be somewhat in error (approximately 30 per cent error in the critical load can be expected at $n = 2$.)

Fig. 16 indicates the rate at which the results of an analysis for which the distributed load is lumped as a ring load approaches the distributed load results. The approximation is seen to degrade with increased velocity.

Finally, it should be noted that convergence of the series (46b), or (61), while acceptable, was observed to be in general worse for the decayed step case than for the ring load case. In some instances four terms of the series (61) were necessary to guarantee no change in the second place of λ_{CRM}. (Four terms of (46b) represent a considerable number of terms when the expression is written out in extensio.)

4.3 Concluding Remarks

The numerical examples treated indicate a rather severe degradation of stability as V_L, the load velocity, approaches V_{co}, the axisymmetric cut-off velocity. It should be recognized that this condition is due primarily to the existence of a "resonance" in the primary response as $V_L \longrightarrow V_{co}$. For example, the Green's function (20a), which is characteristic of a moving ring load, indicates the amplitude of the primary response varies[11] as $\left[1 - (V_L/V_{co})^2\right]^{-1/2}$. It is well known that the effect of even small damping is to significantly reduce this resonance. Thus it can be expected that damping, or an acoustic medium-shell interaction, will alter the in-vacuo results to some extent. Simple viscous damping can be easily accounted for along the lines of the present theory. A rigorous account of an acoustic medium is a somewhat more difficult task.

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[11] One is tempted to conclude therefore, as did Presekin [6], that the stability interaction curve, Fig. , should vary as $[1 - (V_L/V_{co})^2]^{1/2}$. Our results, however, indicate the transition is a curve closely approximated by $[1 - (V_L/V_o)^2]$.

If equations of motion of a more exact nature are employed, the restriction on the load velocity imposed in this paper can be lifted. If the axisymmetric response, utilizing such equations, is obtained as a series of exponential functions, then an analysis similar to the foregoing can be used with minor alterations.

It should be noted that in the load velocity range, $V_L < V_{co}$ and with the equations of motion (1), it can be shown that the steady-state solution (15) is the limiting case of a transient problem. Therefore, an instability in the steady-state primary response implies an unstable transient response also. The converse is not necessarily true, i. e., a stable steady-state response does not necessarily imply a stable transient response.

Naturally one would like to interpret the results of the analysis in terms of shell buckling. As with any infinitesimal stability analysis, however, care must be exercised in this respect. The present analysis implies buckling will not take place under <u>small disturbances</u> in the stable zones. As far as the unstable zones are concerned, one cannot differentiate between those instabilities that lead to buckling and those that lead to nonaxisymmetric oscillations when $M > 0$.

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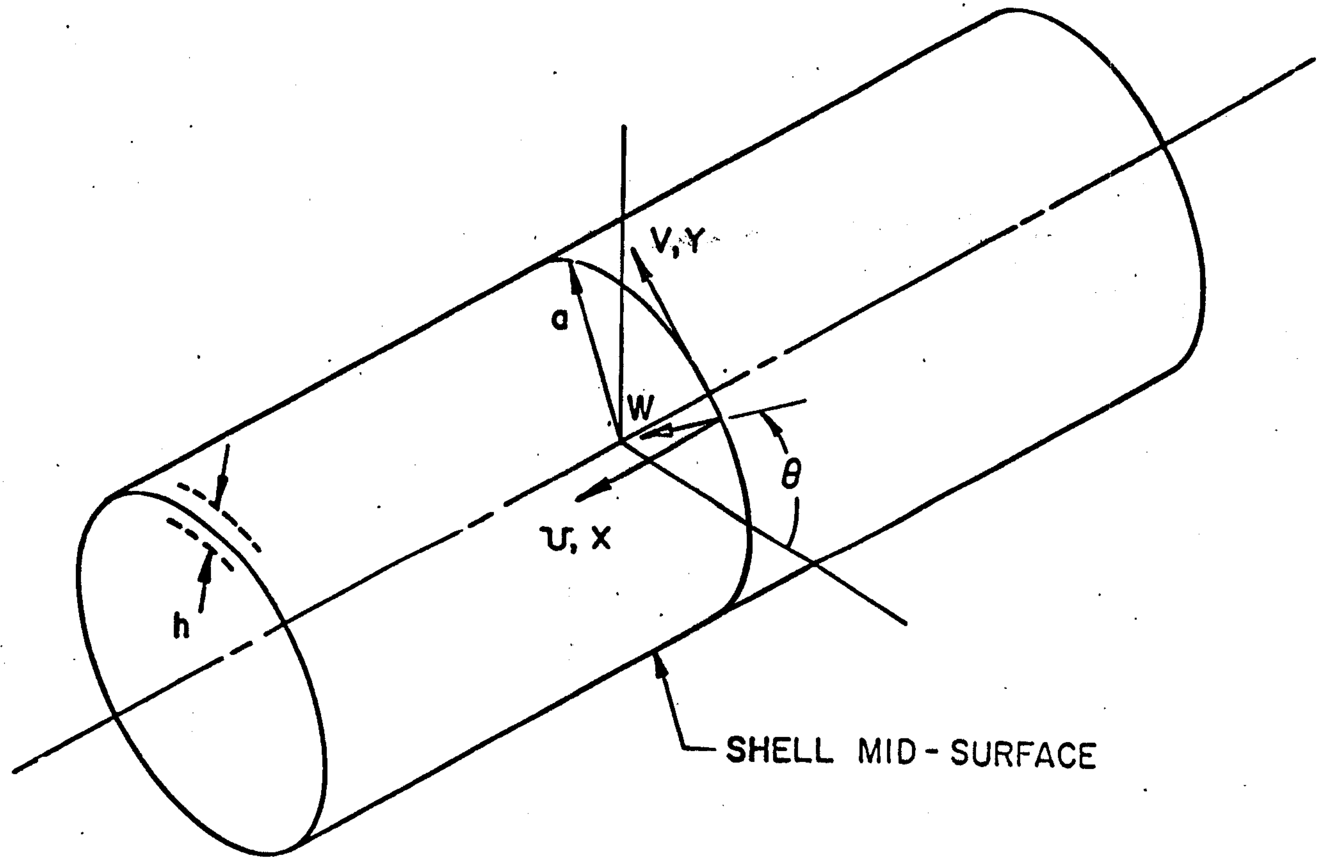

Fig. 1 Coordinate System

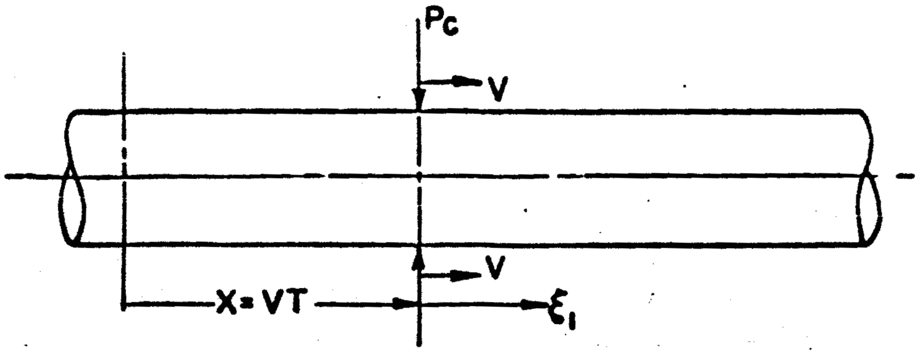

Fig. 2a. Moving Concentrated Load: $P(\xi_1) = P_c\,\delta(\xi_1)$.

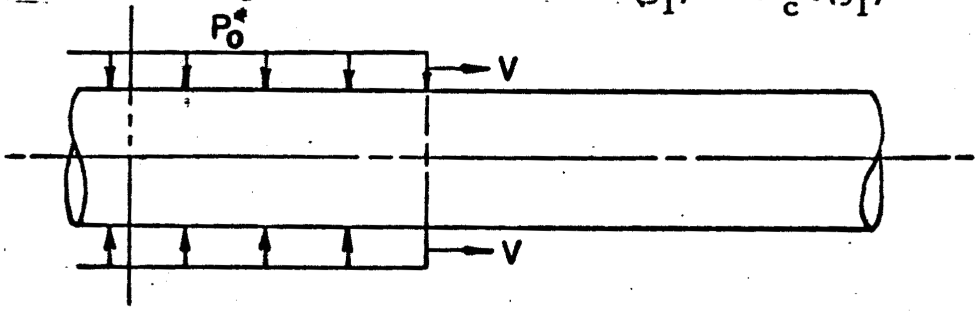

Fig. 2b. Moving Step Load: $P(\xi_1) = P_o^*\,H(-\xi_1)$.

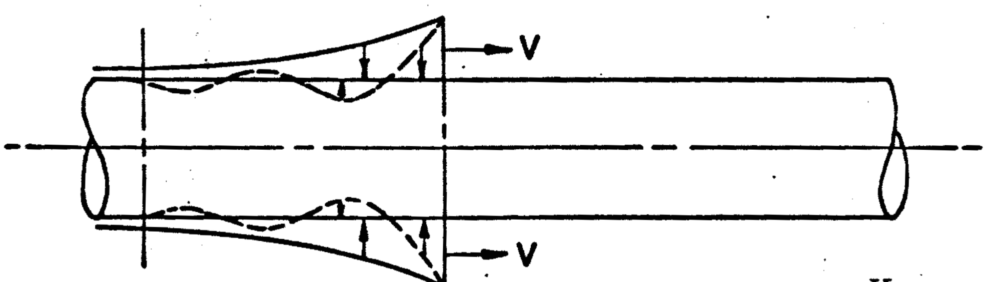

Fig. 2c. Moving Decayed Step Loads: $P(\xi_1) = H(-\xi_1)\sum_{k=1}^{K} P_k^* \exp[\Omega^*\xi_1]$.

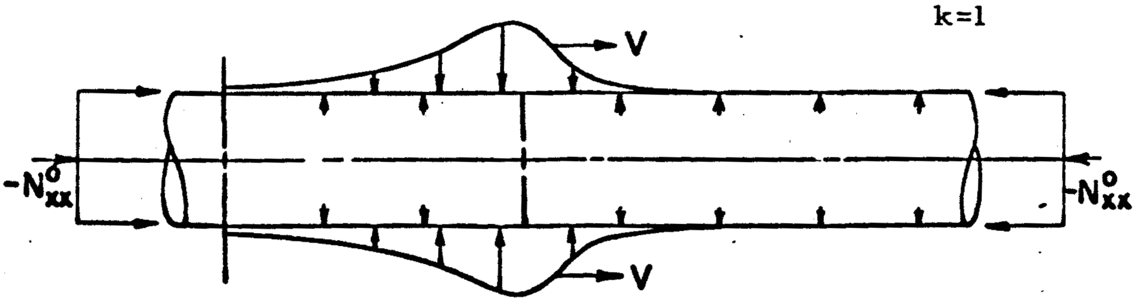

Fig. 2d. General Moving Pulse with Internal Pressure and Axial Compression: $P(\xi_1) = -P_o + H(\xi_1)\sum_{n=1}^{N} P_n \exp[-\Omega_n\xi_1] + H(-\xi_1)\sum_{k=1}^{K} P_k^* \exp[\Omega_k^*\xi_1]$.

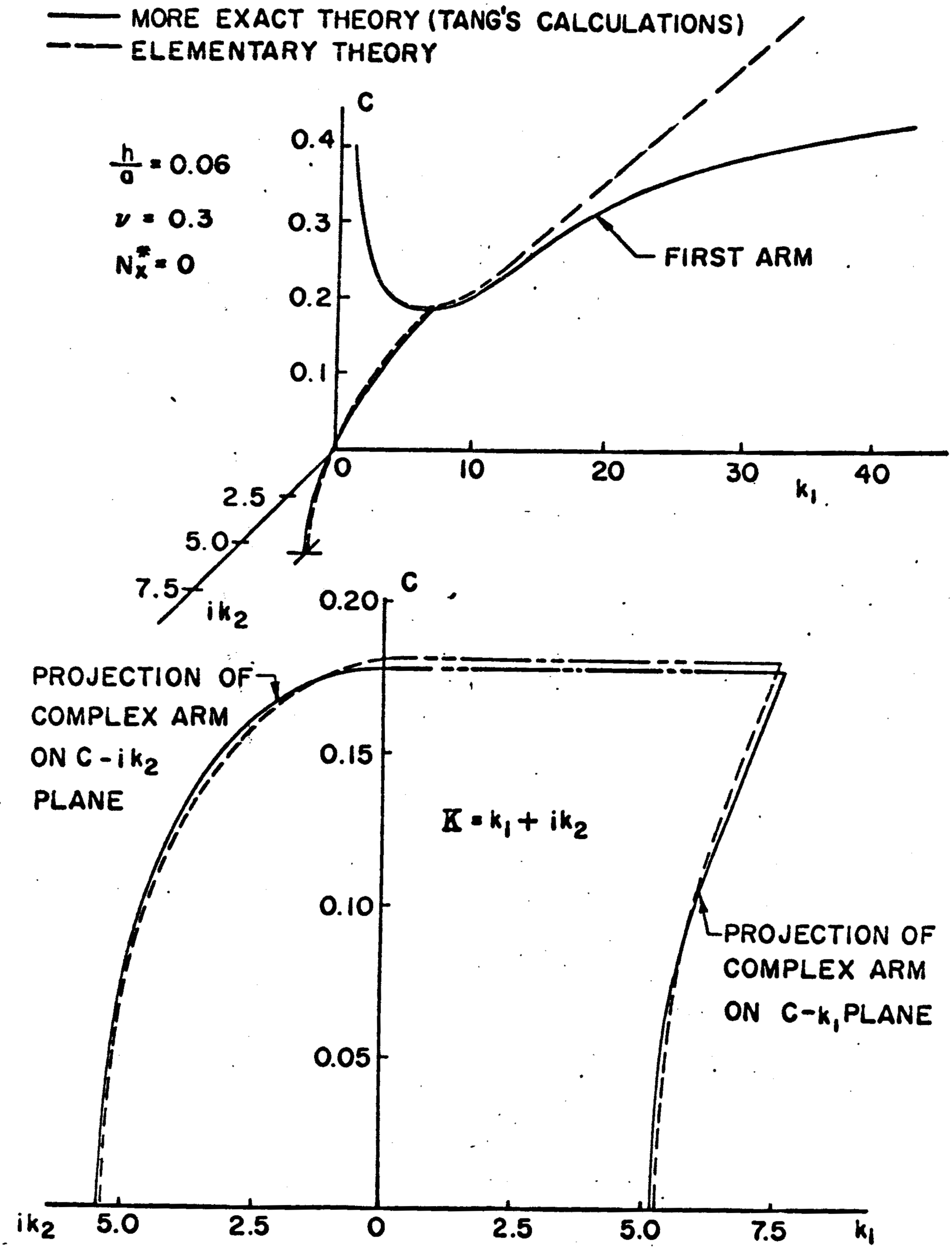

Fig. 3. Phase Velocity Spectrum.

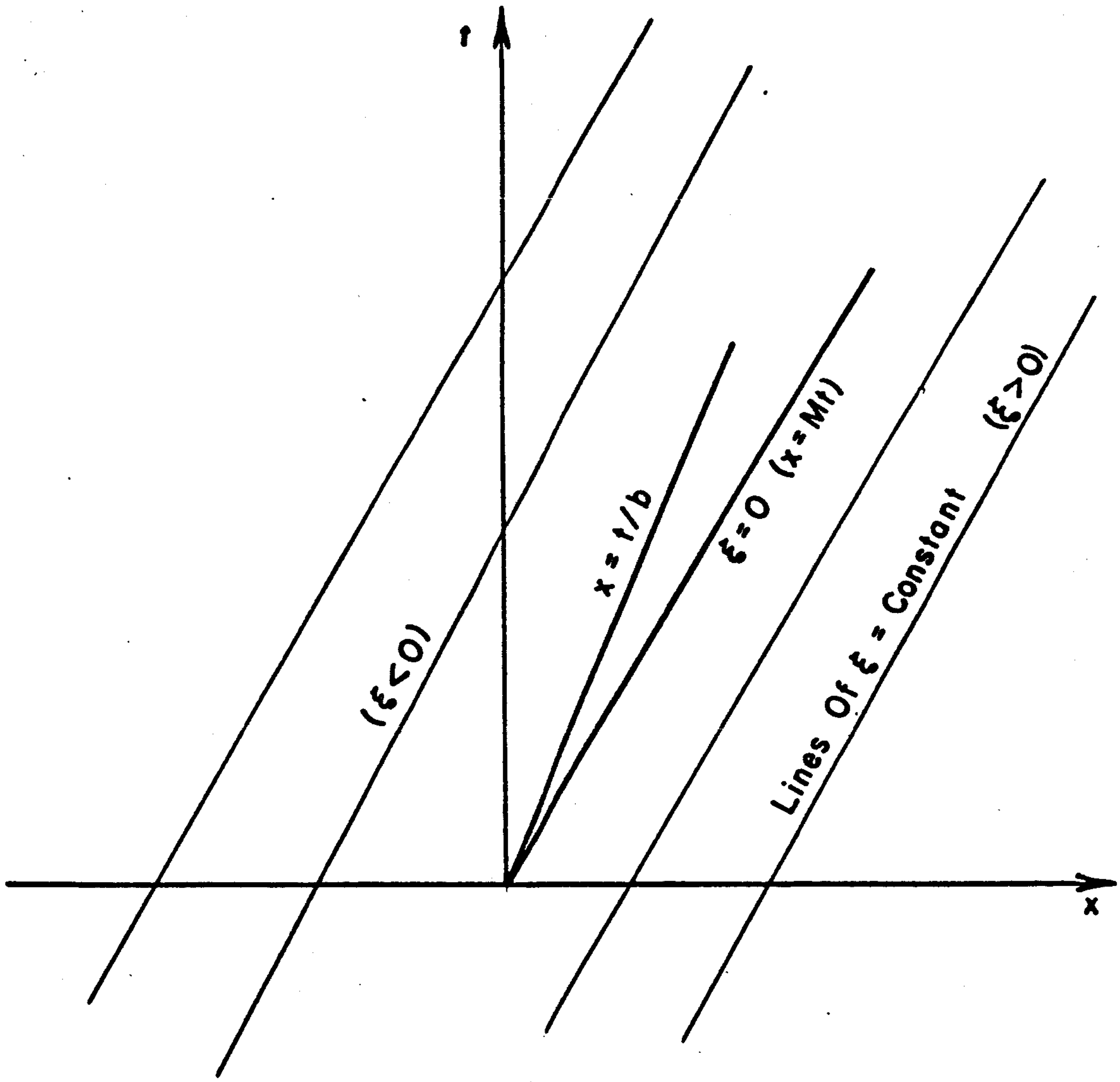

Fig. 4. t-x Plane.

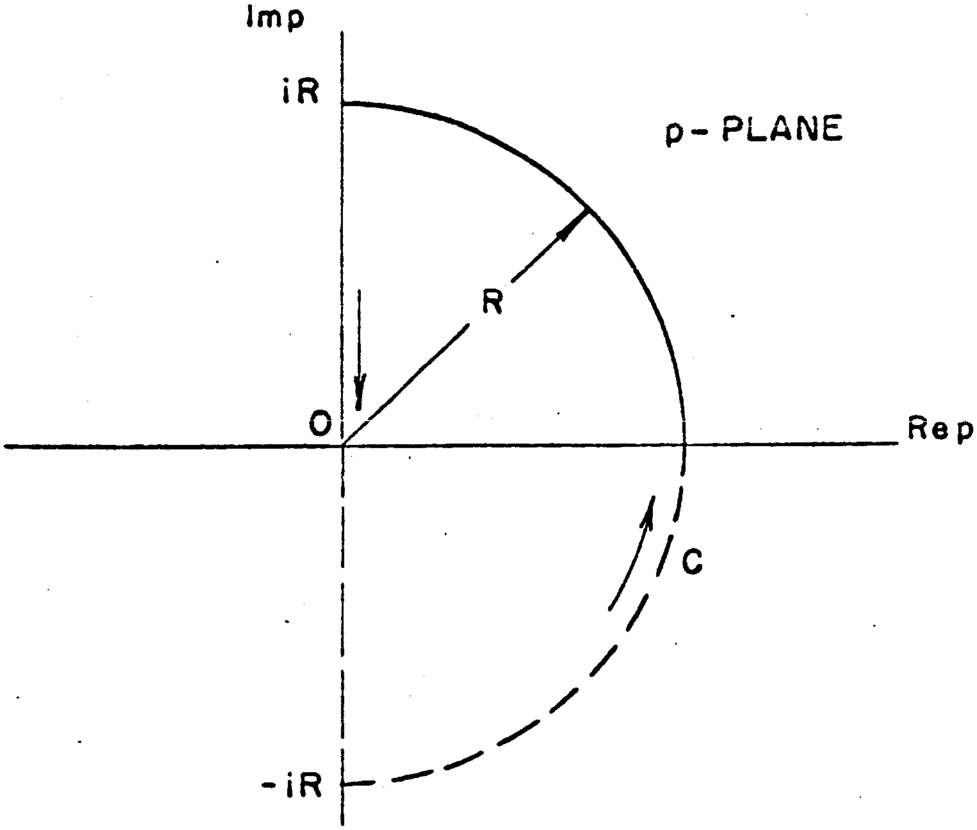

Fig. 5a. ORIGINAL CONTOUR

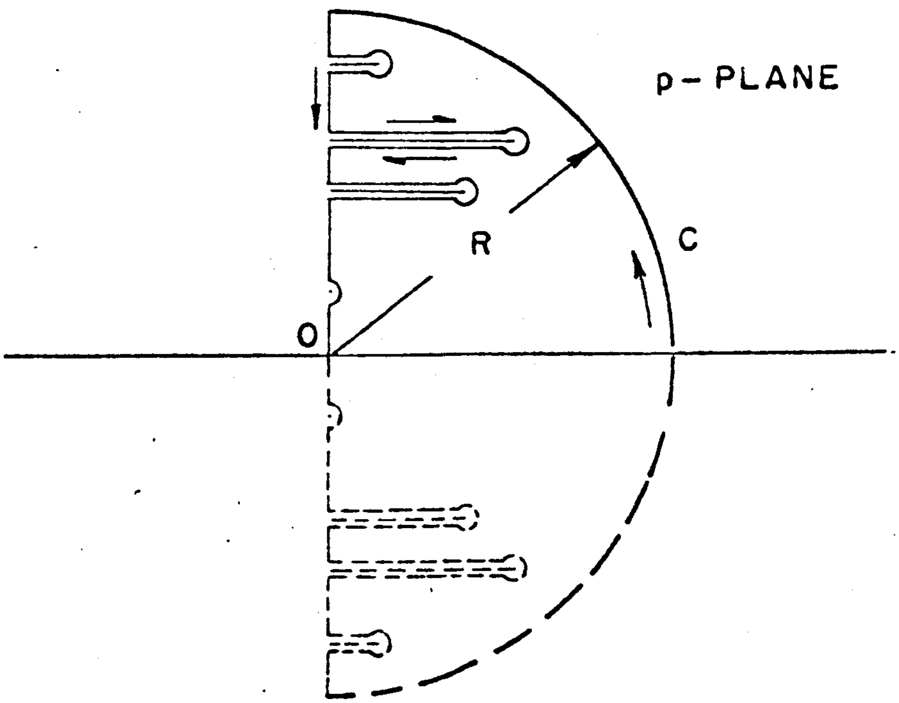

Fig. 5b. DEFORMED CONTOUR - BRANCH POINTS CIRCUMVENTED

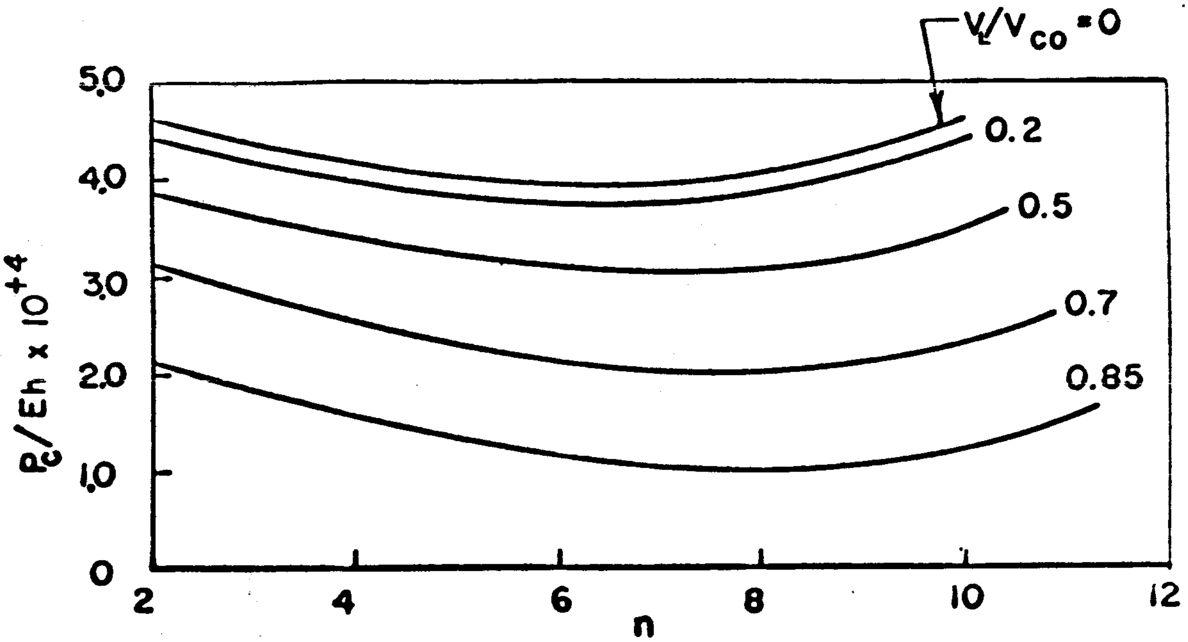

Fig. 6a. Load Parameter versus number of circumferential half waves n, for a/h = 100, $\nu = 0.3$, $N_X^o = 0$.

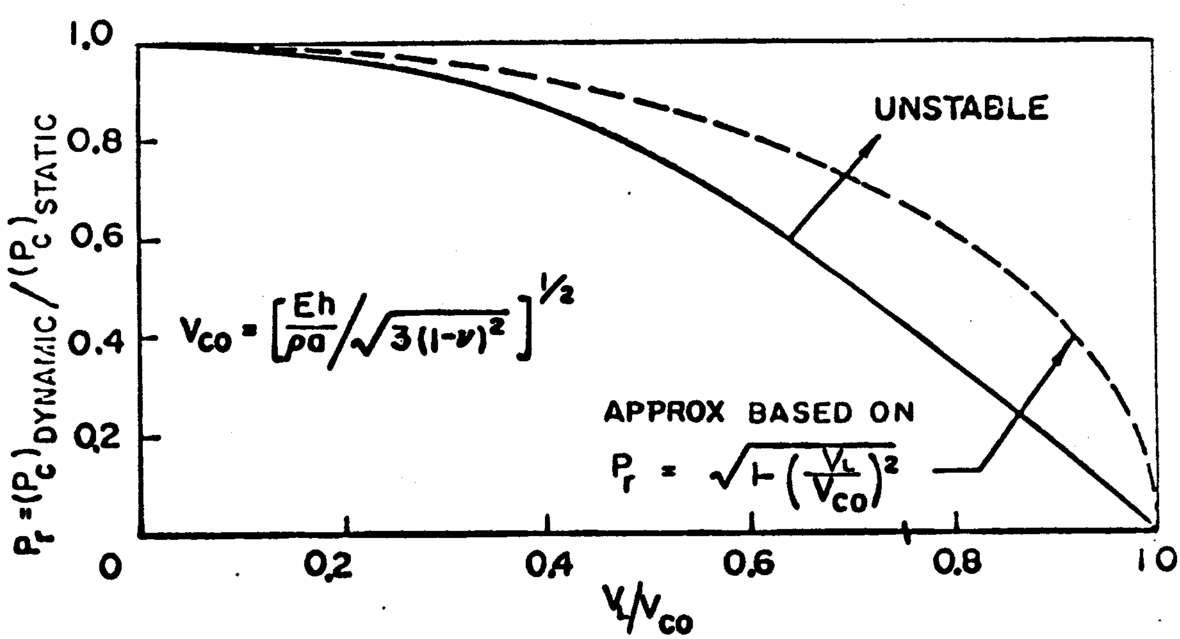

Fig. 6b. Critical Load versus Velocity, $N_X^o = 0$.

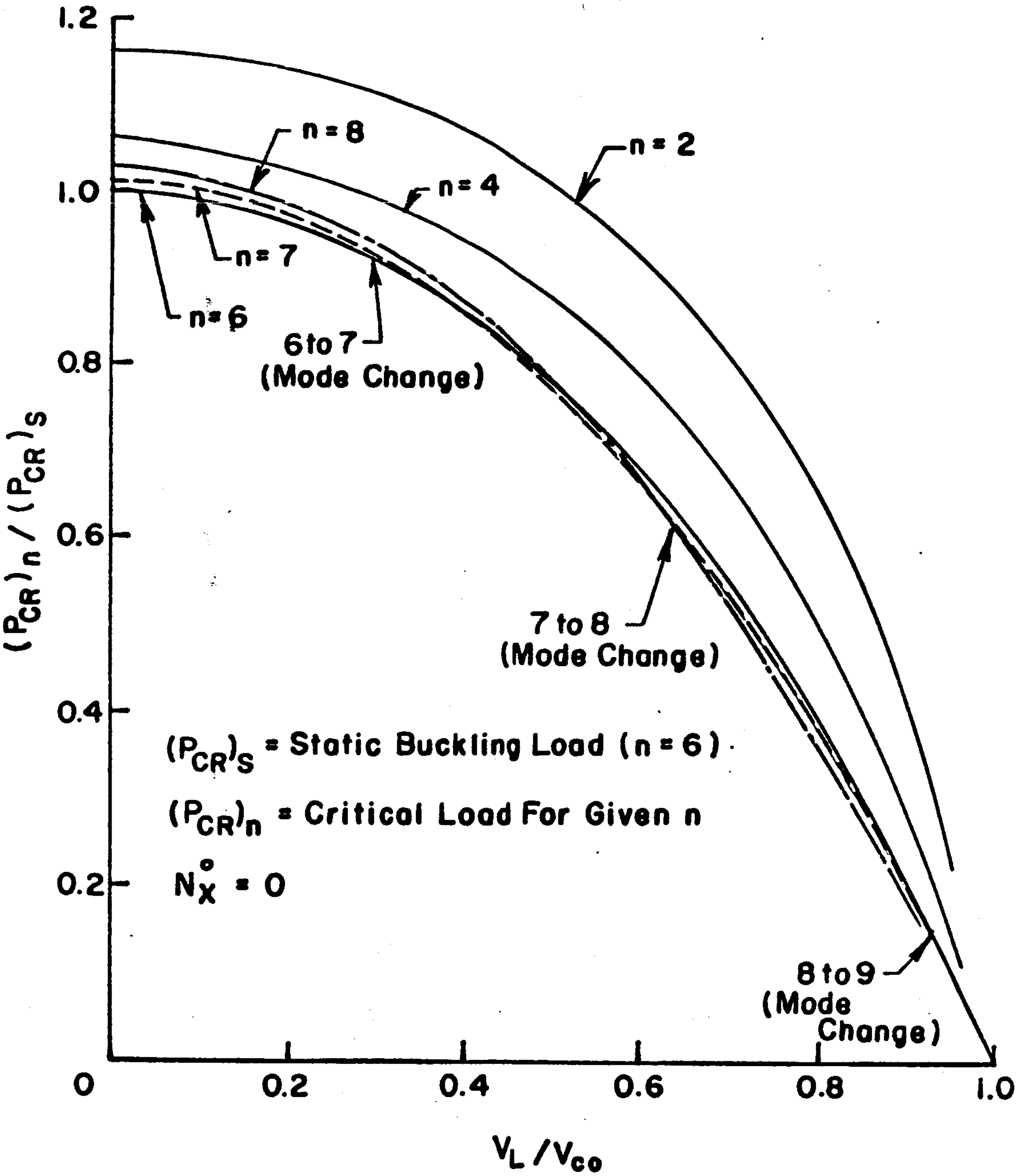

Fig. 7a. Critical Load versus Velocity for $a/h = 100$, $\nu = 0.3$, $N_X^0 = 0$.

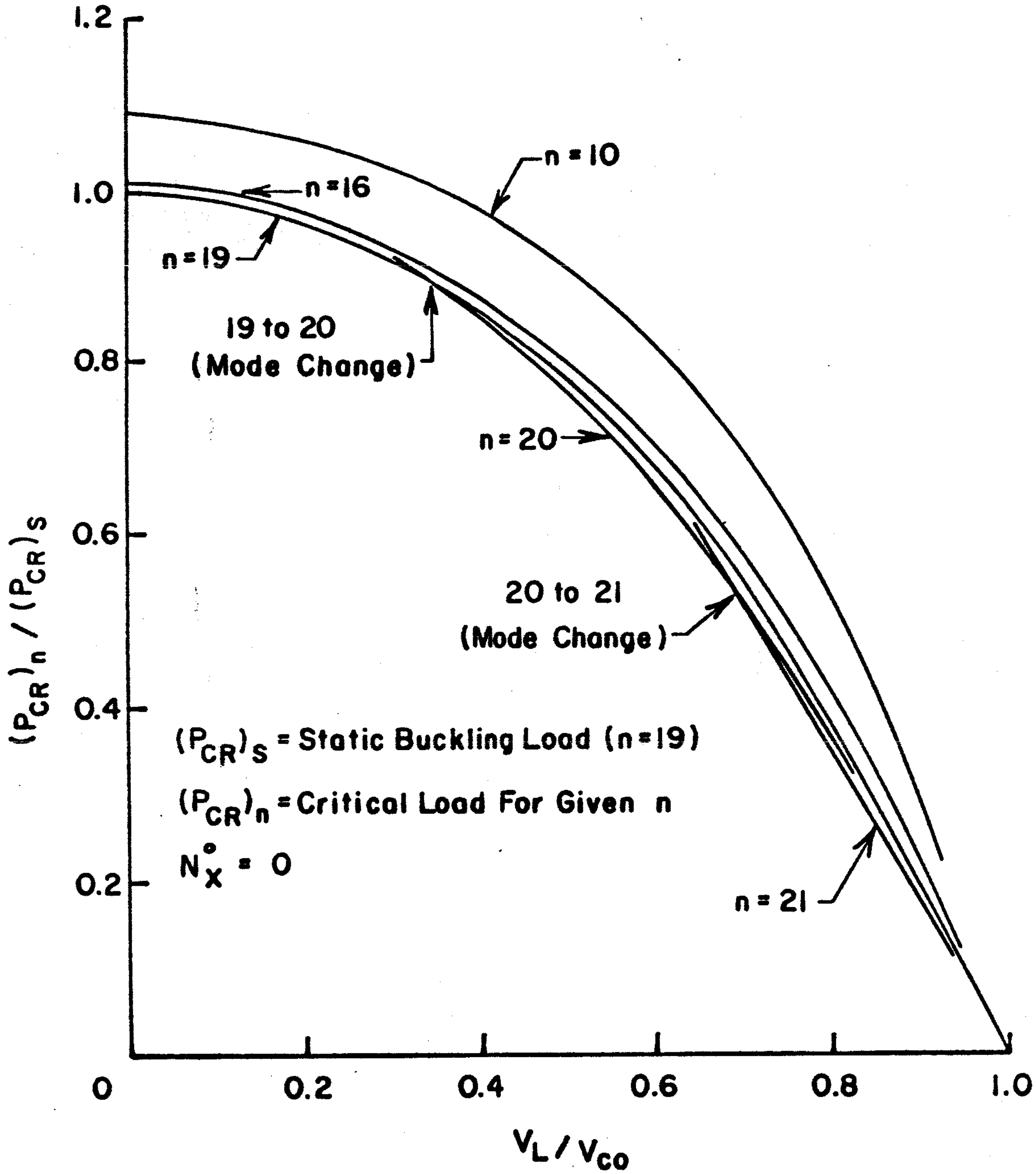

Fig. 7b. Critical Load versus Velocity for $a/h = 800$, $\nu = 0.3$, $N_X^o = 0$.

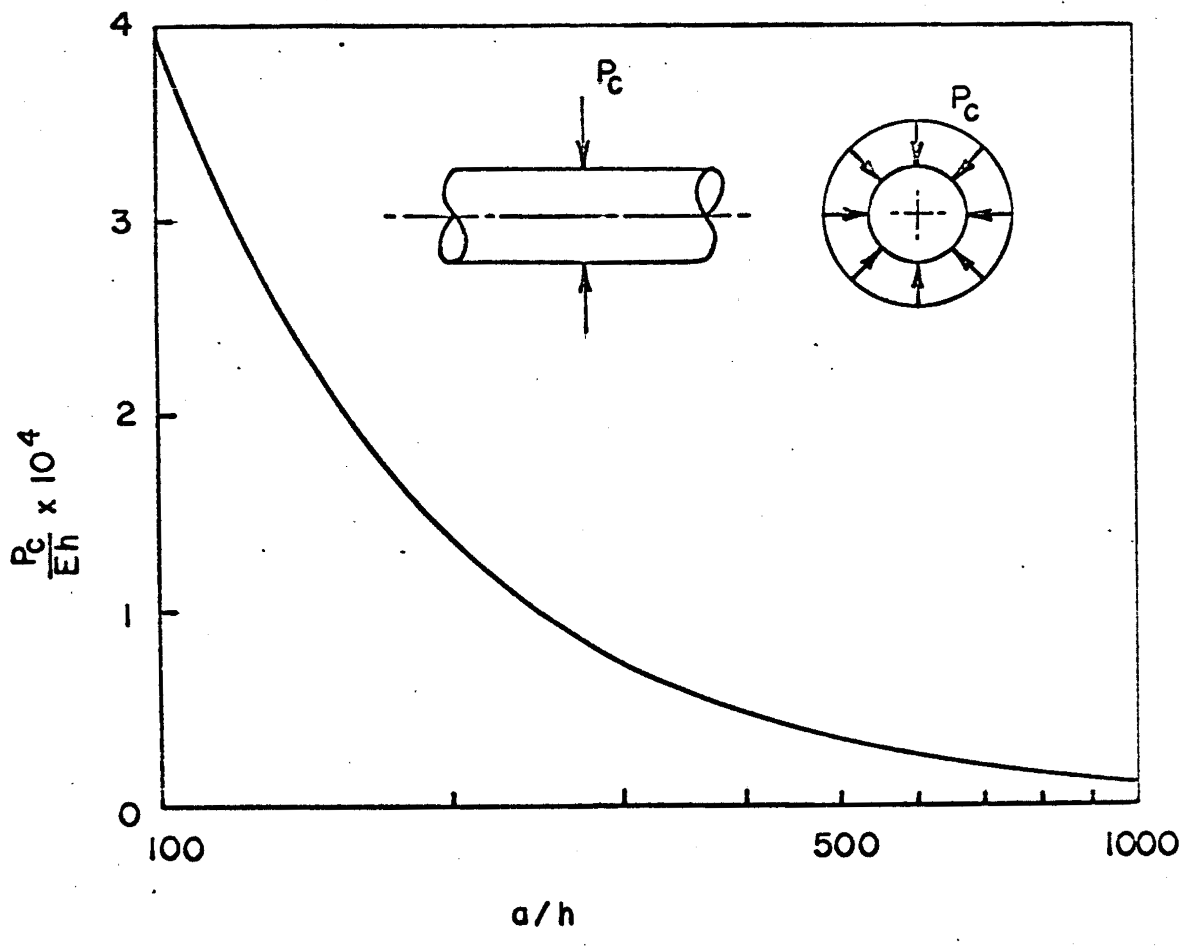

Fig. 8. Dependence of Static Buckling Load on a/h; $\nu = 0.3$, $N_x^o = 0$.

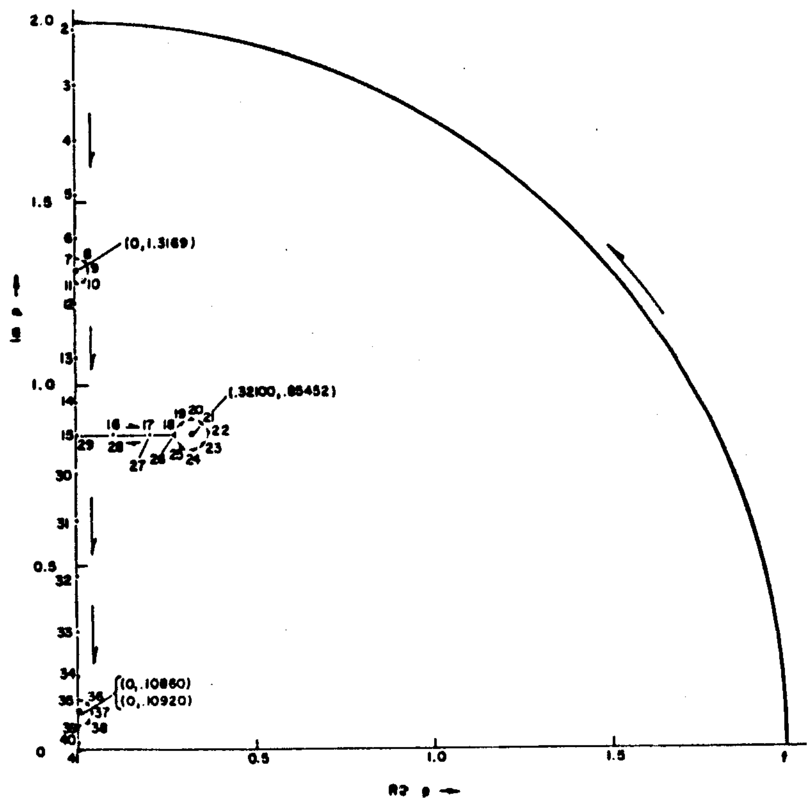

Fig. 9a. Contour C' for $n = 6$, $V_L = 0$, $N_X^o = 0$, $\nu = 0.3$, $a/h = 100$.

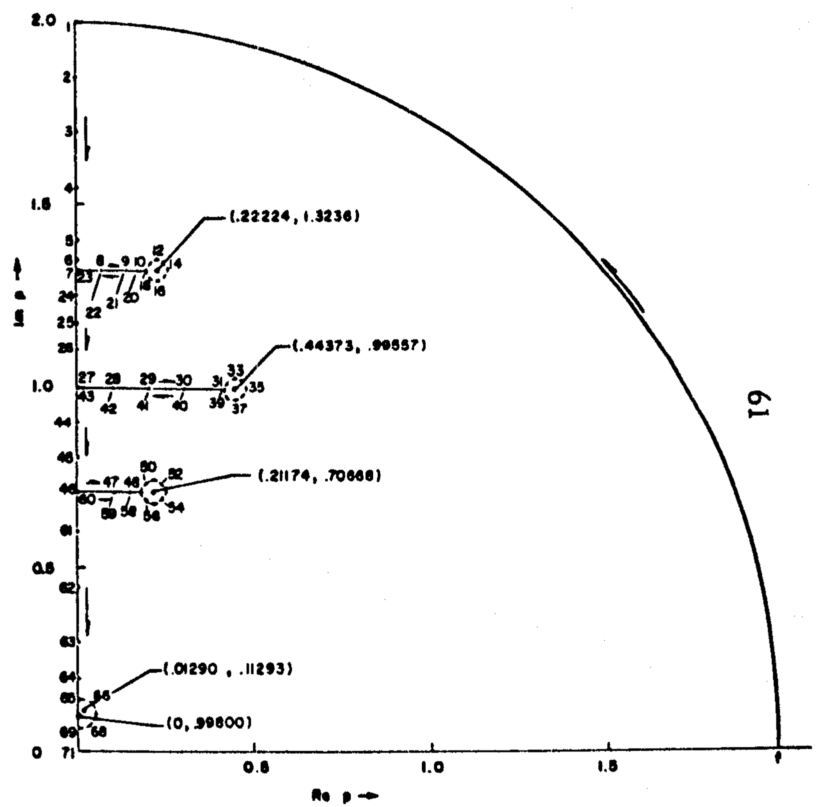

Fig. 9b. Contour C' for $n = 6$, $V_L/V_{co} = 0.2$, $N_X^o = o$, $\nu = 0.3$, $a/h = 100$.

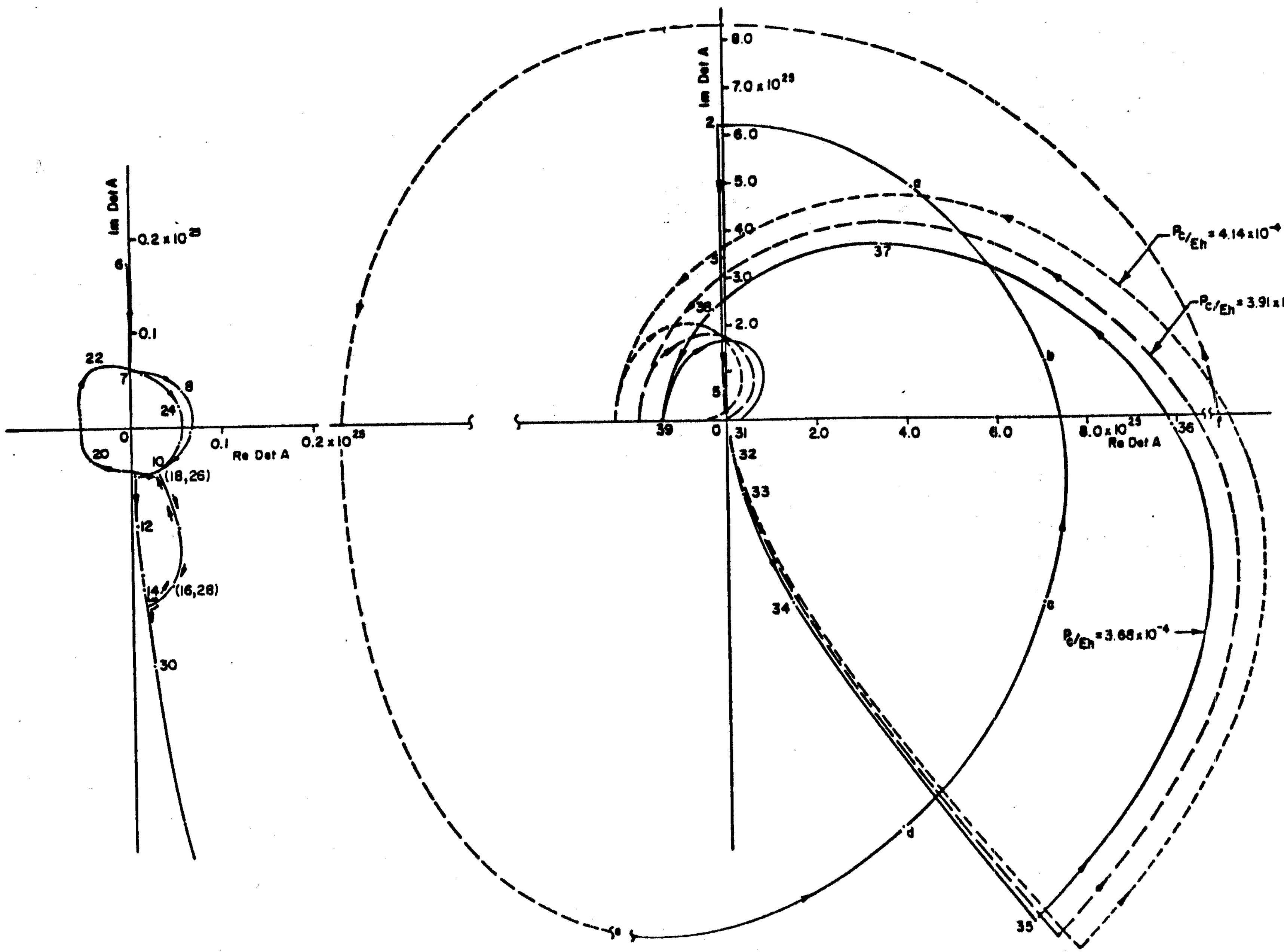

Fig. 10a. Det A Contour, $n = 6$, $V_L = 0$, $N_X^o = 0$, $\nu = 0.3$, $a/h = 100$.

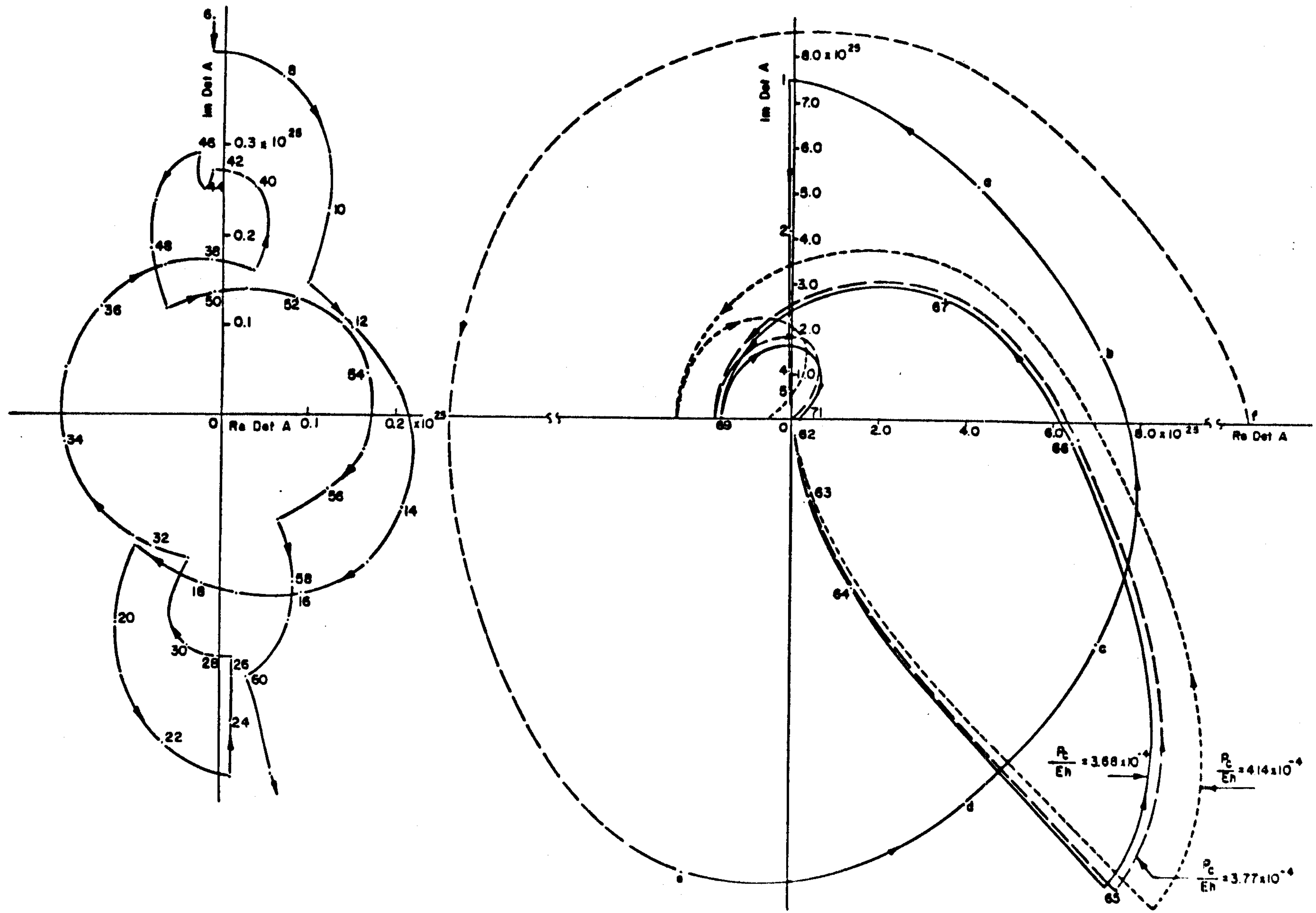

Fig. 10b. Det A Contour, n = 6, $V_L / V_{co} = 0.2$, $N_X^o = 0$, $\nu = 0.3$, a/h = 100.

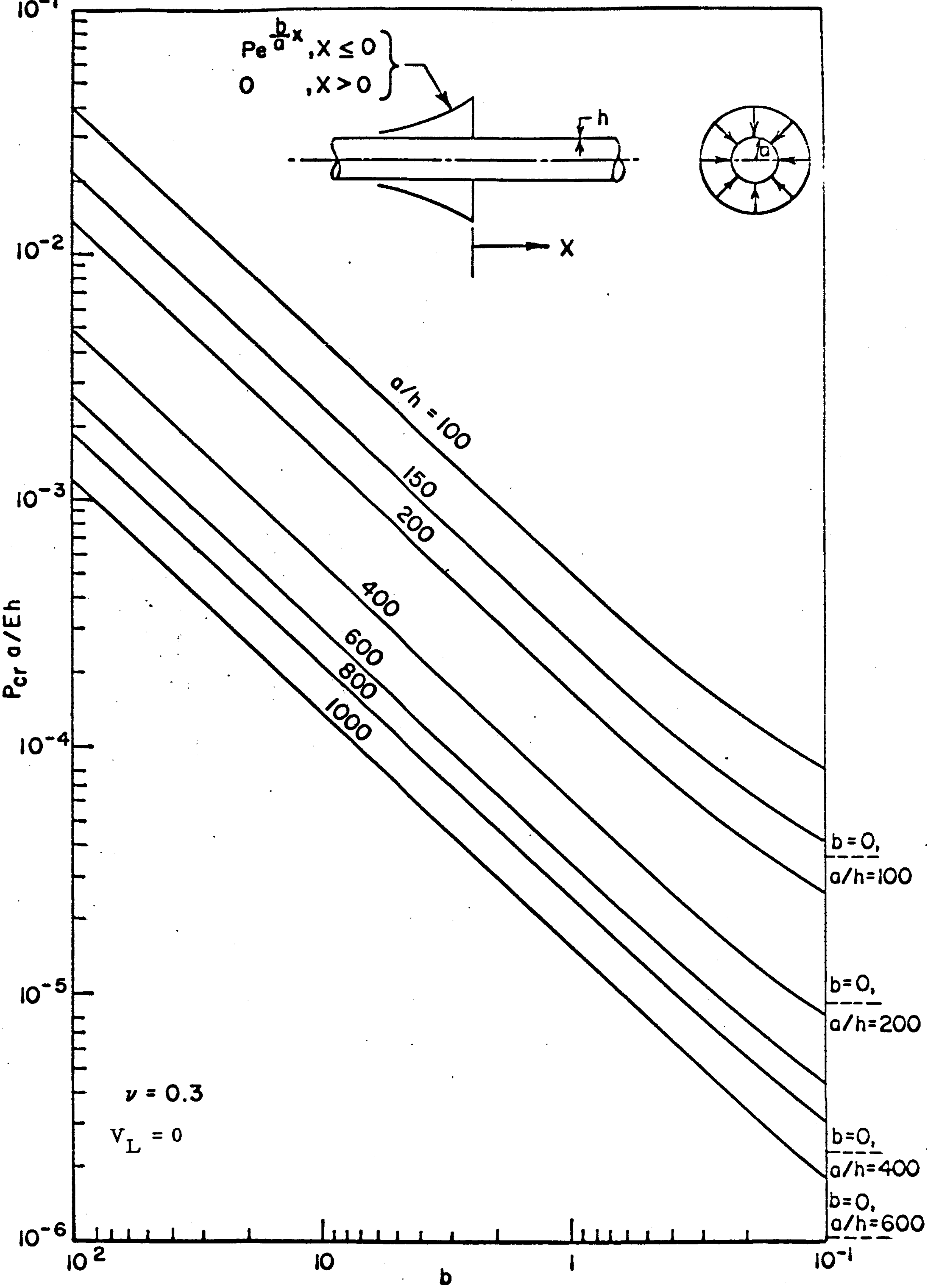

Fig. 11. Critical Load versus Decay Parameter, b.

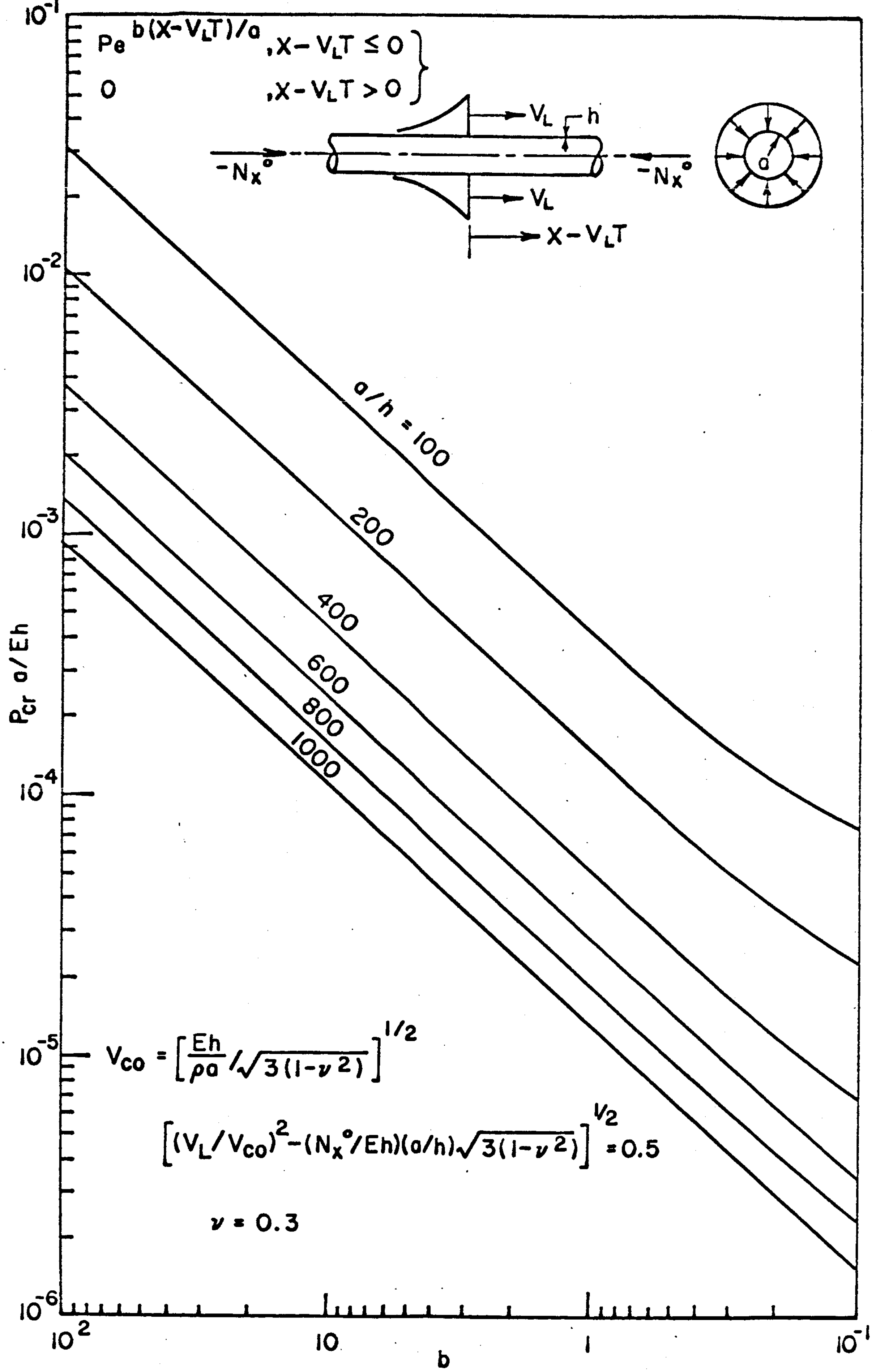

Fig. 12. Critical Load versus Decay Parameter, b.

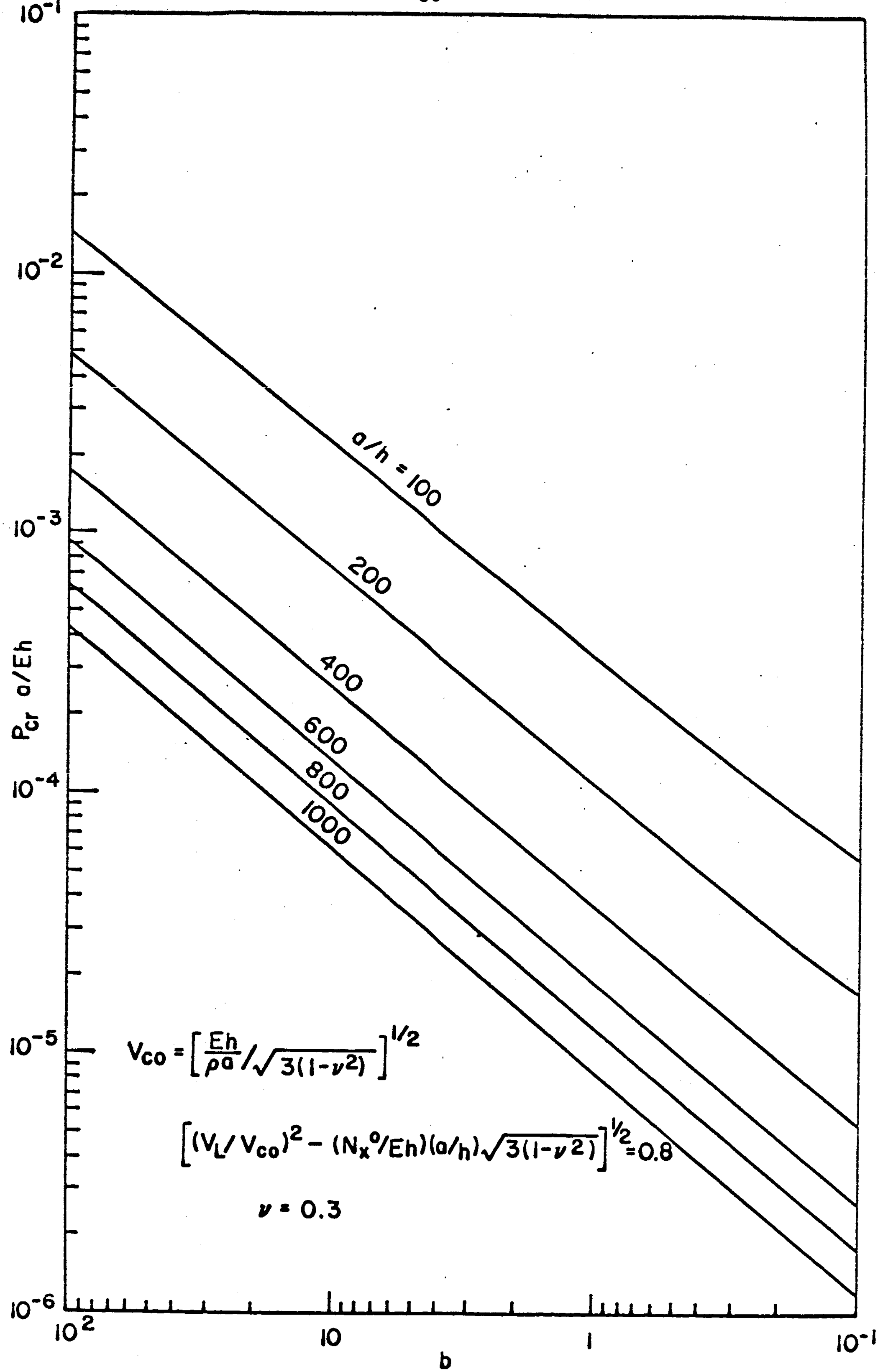

Fig. 13. Critical Load versus Decay Parameter, b.

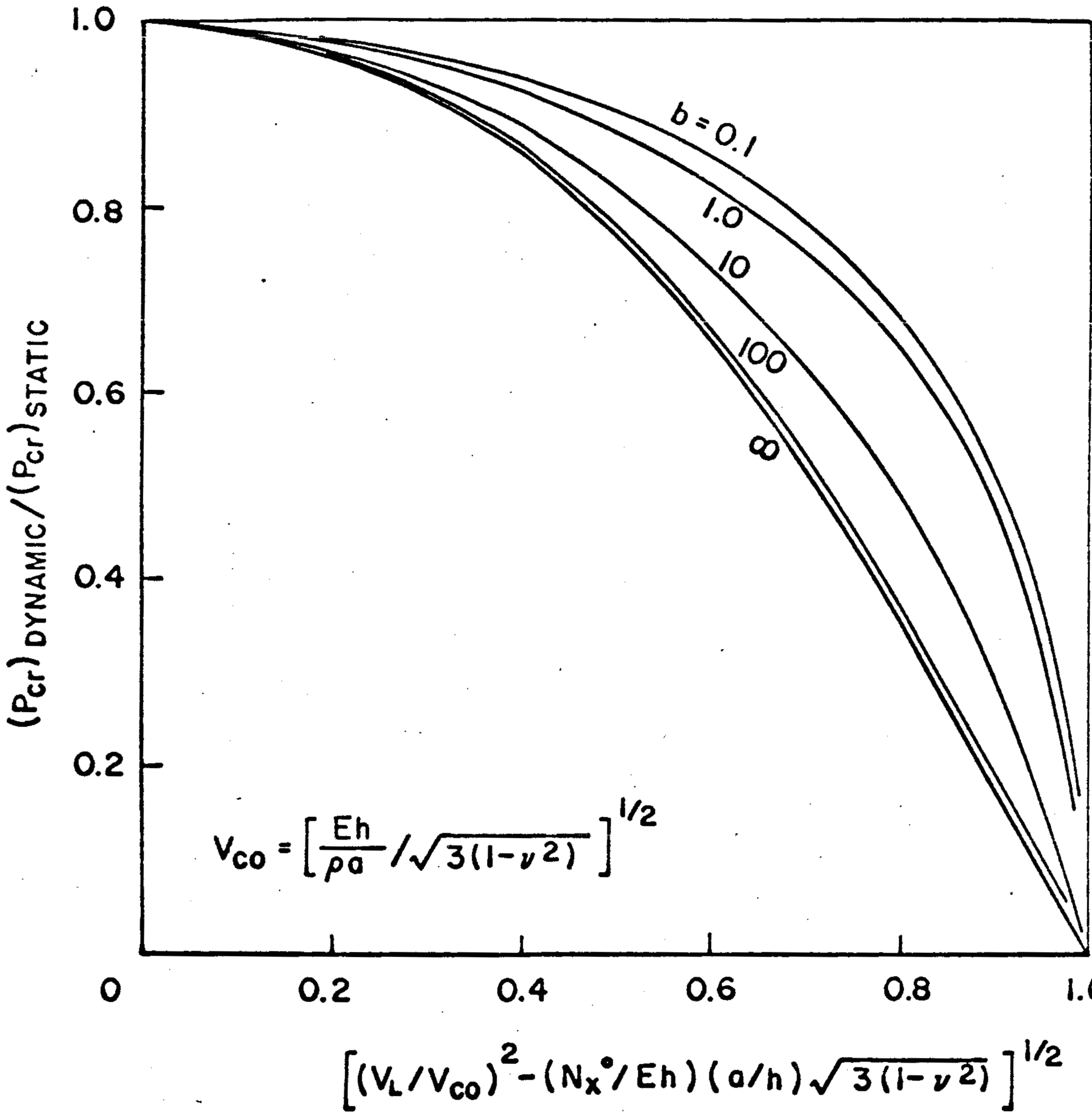

Fig. 14. Decayed Step Load Interaction Curve for $a/h = 100$, $\nu = 0.3$.

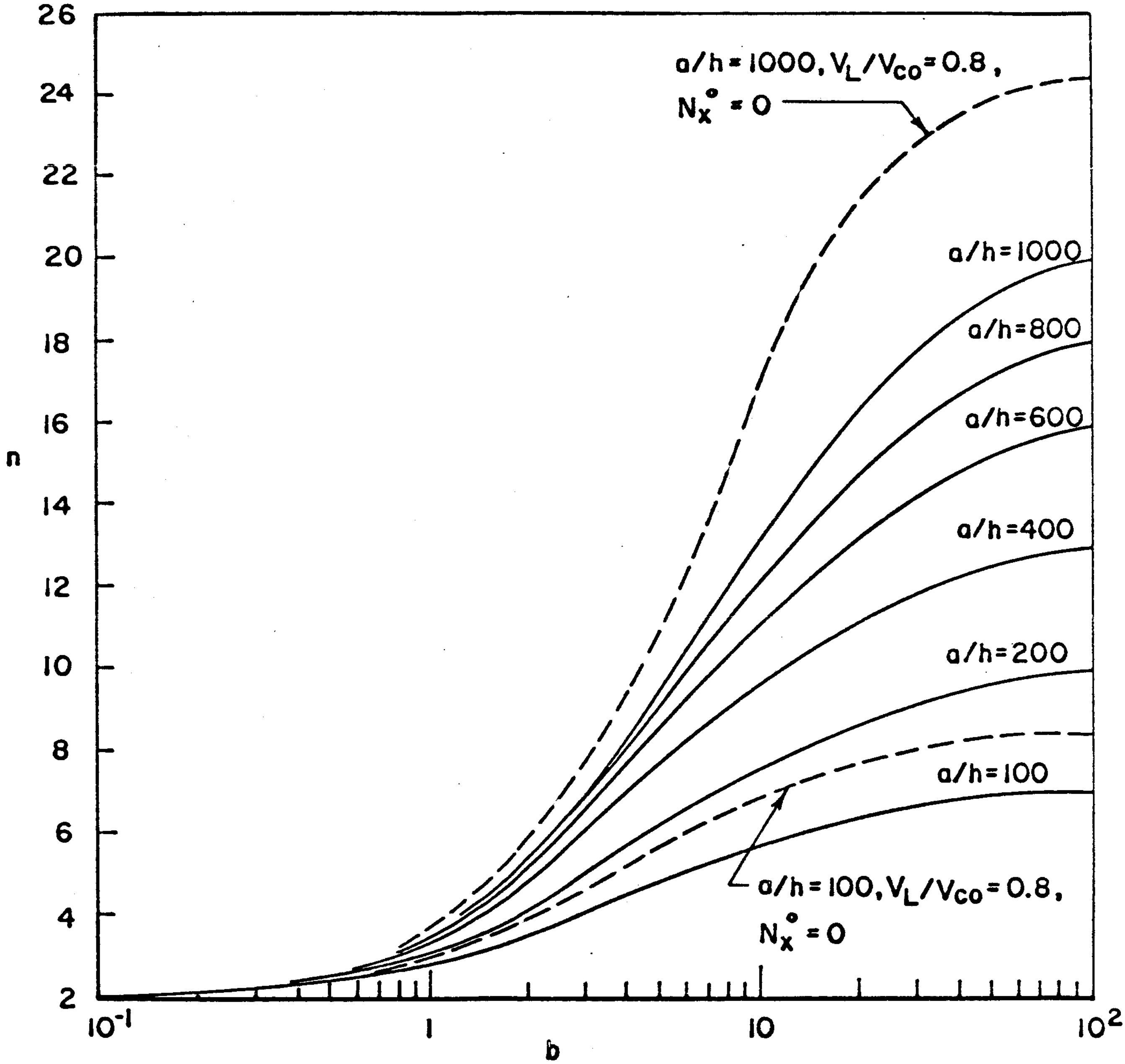

Fig. 15. Critical Mode Number, Decayed Step Load Case.

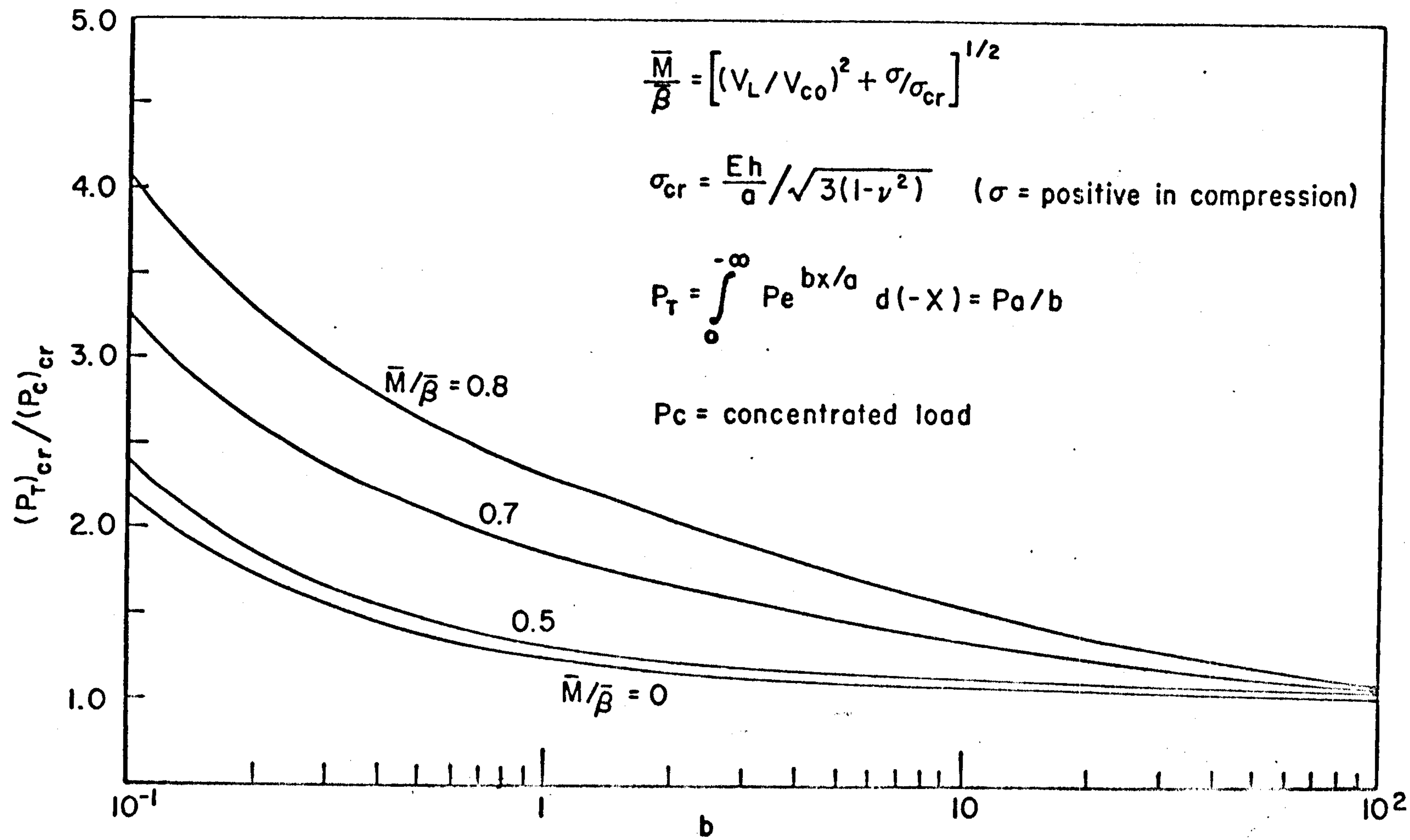

Fig. 16. Equivalent Concentrated Load, a/h = 100, $\nu = 0.3$.