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ORTHOGONAL POLYNOMIALS ON A FINITE POINTSET

by

Charles K. Mesztenyi
Senior Research Programmer
Computer Science Center
University of Maryland, College Park, Maryland

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ABSTRACT

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The report deals with orthogonal polynomials on a finite number of points where orthogonality is defined by

$$\sum_{k=1}^{N} P_i(x_k)\, P_j(x_k)\, W_k = 0 \qquad \text{if } i \neq j$$

Part I derives a relation between the sets of orthogonal polynomials $P_i^{(N)}(x)$ and $P_i^{(N+1)}(x)$ where $P_i^{(N)}(x)$ are orthogonal polynomials on $\{x_k \mid k=1,\dots,N\}$ and $P_i^{(N+1)}(x)$ are orthogonal polynomials on $\{x_k \mid k=1,\dots,N\} \cup x_{N+1}$.

Part II derives an algorithm for polynomial least square fit using the results of Part I.

Author

PART I

1.1 Definitions and Basic Formulae

The polynomials

$$\{ P_i(x) \mid \text{degree of } P_i(x) = i \; ; \; i = 0, 1, \ldots, m \}$$

are called orthogonal relative to a weight function $w(x) > 0$ on the set $\{x_k \mid k = 1, \ldots, N\}$ if they satisfy the relation:

$$(1) \qquad \sum_{k=1}^{N} P_i(x_k) P_j(x_k) w(x_k) = 0 \qquad \text{if} \quad i \neq j$$

Given the set $\{x_k \mid k = 1, \ldots, N\}$ with $\{w_k = w(x_k) > 0 \mid k = 1, \ldots, N\}$ the relation (1) determines the polynomials up to a constant factor. Through out this report the factor will be determined such that the coefficient of the highest power in $P_i(x)$ is one.

By the relation (1) the vectors

$$v_i = (P_i(x_1), P_i(x_2), \ldots, P_i(x_N)) \quad , \qquad i = 0, 1, \ldots, m$$

build an orthogonal set in an N-dimensional vectorspace. Since there are maximum of N linearly independent vectors; $P_j(x)$, $j \geq N$, must be zero vectors.

In particular

$$(2) \qquad \begin{aligned} P_N(x) &= \prod_{k=1}^{N} (x - x_k) \\ P_j(x) &= P_N(x) \cdot Q_{j-N}(x) \qquad \text{for} \quad j > N \end{aligned}$$

The orthogonal polynomials satisfy the recursive relation

$$(3) \qquad P_i(x) = (x - \alpha_i) P_{i-1}(x) - \beta_i P_{i-2}(x) \qquad i = 1, 2, \ldots, N$$

with $\qquad P_{-1} = 0 \quad , \qquad P_0 = 1$

where

$$(4) \qquad \alpha_i = \frac{\sum_{k=1}^{N} x_k \left[P_{i-1}(x_k)\right]^2 w_k}{\sum_{k=1}^{N} \left[P_{i-1}(x_k)\right]^2 w_k} \qquad i = 1, 2, \ldots, N$$

$$(5) \qquad \beta_i = \frac{\sum_{k=1}^{N} x_k P_{i-1}(x_k) P_{i-2}(x_k) w_k}{\sum_{k=1}^{N} \left[P_{i-2}(x_k)\right]^2 w_k} \qquad i = 2, \ldots, N$$

If (3) is multiplied by $P_i(x)\, w(x)$ and summed over k, by orthogonality we get:

$$\sum_{k=1}^{N} \left[P_i(x_k)\right]^2 w_k = \sum_{k=1}^{N} x_k P_{i-1}(x_k) P_i(x_k) w_k$$

and by (5)

$$= \beta_{i+1} \sum_{k=1}^{N} \left[P_{i-1}(x_k)\right]^2 w_k$$

By repeated use of this equation we have:

$$(6) \qquad \sum_{k=1}^{N} \left[P_i(x_k)\right]^2 w_k = \prod_{j=1}^{i+1} \beta_j$$

if we define:

$$(7) \qquad \beta_1 = \sum_{k=1}^{N} \left[P_0(x_k)\right]^2 w_k = \sum_{k=1}^{N} w_k$$

From (6) follows that $\beta_j > 0 \quad , \quad j = 1, 2, \ldots$

1.2. Calculation of the Orthogonal Polynomials

Once the set of points with the corresponding weights $\{x_k, w_k \mid k=1,\ldots,N\}$ is given, using (4), (5) and (3), the orthogonal polynomials $\{P_i(x) \mid i=0,1,\ldots,N\}$ can be determined.

In this report, we derive an other formula for the orthogonal polynomials which is based on the incrementation of the number of points.

Let

$$\underline{X}_N = \{(x_k, w_k) \mid k=1,2,\ldots,N\}$$

be the set of points with weights and

$$\underline{X}_{N+1} = \underline{X}_N \cup (x_{N+1}, w_{N+1})$$

The corresponding orthogonal polynomials are noted as:

$$P_i^{(N)}(x) \quad , \quad P_i^{(N+1)}(x) \quad , \qquad i=0,1,\ldots,m \leq N$$

Theorem 1

With the above notation, $P_i^{(N+1)}(x)$ can be expressed as

$$(8) \qquad P_i^{(N+1)}(x) = P_i^{(N)}(x) - \frac{P_i^{(N)}(x_{N+1})}{1 + \sum_{j=0}^{i-1} \frac{[P_j^{(N)}(x_{N+1})]^2 w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}}} \sum_{j=0}^{i-1} \frac{P_j^{(N)}(x_{N+1})\, P_j^{(N)}(x)\, w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}}$$

$$(i=1,2,\ldots,m)$$

$$P_0^{(N+1)} = P_0^{(N)} = 1$$

Proof

$P_i^{(N+1)}(x)$ can be expressed as a linear combination of $P_j^{(N)}(x)$, $j=0,1,\ldots,i$

$$P_i^{(N+1)}(x) = P_i^{(N)}(x) - \sum_{j=0}^{i-1} c_j P_j^{(N)}(x)$$

where $P_i^{(N)}(x)$ has coefficient one since we normalize the polynomials with the coefficients of $x^i = 1$. Multiply the above equation by $P_j^{(N)}(x)w(x)$, $j < i$ and sum over $\underline{X}_N$. We get

$$\sum_{k=1}^{N} P_i^{(N+1)}(x_k) P_j^{(N)}(x_k) w_k = \sum_{k=1}^{N+1} P_i^{(N+1)}(x_k) P_j^{(N)}(x_k) w_k - P_i^{(N+1)}(x_{N+1}) P_j^{(N)}(x_{N+1}) w_{N+1}$$

$$= -c_j \sum_{k=1}^{N} \left[P_j^{(N)}(x_k)\right]^2 w_k$$

by the orthogonolity of $P_j^{(N)}(x)$.

From the orthogonality of $P_i^{(N+1)}(x)$

$$\sum_{k=1}^{N+1} P_i^{(N+1)}(x_k) P_j^{(N)}(x_k) w_k = 0 \qquad (j < i)$$

so

$$c_j = \frac{P_i^{(N+1)}(x_{N+1}) P_j^{(N)}(x_{N+1}) w_{N+1}}{\sum_{k=1}^{N} \left[P_j^{(N)}(x_k)\right]^2 w_k}$$

$$= \frac{P_i^{(N+1)}(x_{N+1}) P_j^{(N)}(x_{N+1}) w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}} \qquad (j = 0, 1, \dots, i-1)$$

by (6). Inserting this to the original equation, we get

(9)

$$P_i^{(N+1)}(x) = P_i^{(N)}(x) - P_i^{(N+1)}(x_{N+1}) \sum_{j=0}^{i-1} \frac{P_j^{(N)}(x_{N+1}) P_j^{(N)}(x) w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}}$$

At $X = X_{N+1}$ we can express $P_i^{(N+1)}(x_{N+1})$

(10)
$$P_i^{(N+1)}(x_{N+1}) = \frac{P_i^{(N)}(x_{N+1})}{1 + \sum_{j=0}^{i-1} \frac{\left[P_j^{(N)}(x_{N+1})\right]^2 w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}}}$$

(9) and (10) give the desired expression (8) for $P_i^{(N+1)}(x)$.

With the following notations:

(11)
$$\Delta_j = \frac{P_j^{(N)}(x_{N+1})\, w_{N+1}}{\prod_{k=1}^{j+1} \beta_k^{(N)}} \qquad j = 0, 1, \ldots, i-1$$

(12)
$$\varepsilon_i = \frac{P_i^{(N)}(x_{N+1})}{1 + \sum_{j=0}^{i-1} \Delta_j\, P_j^{(N)}(x_{N+1})} = P_i^{(N+1)}(x_{N+1})$$

$i = 1, 2, \ldots, n$

We can write (8)

(13)
$$P_i^{(N+1)}(x) = P_i^{(N)}(x) - \varepsilon_i \sum_{j=0}^{i-1} \Delta_j \cdot P_j^{(N)}(x)$$

$i = 1, 2, \ldots, n$

Theorem 2

In the recurrence formulae of the orthogonal polynomials

$$\left.\begin{aligned} P_i^{(N)}(x) &= (x-\alpha_i^{(N)})\,P_{i-1}^{(N)}(x) - \beta_i^{(N)}\,P_{i-2}^{(N)}(x) \\ P_i^{(N+1)}(x) &= (x-\alpha_i^{(N+1)})\,P_{i-1}^{(N+1)}(x) - \beta_i^{(N+1)}\,P_{i-2}^{(N+1)}(x) \end{aligned}\right\} \quad i=1,2,\ldots,n$$

$$P_{-1}^{(N)} = P_{-1}^{(N+1)} = 0$$

$$P_0^{(N)} = P_0^{(N+1)} = 1$$

the recurrence coefficients satisfy the following relation:

(14) $$\alpha_i^{(N+1)} - \alpha_i^{(N)} = \varepsilon_i \Delta_{i-1} - \varepsilon_{i-1} \Delta_{i-2} \quad , \quad i=1,2,\ldots,n$$

(15) $$\frac{\beta_i^{(N+1)}}{\beta_i^{(N)}} = \frac{1 + \varepsilon_{i-1}\Delta_{i-1}}{1 + \varepsilon_{i-2}\Delta_{i-2}} \quad , \quad i=2,3,\ldots,n$$

and corresponding to (7)

(16) $$\beta_1^{(N+1)} = \beta_1^{(N)} + \omega_{N+1}$$

Proof

(13) can be rewritten by the recurrence relations

$$(x-\alpha_i^{(N+1)})P_{i-1}^{(N+1)}(x)-\beta_i^{(N+1)}P_{i-2}^{(N+1)}(x)$$
$$=(x-\alpha_i^{(N)})P_{i-1}^{(N)}(x)-\beta_i^{(N)}P_{i-2}^{(N)}(x)-\varepsilon_i\sum_{j=0}^{i-1}\Delta_j\cdot P_j^{(N)}(x)$$

and by using (13) again for $P_{i-1}^{(N+1)}$ and $P_{i-2}^{(N+1)}$

$$(x-\alpha_i^{(N+1)})P_{i-1}^{(N)}(x)-\varepsilon_{i-1}(x-\alpha_i^{(N+1)})\sum_{j=0}^{i-2}\Delta_j\cdot P_j^{(N)}(x)-\beta_i^{(N+1)}P_{i-2}^{(N)}(x)$$
$$+\beta_i^{(N+1)}\varepsilon_{i-2}\sum_{j=0}^{i-3}\Delta_j\cdot P_j^{(N)}(x)$$
$$=(x-\alpha_i^{(N)})P_{i-1}^{(N)}(x)-\beta_i^{(N)}P_{i-2}^{(N)}(x)-\varepsilon_i\sum_{j=0}^{i-1}\Delta_j\cdot P_j^{(N)}(x)$$

From the recurrence relation of $P_j^{(N)}$

$$x\,P_j^{(N)}(x)=P_{j+1}^{(N)}(x)+\alpha_{j+1}^{(N)}P_j^{(N)}(x)+\beta_{j+1}^{(N)}P_{j-1}^{(N)}(x)$$

We get the above equation in linear terms of

$$(\alpha_i^{(N)}-\alpha_i^{(N+1)})P_{i-1}^{(N)}(x)+(\beta_i^{(N)}-\beta_i^{(N+1)})P_{i-2}^{(N)}(x)+\varepsilon_i\sum_{j=0}^{i-1}\Delta_j\cdot P_j^{(N)}(x)$$
$$-\varepsilon_{i-1}\sum_{j=0}^{i-2}\Delta_j\left[P_{j+1}^{(N)}(x)+\alpha_{j+1}^{(N)}P_j^{(N)}(x)+\beta_{j+1}^{(N)}P_{j-1}^{(N)}(x)\right]$$
$$+\alpha_i^{(N+1)}\varepsilon_{i-1}\sum_{j=0}^{i-2}\Delta_j\cdot P_j^{(N)}(x)-\beta_i^{(N+1)}\varepsilon_{i-2}\sum_{j=0}^{i-3}\Delta_j\cdot P_j^{(N)}(x)=0$$

Since $P_j^{(N)}(x)$, $j = 0, 1, \ldots, i-1$ are linearly independent, in the above homogeneous equations, the coefficients of $P_j^{(N)}(x)$

have to be zero

Thus for $j = i-1$

$$\alpha_i^{(N)} - \alpha_i^{(N+1)} + \varepsilon_i \Delta_{i-1} - \varepsilon_{i-1} \Delta_{i-2} = 0$$

from which (14) is proved.

For $j = i-2$ we get

$$\beta_i^{(N)} - \beta_i^{(N+1)} + \varepsilon_i \Delta_{i-2} - \varepsilon_{i-1} \left(\Delta_{i-3} + \Delta_{i-2} \alpha_{i-1}^{(N)} - \alpha_i^{(N+1)} \Delta_{i-2} \right) = 0$$

By the definition of Δ_j and ε_j, (11) and (12)

$$\Delta_{i-1} = \frac{P_{i-1}^{(N)}(x_{N+1})\, w_{N+1}}{\prod_{k=1}^{i} \beta_k^{(N)}}$$

$$= \frac{(x_{N+1} - \alpha_{i-1}^{(N)}) P_{i-2}^{(N)}(x_{N+1}) - \beta_{i-1}^{(N)} P_{i-3}^{(N)}(x_{N+1})}{\prod_{k=1}^{i} \beta_k^{(N)}} \; w_{N+1}$$

$$= \frac{1}{\beta_i^{(N)}} \left[(x_{N+1} - \alpha_i^{(N)}) \Delta_{i-2} - \Delta_{i-3} \right]$$

and $$\varepsilon_i = P_i^{(N+1)}(x_{N+1}) = (x_{N+1} - \alpha_i^{(N+1)}) \varepsilon_{i-1} - \beta_i^{(N+1)} \varepsilon_{i-2}$$

we get

$$\beta_i^{(N)}\left[1+\varepsilon_{i-1}\Delta_{i-1}\right] = \beta_i^{(N+1)}\left[1+\varepsilon_{i-2}\Delta_{i-2}\right]$$

which proves (15)

From (14) and (15), by repeated summation and product we get

(17) $$\sum_{j=1}^{i}\left(\alpha_j^{(N+1)}-\alpha_j^{(N)}\right) = \varepsilon_i\,\Delta_{i-1}$$

(18) $$\prod_{j=1}^{i}\frac{\beta_j^{(N+1)}}{\beta_j^{(N)}} = 1+\varepsilon_{i-1}\,\Delta_{i-1}$$

1.3 <u>Christoffel-Darboux Formula and Properties of $P^{(N+1)}(x)$</u>.

The Christoffel-Darboux formula states that

$$\sum_{j=0}^{i}\frac{P_j(t)\,P_j(x)}{\prod_{k=1}^{j+1}\beta_k} = \frac{1}{\prod_{k=1}^{i+1}\beta_k}\;\frac{P_{i+1}(t)\,P_i(x)-P_i(t)\,P_{i+1}(x)}{t-x}$$

and with $t \to x$

$$\sum_{j=0}^{i}\frac{\left[P_j(x)\right]^2}{\prod_{k=1}^{j+1}\beta_j} = \frac{1}{\prod_{k=1}^{i+1}\beta_k}\left[P_{i+1}'(x)\,P_i(x)-P_i'(x)\,P_{i+1}(x)\right]$$

where ' denotes differentiation.

With the help of these formulae we may rewrite (10), (9), (17) and (18):

(19)
$$P_i^{(N+1)}(x_{N+1}) = \frac{P_i^{(N)}(x_{N+1})}{1 + \dfrac{w_{N+1}}{\prod_{k=1}^{i} \beta_k^{(N)}} \left[P_i^{(N)\prime}(x_{N+1}) P_{i-1}^{(N)}(x_{N+1}) - P_{i-1}^{(N)\prime}(x_{N+1}) P_i^{(N)}(x_{N+1}) \right]}$$

$$i = 1, 2, \ldots, N$$

(20)
$$P_i^{(N+1)}(x) = P_i^{(N)}(x) - \frac{P_i^{(N+1)}(x_{N+1})\, w_{N+1}}{\prod_{k=1}^{i} \beta_k^{(N)}} \; \frac{P_i^{(N)}(x) P_{i-1}^{(N)}(x_{N+1}) - P_i^{(N)}(x_{N+1}) P_{i-1}(x)}{x - x_{N+1}}$$

$$i = 1, 2, \ldots, N$$

(20a)
$$P_0^{(N+1)} = 1$$

(21)
$$\sum_{j=1}^{i} \left(\alpha_j^{(N+1)} - \alpha_j^{(N)} \right) = \frac{P_{i-1}^{(N)}(x_{N+1}) P_i^{(N)}(x_{N+1})\, w_{N+1}}{\prod_{k=1}^{i} \beta_k^{(N)} + w_{N+1} \left[P_i^{(N)\prime}(x_{N+1}) P_{i-1}^{(N)}(x_{N+1}) - P_{i-1}^{(N)\prime}(x_{N+1}) P_i^{(N)}(x_{N+1}) \right]}$$

$$i = 1, 2, \ldots, N$$

$$(22) \quad \prod_{j=1}^{i} \frac{\beta_j^{(N+1)}}{\beta_j^{(N)}} = 1 + \frac{\left[P_{i-1}^{(N)}(x_{N+1})\right]^2 w_{N+1}}{\prod_{k=1}^{i} \beta_k^{(N)} + \beta_i^{(N)} w_{N+1} \left[P_{i-1}^{(N)\prime}(x_{N+1}) P_{i-2}^{(N)}(x_{N+1}) - P_{i-2}^{(N)\prime}(x_{N+1}) P_{i-1}^{(N)}(x_{N+1})\right]}$$

$$= 1 + \frac{P_{i-1}^{(N)}(x_{N+1})}{\beta_i^{(N)} P_{i-2}^{(N)}(x_{N+1})} \sum_{j=1}^{i-1} \left(\alpha_j^{(N+1)} - \alpha_j^{(N)}\right)$$

$$i = 2, 3, \ldots, N$$

Lemma 1

$$\left| P_i^{(N+1)}(x_{N+1}) \right| \leq \left| P_i^{(N)}(x_{N+1}) \right|$$

$$\operatorname{sign} P_i^{(N+1)}(x_{N+1}) = \operatorname{sign} P_i^{(N)}(x_{N+1})$$

Proof

This relation follows directly from (10).

Lemma 2.

$$P_i^{(N+1)}(x) \equiv P_i^{(N)}(x) \quad \text{if and only if} \quad P_i^{(N)}(x_{N+1}) = 0$$

Proof

If x_{N+1} is a root of $P_i^{(N)}(x)$ then by (19) $P_i^{(N+1)}(x_{N+1}) = 0$

and by (20) $P_i^{(N+1)}(x) = P_i^{(N)}(x)$

If $P_i^{(N+1)}(x) \equiv P_i^{(N)}(x)$ then (20)

$$P_i^{(N+1)}(x_{N+1})\,\frac{P_i^{(N)}(x)\,P_{i-1}^{(N)}(x_{N+1})-P_{i-1}^{(N)}(x)\,P_i^{(N)}(x_{N+1})}{x-x_{N+1}}=0$$

for all x. At $x=x_{N+1}$ the fraction

$$\lim_{x\to x_{N+1}}\frac{P_i^{(N)}(x)\,P_{i-1}^{(N)}(x_{N+1})-P_{i-1}^{(N)}(x)\,P_i^{(N)}(x_{N+1})}{x-x_{N+1}}=\prod_{k=1}^{i}\beta_k^{(N)}\sum_{j=0}^{i-1}\frac{\left[P_j^{(N)}(x_{N+1})\right]^2}{\prod_{k=1}^{j+1}\beta_k^{(N)}}\neq 0$$

thus $P_i^{(N+1)}(x_{N+1})$ must be zero. And by (19) $P_i^{(N)}(x_{N+1})=0$.

Lemma 3

Let the roots of $P_i^{(N)}(x)$ be

$$\xi_1<\xi_2<\dots<\xi_i$$

Then the roots of $P_i^{(N+1)}(x)$ are located as follows:

1. if $x_{N+1}=\xi_k$ then $P_i^{(N+1)}(x)\equiv P_i^{(N)}(x)$
2. if $x_{N+1}\neq\xi_k$, $k=1,2,\dots,i$, then

   if $x_{N+1}\notin(\xi_k,\xi_{k+1})$ then $P_i^{(N+1)}(x)$ has one root in (ξ_k,ξ_{k+1})

   If $x_{N+1}\in(\xi_k,\xi_{k+1})$ then $P_i^{(N+1)}(x)$ has two roots, one in each of the intervals (ξ_k,x_{N+1}) and (x_{N+1},ξ_{k+1})

Proof

Case 1 is proved by Lemma 2

Case 2 By (20) we have

$$P_i^{(N+1)}(\xi_j)=\frac{P_i^{(N+1)}(x_{N+1})\,P_i^{(N)}(x_{N+1})}{\prod_{j=1}^{i}\beta_j^{(N)}}\quad\frac{P_{i-1}^{(N)}(\xi_j)}{\xi_j-x_{N+1}}\qquad j=1,2,\dots,i$$

and by Lemma 1

$$\operatorname{sign} P_i^{(N+1)}(\xi_j) = \operatorname{sign} \frac{P_{i-1}^{(N)}(\xi_j)}{\xi_j - x_{N+1}}$$

Since the roots of $P_{i-1}^{(N)}(x)$ and $P_i^{(N)}(x)$ are nested we have

$$\operatorname{sign} P_{i-1}^{(N)}(\xi_j) = -\operatorname{sign} P_{i-1}^{(N)}(\xi_{j+1})$$

and

$$\operatorname{sign} P_{i-1}^{(N)}(\xi_j) = -\operatorname{sign} P_i^{(N)}(\xi_j - \varepsilon)$$

with

$$0 < \varepsilon < \xi_j - \xi_{j-1}$$

If $x_{N+1} \notin (\xi_k, \xi_{k+1})$ then

$$\operatorname{sign} P_i^{(N)}(\xi_k) = \operatorname{sign} \frac{P_{i-1}^{(N)}(\xi_k)}{\xi_k - x_{N+1}} = -\operatorname{sign} \frac{P_{i-1}^{(N)}(\xi_{k+1})}{\xi_{k+1} - x_{N+1}}$$

$$= -\operatorname{sign} P_i^{(N+1)}(\xi_{k+1})$$

thus $P_i^{(N+1)}(x)$ has a root in (ξ_k, ξ_{k+1}).

If $x_{N+1} \in (\xi_k, \xi_{k+1})$ then

$$\operatorname{sign} P_i^{(N+1)}(\xi_k) = \operatorname{sign} \frac{P_{i-1}^{(N)}(\xi_k)}{\xi_k - x_{N+1}} = \operatorname{sign} \frac{P_{i-1}^{(N)}(\xi_{k+1})}{\xi_{k+1} - x_{N+1}}$$

$$= \operatorname{sign} P_i^{(N+1)}(\xi_{k+1}) = \operatorname{sign} P_{i-1}^{(N)}(\xi_{k+1}) = -\operatorname{sign} P_i^{(N)}(\xi_{k+1} - \varepsilon)$$

$$= -\operatorname{sign} P_i^{(N)}(x_{N+1}) = -\operatorname{sign} P_i^{(N+1)}(x_{N+1})$$

thus $P_i^{(N+1)}(x)$ has roots in (ξ_k, x_{N+1}) and in (x_{N+1}, ξ_{k+1}).

PART II.

2.1 Polynomial Least Square Approximation

Let $\{P_i^{(N)}(x) \mid i=0,1,\ldots,m\}$ be the orthogonal set of polynomials over $\{x_k \mid k=1,\ldots,N\}$ with weights w_k. Given the set of points $\{(x_k, y_k) \mid k=1,\ldots,N\}$ of a function $y(x)$ then the m-th degree polynomial

(23)
$$f_N(x) = \sum_{i=0}^{m} c_i^{(N)} P_i^{(N)}(x)$$

approximates $y(x)$ such that

(24)
$$E_N = \sum_{k=1}^{N} \left[y_k - f_N(x_k) \right]^2 w_k$$

is minimum for all possible m-th degree polynomials. The coefficients in (23) are

(25)
$$c_i^{(N)} = \frac{\sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k}{\sum_{k=1}^{N} \left[P_i^{(N)}(x_k)\right]^2 w_k} = \frac{\sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k}{\prod_{j=1}^{i+1} \beta_j^{(N)}}$$

(24) may be written as

(26)
$$E_N = \sum_{k=1}^{N} y_k^2 w_k - \sum_{i=0}^{m} \left[(c_i^{(N)})^2 \prod_{j=1}^{i+1} \beta_j^{(N)} \right]$$

Given a new point (x_{N+1}, y_{N+1}) with weight w_{N+1} the new set of orthogonal polynomials $P_i^{(N+1)}(x)$ can be generated by (14), (15). The new least square coefficients are $c_i^{(N+1)}$,

$i = 0, 1, \ldots, m$

$$c_i^{(N+1)} = \frac{\sum_{k=1}^{N+1} y_k P_i^{(N+1)}(x_k)\, w_k}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}}$$

Then

$$c_i^{(N+1)} - c_i^{(N)} = \frac{\sum_{k=1}^{N+1} y_k P_i^{(N+1)}(x_k)\, w_k}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}} - \frac{\sum_{k=1}^{N} y_k P_i^{(N)}(x_k)\, w_k}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}}$$

by (18)

$$c_i^{(N+1)} - c_i^{(N)} = \frac{1}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}} \Big[\sum_{k=1}^{N} y_k P_i^{(N+1)}(x_k) w_k + y_{N+1} P_i^{(N+1)}(x_{N+1}) w_{N+1} - (1 + \varepsilon_i \Delta_i) \sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k \Big]$$

by (13)

$$= \frac{1}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}} \Big[\sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k - \varepsilon_i \sum_{j=0}^{i-1} \Delta_j \sum_{k=1}^{N} y_k P_j^{(N)}(x_k) w_k + y_{N+1} P_i^{(N+1)}(x_{N+1}) w_{N+1} - \sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k - \varepsilon_i \Delta_i \sum_{k=1}^{N} y_k P_i^{(N)}(x_k) w_k \Big]$$

$$= \frac{1}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}} \Big[y_{N+1} P_i^{(N+1)}(x_{N+1}) w_{N+1} - \varepsilon_i \sum_{j=0}^{i} \Delta_j \sum_{k=1}^{N} y_k P_j^{(N)}(x_k) w_k \Big]$$

Finally by (11) and (25)

(27)

$$c_i^{(N+1)} - c_i^{(N)} = \frac{P_i^{(N+1)}(x_{N+1}) w_{N+1}}{\prod_{j=1}^{i+1} \beta_j^{(N+1)}} \Big[y_{N+1} - \sum_{j=0}^{i} c_j P_j^{(N)}(x_{N+1}) \Big]$$

with the new error sum

$$E_{N+1} = \sum_{k=1}^{N+1} y_k^2 w_k - \sum_{i=0}^{m} \left[(c_i^{(N+1)})^2 \prod_{j=1}^{i+1} \beta_j^{(N+1)} \right] \tag{28}$$

$$= E_N + y_{N+1}^2 w_{N+1} + \sum_{i=0}^{m} \left[(c_i^{(N)})^2 \prod_{j=1}^{i+1} \beta_j^{(N)} - (c_i^{(N+1)})^2 \prod_{j=1}^{i+1} \beta_j^{(N+1)} \right]$$

2.2 Numerical example

For a large number of points, the point-wise generation of the orthogonal polynomials and least square coefficients gave very good numerical results.

The following table contains the results of 3 methods with N = 110 points,

Method 1 calculates the orthogonal polynomials for N points by (3), (4) and (5) and the least square coefficients by (25). All operations are in single precision.

Method 2 calculates the same way as Method 1. but it uses double precision arithmetic for calculation of the sums required in (4), (5) and (25).

Method 3 uses pointwise generation as described in Part I and Part II with single precision arithmetic.

TABLE

	Method 1	Method 2	Method 3
α_1	.5045 4535	.5045 4545	.5045 4535
α_2	.5042 7809	.5042 7781	.5042 7802
α_3	.5036 6794	.5036 6822	.5036 6831
α_4	.5026 3852	.5026 3823	.5026 3834
α_5	.5012 5722	.5012 5753	.5012 5769
β_2	.8499 4689 10^{-1}	.8499 4642 10^{-1}	.8499 4578 10^{-1}
β_3	.6827 9868 10^{-1}	.6827 9986 10^{-1}	.6827 9930 10^{-1}
β_4	.6621 7871 10^{-1}	.6621 7729 10^{-1}	.6621 7695 10^{-1}
β_5	.6567 7349 10^{-1}	.6567 7495 10^{-1}	.6567 7424 10^{-1}
c_0	.1727 5155 10	.1727 5160 10	.1727 5148 .10
c_1	.1698 7021 .10	.1698 6992 .10	.1698 7004 .10
c_2	.8430 3060	.8430 4804	.8430 4877
c_3	.2797 1388	.2797 8422	.2797 8262
c_4	.7031 3640 $.10^{-1}$	.6972 7859 $.10^{-1}$	.6972 5276 $.10^{-1}$
c_5	.1509 3968 $.10^{-1}$	.1391 0134 $.10^{-1}$	.1388 1274 $.10^{-1}$

APPENDIX

Interpolation by Orthogonal Polynomials

When the number of points is less than the degree of the fitting polynomial, we have an interpolation problem. Furthermore formulae (19), (20), (21), (22) and (27) define the new recursive coefficients and polynomials when the previous ones were given. These formulae may be extended the following way:

For N points, the values of $\alpha_i^{(N)}, \beta_i^{(N)}; \; i=1,2,\ldots,N$ are given. With the given new point (x_{N+1}, y_{N+1}), the new coefficients $\alpha_i^{(N+1)}, \beta_i^{(N+1)}$ can be calculated by (19), (21) and (22) for i=1, 2, ..., N. The last coefficients can be calculated from

(29) $$\alpha_{N+1}^{(N+1)} = x_{N+1} - \sum_{j=1}^{N} (\alpha_j^{(N+1)} - \alpha_j^{(N)})$$

(30) $$\beta_{N+1}^{(N+1)} = \frac{P_N^{(N+1)}(x_{N+1})\, P_N^{(N}(x_{N+1})\, w_{N+1}}{\prod_{k=1}^{N} \beta_k^{(N+1)}}$$

using (19) for $P_N^{(N+1)}(x_{N+1})$.
The least square coefficients, $c_i^{(N+1)}$, $i=0,1,\ldots,$ N can be calculated from (27) and

(31) $$C_{N+1}^{(N+1)} = 0$$

The least square coefficients are now interpolating coefficients.