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OPTIMIZATION OF ADIABATIC MAGNETIC MIRROR FIELDS FOR CONTROLLED FUSION RESEARCH

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SUMMARY

This report formulates guidelines for the design of magnetic mirror machines used in controlled fusion research apparatus. The following optimization problem is considered: a plasma of density n_0 and kinetic temperature V_0, which is isotropic in velocity space, is released at the midplane of a magnetic mirror machine. The dimensions, field strength, and mirror ratio of the magnetic mirrors are then adjusted in such a way as to adiabatically confine the largest possible number of particles per dollar of investment in the magnetic field coils. In keeping with the current state of controlled fusion research, it is assumed that the numerical magnitudes of the plasma density and kinetic temperature are dependent on the outcome of the experiment and are not independently adjustable parameters which the experimenter can vary at will. Results of a recent experimental investigation are used to provide criteria that will assure adiabatic confinement of the plasma ions. Scaling relations, derived for both conventional and superconducting coils, relate the cost of the coils to the coil dimensions and the magnetic field strength produced by the coil. Such a scaling law for superconducting solenoids was derived from data on 98 superconducting coils that have been sold commercially.

It is shown that under many conditions, including conventional water-cooled copper coils, there exists no optimum design, and the total number of adiabatically confined particles is just proportional to the amount of capital one can invest in the magnetic field coils. For superconducting coils, however, there exist optimum values of the magnetic field strength, coil dimensions, and mirror ratio for which the number of adiabatically confined particles per dollar of investment in the coils will be a maximum. For superconducting coils, the optimum mirror ratios lie between the high values $1.00 \leq B_{min}/B_{max} \leq 0.95$. This optimum mirror ratio is relatively insensitive to the scaling law parameters for magnet cost, provided that superconducting coils are under discussion. This high optimum mirror ratio, and the other guidelines developed in this report, are not qualitatively dependent on the cost of superconducting coils at any given time.

INTRODUCTION

Until the present time, it has not been customary to optimize systematically the mirror ratio, the maximum magnetic field strength, and the dimensions of a proposed magnetic mirror machine intended for controlled fusion research. This state of affairs has come about for several reasons:

(1) There has been no generally accepted figure of merit for mirror machines, the maximization or minimization of which is a desirable goal of the design process.

(2) It has been widely realized that nonadiabatic particle losses, which result from an escape cone in velocity space larger than would otherwise exist (refs. 1 and 2), may be avoided by designing the apparatus with mirror ratios $R_m = B_{min}/B_{max}$ near unity, or with strong magnetic fields, or with small magnetic field gradients. Until recently, however, there has been no quantitative relation among these variables to tell the apparatus designer exactly where nonadiabatic losses will start to occur. Figure 1 shows the adiabatic and nonadiabatic loss cones in velocity space. The particles whose velocity vectors lie between these two loss cones are those whose loss is attributable to nonadiabatic effects.

(3) If an economic constraint is to be applied to the design problem, the cost of the magnetic field coils must be expressed as a function of the magnetic field strength and

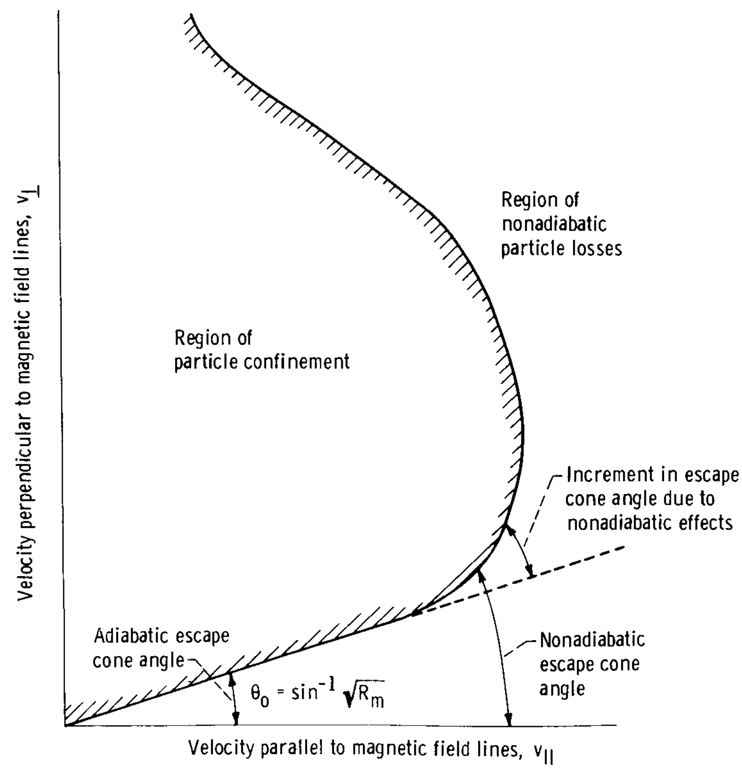


Figure 1. - Regions of adiabatic and nonadiabatic particle losses in velocity space.

geometry. Such scaling laws have not been available for superconducting coils.

In this report, a figure of merit is proposed, the maximization of which is consistent with the requirements of plasma stability, the minimization of particle losses, and the maximization of the product of plasma density and kinetic temperature. A constraint on this figure of merit is given by the results of recent experimental measurements on nonadiabatic particle losses, which are summarized in a formula that predicts the mirror ratio, apparatus dimension, and field strength for which particles begin to be nonadiabatically lost (ref. 2). The second constraint is provided by a scaling law for the cost of superconducting and conventional magnets.

The optimization process is treated as a calculus of variations problem in which the figure of merit is maximized, subject to the constraints imposed by the coil cost and the requirement of adiabatic particle confinement. This procedure yields general expressions for the optimum mirror ratio, magnetic field, and apparatus dimensions as functions of the fixed parameters of the constraint equations. The best available estimates of the values of these parameters are substituted into the general expressions, and the optimum values of mirror ratio, dimensions, and magnetic field are exhibited for mirror machines designed under current conditions. The plasma density and temperature are left as floating variables to be determined by the characteristics or the outcome of the experiment.

The author would like to acknowledge the cooperation of the following organizations and individuals, who provided unpublished data on the dimensions and cost of superconducting magnetic field coils: The Westinghouse Electric Company; A. J. Donius, of the Magnion Corp.; N. S. Freedman of the Radio Corporation of America; Z. J. J. Stekly of the AVCO Corporation; J. F. Howlett of the Cryonetics Corporation; and D. C. Freeman, Jr., of the Union Carbide Corporation. At the request of certain of the organizations which supplied information, the names of the manufacturers are not associated with the specific coils listed in tables I to III.

FIGURE OF MERIT FOR CONTROLLED FUSION RESEARCH APPARATUS

The present state of controlled fusion research is such that a figure of merit can be only empirical at this time. However, a careful study of existing mirror machines and research objectives has led to the conclusion that the following factors should be taken into account in any proposed figure of merit:

(1) The figure of merit should be proportional both to the plasma density n and to the plasma kinetic temperature V ($kT = eV$), since one of the basic objectives of controlled fusion research is to raise the energy density to a level high enough so that controlled fusion reactions take place.

(2) The ions that constitute the fuel of the thermonuclear reaction must, on the average, be confined long enough to react. The figure of merit should, therefore, be proportional to the average characteristic confinement time of an individual ion in cases where this time is equal to or less than the mean free time between fusion reactions at the existing plasma density and temperature.

(3) The figure of merit should be proportional to the volume of the plasma, since large dimensions promote adiabatic confinement (refs. 1 and 2) and reduce the severity of the flute instabilities that are driven by magnetic field gradients (ref. 3). In addition, large dimensions reduce the surface-to-volume ratio. This reduction makes plasma contamination less of a problem and makes the effect of inserting diagnostic probes relatively less disturbing to the bulk of the plasma.

(4) Both theoretical (ref. 4) and experimental (ref. 5) investigations have shown that departures from a Maxwellian distribution along a radius vector in velocity space promote the growth of plasma instabilities. Such departures should be avoided by choosing a suitable injection method or by promoting equilibration among the injected ions. The figure of merit may therefore be based on the assumption of a Maxwellian energy distribution.

(5) Theoretical (ref. 6) and experimental (ref. 7) investigations have shown that angular anisotropies of the velocity distribution in velocity space promote the growth of microinstabilities. Such angular anisotropies are to be avoided for other reasons. The anisotropy characteristic of all magnetic bottle geometries is an escape cone through which particles are lost by collisional scattering, similar to that illustrated in figure 1. The smaller is the escape cone, the more nearly isotropic is the velocity distribution and the smaller will be the losses due to collisional scattering. The figure of merit should, therefore, contain some factor proportional to the angular isotropy of the velocity distribution in velocity space.

The proposed figure of merit may be written as

$$\Omega = \int_0^{\tau} \int_0^{\psi} \int_0^{\infty} \int_0^{2\pi} \int_{\theta_o}^{\pi-\theta_o} V f(v) v^2 \sin\theta \, d\theta \, d\varphi \, dv \, d\psi \, dt \tag{1}$$

where τ is the average e-folding decay constant of the plasma ion density. ψ is the volume of the plasma, v, φ, and θ are the spherical coordinates of velocity space, f(v) is the time-dependent velocity distribution function of the ions in velocity and configuration space, and V is the average particle energy in electron volts. (All symbols are defined in the appendix.)

In equation (1), it has been tacitly assumed that the only anisotropy in velocity space is an adiabatic, axisymmetric escape cone of half-angle θ_o (fig. 1), whose axis is parallel to the magnetic field lines in the plasma.

It is assumed that the plasma is in the steady state and that the distribution function is Maxwellian along a radius vector in velocity space. The distribution function is then

$$f(v, \varphi, \theta, t) = f(v) = n\left(\frac{m}{2\pi eV}\right)^{3/2} \exp\left(-\frac{mv^2}{2eV}\right) \qquad \theta \geq \theta_o \tag{2a}$$

and

$$f(v, \varphi, \theta, t) = 0 \qquad \theta \leq \theta_o \tag{2b}$$

so that the loss cone is entirely free of particles. If equation (2) is substituted into equation (1), the figure of merit may be written as

$$\Omega = \tau \int_0^{\psi} \int_0^{\infty} \int_0^{2\pi} \int_{\theta_o}^{\pi-\theta_o} nV\left(\frac{m}{2\pi eV}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2eV}\right) \sin\theta \, d\theta \, d\varphi \, dv \, d\psi \tag{3}$$

where the integration over confinement time has been performed. Integrating over the coordinates in velocity space gives

$$\Omega = \tau \int_0^{\psi} nV \cos\theta_o \, d\psi \tag{4}$$

It should be noted that the results of this analysis are not affected by the functional dependence of the figure of merit on plasma density and temperature. The energy density nV has been used in equation (4) for the sake of concreteness, thus making the figure of merit proportional to the total plasma energy confined in phase space during a mean confinement time. The thermonuclear reaction rate $n^2 \langle \sigma v \rangle$ could also be used in equation (4) without changing the conclusions of this study. Under these conditions, the figure of merit would then be equal to the total thermonuclear energy released by the plasma during a mean confinement time.

It has been shown (ref. 8) that the loss cone angle in velocity space for a magnetic mirror with maximum field B_{max} is given by

$$\cos\theta_o = \sqrt{1 - \frac{B}{B_{max}}} \tag{5}$$

where B is the local value of the magnetic field. If equation (5) is substituted into equation (4), the figure of merit may be written as

$$\Omega = \tau \int_{\psi} nV \sqrt{1 - \frac{B}{B_{max}}}\, d\psi \qquad (6)$$

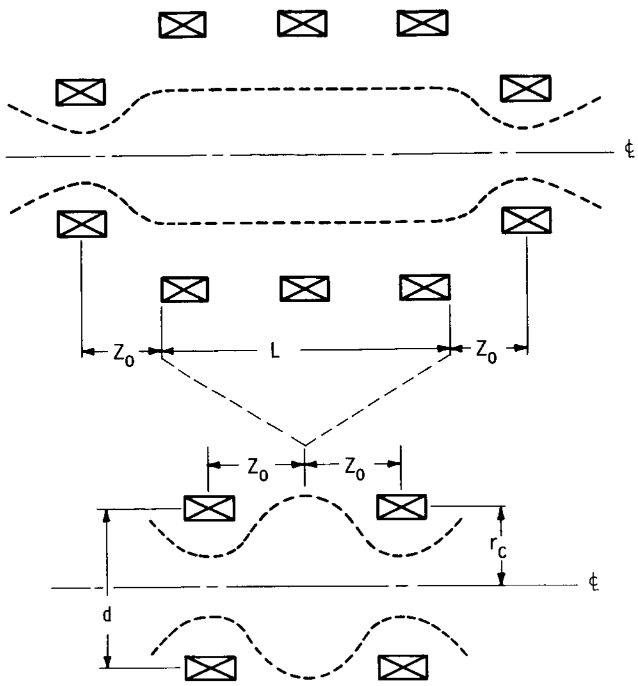


Figure 2. - Assumed geometry of magnetic field.

The geometrical configuration of the plasma has been assumed to be that of the upper half of figure 2 in which a long, cylindrical, uniform-field center section is bounded on both ends by magnetic mirrors. As indicated in figure 2, the uniform field center section will be ignored for two reasons: the uniform field will assure adiabatic confinement of the particles in this region, and, in addition, the optimum placement of the center section coils cannot be found without a long and detailed design study for each geometry considered. The plasma volume assumed in the subsequent discussion is shown in the lower half of figure 2. The limits of integration on equation (6) are, therefore,

$$\Omega = \tau \int_0^{r_p} \int_0^{2\pi} \int_0^{Z_o} nV \sqrt{1 - \frac{B}{B_{max}}}\, r\, dz\, d\theta'\, dr \qquad (7)$$

where r_p is the mean plasma radius, and Z_o is the axial distance from the point where B is equal to B_{min} to the point in the mirror throat where B is equal to B_{max}.

For axisymmetric mirror systems and for nonaxisymmetric magnetic wells of the Ioffe type, the variation of the total magnetic field along the axis is given approximately by

$$B \approx \frac{1}{2} B_{max}\left[(1 + R_m) - (1 - R_m)\cos\frac{\pi Z}{Z_o}\right] \qquad (8)$$

where R_m is B_{min}/B_{max}.

After this definition of R_m is substituted into equation (7) and η, α, and α_o are defined as

$$\eta \equiv \frac{\pi Z}{Z_o} \tag{9a}$$

$$\alpha \equiv \frac{\pi r}{r_p} \tag{9b}$$

$$\alpha_o \equiv \frac{\pi r}{Z_o} \tag{9c}$$

the figure of merit may be written as

$$\Omega = \frac{V_o n_o \pi r_c^2 Z_o}{\pi^3} \left(\frac{1 - R_m}{2} \right)^{1/2} K^2 \int_0^{\pi} \int_0^{2\pi} \int_0^{\pi} \frac{nV}{n_o V_o} \alpha \sqrt{1 + \cos \eta} \; d\alpha \, d\theta' \, d\eta \tag{10}$$

where V_o is the ion temperature expressed in electron volts on the axis at the midplane of the magnetic bottle, and n_o is a fictitious density equal to the density of a distribution isotropic in velocity space located on the axis at the midplane of the magnetic bottle. Maximizing the figure of merit can then be accomplished by minimizing the escape cone angle. This would then increase the fraction of an initially isotropic velocity distribution which would be trapped between the mirrors. The actual density of particles is equal to $n_o \sqrt{1 - R_m}$. The parameter K is the ratio of the mean plasma to the mean coil radius.

A "geometry factor" is then defined by

$$\xi \equiv \frac{K^2}{\pi^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\pi} \frac{nV}{n_o V_o} \sqrt{1 + \cos \eta} \; \alpha \, d\alpha \, d\theta' \, d\eta \tag{11}$$

whose precise value depends on the detailed way in which particle density, particle kinetic temperature, and magnetic field depend on the dimensionless spatial coordinates α, θ', and η.

The geometry factor ξ is of order unity and, for the special case in which the energy density and the magnetic field are independent of position, ξ has the value of $2\sqrt{2}\ K^2$. This factor must be known so that the figure of merit of any two specific devices can be compared. It is sufficient to write the figure of merit as

$$\Omega = n_o V_o \pi r_c^2 Z_o \left(\frac{1 - R_m}{2}\right)^{1/2} \xi \tag{12}$$

in the present study, since only its extremal value is of interest. The optimization of equation (12) requires a knowledge only of the functional dependence of the figure of merit on r_c, Z_o, R_m, and τ and not the absolute value of Ω. Equation (12) contains the desired product of the confinement time with the velocity space volume and physical space volume. The term $\sqrt{1 - R_m}$ in equation (12) is a measure of the size of the confinement region in velocity space. As discussed on page 4, this term should be made as large as possible to minimize losses due to collisional scattering into the escape cone and to reduce the severity of the instabilities that are driven by angular anisotropies of the velocity distribution function in velocity space.

DESIGN CONSTRAINTS ON CONTROLLED FUSION RESEARCH DEVICES

Adiabatic Confinement

The degree to which conditions are adiabatic may be measured by the value of the adiabatic parameter ϵ, which is defined as

$$\epsilon \equiv \frac{mv}{eB_{av}Z_o} = \frac{1}{B_{av}Z_o}\sqrt{\frac{2mV}{e}} \tag{13}$$

where B_{av} is equal to $(1/2)(B_{min} + B_{max})$. If ϵ is small, the ratio of the radius of gyration to the apparatus dimensions is small, and conditions are adiabatic. If ϵ is large, the conditions are nonadiabatic, and a magnetic mirror will not be fully effective in reflecting particles because of the enlarged escape cone.

There are several reasons why ϵ should be as large as possible, but not so large that nonadiabatic losses occur. For a given particle energy, an apparatus with large ϵ will be less expensive than an apparatus with small ϵ, since the cost is proportional to both size Z_o and magnetic field B_{av}. If, however, ϵ is made too large, nonadiabatic losses will occur, and the apparatus will not confine particles effectively. The best economic compromise, therefore, is to make ϵ as large as possible without making it so large that nonadiabatic losses occur.

Stability considerations also imply a compromise value of ϵ. It has been shown by Kuo, Murphy, Petravić, and Sweetman (ref. 9) that large values of ϵ promote plasma stability by the mechanism of "finite Larmor radius stabilization"; therefore, ϵ should be as large as possible. On the other hand, ϵ should not be so large that nonadiabatic losses occur, since the additional anisotropy in velocity space associated with such losses will promote the anisotropy-driven instabilities discussed by Harris (ref. 6). The best compromise value of ϵ, again, is to make ϵ as large as possible without making it so large that nonadiabatic losses occur.

A series of experiments has been performed recently to measure the critical value of ϵ, at a given mirror ratio and radius in the magnetic field, above which nonadiabatic losses will occur (ref. 2). It was found that the critical value of ϵ for a particular mirror ratio R_m is given by

$$R_m = \left(\frac{\epsilon}{\epsilon_A}\right)^A \tag{14}$$

In the axisymmetric geometry studied in the series of experiments in reference 2, the parameters A and ϵ_A for single interactions with a magnetic mirror were given by

$$A = 0.388\ e^{-0.167\alpha_o} \tag{15a}$$

and

$$\epsilon_A = 0.348\ e^{0.075\alpha_o} \tag{15b}$$

where α_o is the dimensionless radius defined by equation (9c). These experiments covered the range of parameters $0 \leq \alpha_o \leq 1.75$, $0.003 \leq \epsilon \leq 0.3$, and $0.175 \leq R_m \leq 0.95$.

In order to assure adiabatic confinement in axisymmetric magnetic mirror machines, therefore, the relation between the axial mirror ratio and the adiabatic parameter should be

$$R_m \geq \left(\frac{\epsilon}{0.348\ e^{0.075\alpha}}\right)^{0.388e^{-0.167\alpha_o}} \tag{16}$$

where the inequality specifies the region of adiabatic confinement, and the equality represents the experimentally derived boundary of the nonadiabatic region.

It was found from an auxiliary series of numerical computations (ref. 2) that, if all other conditions are held constant, the addition of multipolar (Ioffe) windings to an axisymmetric mirror field will make the conditions within the apparatus volume less adiabatic. It was also found that, if the apparatus is designed to be adiabatic at a given radius, it will be adiabatic for all lesser radii. Equations (15) and (16) were obtained for a single interaction of a particle with an axisymmetric magnetic mirror. In designing magnetic bottles that have superimposed multipolar windings and in which multiple reflections occur, equation (16) must be treated as though it puts an upper bound on ϵ and/or a lower bound on R_m, the exceeding of which will certainly result in nonadiabatic particle losses at the radius under consideration.

Scaling Laws for Cost of Superconducting Solenoids

Most of the conventional steady-state magnet facilities currently in use in controlled fusion research consist of water-cooled copper solenoids, which absorb large amounts of direct current electrical power. Recent progress in the art of fabricating superconducting solenoids has made it clear that such coils are not only substantially less expensive to build and operate than conventional magnet facilities, but they also offer many advantages of convenience and flexibility over conventional water-cooled copper coils (ref. 10). In reference 11, it is shown that superconducting solenoids are less expensive to build and operate than conventional water-cooled copper solenoids for all conditions of interest for controlled fusion applications (see figs. 3 and 4 of ref. 11). Progress since this reference was published make superconducting solenoids even more attractive.

In both B_{max} = 25 kg (ref. 10) and B_{max} = 67 kg (private communication with Dr. C. Laverick) superconducting magnet facilities, the capital costs of the superconducting magnet facility were less than half that of an equivalent conventional magnet facility, and the steady-state running costs were no higher than those of a conventional facility. For these reasons, superconducting magnet facilities will be discussed more extensively than conventional magnets in this report.

In order to maximize the figure of merit subject to an economic constraint, it is necessary to have a scaling law for the cost of a single coil as a function, for example, of B_{max}, the maximum magnetic field on the coil axis, and d, the mean diameter of the coil. Information about the dimensions, the maximum magnetic field, and the cost of 98 superconducting coils was obtained from commercial suppliers, and a scaling law was derived by obtaining a best-fitting curve to this data.

The inner radius, outer radius, mean diameter, length of winding, coil volume, maximum magnetic field on the coil axis, and cost of 98 coils are listed in tables I, II, and III. The companies and individuals that supplied this data have been mentioned in the INTRODUCTION. Table I contains data from 6 coils wound with niobium-tin superconductors, table II contains 85 single, plain coils wound with niobium zirconium, and table III lists 7 single niobium-zirconium coils enclosed in cryogenic Dewars.

An attempt was made to fit the data in the tables to three candidate scaling laws,

$$\$ = \$_0 + \$_1 B_{max}^{\gamma_1} d^{\gamma_2} \tag{17}$$

$$\$ = \$_0 + \$_1 B_{max}^{\gamma_1} V_s^{\gamma_2} \tag{18}$$

$$\$ = \$_0 + \$_1 B_{max}^{\gamma_1} + \$_2 d^{\gamma_2} \tag{19}$$

where B_{max} is the maximum magnetic field strength on the coil axis, and V_s is the volume of the superconducting windings. The data in tables I, II, and III were used as input to a least-squares curve-fitting computer program that calculated the values of $\$_0$, $\$_1$, and $\$_2$ for given values of γ_1 and γ_2. The computer program also calculated the mean error

$$\Delta_1 \$ = \frac{\sum_{k=1}^{N} \left| \$_k - \$_{ak} \right|}{N} \tag{20}$$

and the mean square error,

$$\Delta_0 \$ = \sqrt{\frac{\sum_{k=1}^{N} \left| \$_k - \$_{ak} \right|^2}{N}} \tag{21}$$

TABLE I. - PROPERTIES OF HIGH-FIELD NIOBIUM-TIN COILS

Coil	Inner radius, r_1, m	Outer radius, r_2, m	Mean diameter, d, m	Coil volume, V_s, m^3	Axial length of winding, L, m	Maximum field on axis, B_{max}, W/m^2	Actual cost, $\$_A$
1	0.0159	0.0858	0.102	0.0022	0.098	11.0	30 000
2	.042	.105	.147	.0035	.121	8.0	100 000
3	.0762	.242	.318	.0484	.292	15.0	330 000
4	.254	.406	.66	.0322	.102	4.0	150 025
5	.254	.406	.66	.0322	.102	4.0	167 700
6	.254	.406	.66	.0322	.102	4.0	125 500

TABLE II. - PROPERTIES OF BARE NIOBIUM ZIRCONIUM COILS

Coil	Inner radius, r_1, m	Outer radius, r_2, m	Mean diameter, d, m	Coil volume, V_s, m	Axial length of winding, L, m	Maximum field on axis, B_{max}, W/m^2	Actual cost, $\$_A$	Coil	Inner radius, r_1, m	Outer radius, r_2, m	Mean diameter, d, m	Coil volume, V_s, m	Axial length of winding, L, m	Maximum field on axis, B_{max}, W/m^2	Actual cost, $\$_A$
1	0.032	0.089	0.121	0.00145	0.067	3.2	6 000	46	0.038	0.060	0.098	0.00148	0.216	2.50	7 100
2	.076	.146	.222	.00294	.060	1.65	7 980	47	.0064	.035	.041	.00014	.038	3.00	1 365
3	.065	.158	.223	.00367	.056	1.5	10 700	48	.0064	.025	.032	.00011	.060	↓	1 590
4	.138	.214	.352	.00360	.043	.56	18 500	49	.013	.035	.048	.00023	.070	↓	1 955
5	.026	.051	.077	.00069	.114	3.0	4 000	50	.013	.035	.048	.00030	.089	↓	2 745
6	.137	.304	.441	.0706	.305	3.2	125 000	51	.014	.033	.048	.00044	.155	3.00	2 850
7	.0064	.029	.035	.00011	.045	3.0	1 350	52	.025	.051	.076	.00033	.054	↓	3 980
8	↓	.032	.038	.00023	.076	3.0	1 950	53	.025	.045	.071	.00116	.264	↓	7 095
9	↓	.045	.051	.00047	.076	5.0	3 950	54	.052	.079	.131	.00274	.242	↓	18 450
10	↓	.045	.051	.00074	.121	5.0	4 500	55	.019	.045	.064	.00100	.190	3.50	5 680
11	.0064	.057	.063	.00077	.076	6.0	4 900	56	.013	.048	.061	.00107	.159	4.00	6 150
12	.011	.070	.081	.00238	.159	6.0	12 500	57	.013	.045	.057	.00082	.143	4.00	4 965
13	.013	.038	.051	.00041	.102	3.0	2 700	58	.0064	.043	.049	.00027	.048	5.00	2 560
14	.013	.032	.044	.00034	.127	3.0	3 300	59	.0064	.038	.044	.00034	.076	5.00	2 830
15	.013	.046	.059	.00078	.127	4.0	5 650	60	.011	.045	.056	.00088	.146	5.00	4 810
16	.013	.076	.089	.00191	.108	5.5	6 700	61	.013	.057	.070	.00080	.083	5.00	4 535
17	↓	.057	.070	.00099	.102	6.0	7 100	62	↓	.061	.074	.00100	.089	↓	5 085
18	↓	.054	.067	.00132	.153	5.5	9 500	63	↓	.049	.062	.00077	.108	↓	4 660
19	↓	.054	.067	.00176	.203	5.5	12 500	64	↓	.051	.064	.00106	.140	↓	5 665
20	↓	.078	.090	.00223	.121	8.0	16 300	65	↓	.051	.064	.00135	.178	↓	7 200
21	.013	.076	.089	.00248	.140	8.0	18 890	66	.013	.046	.059	.00097	.159	5.00	6 770
22	.016	.029	.045	.00017	.095	1.5	1 100	67	.014	.059	.073	.00064	.064	↓	4 710
23	↓	.067	.083	.00167	.127	6.0	11 300	68	↓	.059	.073	.00077	.076	↓	4 895
24	↓	.067	.083	.00234	.178	6.0	14 500	69	↓	.051	.066	.00063	.083	↓	6 165
25	↓	.067	.083	.00276	.210	6.0	18 100	70	↓	.054	.068	.00087	.102	↓	6 165
26	.027	.041	.068	.00016	.051	.20	765	71	.014	.051	.065	.00105	.140	5.00	6 950
27	.0254	.039	.065	.00021	.076	.50	955	72	.018	.052	.069	.00113	.152	↓	7 550
28	.095	.108	.203	.00104	.127	.50	3 010	73	.019	.056	.075	.00103	.121	↓	8 265
29	.0064	.014	.021	.00002	.045	.80	805	74	.025	.067	.092	.00196	.165	↓	10 940
30	.014	.029	.043	.00013	.070	1.00	955	75	.025	.068	.094	.00140	.111	↓	8 250
31	.0222	.038	.060	.00021	.070	1.00	1 175	76	.013	.056	.068	.00134	.146	5.50	7 715
32	.025	.044	.070	.00106	.254	1.00	2 390	77	.013	.052	.065	.00184	.228	↓	9 485
33	.018	.035	.053	.00043	.152	1.50	2 445	78	.014	.060	.075	.00167	.155	↓	9 920
34	.018	.035	.053	.00055	.194	1.50	2 745	79	.014	.054	.068	.00087	.102	↓	6 715
35	.020	.033	.053	.00011	.051	1.50	1 455	80	.018	.060	.078	.00253	.241	↓	12 090
36	.025	.038	.064	.00026	.102	1.50	2 560	81	.018	.060	.078	.00160	.152	5.50	10 030
37	.025	.038	.064	.00051	.203	↓	3 055	82	.0064	.048	.054	.00058	.084	6.0	5 290
38	.032	.048	.079	.00079	.200	↓	3 895	83	.019	.067	.086	.00204	.159	↓	11 530
39	.032	.044	.076	.00069	.228	↓	3 895	84	.025	.073	.098	.00252	.171	↓	14 535
40	.013	.032	.044	.00012	.045	2.00	1 450	85	.025	.083	.108	.00215	.111	↓	12 445
41	.013	.032	.044	.00074	.280	2.00	3 480								
42	.025	.041	.066	.00063	.203	2.00	3 576								
43	.038	.060	.098	.00061	.089	2.30	4 140								
44	.027	.048	.075	.00133	.267	2.50	6 910								
45	.029	.046	.075	.00099	.242	2.50	6 625								

TABLE III. - PROPERTIES OF NIOBIUM-ZIRCONIUM COILS IN DEWARS

Coil	Inner radius, r_1, m	Outer radius, r_2, m	Mean diameter, d, m	Coil volume, V_s, m^3	Axial length of winding, L, m	Maximum field on axis, B_{max}, W/m^2	Actual cost, $\$_A$
1	0.035	0.059	0.094	0.00105	0.152	2.0	8 300
2	.038	.076	.115	.00208	.152	5.0	21 000
3	.029	.070	.099	.00391	.305	5.0	37 000
4	.070	.134	.204	.00287	.070	2.5	26 880
5	.083	.147	.229	.00322	.070	2.5	30 650
6	.095	.159	.254	.00357	.070	2.5	34 335
7	.095	.159	.254	.00357	.070	2.5	32 920

TABLE IV. - BEST-FITTING PARAMETERS CALCULATED FROM DATA OF TABLES I TO III

(a) For relation $\$ = \$_0 + \$_1 B_{max}^{\gamma_1} V_s^{\gamma_2}$

Table	Number of coils	Scaling law parameter		Scaling law exponent		Root mean square error, $\Delta_0\$$	Mean error, $\Delta_1\$$	Mean coil cost, \$	
		$\$_0$	$\$_1$	γ_1	γ_2				
I	6	8 714	5.005×10^5	0.45	0.55	22 200	17 920	150 537	
II	85	-900	5.976×10^5	.200	.675	1 630	1 029	7 729	
III	7	-13 500	1.598×10^6	0.000	.625	613	548	27 297	
I to III	98	-4 452	5.392×10^5	.550	.650	11 520	6 272	17 870	

(b) For relation $\$ = \$_0 + \$_1 B_{max}^{\gamma_1} d^{\gamma_2}$ of equation (17)

Table	Number of coils	Scaling law parameter		Scaling law exponent		Root mean square error, $\Delta_0\$$	Mean error, $\Delta_1\$$	Mean coil cost, \$	Ratio of scaling law exponents, $\delta = \gamma_2/\gamma_1$
		$\$_0$	$\$_1$	γ_1	γ_2				
I	6	33 840	2.426×10^4	1.60	1.60	23 040	18 620	150 537	1.00
II	85	1 655	1.567×10^5	1.20	2.00	2 343	1 601	7 729	1.67
I to III	98	-1 400	4.897×10^4	1.250	1.300	9 581	5 025	17 870	1.04

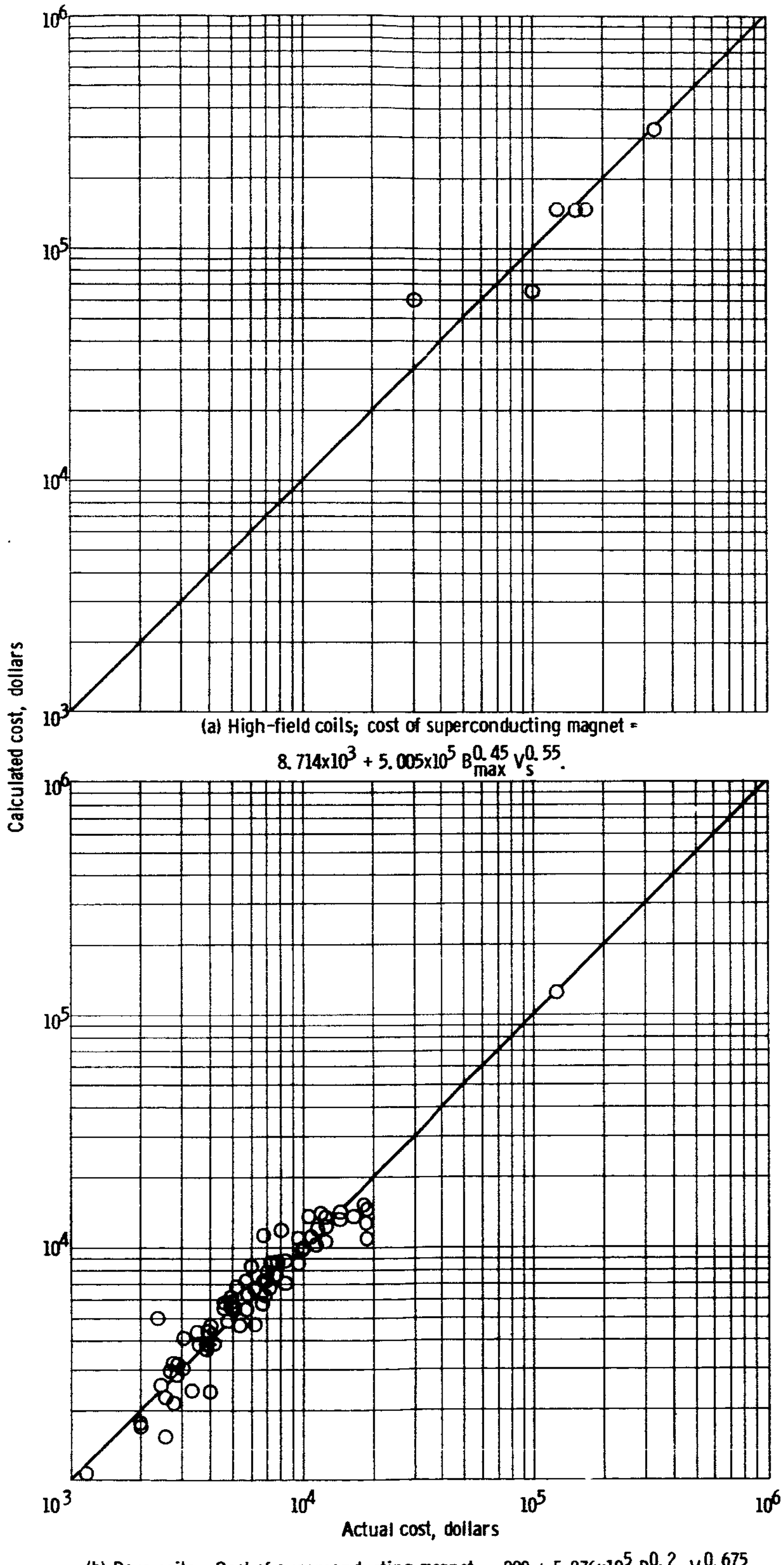
where $\$_k$ is the cost of the k^{th} coil calculated from the scaling laws, and $\$_{ak}$ is the actual cost of the k^{th} coil from the tables. The exponents γ_1 and γ_2 were systematically varied to find the values of $\$_0$, $\$_1$, γ_1, and γ_2 that gave the lowest possible mean square error for a particular scaling law. The scaling law of equation (19) gave a far worse fit than the other two under all situations, and will not be discussed further in this report.

The values of $\$_0$, $\$_1$, γ_1, and γ_2 that give a best fit to equation (18) are shown in table IV(a). The four rows of this table correspond to the data of tables I, II, and III, individually and combined. The mean error and the mean square error of the best fit to these sets of data are shown, as are the average coil costs for each group. Table IV(b) shows the same parameters for equation (17). The computer program would not converge to a best-fitting solution for the seven coils of table III in this case.

The parameters that give a best fit to equation (18) were used to obtain calculated costs for the coils of tables I to III. These calculated costs have been compared with the actual costs of the various classes of superconducting coils in figure 3. The parameters from table IV(b) were substituted into equation (17) to obtain calculated costs of the various classes of coils listed in tables I to III. These calculated costs are compared with the actual costs in figure 4. In this figure, the scaling laws would give a perfectly accurate estimate of the coil cost if the points lay on the diagonal line. In figures 3 and 4, it must be remembered that the least-squares curve-fitting procedure used tends to force a fit to the larger, more expensive coils rather than the larger number of less expensive coils. The dispersion of the calculated costs from the actual costs, even for the best-fitting scaling laws, results from the pressure of many marketplace considerations, which have little to do with the size of the coil. The magnitude of this dispersion may be estimated by comparing the mean error with the mean coil costs in table IV.

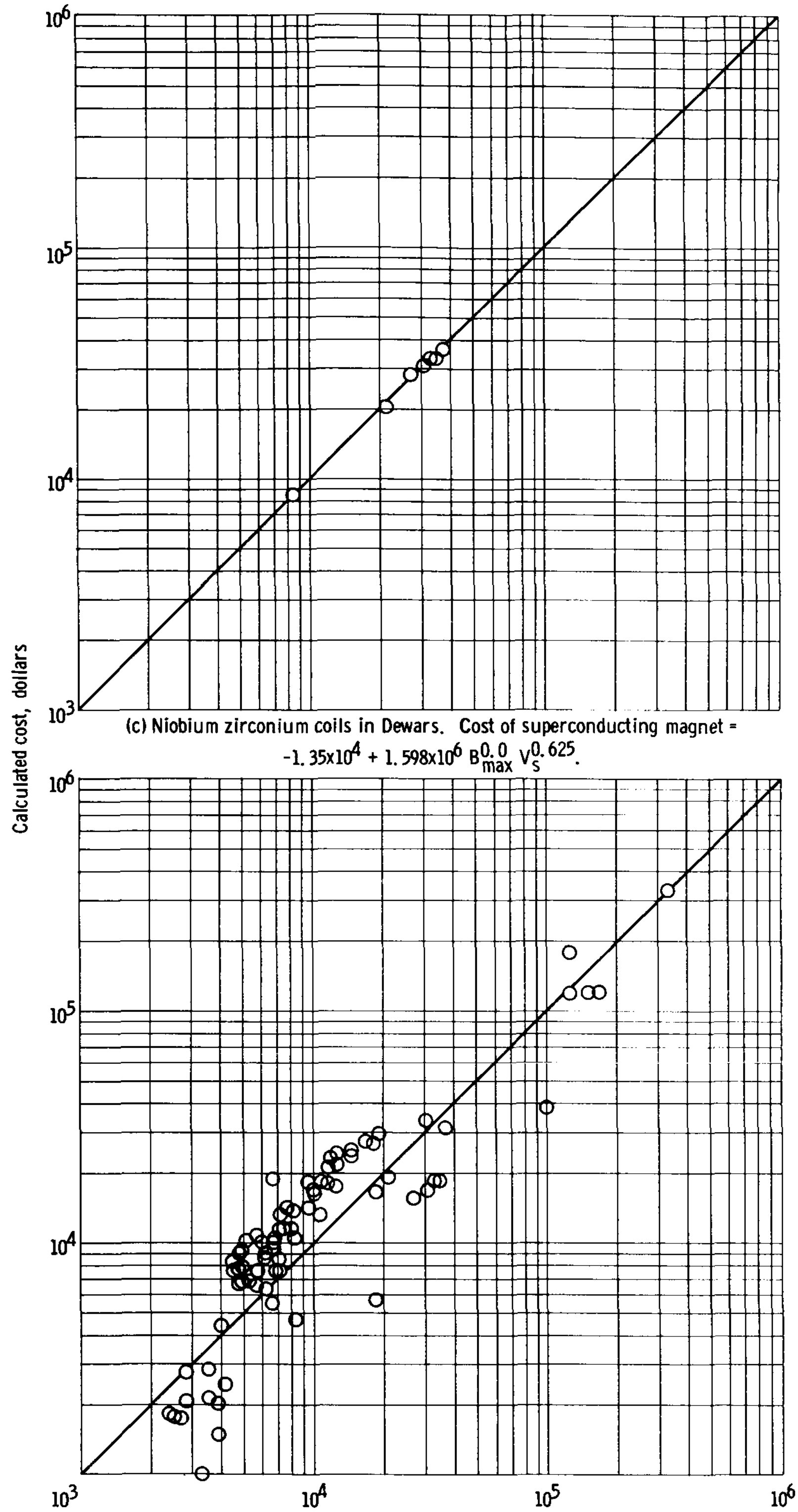
From table IV, it is clear that the mean errors and mean square errors of the scaling laws of equations (17) and (18) are not significantly different, and either one could be employed to provide a rough estimate of the cost of a given coil. In the subsequent discussion, equation (17) will be used, since the mean coil diameter is apt to be known at a much earlier stage in the design process than the volume of the coil windings, which is required in equation (18).

The parameters of table I(b) represent the state of the marketplace as of August 1965. It is probably not unreasonable to expect that the cost calculated by equation (17) is an upper limit to the future cost of superconducting coils, since increased experience and improved superconducting materials should tend in the direction of decreased costs. The constants $\$_0$ and $\$_1$ would probably be most sensitive to advances in the state of the art, but the values of the parameters γ_1 and γ_2 ought to be less sensitive.



(b) Bare coils. Cost of superconducting magnet = $-900 + 5.976\times10^5\ B_{max}^{0.2}\ V_s^{0.675}$.

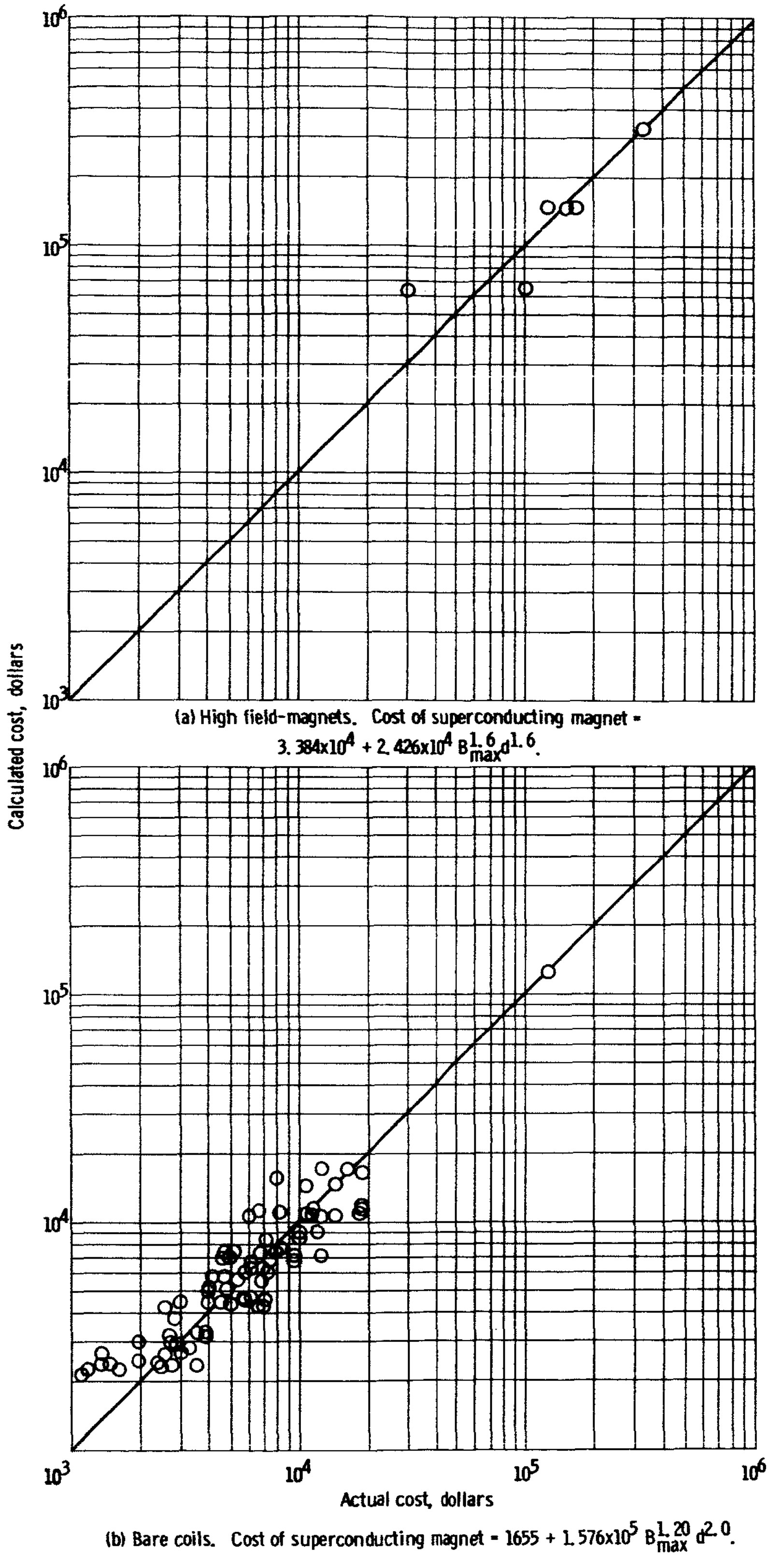
Figure 3. - Scaling laws calculated from equation (18). B_{max} is maximum magnetic field on axis in Webers per square meter; V_s is coil volume in meters.



(c) Niobium zirconium coils in Dewars. Cost of superconducting magnet = $-1.35\times10^4 + 1.598\times10^6\ B_{max}^{0.0}\ V_s^{0.625}$.

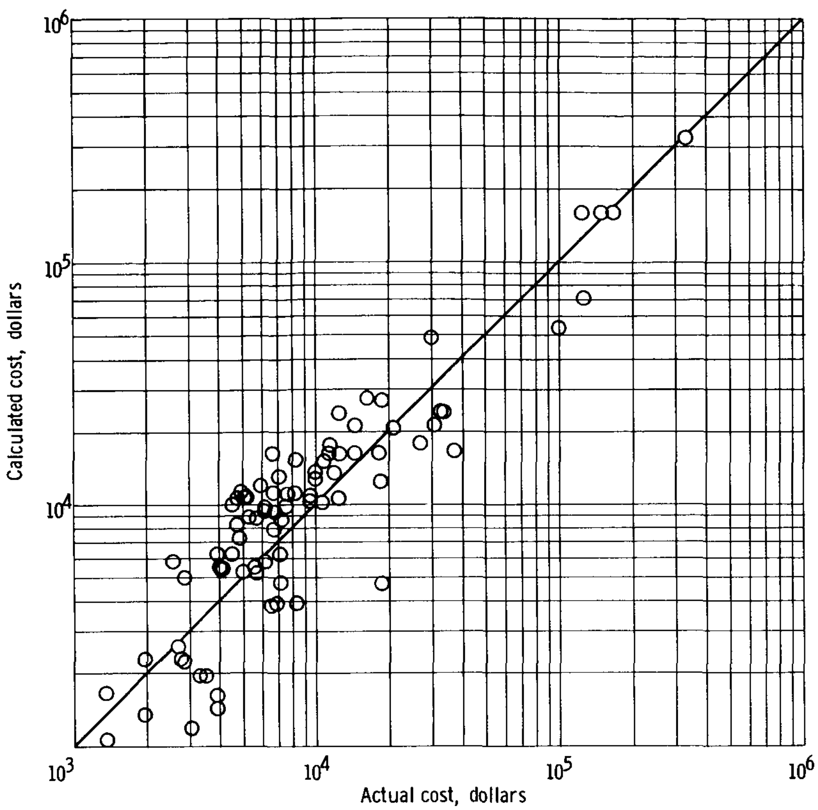
(d) All coils. Cost of superconducting magnet = $-4452 + 5.392\times10^5\ B_{max}^{0.55}\ V_s^{0.65}$.

Figure 3. - Concluded.



(b) Bare coils. Cost of superconducting magnet = $1655 + 1.576\times10^5\, B_{max}^{1.20} d^{2.0}$.

Figure 4. - Scaling laws calculated from equation (17). Maximum magnetic field on axis of mirror machine, B_{max}; mean diameter of superconducting coil, d. B_{max} in Webers per square meter, d in meters.



(c) All coils. Cost of superconducting magnet = $-1400 + 4.897\times10^4\ B_{max}^{1.25}\ d^{1.30}$.

Figure 4. - Concluded.

Scaling Law for Cost of Conventional, Water-Cooled Copper Coils

The scaling laws for the cost of conventional water-cooled copper coils have been discussed extensively in reference 11. The power required by a conventional coil is given in reference 12 as

$$W = \frac{\rho B_{max}^2 d}{G^2 \lambda(1 + P)} \text{ watts} \tag{22}$$

where ρ is the conductivity of the conductor, G is the Fabry G factor, λ is the packing fraction of the conductor, and P is the ratio of the outer to the inner coil radius. The major items of cost in conventional magnet systems are the alternating- to direct-current conversion equipment, and the equipment required to supply cooling water. Both are proportional to the power dissipated by the coils. An approximate scaling law for the cost of geometrically similar coils made of the same type of conductor and using the same packing fraction may be written as

$$\$_{w} = \$_{ow} + \$_{1w} B_{max}^{2} d_{w} \tag{23}$$

where $\$_{ow}$ is the fixed costs, the constant $\$_{1w}$ is given by

$$\$_{1w} \equiv \frac{C\rho}{G^{2}\lambda(1 + P)} \tag{24}$$

and C is the cost of the power supply and water supply per watt. It should be noted that equation (24) is of the same form as the scaling law of equation (17) with γ_1 and γ_2 equal to 2.0 and 1.0, respectively.

GENERAL EXPRESSIONS FOR OPTIMIZED MIRROR MACHINE

This report is concerned with the optimum design of apparatus for an experiment, and not with the plasma properties n_o and V_o, which must be determined by the objectives (or the results) of the experiment itself. The coil geometry that will be assumed for the remainder of this analysis is that shown in figure 5 in which a magnetic bottle is formed by two coils of mean diameter d separated by a distance $2Z_o$. The magnetic field on the axis of the coils is approximated by that of a current loop of radius $r_c = d/2$,

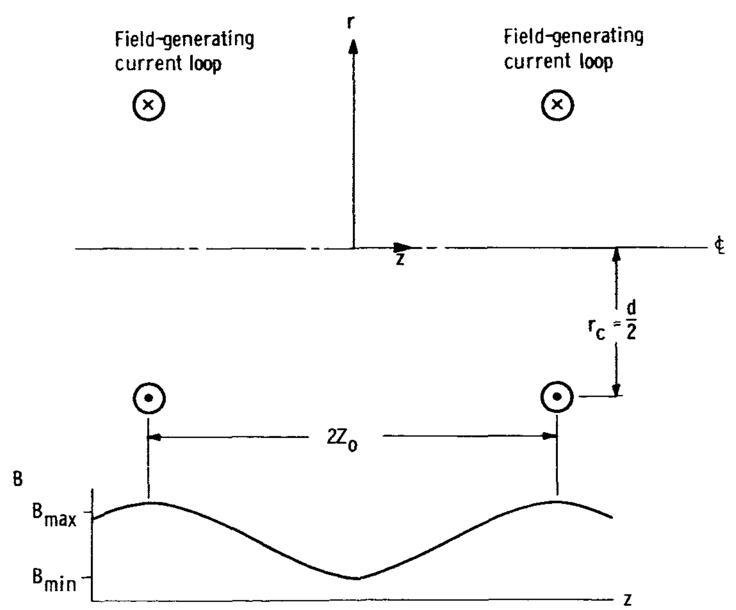


Figure 5. - Assumed magnetic bottle geometry.

given in reference 13 as

$$B_z = \frac{\mu_o I d^2}{8\left(\frac{d^2}{4} + Z^2\right)^{3/2}} \tag{25}$$

The field B_{min} on the axis of the midplane of the magnetic bottle is

$$B_{min} = \frac{2\mu_o I d^2}{\left(d^2 + 4Z_o^2\right)^{3/2}} \tag{26}$$

and the maximum field B_{max} can be expressed as

$$B_{max} = \frac{\mu_o I}{d} + \frac{\mu_o I d^2}{\left(d^2 + 16Z_o^2\right)^{3/2}} \approx \frac{\mu_o I}{d} \tag{27}$$

Equations (26) and (27) can then be used to relate the mean coil diameter d to the characteristic length of the magnetic field Z_o:

$$R_m \equiv \frac{B_{min}}{B_{max}} \approx \frac{2d^3}{\left(d^2 + 4Z_o^2\right)^{3/2}} \approx \frac{2}{\left(1 + \frac{4Z_o^2}{d^2}\right)^{3/2}} \tag{28}$$

Therefore,

$$r_c = \frac{d}{2} = Z_o\left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-1/2} \tag{29}$$

The average magnetic field appearing in equation (13) may be written in terms of the mirror ratio $R_m \equiv B_{min}/B_{max}$ as

$$B_{av} = \frac{1}{2}(B_{min} + B_{max}) = \frac{1}{2} B_{max}(R_m + 1) \tag{30}$$

If equation (29) is substituted into the figure of merit of equation (12),

$$\Omega = n_o V_o \tau \xi Z_o^3 \left(\frac{1 - R_m}{2}\right)^{1/2} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-1} \tag{31}$$

is obtained. When equations (13) and (30) are substituted into equation (14), the adiabatic constraint may be written as

$$R_m = \left[\frac{2}{B_{max} Z_o (R_m + 1)\epsilon_A} \left(\frac{2mV_o}{e}\right)^{1/2}\right]^A \tag{32}$$

where V_o is taken as the typical particle energy. If equation (29) is substituted into equation (17), the scaling law for the cost of a single superconducting coil may then be written as

$$\$ = \$_0 + \$_1 B_{max}^{\gamma_1} Z_o^{\gamma_2} 2^{\gamma_2} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-\gamma_2/2} \tag{33}$$

The figure of merit in equation (31) is maximized subject to the constraints of equations (32) and (33).

If B_{max} is eliminated between equations (32) and (33),

$$\$ = \$_0 + \frac{\$_1 2^{\gamma_1+\gamma_2} Z_o^{\gamma_2-\gamma_1}}{\epsilon_A^{\gamma_1} R_m^{\gamma_1/A} (R_m + 1)^{\gamma_1}} \left(\frac{2mV_o}{e}\right)^{\gamma_1/2} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-\gamma_2/2} \tag{34}$$

If the scaling law exponents are unequal, $\gamma_1 \neq \gamma_2$, Z_o can be eliminated between equations (31) and (34) and the figure of merit can be written entirely in terms of the mirror ratio as

$$\Omega = \frac{n_o V_o \xi}{\sqrt{2}} \left\{ \left(\frac{\epsilon_A^{\gamma_1}}{2^{\gamma_1+\gamma_2}} \right) \left[\frac{\$ - \$_0}{\$_1} \left(\frac{e}{2mV_o} \right)^{\gamma_1/2} \right] \right\}^{3/(\gamma_2-\gamma_1)}$$

$$\times \tau(R_m)(1 - R_m)^{1/2} \left[R_m^{1/A}(1 + R_m) \right]^{3\gamma_1/(\gamma_2-\gamma_1)} \left[\left(\frac{2}{R_m} \right)^{2/3} - 1 \right]^{(\gamma_2+2\gamma_1)/[2(\gamma_2-\gamma_1)]} \tag{35}$$

After inspection of the terms containing the mirror ratio, it can be verified that, for $R_m = 1.0$, the figure of merit Ω is zero. This is expected, since the escape cone occupies all of velocity space, and no particles are trapped. If the experimental results of equation (16) can be extrapolated below $R_m = 0.175$ to $R_m \approx 0$, the figure of merit becomes proportional to

$$\Omega \underset{R_m \to 0}{\simeq} \left[R_m^{3\gamma_1/A-(\gamma_2+2\gamma_1)/3} \right]^{1/(\gamma_2-\gamma_1)} \tau(R_m) \tag{36}$$

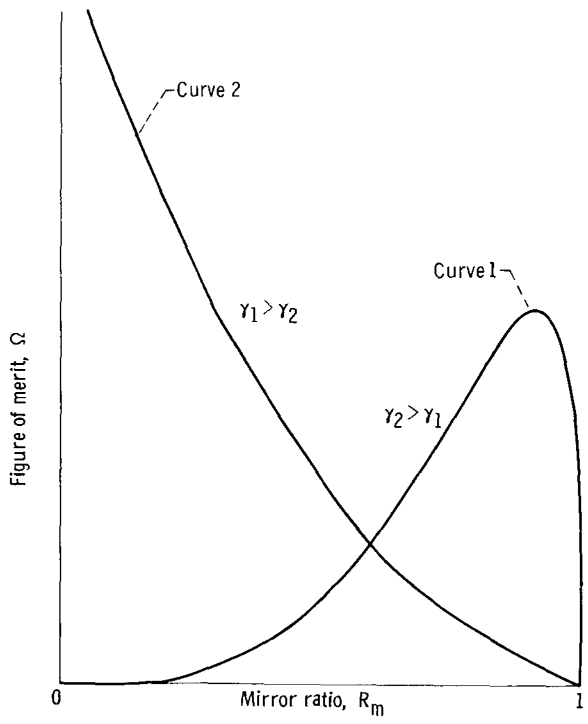


Figure 6. - Schematic variation of figure of merit with mirror ratio.

The strongest dependence of τ on R_m is such that, unless A is larger than unity, the figure of merit will approach zero as the mirror ratio approaches zero, provided that $\gamma_2 > \gamma_1$. From the mean value theorem, it is known that Ω must have a maximum somewhere in the range $0 \leq R_m \leq 1$.

When A is less than unity and $\gamma_2 > \gamma_1$, the figure of merit has the general form shown in curve 1 on figure 6 and possesses a very sharp maximum. If $\gamma_1 > \gamma_2$, however, the figure of merit becomes infinite as R_m approaches zero, as shown on curve 2 of figure 6. In the latter case, there exists no optimum mirror ratio, and the figure of merit then depends entirely on how low a mirror ratio one can afford to pay for. In other words, if the coil cost is more strongly dependent on magnetic field than on size, there exists no economically optimum mirror ratio. Inspection of equation (23) shows that this is the case for conventional, water-cooled copper coils. Only if the cost depends more strongly on size than on magnetic field will an economically optimum mirror ratio exist, and an inspection of table IV(b) shows this to be the case for superconducting coils. For reasons discussed on page 30, one would expect future progress in the state of the art to result in values of γ_2/γ_1 larger than those values shown in table IV(b).

If equation (35) is differentiated with respect to the mirror ratio and set equal to zero, the value of R_m for which the figure of merit is a maximum is given by

$$\frac{1}{A} = \frac{R_m(\delta - 1)}{6(1 - R_m)} - \frac{R_m}{1 + R_m} + \frac{2^{2/3}(\delta + 2)}{9\left(2^{2/3} - R_m^{2/3}\right)} - \frac{R_m(\delta - 1)}{3\tau}\frac{\partial \tau}{\partial R_m} \tag{37}$$

where δ is defined as

$$\delta \equiv \frac{\gamma_2}{\gamma_1} \tag{38}$$

Two assumptions are made regarding the variation of confinement time τ with mirror ratio. If it is assumed that the confinement time is independent of mirror ratio, the optimum value of the mirror ratio is given by solving equation (37), the last term equal to zero:

$$\delta = \left\{\frac{1}{A} + \frac{R_m}{1 + R_m} + \frac{R_m}{6(1 - R_m)} - \frac{2}{9\left[1 - \left(\frac{R_m}{2}\right)^{2/3}\right]}\right\}\left\{\frac{R_m}{6(1 - R_m)} + \frac{1}{9\left[1 - \left(\frac{R_m}{2}\right)^{2/3}\right]}\right\}^{-1} \tag{39}$$

A theoretical analysis by Lehnert (ref. 14) indicates that this assumption may prove to be the case in actual mirror machines, if particles are lost by instabilities or by rapid diffusion across the field lines in physical space, before they are lost by diffusion into the escape cone in velocity space.

If cross-field diffusion in physical space does not occur, the confinement time of an ion may be as long as

$$\tau(R_m) = \tau_{coll} P_o \tag{40}$$

where τ_{coll} is a self-collision time that is a function only of the plasma density and temperature, and P_o is the loss probability per collision. The confined plasma volume in a velocity space of unit radius is just 4π, and the escape cone occupies a volume $4\pi(1 - \cos\theta_o)$. The fractional volume of velocity space occupied by the escape cone gives a rough measure of the loss probability per collision, which is

$$P_o \equiv \frac{1}{1 - \cos\theta_o} \tag{41}$$

If equations (5) (using $B = B_{min}$ at the midplane) and (41) are substituted into equation (40), the confinement time is given by

$$\tau(R_m) = \frac{\tau_{coll}}{1 - (1 - R_m)^{1/2}} \tag{42}$$

The last term in equation (37) is then

$$\frac{1}{\tau}\frac{d\tau}{dR_m} = -\frac{1}{2}\left[\frac{1}{(1 - R_m)^{1/2} - (1 - R_m)}\right] \tag{43}$$

Under this assumption as to the variation of τ with R_m, the optimum value of R_m is given by

$$\delta = \left\{\frac{1}{A} + \frac{R_m}{1 + R_m} + \frac{R_m}{6(1 - R_m)} - \frac{2}{9\left[1 - \left(\frac{R_m}{2}\right)^{2/3}\right]} + \frac{R_m}{6\left[(1 - R_m)^{1/2} - (1 - R_m)\right]}\right\}$$

$$\times\left\{\frac{R_m}{6(1 - R_m)} + \frac{1}{9}\left[1 - \left(\frac{R_m}{2}\right)^{2/3}\right]^{-1} + \frac{R_m}{6\left[(1 - R_m)^{1/2} - (1 - R_m)\right]}\right\}^{-1} \tag{44}$$

For given values of δ and A, equation (39) determines an upper bound on the optimum mirror ratio, and equation (44) determines a lower bound on the optimum mirror ratio.

The true dependence of the confinement time τ on R_m will undoubtedly lie somewhere between that given by equation (42) and the assumption that τ is independent of mirror ratio. If cross-field ion diffusion is a significant process, the confinement time will be somewhat more independent of R_m than given by equation (42).

The optimum values of R_m given by equations (39) and (44) are of interest to the apparatus designer, since they are independent of the apparatus dimensions, cost, and maximum magnetic field. As pointed out on page 22, however, this optimum, will not exist when $\gamma_1 > \gamma_2$, that is, when $\delta \leq 1.0$.

The optimum value of R_m from either equation (39) or (44) may be used to determine the optimum B_{max} and Z_o as a function of the particle energy, particle mass, and the available amount of investment capital. Equation (34) can be solved for the optimum value of Z_o, which yields

$$Z_o = \left(\frac{\$ - \$_0}{\$_1}\right)^{1/(\gamma_2-\gamma_1)} \left\{\frac{\epsilon_A R_m^{1/A}(R_m + 1)}{2^{\delta+1}} \left(\frac{e}{2mV_o}\right)^{1/2} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{\delta/2}\right\}^{1/(\delta-1)} \tag{45}$$

By substituting this into equation (32), the optimum value of B_{max} is seen to be

$$B_{max} = \left\{\frac{4\left(\frac{2mV_o}{e}\right)^{1/2}}{\epsilon_A R_m^{1/A}(1 + R_m)\left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{1/2}}\right\}^{\delta/(\delta-1)} \left(\frac{\$_1}{\$ - \$_0}\right)^{1/(\gamma_2-\gamma_1)} \tag{46}$$

Under no circumstances should this optimum value of magnetic field violate the condition $\beta \leq 1.0$, where β is the ratio of plasma to magnetic energy density (to be defined in eq. (48)).

For small coils and some experimental circumstances, the coil cost may not be a primary consideration. It is then of interest to optimize the mirror ratio subject only to the adiabatic confinement constraint. If Z_o is eliminated between equations (31) and (32), it is found that

$$\Omega = \frac{4\sqrt{2}\, n_o V_o \xi}{B_{min}^3 \epsilon_A^3} \left(\frac{2mV_o}{e}\right)^{3/2} \frac{R_m^3 \tau(R_m)(1 - R_m)^{1/2}}{R_m^{3/A}(1 + R_m)^3} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-1} \tag{47}$$

This equation suggests that the figure of merit may be made as large as desired by decreasing the magnetic field intensity. However, B_{max} cannot be made arbitrarily small, since the magnetic field must be at least large enough to satisfy the condition (ref. 8) for the ratio of plasma to magnetic field pressure, which is

$$\beta \equiv \frac{2\mu_o n_o e V_o}{B_{min}^2} \leq \beta_{max} \tag{48}$$

where β_{max} is some number less than unity and is determined by stability considerations. If the confinement time is independent of the mirror ratio, it will be given approximately by the self-collision time (ref. 15) as

$$\tau(R_m) \approx \tau_{coll} = \frac{25.8\sqrt{\pi}\,\epsilon_o^2 m^{1/2}(eV_o)^{3/2}}{e^4 n_o \ln \Lambda} \tag{49}$$

where Λ is a slowly varying quantity defined by

$$\Lambda \equiv \frac{12\pi}{n_e^{1/2}}\left(\frac{\epsilon_o V_o}{e}\right)^{3/2} \tag{50}$$

If B_{max} is eliminated between equations (47) and (48) and equation (49) is substituted into equation (47),

$$\Omega = \frac{103.2\sqrt{2\pi}\,\epsilon_o^2}{\epsilon_A^3 \mu_o^{3/2} e^{5.5}}\left[\frac{V_o^{5/2}\xi\beta_{max}^{3/2}m^2}{n_o^{3/2}\ln \Lambda} f(R_m)\right] \tag{51}$$

where the mirror ratio dependence is given by

$$f(R_m) \equiv \frac{R_m^3(1 - R_m)^{1/2}}{R_m^{3/A}(1 + R_m)^3}\left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-1} \tag{52}$$

It should be noted that, for small values of R_m, equation (52) is dominated by the $R_m^{3/A}$

dependence. Therefore, when the coil costs are not a consideration, as small a mirror ratio as possible should be used to achieve a high figure of merit.

Possible Extension of General Analysis

It is tempting to consider the possibility of a completely closed-form optimization study in which n_0 and V_0 are also optimized in such a way as to maximize the figure of merit. In order to include these two variables in the optimization process, however, two additional constraint equations are needed, one similar to equation (32) for the adiabatic confinement constraint and the other similar to equation (33) for the coil cost constraint. Unfortunately, various candidate constraint equations are not as soundly based on observational evidence and laboratory experiment as are equations (32) and (33). If they are verified by future experimental investigation, various stability criteria might furnish the required constraint equations. Equation (48), which states that $\beta \leq \beta_{max}$, might be used as a constraint equation, if and when it becomes known what value of β_{max} is compatible with plasma stability and whether and how β_{max} varies with R_m, Z_0, V_0, n_0, etc.

As soon as valid, experimentally established constraint equations become available, it will be possible to extend the present analysis to incorporate an optimization of n_0 and V_0. The conclusions to be drawn from such an analysis may well differ from the present ones, particularly with regard to optimum values of R_m and Z_0; the present state of the experimental art, however, does not permit particular values of n_0 and V_0 to be obtained at will nor does the present understanding of plasma theory furnish valid constraint equations. As long as this situation lasts, n_0 and V_0 must be left as free parameters to be determined by the outcome of experimental investigation.

CHARACTERISTICS OF OPTIMUM MIRROR MACHINE UNDER CURRENT CONDITIONS

It has been shown that if $\gamma_2 > \gamma_1$, there exists an economically optimum mirror ratio that is independent of the amount of money expended, the maximum magnetic field, and the apparatus dimensions. It is of interest to determine the value of this mirror ratio for the specific adiabatic and cost parameters given earlier. If the apparatus is designed so that the plasma is adiabatic for $\alpha_0 \leq \pi/2$, a generous allowance for the radius of the plasma, equations (15a) and (15b) give

$$A = 0.298 \tag{53a}$$

$$\epsilon_A = 0.386 \tag{53b}$$

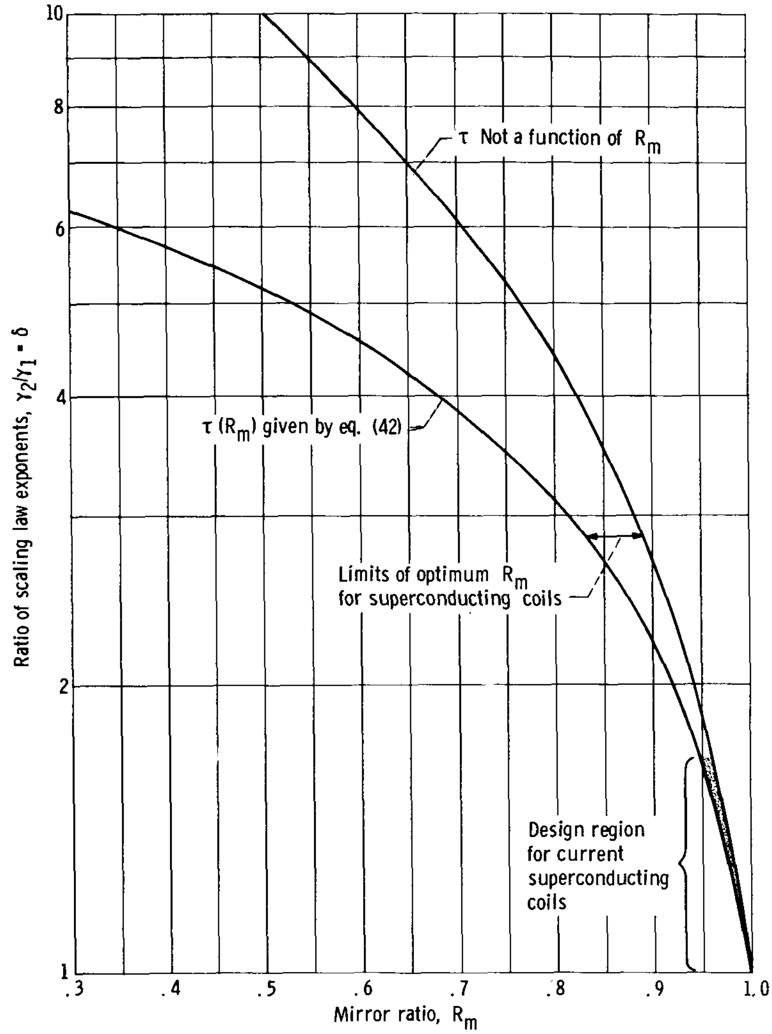


Figure 7. - Optimum values of mirror ratio for parameter A of 0.298.

If this value of A is substituted into equation (39), the upper curve of figure 7 of δ VsR$_m$, labeled "τ not a function of R_m," is obtained. The substitution of this value of A into equation (44) yields the curve on figure 7 labeled "τ given by equation (42)," in which equation (42) gives the dependence of τ on R_m.

For a given value of δ, the maximum value of R_m is given by the upper curve and the minimum by the lower curve. Until experimental evidence on the variation of τ as a function of R_m becomes available, it is not possible to specify the optimum value of R_m any more closely than to give the range between the two curves on figure 7.

If the scaling relation of table IV(b) (p. 13) is used for all 98 superconducting coils, the scaling law parameters are

$$\gamma_1 = 1.250 \tag{54a}$$

$$\gamma_2 = 1.300 \quad (54b)$$

$$\delta = \frac{1.30}{1.25} = 1.04 \quad (54c)$$

$$\$_0 = -1400 \quad (54d)$$

$$\$_1 = 4.897\times10^4 \quad (54e)$$

For the values of δ shown in table IV(b), the optimum mirror ratio lies in the cross-hatched region shown in figure 7. The surprising feature of figure 7 is that the optimum mirror ratios are quite high, in the range

$$0.95 \leq R_m \leq 1.0 \quad (55)$$

which implies that the economically optimum mirror fields are nearly uniform.

These high mirror ratios have a large escape cone, and it is unexpected to find that such large mirror ratios are in any sense optimum. The basic reason for these high mirror ratios is that the object of this investigation was not to confine the maximum number of particles in a given volume, but rather to confine the maximum number of particles of a given energy and density per dollar of investment in the apparatus. The fact that these high mirror ratios give a least expensive apparatus should be taken into account in the designing of superconducting magnet facilities, especially since current design practice for conventional magnet facilities favors mirror ratios of $R_m \leq 0.5$.

Future progress in reducing the absolute costs of superconducting coils probably will not result in the parameter δ lying outside the range $1.0 \leq \delta \leq 2.0$. For $\delta = 1.0$, the cost will be proportional to the magnetic field and diameter raised to the same power. It is difficult to imagine a situation in which the cost of a coil depends so much more strongly on the dimensions than on the magnetic field that δ would be greater than 2.0, unless the coil costs are much smaller than the cost of the cryogenic Dewars which enclose them. As can be seen in figure 7, if $1.0 \leq \delta \leq 2.0$, the economically optimum mirror ratio will lie in the range $0.92 \leq R_{m,opt} \leq 1.00$.

When $\delta < 1.0$, no optimum mirror ratio exists, but this situation is not likely to occur for superconducting magnets. The maximum magnetic field of a coil is approximately

$$B_{max} \sim \frac{I}{d} = \frac{jS}{d} \quad (56)$$

where j is the current density in the superconducting material, and S is the crosssectional area of the superconducting material. The cost of the coil is approximately equal to the cost of the superconducting material, which is

$$\$ \sim V_s \sim dS = \frac{B_{max} d^2}{j} \tag{57}$$

It would be expected, therefore, that $\delta = \gamma_2/\gamma_1 = 2.0$ when j is not a function of the magnetic field. In fact, it can be seen in table IV(b) that $\delta \approx 1.67$ for 85 coils of niobium - 25-percent zirconium. The value of δ near unity found for all 98 coils results from the influence of the larger, more expensive coils. These coils are designed to operate in the region where j is a rapidly decreasing function of B_{max}. This dependence of j on B_{max} results in values of γ_1 that are nearly equal to γ_2 for marginally designed coils, whose field at the coil windings is so high that j is a rapidly decreasing function of the magnetic field. As superconducting materials improve, it should be possible to use a material in any given application for which j is not strongly dependent on the magnetic field strength. Therefore, values of δ as large as 2.0 can be expected in the future. Even higher values of δ may occur if the coil Dewars are more expensive than the coils. These large values of δ could come about, because the Dewar costs are a function of the coil dimensions and are almost independent of the magnetic field strength.

It is not difficult to understand now, the appearance of high mirror ratios. The construction of the figure of merit to represent the total number of confined particles results in its proportionality to the volume of the apparatus, that is, to Z^3. The cost of a coil is shown to increase only as a power of Z between 1.3 and 2.0, even when B_{max} is fixed. The adiabatic constraint, however, requires that $B_{max} Z_o$ is about constant, so that the cost of a barely adiabatic device becomes a weak function of Z_o. The maximization of the figure of merit is therefore dominated by the increasing volume obtainable at high mirror ratios up to the point at which the reduction in velocity space volume becomes counterbalancing.

It is appropriate to examine the effect on the analysis of the assumptions made concerning the apparatus geometry. In all previous cases, the cost of a single mirror coil, which forms one-half of a magnetic bottle (fig. 2, p. 6), has been discussed. It has also been assumed that the magnetic field of the second coil does not contribute to B_{max} on the axis of the coil in question. This latter assumption is not a good one at the high mirror ratios that have been shown to be optimum: it is a conservative assumption, in the sense that one will, in all cases, actually get more gauss per dollar than the optimum curves imply. Equation (37) shows that the assumption of the magnetic field geometry does not significantly affect the optimum mirror ratio. The geometry assumption is embodied in the third term on the right side of the equation, and this term is small compared with the $1/A$ term and the term containing τ, which dominate equation (37).

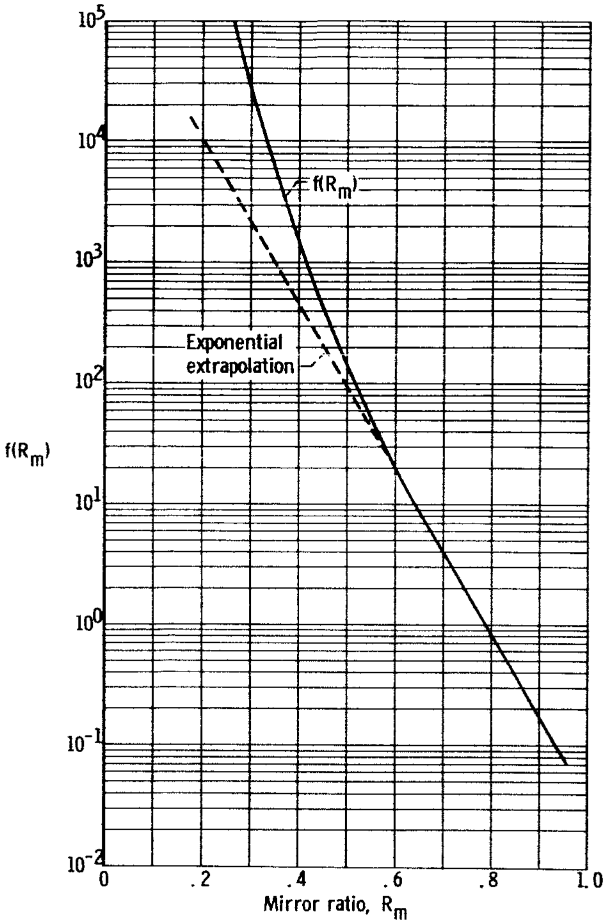


Figure 8. - Dependence of figure of merit on mirror ratio.

So far, the implied assumption has been made that there will be no impediment to using the figure of merit of equation (35). It might be the case that the large escape cone angles resulting from the large optimum mirror ratios will result in plasma instabilities. These instabilities may make it necessary to design apparatus with low mirror ratios. It is therefore of interest to use equations (51) and (52) to investigate how the figure of merit varies when an economic constraint is not applied. Equation (51) implies that one can achieve a large figure of merit in an adiabatic mirror machine by operating at high particle kinetic energies and low densities. Equation (52), which contains the dependence of the figure of merit on mirror ratio, is plotted as a function of R_m in figure 8, where $A = 0.298$. There exists no optimum mirror ratio, since the figure of merit decreases monotonically from infinity at $R_m = 0$ to zero at $R_m = 1.0$. The dependence of Ω on R_m is strong, approximately exponential in the range $0.5 \leq R_m \leq 1.0$ and steeper than exponential for $R_m \leq 0.5$.

This analysis shows the sharp contrast between the case in which economic factors are an important design consideration and the case in which they are not. When $\gamma_1 > \gamma_2$ and/or economic constraints are not important, figure 8 makes it clear that one should go to the smallest possible mirror ratio. On the other hand, if $\gamma_2 > \gamma_1$, as is the case with superconducting coils, and if cost is significant, the maximum figure of merit occurs at a high mirror ratio $0.95 \leq R_{m,\,opt} \leq 1.00$.

Since there exist some circumstances in which one will not operate at the optimum mirror ratio, it is of some interest to use equation (34) to estimate the cost of a coil of given mirror ratio used to confine particles of energy V_o. If the parameters of equations (53) and (54) are substituted in equation (34),

$$\$ = -1400 + 14.9(\ell V_o)^{0.625} \frac{Z_o^{0.05}}{R_m^{4.2}(1 + R_m)^{1.250}} \left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{-0.65} \tag{58}$$

where ℓ is the ion mass in atomic mass units. The dimension Z_o is of the order unity in the mks system of units adopted, and the dependence is so weak that one can define $Z_o^{0.05} \approx 1.0$. Equation (58) is plotted on figure 9, which shows the cost of an axisymmetric superconducting solenoid required to confine adiabatically particles of mass ℓ (amu) and of energy V_o (eV) in a mirror field of the indicated mirror ratios. It is obvious at once that it is very expensive to confine energetic particles adiabatically in mirrors with low mirror ratios. For example, in order to confine 50 keV deuterons ($V_o = 5\times10^4$, $\ell = 2.0$) in a 2 to 1 mirror ratio field ($R_m = 0.5$), approximately \$170 000 must be spent on a single mirror coil.

The information presented in figure 9 can be presented a little differently by rewriting equation (56) as

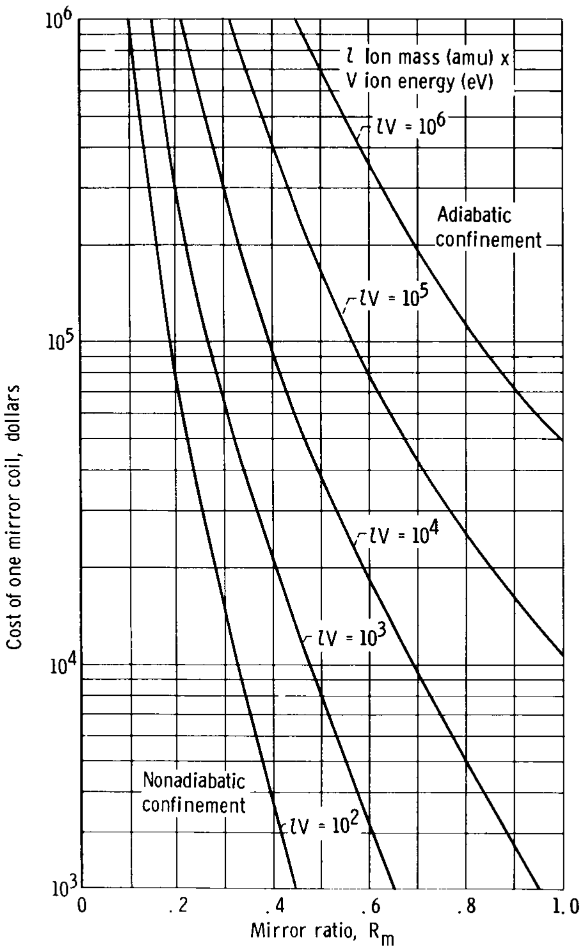


Figure 9. - Cost of axisymmetric superconducting solenoid as function of mirror ratio.

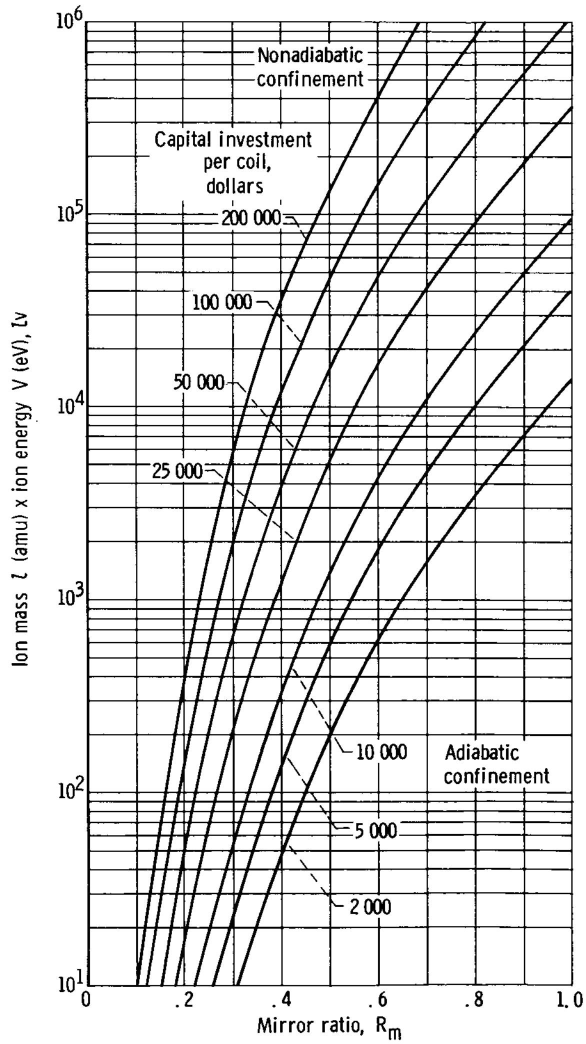


Figure 10. - Critical nonadiabatic energy as function of mirror ratio and cost.

$$\ell V_o = \left\{\left(\frac{\$ + 1400}{14.9}\right) R_m^{4.2}(1 + R_m)^{1.25}\left[\left(\frac{2}{R_m}\right)^{2/3} - 1\right]^{0.65}\right\}^{1.60} \tag{59}$$

The particle mass-energy product ℓV_o is plotted as a function of mirror ratio for several values of capital investment on figure 10. In order that a particle of a given ℓV confined by a given R_m be adiabatically confined, the amount invested must be at least as much as indicated on the curves. Again, it is clear that adiabatic confinement at low mirror ratios is very expensive. In figure 9, adiabatic trapping will occur if the amount of capital invested is equal to or greater than that shown on the energy curves; in figure 10, adiabatic trapping will occur only if the particle energy is less than that indicated on the capital investment curves.

CONCLUSIONS

A figure of merit for controlled fusion research apparatus has been proposed, the maximization of which is consistent with the needs, objectives, and capabilities of current controlled fusion research. The proposed figure of merit is proportional to the apparatus volume, the degree of isotropy in velocity space, and the charged particle density, energy, and confinement time.

A scaling law for the cost of superconducting solenoids has been derived by a least-squares curve fitting to the actual cost, dimensions, and maximum magnetic field of 98 solenoids that have been sold commercially. This scaling law may be used for preliminary design studies and refined on the receipt of actual bids for a specific facility.

The figure of merit has been maximized subject to an economic constraint, which was provided by the scaling law for coil cost, and an adiabatic confinement constraint, which was provided by the results of a recent experimental investigation. It was found that, when superconducting coils are used, there exists an economically optimum mirror ratio that is independent of the dimensions and magnetic field of the mirror coil and of the amount of capital investment in the superconducting coils. This optimum mirror ratio permits the adiabatic confinement of the maximum total number of ions of a given energy and equivalent isotropic density per dollar of capital investment in the magnet facility. In the optimization process, it was assumed that the ion energy and equivalent isotropic density were not functions of the mirror ratio, of B_{max}, or of Z_o. It was also assumed (in keeping with the current state of the art) that the ion energy and density were dependent variables whose numerical values depend on the outcome of an experiment, and not independent variables, whose values can be adjusted at will.

The economically optimum mirror ratio was calculated for adiabatic confinement in an axisymmetric mirror field generated by superconducting coils. It was found that the economically optimum mirror ratio was quite high, in the range $0.95 \leq B_{min}/B_{max} \leq 1.00$. This optimum is a sharp maximum in the figure of merit. Thus, if stability considerations and other factors permit, designers of superconducting magnet facilities for controlled fusion research should aim for much higher mirror ratios than have been customary in existing conventional magnet facilities.

The large anisotropy in velocity space associated with the near-unity optimum mirror ratios may cause such severe instabilities that lower mirror ratios (and, hence, greater isotropy in velocity space) will be mandatory. It may also be desirable to use off-optimum values of R_m, B_{max}, or Z_o for other reasons. The two constraint equations are written in such a way as to give the cost of a superconducting mirror coil as a function of the confined particle mass and energy and of the mirror ratio (not necessarily optimum) of the apparatus. It is shown that the cost of going to adiabatic mirror ratios less than 0.50 is quite high for ions of thermonuclear temperatures and will remain so even if the cost of superconducting coils is substantially reduced.

When cost is not a major consideration in designing a magnet facility, the facility should be designed with as low a mirror ratio as is consistent with adiabatic confinement. When cost is a major consideration, superconducting coils should be designed with a mirror ratio of about $B_{min}/B_{max} = 0.95$. This conclusion is virtually independent of the cost of superconducting coils at any given time.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, March 31, 1966.

APPENDIX - SYMBOLS

[Unless otherwise noted, the rationalized mks system of units has been used throughout this paper.]

A	parameter given by equation (15)
B	magnetic field strength, W/m^2
B_{av}	average magnetic field on axis, $1/2\,(B_{min} + B_{max})$, W/m^2
B_{max}	maximum magnetic field on axis of mirror machine, W/m^2
B_{min}	minimum magnetic field on axis of mirror machine, W/m^2
C	cost of power supply, dollars per watt
d	mean diameter of superconducting coil, m
d_w	mean diameter of conventional coil, m
e	electric charge, C
f	velocity distribution function of ions
G	Fabry G factor (see ref. 12)
I	current, A
j	current density, A/m^2
K	ratio of mean coil to mean plasma radius, r_c/r_p
ℓ	particle mass, amu
m	particle mass
n	ion density (see p. 5)
n_o	ion density on axis where $B = B_{min}$ (see SUMMARY)
P	ratio of outer to inner coil radius
P_o	defined by equation (41)
R_m	mirror ratio B_{min}/B_{max} on apparatus axis
r	radial distance in configuration space
r_c	mean radius of mirror coils
r_p	mean radius of plasma
t	time
V	particle energy, eV
V_o	particle energy on axis where $B = B_{min}$, eV
V_s	volume of superconducting windings
v	particle velocity
W	power required for conventional coils
Z	axial distance in configuration space
Z_o	axial distance between B_{min} and B_{max}
α	dimensionless radial coordinate defined by equation (9b)
α_o	dimensionless radial coordinate defined by equation (9c)
β	ratio of plasma to magnetic pressure, equation (48)

β_{max} defined by equation (48)

γ_1 scaling law exponent in equations (17) to (19)

γ_2 scaling law exponent in equations (17) to (19)

$\Delta_1 \$$ mean error of scaling law, defined by equation (20)

$\Delta_o \$$ mean square error of scaling law, defined by equation (21)

δ parameter defined by equation (38)

ϵ adiabatic parameter defined by equation (13)

ϵ_A parameter defined by equation (15b)

ϵ_o permittivity of free space, 8.854×10^{-12} F/m

η dimensionless axial coordinate defined in equation (9a)

θ polar angle of spherical coordinate system in velocity space

θ_o angle of adiabatic escape cone in velocity space given by equation (5)

θ' azimuthal angle in configuration space

λ packing factor of conductor

μ_o permeability of free space, $4\pi \times 10^{-7}$ H/M

ξ geometry factor defined by equation (11)

ρ conductivity of conductor

τ characteristic confinement time of ions

φ azimuthal angle of spherical coordinate system in velocity space

ψ volume of plasma

Ω figure of merit (defined by eq. (1))

$\$$ cost of superconducting magnet

$\$_A$ actual cost of superconducting magnet

$\$_o$ scaling law parameter in equations (17) to (19)

$\$_w$ cost of conventional magnet

$\$_1$ scaling law parameter in equations (17) to (19)

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