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THEORY OF OPEN-ENDED CIRCULAR CYLINDRICAL MICROWAVE CAVITY

by Norman C. Wenger

Lewis Research Center

Cleveland, Ohio

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SUMMARY

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The TE_{011} mode of oscillation in an open-ended circular cylindrical microwave cavity is discussed. The cavity consists of a circular waveguide terminated at each end with a thin cylindrical partition coaxial with the circular waveguide. The resonant frequency of the cavity is computed by using Laplace transform and Wiener-Hopf techniques. Numerical values for the resonant frequency are presented.

INTRODUCTION

Microwave cavities have been used for a number of years to measure the relative dielectric constants of gases (refs. 1 to 3). The relative dielectric constant is given by the square of the ratio of the resonant frequency when the cavity is evacuated to the resonant frequency when filled with gas.

In many applications, such as in atmospheric research (ref. 4), it is desirable to make dynamic measurements of the dielectric constant by having the gas pass through the cavity. This requires replacing the solid end walls of the cavity by a termination that will totally reflect the microwave energy and yet present very little obstruction to the flow of gas.

Adey (ref. 5) developed a cylindrical cavity with a high Q in which the termination took the form of a ring concentric with the body of the cavity and attached to the body of the cavity by three spokes. In this design, 67 percent of the end area was removed. Thompson, et al. (ref. 6) found that, by judiciously selecting the diameter and thickness of this terminating ring, over 90 percent of the end area can be removed without significantly reducing the cavity Q.

Thorn and Straiton (ref. 7) have shown that an open-ended cavity can be constructed by simply terminating a section of waveguide at each end with partitions so that each subdivision formed by the partitions is a waveguide operating at a frequency below cutoff. The partitioned section acts like a perfect reflector to the microwave energy if the parti-

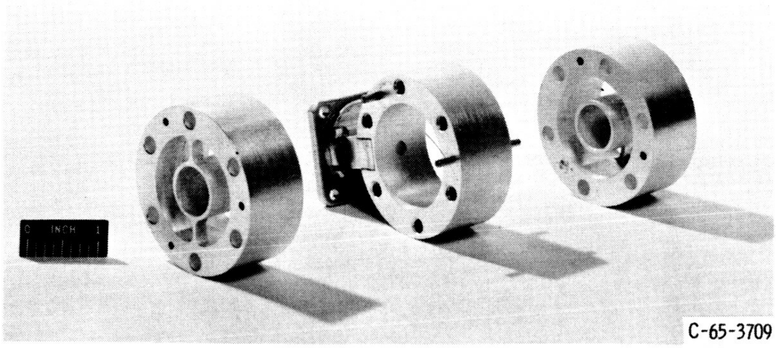

Figure 1. - Open-ended microwave cavity.

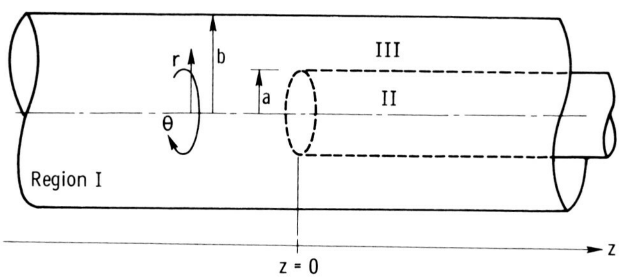

Figure 2. - Cylindrical partition in waveguide.

tions are sufficiently long. The cross-sectional obstruction of the terminations for this type of cavity is a result entirely of the finite wall thickness of the partitions plus any supporting structure. A cross-sectional obstruction of less than 2 percent of the end area has been reported (ref. 7).

The open-ended microwave cavity shown in figure 1 was recently developed at the Lewis Research Center to measure the density of liquid hydrogen under flow conditions. The density of liquid hydrogen can be computed from its relative dielectric constant by using an equation such as the Clausius-Mossotti equation. The cavity consists of a circular waveguide terminated at each end with a coaxial cylindrical partition. The effect of the cylindrical partition is to separate a portion of the waveguide into a coaxial waveguide plus a smaller circular waveguide that also serves as the inner conductor for the coaxial waveguide.

This report presents an analysis of open-ended circular cylindrical cavities of the type shown in figure 1 to determine the relation between the resonant frequency and the cavity dimensions for the case where the cavity is evacuated. The analysis is restricted to the TE_{011} resonant mode since this mode of oscillation has a very high Q and therefore is widely used. The solution for the resonant frequency of the TE_{011} mode can be easily computed if the reflection coefficient of the TE_{01} circular waveguide mode incident on the cylindrical partition is known.

The model that will be used to compute the reflection coefficient is shown in figure 2. The various regions of interest have been numbered for ease in reference. The model consists of a perfectly conducting circular waveguide of radius b and of infinite extent in the z-direction. Coaxial with the waveguide is an infinitely thin, perfectly conducting circular waveguide of radius a that extends over the range $0 < z < \infty$. The reflection coefficient will be computed for the case of the TE_{01} mode incident from the left.

Before proceeding with the analysis, it will be instructive to consider the various types of waves that can exist in the different regions. It will only be necessary, however, to consider the circularly symmetric TE_{on} modes since both the TE_{01} incident mode and the model possess circular symmetry. Thus, the field components of interest will be the θ-component of the electric field and the r- and z-components of the magnetic field. All other field components will vanish.

SYMBOLS

a	radius of circular partition (fig. 2)
b	radius of circular waveguide (fig. 2)
c	velocity of light
$E_\theta(r, z)$	θ-component of electric field
$E_\theta^i(r, z)$	θ-component of incident electric field
$E_\theta^s(r, z)$	θ-component of scattered electric field
$\mathcal{E}(r, \beta)$	bilateral Laplace transformation of E_θ^s
$\mathcal{E}^+(r, \beta)$	single-sided Laplace transformation of E_θ^s over the interval $z > 0$
$\mathcal{E}^-(r, \beta)$	single-sided Laplace transformation of E_θ^s over the interval $z < 0$
$G(r, r_o, z-z_o)$	Green's function
$\mathcal{G}(r, a, \beta)$	bilateral Laplace transformation of $G(r, a, z)$
$\mathcal{G}^+(a, a, \beta)$	Wiener-Hopf factor of $\mathcal{G}(a, a, \beta)$
$\mathcal{G}^-(a, a, \beta)$	Wiener-Hopf factor of $\mathcal{G}(a, a, \beta)$
$H_r(r, z)$	r-component of magnetic field
$H_z(r, z)$	z-component of magnetic field
$\mathcal{I}m$	imaginary part of
$J_\theta(a, z)$	θ-component of electric surface current density on surface $r = a$ for $z > 0$
j	$\sqrt{-1}$
$\mathcal{J}(a, \beta)$	bilateral Laplace transformation of J_θ
k_o	free-space wave number

L	length between partitions (fig. 6)
$p(\beta)$	function of β that gives $\mathscr{G}^+$ and $\mathscr{G}^-$ proper behavior as $\lvert\beta\rvert \to \infty$
R	reflection coefficient of TE_{01} mode
$\lvert R\rvert$	modulus of reflection coefficient
r	radial coordinate (fig. 2)
r_o	dummy coordinate
$\mathscr{R}e$	real part of
t	time
z	axial coordinate (fig. 2)
z_o	dummy coordinate
α_n	propagation constant of TE_{on} mode in region II
β	Laplace transformation variable
β_n	propagation constant of TE_{on} mode in region I
β_1'	real part of β_1
β_1''	negative imaginary part of β_1
Γ_n	eigenvalue of TE_{on} mode in region I
γ_n	eigenvalue of TE_{on} mode in region II
δ_n	eigenvalue of TE_{on} mode in region III
θ	polar angle (fig. 2)
λ	$\left(k_o^2 + \beta^2\right)^{1/2}$
μ_o	magnetic permeability of free space
ρ_n	propagation constant of TE_{on} mode in region III
φ	phase of reflection coefficient
ω	angular frequency

WAVEGUIDE MODES

Region I

The general solution for the electromagnetic field in region I is of the form (ref. 8, pp. 69 to 72)

$$E_\theta(r,z) = A_1 J_1(\Gamma_1 r)e^{-j\beta_1 z} + RA_1 J_1(\Gamma_1 r)e^{j\beta_1 z} + \sum_{n=2}^{\infty} A_n J_1(\Gamma_n r)e^{\beta_n z} \tag{1a}$$

$$H_r(r,z) = -\frac{A_1\beta_1}{\omega\mu_o} J_1(\Gamma_1 r)e^{-j\beta_1 z} + \frac{RA_1\beta_1}{\omega\mu_o} J_1(\Gamma_1 r)e^{j\beta_1 z} + \sum_{n=2}^{\infty} \frac{A_n\beta_n}{j\omega\mu_o} J_1(\Gamma_n r)e^{\beta_n z} \tag{1b}$$

$$H_z(r,z) = -\frac{A_1\Gamma_1}{j\omega\mu_o} J_0(\Gamma_1 r)e^{-j\beta_1 z} - \frac{RA_1\Gamma_1}{j\omega\mu_o} J_0(\Gamma_1 r)e^{j\beta_1 z} - \sum_{n=2}^{\infty} \frac{A_n\Gamma_n}{j\omega\mu_o} J_0(\Gamma_n r)e^{\beta_n z} \tag{1c}$$

where the A_n are complex amplitude constants. A time dependence of $e^{j\omega t}$ is implicit in these equations. The propagation constants β_n must satisfy the equations

$$\beta_1^2 = k_o^2 - \Gamma_1^2 \tag{2a}$$

$$\beta_n^2 = \Gamma_n^2 - k_o^2 \qquad n > 1 \tag{2b}$$

where $k_o \equiv \omega/c$, since the general solution must satisfy Maxwell's equations. The eigenvalues Γ_n are determined by the boundary condition that requires $E_\theta(r,z)$ to vanish at the perfectly conducting surface $r = b$ (see fig. 2). Thus,

$$J_1(\Gamma_n b) = 0 \tag{3}$$

which has the roots $\Gamma_1 b = 3.8317$, $\Gamma_2 b = 7.0156$, . . ., $\Gamma_n b \approx (n + 1/4)\pi$ (ref. 9, p. 166).

The first two terms in the solutions (1a) to (1c) are recognized as the incident and reflected TE_{01} circular waveguide modes, respectively, where R is the reflection coefficient. The remaining terms correspond to the TE_{on} circular waveguide modes. It has been assumed that k_o is in the range $\Gamma_1 < k_o < \Gamma_2$ so that the TE_{01} mode is a propagating mode and the TE_{on} $(n > 1)$ modes are evanescent or cutoff modes.

Region II

The general solution for the electromagnetic field in region II is of the form (ref. 8, pp. 69 to 72)

$$E_\theta(r, z) = \sum_{n=1}^{\infty} B_n J_1(\gamma_n r) e^{-\alpha_n z} \tag{4a}$$

$$H_r(r, z) = -\sum_{n=1}^{\infty} \frac{B_n \alpha_n}{j\omega\mu_o} J_1(\gamma_n r) e^{-\alpha_n z} \tag{4b}$$

$$H_z(r, z) = -\sum_{n=1}^{\infty} \frac{B_n \gamma_n}{j\omega\mu_o} J_0(\gamma_n r) e^{-\alpha_n z} \tag{4c}$$

where the B_n are complex amplitude constants. Since the general solutions must satisfy Maxwell's equations, the α_n are given by

$$\alpha_n^2 = \gamma_n^2 - k_o^2 \tag{5}$$

The eigenvalues γ_n are determined by the boundary condition that requires $E_\theta(r, z)$ to vanish at $r = a$ (see fig. 2). Thus,

$$J_1(\gamma_n a) = 0 \tag{6}$$

which has the roots $\gamma_1 a = 3.8317$, $\gamma_2 a = 7.0156$, . . ., $\gamma_n a \approx (n + 1/4)\pi$ (ref. 9, p. 166).

The terms in the solution are recognized as the TE_{on} circular waveguide modes. It has been assumed that k_o is less than γ_1 so that all the TE_{on} modes in region II are cutoff modes.

Region III

The solution for the electromagnetic field in region III is of the form (ref. 8, pp. 77 to 80)

$$E_\theta(r,z) = \sum_{n=1}^{\infty} C_n\left[N_1(\delta_n a)J_1(\delta_n r) - J_1(\delta_n a)N_1(\delta_n r)\right]e^{-\rho_n z} \tag{7a}$$

$$H_r(r,z) = -\sum_{n=1}^{\infty} \frac{C_n\rho_n}{j\omega\mu_o}\left[N_1(\delta_n a)J_1(\delta_n r) - J_1(\delta_n a)N_1(\delta_n r)\right]e^{-\rho_n z} \tag{7b}$$

$$H_z(r,z) = -\sum_{n=1}^{\infty} \frac{C_n\delta_n}{j\omega\mu_o}\left[N_1(\delta_n a)J_0(\delta_n r) - J_1(\delta_n a)N_0(\delta_n r)\right]e^{-\rho_n z} \tag{7c}$$

where the C_n are complex amplitude constants. The condition that the solution must satisfy Maxwell's equations requires

$$\rho_n^2 = \delta_n^2 - k_o^2 \tag{8}$$

The boundary conditions require $E_\theta(r,z)$ to vanish at $r = a$ and $r = b$ (see fig. 2). The boundary condition at $r = a$ is satisfied automatically by equation (7a). The boundary condition at $r = b$ requires the eigenvalues δ_n to satisfy the equation

$$N_1(\delta_n a)J_1(\delta_n b) - J_1(\delta_n a)N_1(\delta_n b) = 0 \tag{9}$$

The roots of this equation are dependent on the ratio b/a. For example, if $b/a = 1.2$, $\delta_1 a = 15.7277$, $\delta_2 a = 31.4259$, . . ., $\delta_n a \approx n\pi/[(b/a) - 1]$ (ref. 9, p. 204).

The terms in solution (7) correspond to the TE_{on} coaxial waveguide modes. It has been assumed that k_o is less than δ_1 so that all the TE_{on} modes in region III are cutoff modes.

SOLUTION FOR REFLECTION COEFFICIENT

Integral Equation for Electric Field

Since the electric and magnetic fields are related by Maxwell's equations, only the electric field needs to be determined to specify the total field uniquely. Thus, the θ-component of the electric field, which is the only nonvanishing component of the electric field, can be regarded as a potential function from which the magnetic field components can be derived.

The formal solution for the reflection coefficient R will be obtained by setting up an integral equation for the scattered electric field due to the TE_{01} mode incident on the cylindrical partition.

Green's function. - The integral equation will be formulated by employing the Green's function, $G(r, r_o, z-z_o)$, defined to be the solution of the differential equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} + \left(k_o^2 - \frac{1}{r^2}\right)G = -\delta(z - z_o)\,\frac{\delta(r - r_o)}{r_o} \tag{10}$$

that satisfies the boundary condition

$$G(b, r_o, z-z_o) = 0 \tag{11}$$

The Green's function can be thought of as being the θ-component of the electric field radiated from a filamentary loop antenna of radius r_o located at $z = z_o$ in an infinitely long circular waveguide of radius b.

The solution for the Green's function will be obtained by using a mode-expansion technique. A general solution of equation (10) that satisfies boundary condition (11) is of the form

$$G(r, r_o, z-z_o) = \sum_{n=1}^{\infty} D_n(r_o, z-z_o) J_1(\Gamma_n r) \tag{12}$$

where the D_n are expansion coefficients to be determined, and the Γ_n are given by equation (3).

Substituting the general solution into equation (10) gives

$$\sum_{n=1}^{\infty}\left\{D_n\left[\frac{\partial^2 J_1(\Gamma_n r)}{\partial r^2} + \frac{1}{r}\frac{\partial J_1(\Gamma_n r)}{\partial r} + \left(k_o^2 - \frac{1}{r^2}\right)J_1(\Gamma_n r)\right] + J_1(\Gamma_n r)\,\frac{\partial^2 D_n}{\partial z^2}\right\} = -\delta(z - z_o)\frac{\delta(r - r_o)}{r_o} \tag{13}$$

It can easily be shown that the quantity in brackets is equal to $\left(k_o^2 - \Gamma_n^2\right) J_1(\Gamma_n r)$. Thus, equation (13) can be reduced to the form

$$\sum_{n=1}^{\infty} \left[\frac{\partial^2 D_n}{\partial z^2} + \left(k_o^2 - \Gamma_n^2\right) D_n\right] J_1(\Gamma_n r) = - \delta(z - z_o) \frac{\delta(r - r_o)}{r_o} \tag{14}$$

A differential equation for each of the D_n can be obtained by using the orthogonal properties of the Bessel functions. Multiplying both sides of equation (14) by $rJ_1(\Gamma_n r)$ and integrating with respect to r over the interval $0 \leq r \leq b$ give

$$\frac{\partial^2 D_n}{\partial z^2} + \left(k_o^2 - \Gamma_n^2\right) D_n = - \frac{2}{b^2} \frac{J_1(\Gamma_n r_o)}{[J_0(\Gamma_n b)]^2} \delta(z - z_o) \tag{15}$$

The solution to equation (15) must be continuous at $z = z_o$ and have a discontinuity in the derivative with respect to z at $z = z_o$ of an amount $-(2/b^2)\left\{J_1(\Gamma_n r_o)\Big/[J_0(\Gamma_n b)]^2\right\}$. In addition, the solution must correspond to an outward propagating or decaying wave at $z = \pm \infty$. The particular solution that meets all these requirements is

$$D_n(r_o, z-z_o) = \frac{J_1(\Gamma_n r_o)}{\left(\Gamma_n^2 - k_o^2\right)^{1/2} b^2 [J_0(\Gamma_n b)]^2} e^{-\left(\Gamma_n^2 - k_o^2\right)^{1/2} |z - z_o|} \tag{16}$$

where the branch of $\left(\Gamma_n^2 - k_o^2\right)^{1/2}$ must be selected so that $\mathcal{R}e\left(\Gamma_n^2 - k_o^2\right)^{1/2} \geq 0$ and $\mathcal{I}m\left(\Gamma_n^2 - k_o^2\right)^{1/2} \geq 0$ to satisfy the conditions at infinity. If the expression for D_n is substituted into the general solution (12), the Green's function can be expressed in the form

$$G(r, r_o, z-z_o) = \frac{J_1(\Gamma_1 r_o) J_1(\Gamma_1 r)}{j\beta_1 b^2 [J_0(\Gamma_1 b)]^2} e^{-j\beta_1 (z-z_o)} + \sum_{n=2}^{\infty} \frac{J_1(\Gamma_n r_o) J_1(\Gamma_n r)}{\beta_n b^2 [J_0(\Gamma_n b)]^2} e^{-\beta_n (z-z_o)} \qquad z > z_o$$

(17a)

$$G(r, r_o, z-z_o) = \frac{J_1(\Gamma_1 r_o) J_1(\Gamma_1 r)}{j\beta_1 b^2 [J_0(\Gamma_1 b)]^2} e^{j\beta_1(z-z_o)} + \sum_{n=2}^{\infty} \frac{J_1(\Gamma_n r_o) J_1(\Gamma_n r)}{\beta_n b^2 [J_0(\Gamma_n b)]^2} e^{\beta_n(z-z_o)} \qquad z < z_o \tag{17b}$$

As $|z - z_o|$ becomes large, the value of G is given by the first term in either equation (17a) or (17b).

Electric field. - The θ-component of the electric field $E_\theta(r, z)$ must satisfy the differential equation

$$\frac{\partial^2 E_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial E_\theta}{\partial r} + \frac{\partial^2 E_\theta}{\partial z^2} + \left(k_o^2 - \frac{1}{r^2}\right) E_\theta = 0 \tag{18}$$

and the following boundary conditions (see fig. 2):

$$E_\theta(b, z) = 0 \qquad \text{all } z \tag{19a}$$

$$E_\theta(a, z) = 0 \qquad \text{for } z > 0 \tag{19b}$$

In addition, the electric field must correspond to the incident TE_{01} mode as $z \rightarrow -\infty$. Thus,

$$\lim_{z \rightarrow -\infty} E_\theta(r, z) = J_1(\Gamma_1 r) e^{-j\beta_1 z} \tag{20}$$

The reflected TE_{01} mode does not appear in equation (20) since it will later be assumed that the medium within the waveguide has small electrical losses so that the reflected field will have a negligible amplitude at $z = -\infty$.

Integral equation. - Multiplying equation (10) by E_θ and equation (18) by G and then subtracting give

$$E_\theta \nabla^2 G - G \nabla^2 E_\theta = - \delta(z - z_o) \frac{\delta(r - r_o)}{r_o} E_\theta(r, z) \tag{21}$$

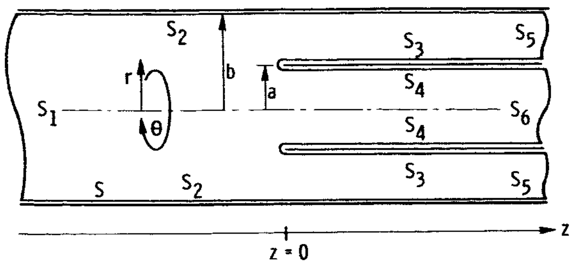

Figure 3. - Surface S.

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Next, each term in equation (21) will be integrated over the volume V bounded by the surface S, as shown in figure 3. The various portions of the surface have been denoted S_1 to S_6 for ease in reference. The surfaces S_1, S_5, and S_6 are at infinity. Performing the integration over the volume V gives

$$\int_V \left(E_\theta \nabla^2 G - G\nabla^2 E_\theta\right) dV = -2\pi E_\theta(r_o, z_o) \tag{22}$$

The integral over the volume V can be reduced to an integral over the closed bounding surface S by using Green's theorem (ref. 10, pp. 803 and 804) with the result

$$\oint_S \left(E_\theta \frac{\partial G}{\partial n} - G \frac{\partial E_\theta}{\partial n}\right) dS = -2\pi E_\theta(r_o, z_o) \tag{23}$$

where n is the outward normal coordinate to the surface S.

Since both E_θ and G are zero along the perfectly conducting surface $r = b$ (boundary conditions eqs. (11) and (19a)) the portion of the surface integral over S_2 will vanish. The surface integral over S_5 and S_6 will also vanish since E_θ is zero at $z = +\infty$ due to all the TE_{on} modes in regions II and III being cutoff modes. In addition, the first term in the surface integral must vanish along the surfaces S_3 and S_4 since E_θ is zero along the perfectly conducting surface $r = a$ for $z > 0$ (boundary condition eq. (19b)). Thus, the only contributions to the surface integral will come from the integration over S_1 and from the second term in the integrand over S_3 and S_4.

It can easily be shown by using the asymptotic forms of E_θ and G as $z \to -\infty$ that the integral over the surface S_1 is equal to $-2\pi J_1(\Gamma_1 r_o)e^{-j\beta_1 z_o}$. Along the surfaces S_3 and S_4, the quantity $\partial E_\theta/\partial n$ is equal to $-\partial E_\theta/\partial r$ and $\partial E_\theta/\partial r$, respectively. Equation (23) can therefore be expressed in the following form:

$$E_\theta(r_o, z_o) = J_1(\Gamma_1 r_o)e^{-j\beta_1 z_o} - a\int_0^\infty G(a, r_o, z-z_o)\left[\frac{\partial E_\theta}{\partial r}(r, z)\Bigg|_{r=a+} - \frac{\partial E_\theta}{\partial r}(r, z)\Bigg|_{r=a-}\right]dz \tag{24}$$

In order to simplify the form of equation (24), the function $J_\theta(a, z)$ will be defined as

$$J_\theta(a, z) = \frac{1}{j\omega\mu_o}\left[\frac{\partial E_\theta(r, z)}{\partial r}\Bigg|_{r=a+} - \frac{\partial E_\theta(r, z)}{\partial r}\Bigg|_{r=a^-}\right] \tag{25}$$

It can be easily shown by using Maxwell's equations that $J_\theta(a, z)$ is the electric surface current density on the surface $r = a$ for $z > 0$. The value of $J_\theta(a, z)$ is zero, of course, for $z < 0$. Thus, equation (24) can be rewritten in the form

$$E_\theta(r, z) = J_1(\Gamma_1 r)e^{-j\beta_1 z} - j\omega\mu_o a \int_0^\infty G(r, a, z-z_o)J_\theta(a, z_o)dz_o \tag{26}$$

where r has been interchanged with r_o, and z with z_o for clarity. The symmetry property of the Green's function $G(a, r, z_o-z) = G(r, a, z-z_o)$ (see eqs. (17)) was used to obtain equation (26)). Equation (26) shows that the total electric field E_θ is the sum of the TE_{01} mode incident field given by the first term on the right plus an integral that corresponds to the field radiated by the electric current on the surface $r = a$ for $z > 0$.

It will be convenient in the following analysis to split the electric field E_θ into two parts: an incident field E_θ^i and a scattered field E_θ^s so that

$$E_\theta(r, z) = E_\theta^i(r, z) + E_\theta^s(r, z) \tag{27}$$

where

$$E_\theta^i(r, z) = J_1(\Gamma_1 r)e^{-j\beta_1 z} \tag{28}$$

Combining equations (26) to (28) gives the desired integral equation for the scattered electric field E_θ^s:

$$E_\theta^s(r,z) = -j\omega\mu_o a \int_{-\infty}^{\infty} G(r,a,z-z_o)J_\theta(a,z_o)dz_o \tag{29}$$

In going from equation (26) to equation (29), the lower limit on the integral was changed from 0 to $-\infty$ since $J_\theta(a,z_o)$ is zero in the range $-\infty < z_o < 0$.

Solution of Integral Equation

Laplace transforms. - The solution of the integral equation for the scattered electric field will be obtained by using Laplace transform and Wiener-Hopf techniques. Let the functions $\mathcal{E}(r,\beta)$, $\mathcal{G}(r,a,\beta)$, and $\mathcal{J}(a,\beta)$ be the bilateral Laplace transforms with respect to z of $E_\theta^s(r,z)$, $G(r,a,z)$, and $J_\theta(a,z)$, respectively:

$$\mathcal{E}(r,\beta) = \int_{-\infty}^{\infty} E_\theta^s(r,z)e^{-\beta z}\,dz \tag{30a}$$

$$\mathcal{G}(r,a,\beta) = \int_{-\infty}^{\infty} G(r,a,z)e^{-\beta z}\,dz \tag{30b}$$

$$\mathcal{J}(a,\beta) = \int_{-\infty}^{\infty} J_\theta(a,z)e^{-\beta z}\,dz \tag{30c}$$

In order to make all the Laplace transforms exist in a common region in the complex β-plane, the propagation constant β_1 will be made complex. This is equivalent to introducing an energy loss mechanism into the medium interior to the waveguide. Let $\beta_1 = \beta_1' - j\beta_1''$ where $\beta_1' > 0$ and $\beta_1'' > 0$. In the final solution, β_1'' will be set equal to zero to recover the solution for the lossless case.

The functions $\mathcal{E}$ and $\mathcal{J}$ are, at this point, unknown since E_θ^s and J_θ are unknown. However, it can be shown by using the expressions for the general forms of the solutions given by equations (1), (4), and (7) that $\mathcal{E}$ is analytic in the strip $-\beta_1'' < \mathcal{R}e\beta < \beta_1''$ and that $\mathcal{J}$ is analytic in the region $\mathcal{R}e\beta > -\beta_1''$.

The transform of the Green's function can be computed at once by taking the transform of the solution for G given by equations (17). This approach will yield the transformed Green's function in the form of an infinite series. It will be more convenient in the following analyses, however, to compute the Green's function by first transforming the basic equation and boundary condition for the Green's function given by equations (10) and (11) and then solve the transformed equations. This approach will yield the solution for the transformed Green's function in closed form.

The Laplace transforms of equations (10) and (11) are

$$\frac{\partial^2 \mathscr{G}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathscr{G}}{\partial r} + \left(k_o^2 + \beta^2 - \frac{1}{r^2}\right)\mathscr{G} = -\frac{\delta(r - a)}{a} \tag{31a}$$

$$\mathscr{G}(b, a, \beta) = 0 \tag{31b}$$

where the variable r_o has been set equal to a to conform with equation (30b). It should be noted that the transform of G is taken with respect to z with $z_o = 0$. The solution of equation (31a) that is bounded in the interval $0 \leq r \leq b$ and satisfies equation (31b) is of the form

$$\mathscr{G}(r, a, \beta) = A(a, \beta)J_1(\lambda r) \qquad r \leq a \tag{32a}$$

$$\mathscr{G}(r, a, \beta) = B(a, \beta)\Big[J_1(\lambda b)N_1(\lambda r) - J_1(\lambda r)N_1(\lambda b)\Big] \qquad r \geq a \tag{32b}$$

where $\lambda = (k_o^2 + \beta^2)^{1/2}$.

The functions $A(a, \beta)$ and $B(a, \beta)$ are determined from the conditions imposed by the right side of equation (31a), which requires $\mathscr{G}$ to be continuous at $r = a$ and to have a discontinuity in the first derivative with respect to r at $r = a$ of an amount $-1/a$. Solving for A and B and then substituting the results into equations (32) give

$$\mathscr{G}(r, a, \beta) = -\frac{\pi}{2}\left[\frac{J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b)}{J_1(\lambda b)}\right] J_1(\lambda r) \qquad r \leq a \tag{33a}$$

$$\mathscr{G}(r, a, \beta) = -\frac{\pi}{2}\left[\frac{J_1(\lambda b)N_1(\lambda r) - J_1(\lambda r)N_1(\lambda b)}{J_1(\lambda b)}\right] J_1(\lambda a) \qquad r \geq a \tag{33b}$$

Since $\mathscr{G}$ is an even function of λ, the choice of either branch of λ will yield the same result for $\mathscr{G}$. It can be shown that $\mathscr{G}$ is analytic in the strip $-\beta_1'' < \mathscr{R}e\beta < \beta_1''$.

The Laplace transform of the integral equation for the scattered electric field given by equation (29) is

$$\mathcal{E}(r, \beta) = -j\omega\mu_0 a\mathcal{G}(r, a, \beta)\mathcal{J}(a, \beta) \tag{34}$$

where the faltung theorem (ref. 10, pp. 464 and 465) has been used to take the transform of the integral. It will be convenient to express the transform of the scattered electric field $\mathcal{E}$ as the sum of two single-sided transforms $\mathcal{E}^+(r, \beta)$ and $\mathcal{E}^-(r, \beta)$ where

$$\mathcal{E}^+(r, \beta) = \int_0^{\infty} E_\theta^s(r, z)e^{-\beta z}\, dz \tag{35a}$$

$$\mathcal{E}^-(r, \beta) = \int_{-\infty}^{0} E_\theta^s(r, z)e^{-\beta z}\, dz \tag{35b}$$

The function $\mathcal{E}^-$ is, at this point, unknown. The function $\mathcal{E}^+$, however, can be computed at the radius $r = a$ by using the fact that the scattered electric field at $r = a$ for $z > 0$ must simply be the negative of the incident electric field since the total electric field must be zero to satisfy the proper boundary condition. Using the expression for the incident field given by equation (28) gives

$$\mathcal{E}^+(a, \beta) = -\int_0^{\infty} E_\theta^i(a, z)e^{-\beta z}\, dz = -\int_0^{\infty} J_1(\Gamma_1 a)e^{-j\beta_1 z}\, e^{-\beta z}\, dz = -\frac{J_1(\Gamma_1 a)}{\beta + j\beta_1} \tag{36}$$

The transform of $E_\theta^i(a, z)$ is only defined in the region $\mathcal{R}e\beta > -\beta_1''$.

Combining the result for $\mathcal{E}^+$ with equations (34) and (35) yields a form for the transformed integral equation that can be solved by using the Wiener-Hopf technique:

$$\mathcal{E}^-(a, \beta) - \frac{J_1(\Gamma_1 a)}{\beta + j\beta_1} = -j\omega\mu_0 a\mathcal{G}(a, a, \beta)\mathcal{J}(a, \beta) \tag{37}$$

It should be noted that, when the previous equations were combined to obtain equation (37), the radius r had to be set equal to a in each of the equations since $\mathcal{E}^+$ could only be computed at $r = a$.

Wiener-Hopf technique. - Equation (37) is a transformed integral equation of the Wiener-Hopf type (ref. 10, pp. 978 to 980). This equation can be solved for $\mathcal{E}^-$ if the transformed Green's function $\mathcal{G}$ can be decomposed into the factors $\mathcal{G}^+$ and $\mathcal{G}^-$ so that

$\mathscr{G} = \mathscr{G}^+/\mathscr{G}^-$, where $\mathscr{G}^+$ is analytic and nonzero for $\mathscr{R}e\beta > -\beta_1''$ and $\mathscr{G}^-$ is analytic and nonzero for $\mathscr{R}e\beta < \beta_1''$. This step allows equation (37) to be rewritten in the form

$$\mathscr{E}^-(a,\beta)\mathscr{G}^-(a,a,\beta) - \frac{J_1(\Gamma_1 a)\mathscr{G}^-(a,a,\beta)}{\beta + j\beta_1} = -j\omega\mu_0 a\mathscr{G}^+(a,a,\beta)\mathscr{J}(a,\beta) \tag{38}$$

The left side of equation (38) is analytic in the region $\mathscr{R}e\beta < \beta_1''$ except for the pole at $\beta = -j\beta_1$ introduced by the second term, whereas the right side of equation (38) is analytic for $\mathscr{R}e\beta > -\beta_1''$. The left side can be made analytic everywhere in the region $\mathscr{R}e\beta < \beta_1''$ by adding the term $[J_1(\Gamma_1 a)\mathscr{G}^-(a,a,-j\beta_1)]\big/(\beta + j\beta_1)$ to each side of the equation. Thus,

$$\mathscr{E}^-(a,\beta)\mathscr{G}^-(a,a,\beta) + J_1(\Gamma_1 a)\frac{\mathscr{G}^-(a,a,-j\beta_1) - \mathscr{G}^-(a,a,\beta)}{\beta + j\beta_1}$$

$$= \frac{J_1(\Gamma_1 a)\mathscr{G}^-(a,a,-j\beta_1)}{\beta + j\beta_1} - j\omega\mu_0 a\mathscr{G}^+(a,a,\beta)\mathscr{J}(a,\beta) \tag{39}$$

Now, the left side is analytic for $\mathscr{R}e\beta < \beta_1''$ and the right side is analytic for $\mathscr{R}e\beta > -\beta_1''$. The equality in equation (39) holds only in the strip $-\beta_1'' < \mathscr{R}e\beta < \beta_1''$.

A function $\mathscr{F}(\beta)$ can be defined from equation (39) so that $\mathscr{F}(\beta)$ is equal to the left side of equation (39) for $\mathscr{R}e\beta \leq -\beta_1''$, the right side for $\mathscr{R}e\beta \geq \beta_1''$, and to either side in the strip, $-\beta_1'' < \mathscr{R}e\beta < \beta_1''$. Thus, $\mathscr{F}(\beta)$ is analytic everywhere in the finite complex β-plane.

It can be shown by using the asymptotic expressions for $\mathscr{E}^-, \mathscr{J}, \mathscr{G}^+$, and $\mathscr{G}^-$ for large β, derived in the following section, that $\mathscr{F}(\beta)$ must also approach zero as $\beta \to \infty$. Consequently, $\mathscr{F}(\beta)$ must be zero everywhere since zero is the only function that is analytic everywhere and vanishes at infinity (Liouville's theorem). Since $\mathscr{F}(\beta)$ is zero, each side of equation (39) must also be zero. Thus,

$$\mathscr{E}^-(a,\beta) = -\frac{J_1(\Gamma_1 a)\left[\mathscr{G}^-(a,a,-j\beta_1) - \mathscr{G}^-(a,a,\beta)\right]}{\mathscr{G}^-(a,a,\beta)(\beta + j\beta_1)} \tag{40a}$$

$$\mathscr{J}(a,\beta) = \frac{J_1(\Gamma_1 a)\mathscr{G}^-(a,a,-j\beta_1)}{j\omega\mu_0 a\mathscr{G}^+(a,a,\beta)(\beta + j\beta_1)} \tag{40b}$$

Edge conditions. - The conditions that have been specified for $\mathscr{G}^+$ and $\mathscr{G}^-$ are that $\mathscr{G} = \mathscr{G}^+/\mathscr{G}^-$, $\mathscr{G}^+$ be analytic and nonzero for $\mathscr{R}e\beta > -\beta_1''$, and $\mathscr{G}^-$ be analytic and nonzero for $\mathscr{R}e\beta < \beta_1''$. These conditions do not uniquely specify this pair of functions since both $\mathscr{G}^+$ and $\mathscr{G}^-$ can be multiplied by any function that is analytic and nonzero everywhere in the finite β-plane to yield a new pair of functions that will satisfy all the original conditions. Any such pair of functions when combined with equations (40) will yield a solution for the scattered electric field that satisfies all the conditions imposed on the field up to this point.

The remaining boundary condition that must be imposed to specify the field uniquely is commonly called the edge condition (ref. 11). The edge condition requires the total electric field E_θ when evaluated at $r = a$ to be of the order $z^{1/2}$ as $z \to 0$. The edge condition also requires the current density J_θ to be of the order $z^{-1/2}$ as $z \to 0$.

Equation (28) shows that the incident electric field E_θ^i is of order 1 as $z \to 0$. Thus, the scattered electric field must also be of order 1 as $z \to 0$ since the scattered field must have a component of this order to nullify the effect of the incident field so that the total field can be of the order $z^{1/2}$ as $z \to 0$. The edge conditions can therefore be stated as

$$E_\theta^s(a, z) = \mathscr{O}(1) \qquad z \to 0 \tag{41a}$$

$$J_\theta(a, z) = \mathscr{O}(z^{-1/2}) \qquad z \to 0 \tag{41b}$$

It can be easily shown that, if a function is of the order z^α as $z \to 0$ $(\alpha > -1)$, its Laplace transform with respect to z will be of the order $\beta^{-(\alpha+1)}$ as $\beta \to \infty$. Consequently, the edge conditions require the transformed quantities $\mathscr{E}^-$ and $\mathscr{J}$ to satisfy the conditions

$$\mathscr{E}^-(a, \beta) = \mathscr{O}(\beta^{-1}) \qquad \beta \to \infty \tag{42a}$$

$$\mathscr{J}(a, \beta) = \mathscr{O}(\beta^{-1/2}) \qquad \beta \to \infty \tag{42b}$$

Combining equations (40) and (42) reveals that the functions $\mathscr{G}^+$ and $\mathscr{G}^-$ must be of the order $\beta^{-1/2}$ and $\beta^{1/2}$, respectively, as $|\beta| \to \infty$.

Wiener-Hopf factors of Green's function. - The transformed Green's function evaluated at $r = a$ is given by (see eq. (33))

$$\mathscr{G}(a, a, \beta) = -\frac{\pi}{2}\frac{J_1(\lambda a)}{J_1(\lambda b)}\left[J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b)\right] \tag{43}$$

where $\lambda = \left(k_0^2 + \beta^2\right)^{1/2}$. The decomposition of $\mathscr{G}$ into the ratio of $\mathscr{G}^+$ to $\mathscr{G}^-$ is best carried out by expressing $\mathscr{G}$ in the form of an infinite product.

The infinite product expansion of $J_1(\lambda a)/\lambda a$ is given by (ref. 10, pp. 382 to 385):

$$\frac{J_1(\lambda a)}{\lambda a} = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\gamma_n^2}\right) \tag{44}$$

since the function $J_1(\lambda a)/\lambda a$ is equal to 1 at $\lambda = 0$, has a zero derivative with respect to λ at $\lambda = 0$, and has simple zeros at $\lambda = \pm\gamma_n$ (see eq. (6)). The infinite product expansion of $J_1(\lambda b)/\lambda b$ can be obtained from equation (44) simply by replacing a with b and γ_n with Γ_n. Thus,

$$\frac{J_1(\lambda b)}{\lambda b} = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\Gamma_n^2}\right) \tag{45}$$

The infinite product expansion of the function $J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b)$ is given by

$$J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b) = \frac{2}{\pi}\left(\frac{b}{a} - \frac{a}{b}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\delta_n^2}\right) \tag{46}$$

since this function is equal to $2/\pi(b/a - a/b)$ at $\lambda = 0$, has a zero derivative with respect to λ at $\lambda = 0$, and has simple zeros at $\lambda = \pm\delta_n$ (see eq. (9)).

Combining equations (44) to (46) reveals that the transformed Green's function can be written in the infinite product form

$$\mathscr{G}(a, a, \beta) = -\left(1 - \frac{a^2}{b^2}\right) \frac{\prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\gamma_n^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\delta_n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{\gamma^2}{\Gamma_n^2}\right)} \tag{47}$$

Equation (47) can be put into a more useful form by reintroducing the parameters β_n, α_n, and ρ_n from equations (2), (5), and (8), respectively. Thus,

$$\mathscr{G}(a, a, \beta) = \left(1 - \frac{a^2}{b^2}\right) \frac{\prod_{n=1}^{\infty}\left(\frac{\alpha_n^2 - \beta^2}{\gamma_n^2}\right)\prod_{n=1}^{\infty}\left(\frac{\rho_n^2 - \beta^2}{\delta_n^2}\right)}{\left(\frac{\beta_1^2 + \beta^2}{\Gamma_1^2}\right)\prod_{n=2}^{\infty}\left(\frac{\beta_n^2 - \beta^2}{\Gamma_n^2}\right)} \tag{48}$$

Equation (48) shows that the transformed Green's function has simple zeros at $\beta = \pm\alpha_n$ and $\beta = \pm\rho_n$ and simple poles at $\beta = \pm j\beta_1$ and $\beta = \pm\beta_n (n > 1)$. Each of the infinite products can now be easily factored into the product of two infinite products so that one infinite product is analytic and nonzero in the region $\mathscr{R}e\beta < \beta_1''$, and the other is analytic and nonzero in the region $\mathscr{R}e\beta > -\beta_1''$. The result of this operation allows $\mathscr{G}^+$ and $\mathscr{G}^-$ to be easily identified as

$$\mathscr{G}^+(a, a, \beta) = \left(1 - \frac{a^2}{b^2}\right) \frac{\prod_{n=1}^{\infty}\left(\frac{\alpha_n + \beta}{\gamma_n}\right)e^{-\beta a/n\pi}\prod_{n=1}^{\infty}\left(\frac{\rho_n + \beta}{\delta_n}\right)e^{-\beta(b-a)/n\pi}}{\left(\frac{\beta + j\beta_1}{\Gamma_1}\right)e^{-\beta b/\pi}\prod_{n=2}^{\infty}\left(\frac{\beta_n + \beta}{\Gamma_n}\right)e^{-\beta b/n\pi}}\, p(\beta) \tag{49a}$$

$$\mathscr{G}^-(a, a, \beta) = \frac{\left(\frac{\beta - j\beta_1}{\Gamma_1}\right)e^{\beta b/\pi}\prod_{n=2}^{\infty}\left(\frac{\beta_n - \beta}{\Gamma_n}\right)e^{\beta b/n\pi}}{\prod_{n=1}^{\infty}\left(\frac{\alpha_n - \beta}{\gamma_n}\right)e^{\beta a/n\pi}\prod_{n=1}^{\infty}\left(\frac{\rho_n - \beta}{\delta_n}\right)e^{\beta(b-a)/n\pi}}\, p(\beta) \tag{49b}$$

The quantity $1 - a^2/b^2$ has been arbitrarily associated with the factor $\mathscr{G}^+$. The exponential terms in equations (49) were introduced to make the infinite products converge. The logarithmic derivative of each of the infinite products generates an infinite series. If the exponential terms were absent, the n^{th} term in each series would be of the order of $1/n$; consequently, each series and the associated infinite products would diverge. The exponential terms require the n^{th} term of the series to be of the order of $1/n^2$, which ensures that both the infinite series and infinite products will converge. The form of these exponential terms is not unique since they are only required to generate terms of the order $1/n$ that will exactly cancel the existing terms of order $1/n$ in the infinite series.

The function $p(\beta)$ in equations (49) must be selected to give the functions $\mathcal{G}^+$ and $\mathcal{G}^-$ the proper algebraic behavior for large β, as discussed in the previous section. In order to solve for $p(\beta)$ it will be necessary to determine the asymptotic forms of the infinite products for large β.

Consider the infinite product

$$\prod_{n=1}^{\infty} \left(\frac{\alpha_n + \beta}{\gamma_n}\right) e^{-\beta a/n\pi}$$

Since α_n and γ_n approach $n\pi/a$ as $n \to \infty$, it is apparent that the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{\beta a}{n\pi}\right) e^{-\beta a/n\pi}$$

only differs from the previous infinite product by a bounded function of β. Consequently, the asymptotic forms of the previous two infinite products will be identical for large β.

The gamma function $\Gamma(x)$ can be expressed in infinite product form as (ref. 10, pp. 421 and 422)

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \tag{50}$$

where γ is Euler's constant ($\gamma = 0.5772157\ldots$). The asymptotic form of $\Gamma(x)$ is (ref. 10, p. 424)

$$\Gamma(x) \simeq (2\pi)^{1/2}\, x^{x-(1/2)} e^{-x} \tag{51}$$

as $x \to \infty$. Combining equations (50) and (51) shows that

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \simeq (2\pi)^{-1/2}\, x^{-[x+(1/2)]} e^{(1-\gamma)x} \tag{52}$$

as $x \to \infty$. Thus, the initial infinite product

$$\prod_{n=1}^{\infty} \left(\frac{\alpha_n + \beta}{\gamma_n}\right) e^{-\beta a / n\pi}$$

is of the order

$$\left(\frac{\beta a}{\pi}\right)^{-[(\beta a/\pi)+(1/2)]} e^{(1-\gamma)\beta a/\pi}$$

as $\beta \to \infty$. This same procedure can be applied to each of the infinite products appearing in equations (49) with the result

$$\mathscr{G}^{+}(a, a, \beta) = \mathcal{O}\beta^{-1/2} \exp\left[\frac{\beta b}{\pi} \ln\left(\frac{b}{b - a}\right) - \frac{\beta a}{\pi} \ln\left(\frac{a}{b - a}\right)\right] p(\beta) \tag{53a}$$

as $\beta \to \infty$, and

$$\mathscr{G}^{-}(a, a, \beta) = \mathcal{O}\beta^{1/2} \exp\left[\frac{\beta b}{\pi} \ln\left(\frac{b}{b - a}\right) - \frac{\beta a}{\pi} \ln\left(\frac{a}{b - a}\right)\right] p(\beta) \tag{53b}$$

as $\beta \to -\infty$. Since $\mathscr{G}^{+}$ and $\mathscr{G}^{-}$ must be of the order of $\beta^{-1/2}$ and $\beta^{1/2}$, respectively, as $|\beta| \to \infty$ it is apparent from equations (53) that a suitable choice for $p(\beta)$ is

$$p(\beta) = \exp -\left[\frac{\beta b}{\pi} \ln\left(\frac{b}{b - a}\right) - \frac{\beta a}{\pi} \ln\left(\frac{a}{b - a}\right)\right] \tag{54}$$

Thus, the solution for the Wiener-Hopf factors of the transformed Green's function is completed.

Scattered electric field. - The scattered electric field evaluated at $r = a$ can now be computed by inverting the transformed field $\mathscr{E}$:

$$E_{\theta}^{s}(a, z) = \frac{1}{2\pi j} \int_{C} \mathscr{E}(a, \beta) e^{\beta z}\, d\beta \tag{55}$$

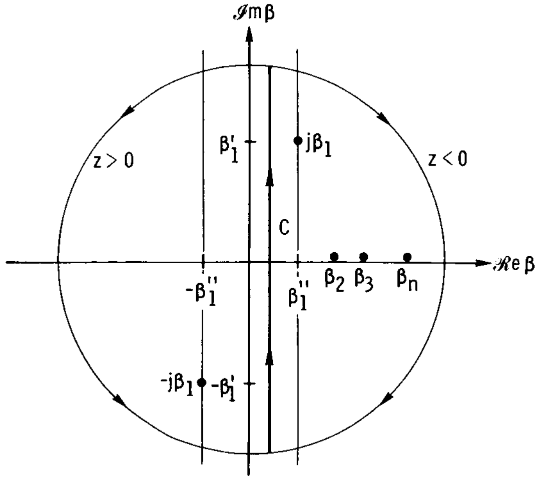

Figure 4. - Complex β-plane.

The inversion contour C must be located in the strip $-\beta_1'' < \mathcal{R}e\,\beta < \beta_1''$, as shown in figure 4, since this is the only common region in the β-plane in which all the transforms are analytic.

The transformed field $\mathcal{E}(a, \beta)$ is the sum of $\mathcal{E}^+(a, \beta)$ given by equation (36) and $\mathcal{E}^-(a, \beta)$ given by equation (40a). Combining equations (36) and (40a) with equation (55) gives

$$E_\theta^s(a, z) = -\frac{1}{2\pi j}\int_C \frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{(\beta + j\beta_1)\mathcal{G}^-(a, a, \beta)}\, e^{\beta z}\, d\beta \tag{56}$$

To evaluate the field for $z < 0$, the contour C can be closed in the right half β-plane with a semicircle of infinite radius, as shown in figure 4. It can be easily shown that the integral over this semicircle is zero. Thus, the integral over the original contour C must equal $-2\pi j$ times the sum of the residues of the poles of the integrand in the right half β-plane. The integrand has poles in the right half β-plane due to the zeros of $\mathcal{G}^-$ at $\beta = j\beta_1$ and at $\beta = \beta_n (n > 1)$. Performing the integration gives

$$E_\theta^s(a, z) = +\frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{2j\beta_1 \left.\dfrac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta=j\beta_1}}\, e^{j\beta_1 z}$$

$$+\sum_{n=2}^{\infty} \frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{(\beta_n + j\beta_1)\left.\dfrac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta=\beta_n}}\, e^{\beta_n z} \qquad z < 0 \tag{57}$$

Comparing equations (1a) and (57) shows that the first term in equation (57) corresponds to the reflected TE_{01} mode, and the terms in the summation correspond to the evanescent $TE_{0n} (n > 1)$ modes.

To evaluate the field for $z > 0$, the contour C can be closed in the left half β-plane with a semicircle of infinite radius, as shown in figure 4. It can be shown that the integral over this semicircle is also zero. Thus, the integral over the original contour must equal $2\pi j$ times the sum of the residues of the poles of the integrand in the left half β-plane.

The integrand in equation (56) has only one pole in this half plane located at $\beta = -j\beta_1$. Thus,

$$E_\theta^S(a, z) = -J_1(\Gamma_1 a)e^{-j\beta_1 z} \qquad z > 0 \tag{58}$$

Equation (58) shows that the scattered electric field evaluated at $r = a$ for $z > 0$ is simply the negative of the incident field. This result should be expected. Since the incident field does not satisfy the proper boundary condition at $r = a$ for $z > 0$, the scattered field must contain a term of the same form as the incident field for $z > 0$ to nullify this improper solution. Terms that correspond to the various TE_{on} modes that exist in the region $z > 0$ do not appear in equation (58) since the electric field associated with these modes must vanish when evaluated along the perfectly conducting surface $r = a$ for $z > 0$.

From this point on, the propagation constant β_1 will be regarded as being real so that the subsequent results will apply for the case where there are no losses.

Reflection Coefficient

The total electric field in the region $z < 0$ can now be obtained by adding the incident field given by equation (28) to the scattered field given by equation (57) with the result

$$E_\theta(r, z) = J_1(\Gamma_1 r)e^{-j\beta_1 z} + \frac{\mathcal{I}^-(a, a, -j\beta_1)}{2j\beta_1 \left.\frac{\partial \mathcal{I}^-}{\partial \beta}(a, a, \beta)\right|_{\beta = j\beta_1}} J_1(\Gamma_1 r)e^{j\beta_1 z}$$

$$+ \sum_{n=2}^{\infty} \frac{\mathcal{I}^-(a, a, -j\beta_1)J_1(\Gamma_1 a)}{(\beta_n + j\beta_1) \left.\frac{\partial \mathcal{I}^-}{\partial \beta}(a, a, \beta)\right|_{\beta = \beta_n}} \frac{J_1(\Gamma_n r)}{J_1(\Gamma_n a)} e^{\beta_n z} \qquad z < 0 \tag{59}$$

In obtaining equation (59), the amplitude factor of each TE_{on} mode in the scattered field was multiplied by $J_1(\Gamma_n r)/J_1(\Gamma_n a)$ to reintroduce the radial dependence of the modes.

If equation (59) is compared with the general solution for the field in the region $z < 0$ given by equation (1a), the reflection coefficient R of the TE_{01} mode can be easily identified as

$$R = \frac{\mathscr{G}^-(a, a, -j\beta_1)}{2j\beta_1 \left. \frac{\partial \mathscr{G}^-}{\partial \beta}(a, a, \beta) \right|_{\beta = j\beta_1}} \tag{60}$$

Combining equation (60) with the expression for $\mathscr{G}^-$ given by equation (49b) allows R to be put into the form

$$R = -e^{-2j\beta_1 b/\pi} \frac{\prod_{n=2}^{\infty} \left(\frac{\beta_n + j\beta_1}{\Gamma_n}\right) e^{-j\beta_1 b/n\pi} \prod_{n=1}^{\infty} \left(\frac{\alpha_n - j\beta_1}{\gamma_n}\right) e^{j\beta_1 a/n\pi} \prod_{n=1}^{\infty} \left(\frac{\rho_n - j\beta_1}{\delta_n}\right) e^{j\beta_1 (b-a)/n\pi} \; p(j\beta_1)}{\prod_{n=2}^{\infty} \left(\frac{\beta_n - j\beta_1}{\Gamma_n}\right) e^{j\beta_1 b/n\pi} \prod_{n=1}^{\infty} \left(\frac{\alpha_n + j\beta_1}{\gamma_n}\right) e^{-j\beta_1 a/n\pi} \prod_{n=1}^{\infty} \left(\frac{\rho_n + j\beta_1}{\delta_n}\right) e^{-j\beta_1 (b-a)/n\pi} \; p(j\beta_1)} \tag{61}$$

A careful examination of equation (61) shows that, apart from the first term $-\exp(-2j\beta_1 b/\pi)$, which has unit modulus, the expression for R is simply the ratio of two complex numbers with the numerator being the complex conjugate of the denominator. Thus, it must follow that $|R| = 1$.

This result should be expected. The TE_{01} mode incident on the coaxial cylindrical partition shown in figure 2 (p. 2) must be perfectly reflected since it has been assumed that all the TE_{on} modes in regions II and III are cutoff modes and that there are no losses. Thus, the reflected TE_{01} mode must have the same amplitude as the incident mode.

The argument or phase of R that will be denoted by φ must equal the sum of twice the phase of the numerator in equation (61), since the numerator is the complex conjugate of the denominator, plus π, which results from the initial minus sign. Thus,

$$\varphi = \pi + 2\left[\sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\beta_1}{\beta_n}\right) - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\beta_1}{\alpha_n}\right) - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\beta_1}{\rho_n}\right)\right]$$

$$+ 2\left[\frac{\beta_1 b}{\pi} \ln\left(\frac{b}{b - a}\right) - \frac{\beta_1 a}{\pi} \ln\left(\frac{a}{b - a}\right)\right] \qquad (62)$$

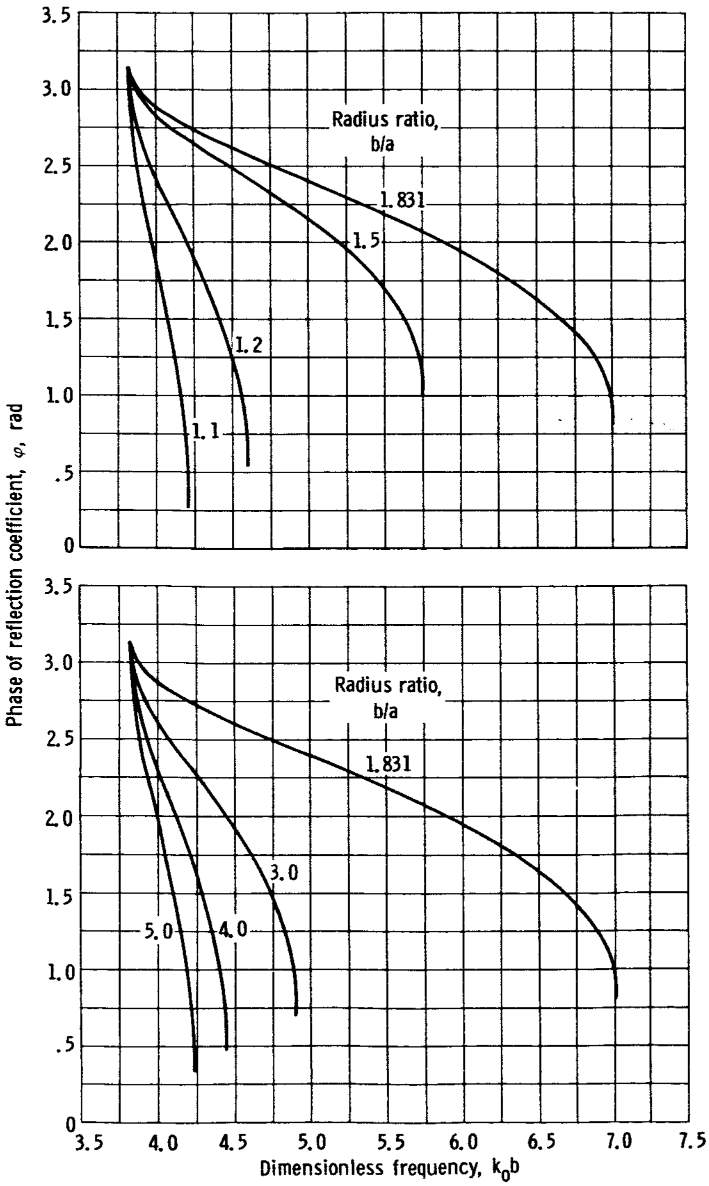

Figure 5. - Phase of reflection coefficient against dimensionless frequency for constant values of radius ratio.

The terms in the summations are recognized as the phases of the various infinite products that appear in equation (61). The last term corresponds to twice the phase of $p(-j\beta_1)$. It should be noted that the exponential terms introduced to make the infinite products converge did not appear in the final solutions for the magnitude and phase of the reflection coefficient.

The phase of the reflection coefficient is not unique since any integral multiple of 2π can be added to φ without changing the numerical value of the reflection coefficient. The values of φ given by equation (62) will be in the range $0 < \varphi < 2\pi$.

Numerical values for φ were obtained by using an IBM 7094 computer. The first 100 terms in each of the infinite series were retained. It was estimated by using an integral technique that truncating the series at 100 terms introduces an error in φ of less than 0.01 radian. The numerical results are shown in figure 5 in the form of curves of φ against $k_0 b$ for typical values of the parameter b/a. The

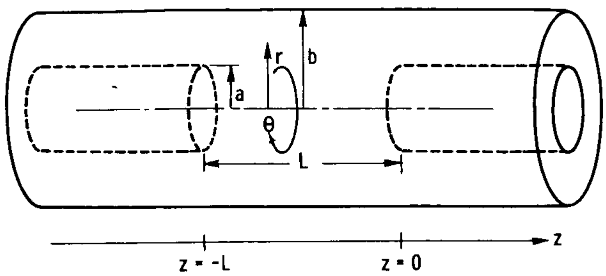

Figure 6. - Model of open-ended cavity.

range of $k_0 b$ is bounded. The lower limit is determined by the condition that $k_0 b$ must exceed $\Gamma_1 b$, which has the constant value 3.8317 so that the TE_{01} mode in region I can propagate. The upper limit is determined by the condition that $k_0 b$ must be less than the smaller of $\gamma_1 b$ or $\delta_1 b$. This condition ensures that neither the TE_{01} mode in region II nor the TE_{01} mode in region III will propagate. The values of $\gamma_1 b$ and $\delta_1 b$ are dependent on the parameter b/a. It can be shown that, if b/a is less than 1.831, $\gamma_1 b < 7.0153 < \delta_1 b$, and if b/a exceeds 1.831, $\delta_1 b < 7.0153 < \gamma_1 b$. For the case where b/a = 1.831, the TE_{01} modes in regions II and III have identical low-frequency cutoff points, which correspond to the values $\gamma_1 b = \delta_1 b = 7.0153$.

The numerical results for φ show that at the lower limit of $k_0 b$ the value of φ is equal to π for all values of the parameter b/a. Thus, at the lower limit of $k_0 b$ the cylindrical partition reflects the incident TE_{01} mode as if a perfectly conducting surface spanned the waveguide cross section at z = 0 (see fig. 2, p. 2) since a perfectly conducting surface gives a reflection coefficient of unit modulus and argument π. As $k_0 b$ increases, the incident TE_{01} mode is still perfectly reflected since the modulus of R remains at 1. However, since the phase of the reflection coefficient decreases with increasing $k_0 b$, it appears as though the position of the perfectly conducting surface moves. Thus, the cylindrical partition acts like a "virtual" perfectly conducting surface that spans the entire waveguide cross section whose position is a function of $k_0 b$ and the parameter b/a.

RESONANT FREQUENCY OF CAVITY

An open-ended cavity can be constructed by placing two cylindrical partitions of finite length within a circular waveguide as shown in figure 6. The partitions need only be long compared with $1/\alpha_1$ and $1/\rho_1$ for the formula for the reflection coefficient of the infinitely long partition to apply. To determine the conditions for resonance in the cavity, consider the electric field in the region between the partitions to be the sum of TE_{01} modes propagating in the positive and negative z-directions. Thus,

$$E_\theta(r,z) = AJ_1(\Gamma_1 r)e^{-j\beta_1 z} + BJ_1(\Gamma_1 r)e^{j\beta_1 z} \tag{63}$$

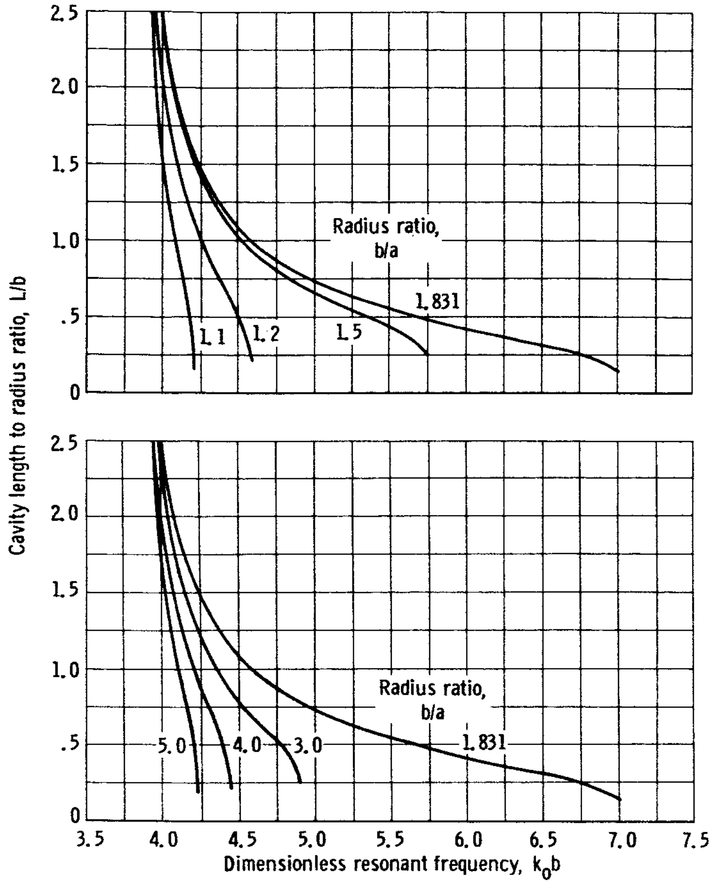

Figure 7. - Ratio of cavity length to radius against dimensionless resonant frequency for constant values of radius ratio.

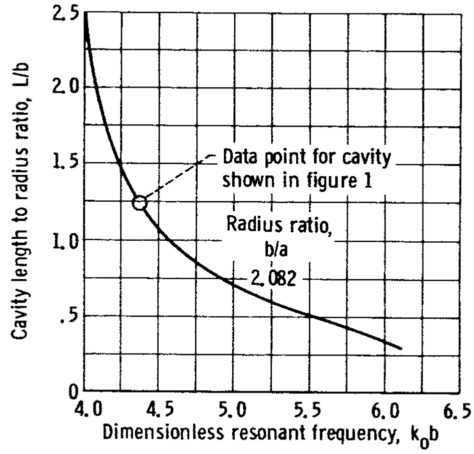

Figure 8. - Ratio of cavity length to radius against dimensionless resonant frequency for radius ratio of 2.082.

At $z = 0$, the ratio of the reflected wave $BJ_1(\Gamma_1 r)$ to the incident wave $AJ_1(\Gamma_1 r)$ must equal the reflection coefficient R. Likewise, at $z = -L$, the ratio of the reflected wave $AJ_1(\Gamma_1 r)e^{j\beta_1 L}$ to the incident wave $BJ_1(\Gamma_1 r)e^{-j\beta_1 L}$ must also equal R. Since R is equal to $e^{j\varphi}$ these conditions are simply

$$\frac{B}{A} = e^{j\varphi} \tag{64a}$$

and

$$\frac{Ae^{j\beta_1 L}}{Be^{-j\beta_1 L}} = e^{j\varphi} \tag{64b}$$

It can be easily shown that, if there is to be a nontrivial solution for the amplitudes A and B, φ and L must be related by the equation

$$e^{j\beta_1 L} - e^{2j\varphi_e} e^{-j\beta_1 L} = 0 \tag{65}$$

which has the solution

$$\beta_1 L = \varphi \tag{66}$$

The required spacing between the partitions to resonate the TE_{011} mode can be found by solving equation (66) for L by using the values of φ given in figure 5. The spacing for the TE_{01n} resonant mode can be obtained simply by adding $(n - 1)2\pi$ to φ before solving for L.

The numerical results for the TE_{011} mode are shown in figures 7 and 8 in the form of curves of L/b against $k_0 b$ for typical values of the parameter b/a. The range of frequencies over which the cavity can be tuned by varying L/b is a strong function of b/a. The value b/a = 1.831 gives the maximum tuning range.

The numerical results shown in figure 8 are for b/a = 2.082. This value is commonly used in practice since it corresponds to locating the cylindrical partition at the radius where the electric field of the TE_{01} mode has its maximum intensity (i.e., $J_1(\Gamma_1 r)$ has a maximum at $r = b/2.082$). This value of b/a was used in the design of the cavity shown in figure 1 (p. 2) where the radius a was taken as the mean radius of the cylindrical partition. A data point corresponding to the measured values of L/b and $k_0 b$ for this cavity, as shown in figure 8, indicates that the theoretical and experimental results are in good agreement.

CONCLUSIONS

A theory has been developed for a class of open-ended circular cylindrical microwave cavities by using Laplace transform and Wiener-Hopf techniques. Numerical results were presented for the resonant frequency of the TE_{011} mode. The theory was shown to be in good agreement with experimental data.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, April, 6, 1966.

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