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THE EQUATORIAL ELLIPTICITY OF THE EARTH FROM TWO MONTHS OF SYNCOM II DRIFT OVER THE CENTRAL PACIFIC

by C. A. Wagner

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ABSTRACT

This is the second in a series of reports on the use of tracking information from NASA's Syncom II satellite to define the longitude-dependent gravity field of the earth. This tracking information shows that Syncom II drifted freely in the gravity fields of the earth, sun and moon from 25 April to 3 July 1964. In early May, the ascending Equator crossing of the orbit, as determined from radar range and range-rate tracking at the Goddard Space Flight Center, was at 121° West, with a westward drift rate of 0.81 degrees/day. During the following two-month period, this drift rate decreased steadily until late in June when, with the ascending Equator crossing at 161° West, the westward drift rate was 0.75 degrees/day. This decelerating drift rate is entirely consistent with the magnitude and phase angle of the equatorial ellipticity of the earth, as previously determined by this author. These ellipticity constants were responsible for the accelerated drift of this satellite in the longitudes from 55° to 64° West. By utilizing the first (energy) integral of the drift motion of a 24-hour satellite in a second-order longitude-dependent earth gravity field, the drift of Syncom II over the Central Pacific during the two months in question was seen to be sensitive to the following parameters of the earth's equatorial ellipticity.

$$J_{22} = -(1.71 \pm 0.22) \times 10^{-6} ,$$

which corresponds to a difference in the major and minor equatorial axes of the earth of $a_0 - b_0 = 214 \pm 28$ feet, and

$$\lambda_{22} = -(17.1 \pm 4.9) \text{ degrees},$$

which gives the longitude of the major equatorial axis with respect to Greenwich. These new results confirm the postulate (made in an earlier study) that higher order longitude-dependent earth gravity has small influence on the long term drift of the high-altitude 24-hour satellite. The consistency of the new drift data indicates that the earth's mass inhomogeneities of third order are of smaller magnitude than previously supposed by most geodesists.

CONTENTS

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INTRODUCTION

This paper is one of a series dealing with the utilization of the Syncom II satellite to gain insight into the exact nature of the earth's gravity field. During long periods between gross orbit change maneuvers initiated by on-board propulsion, this nearly synchronous satellite has remained over limited longitude regions of the earth for a time sufficient for it to accumulate noticeable drift perturbations caused by small longitude mass anomalies within the earth. The first gravity experiment with this satellite (Reference 1) was designed to test the sensitivity of its observed drift to the simplest kind of longitude-dependent gravity; namely, that associated with an earth whose equator is elliptical, not circular. As a result of that experiment the very strong assertion was made that the earth is, in fact, a triaxial ellipsoid whose gross elliptic equatorial "bulge" amounts to between 200 and 250 feet. This new gravity investigation, which essentially confirms the previous result, examines drift data from Syncom II for two months during the period April-June 1964. The mean daily drift rate of the satellite was 0.8 degrees/day during these months, and the nearly figure eight ground track progressively swept over 45° of longitude from 117° West to 163° West.

It was shown in Reference 1 that at a drift rate larger than 0.39 degrees/day a 24-hour satellite with the inclination of Syncom II would circulate completely around the world if left to drift freely. Clearly, Syncom II in April-June 1964 had a period far from "very close to synchronous." The approximate theory of *drift* as a function of *time* for such a satellite, used in Reference 1 to analyse Syncom II data from August-February 1963-1964, was not serviceable for this "fast-drift" regime. The essential point is that the "very close to synchronous" theory dealt with long term drift in a limited longitude range. In fact, it assumed that the perturbation force averaged over the orbit referred to a single mean longitude position of the satellite, the longitude at synchronism. For data over a large longitude arc, it has been found most convenient and efficient to use a data reduction theory stemming from the exact expression of *drift rate* as a function of *longitude position* for a 24-hour satellite. The important point in the applicability of this theory is that the net longitude drift of the satellite is limited over a single orbit. In the case of Syncom II during

April-June 1964, the small drift of 0.8° per orbit allowed the direct application of the theory to the ascending equator crossing of the satellite. For drift regimes with rates much larger than this, this simple theory for the drift of the ascending equator crossing probably should be modified to apply to the drift of the mean of the longitudes of the ascending equator crossings over a single orbit. Moreover, if the eccentricity of the 24-hour satellite is fairly large, R. R. Allan* suggests that this simple theory for the 24-hour circular orbit satellite would still apply with the longitude position taken as the mean of ascending and descending equator crossings.

At any rate, aside from the use of the first integral of the drift motion instead of the second, the analysis for longitude-dependent earth gravity in this report proceeds with the same basic assumptions as those of the previous report, namely:

1. Only second-order longitude-dependent earth gravity (associated with an earth whose equator is elliptical) was sensed during the long term drift of Syncom II in April-June 1964;
2. All non-gravity perturbations on the orbit of Syncom II during this period were negligible;
3. The earth zonal and sun and moon gravity effects on Syncom II drift can be adequately represented over the span of a few months by a small bias in the reduced results. The bias can be estimated from simulated Syncom II trajectories, numerically integrated, as explained below.

Assumptions 2 and 3 proved good for the previous data analysis. Assumption 1 was not evaluated in that analysis. Calculations of probable higher order earth effects on the bias of recently reported geoids shows (Reference 2) that the Syncom II reduction in Reference 1 for the second-order effect only has a bias of no more than 25% in the reported magnitude of that effect. The major conclusion of this report is that assumption 1 is probably satisfied to a greater degree than this for *all* the Syncom II drift data thus far available.

The theory of Syncom II drift rate variation in a second order field is presented. The application of this theory to the calculation of the second order field from equator crossing data is then discussed. As shown in Reference 1, this second order drift theory is only strictly exact for a perfectly synchronous satellite, and ignores sun, moon and other earth perturbations. While the errors in the theory due to these sources were proved to be small in the previous Syncom analysis, there is no guarantee that they will be similarly small for the fast drift Syncom II in April-June 1964. Therefore, in order both to verify the theory for this Syncom drift orbit and gain insight into the likely magnitude of these "model errors," the field calculating method stemming from the drift rate theory is tested on simulated Syncom data. This simulated data consists of equator crossings closely paralleling the observed Syncom crossings for this period.

The simulated data is derived by numerical integration of the motion of Syncom II starting from the observed elements in late April 1964. The closely paralleling numerically integrated Syncom trajectory is disturbed by the sun and moon. It is also disturbed by an earth which

*Private communication.

includes, as inputs to the integrating program, two second order longitude gravity constants believed to be close to those for the real earth. The proposed field calculating method, applied to this simulated data, estimates these input constants. The discrepancy between this estimate and the input constants is found to be small. This discrepancy is then taken as a good measure of the model error in the constants derived, in the final section of the report, from the observed Syncom crossing data.

THEORY OF THE REDUCTION

Equation 36 in Reference 1 gives the first integral of the drift motion of a 24-hour satellite in a second-order longitude-dependent earth gravity field as:

$$(\dot{\gamma})^2 = A_{22} \cos 2\gamma + C_1 \quad , \tag{1}$$

where γ is the geographic longitude of the ascending equator crossing of the 24-hour satellite east of the equatorial *minor* axis, C_1 is an integration constant for the motion, and A_{22} is a function of the semimajor axis (a_s) and the inclination (i_s) of the satellite's orbit, the mean radius (R_0) of the earth's equator, and the magnitude (J_{22}) of second-order longitude gravity. A_{22} is given by

$$A_{22} = -72\pi^2 J_{22} \left(\frac{R_0}{a_s}\right)^2 \frac{(\cos^2 i_s + 1)}{2} \quad . \tag{2}$$

Let λ be the geographic longitude of the ascending equator crossing of the 24-hour satellite at a certain time. If λ_{22} is the geographic longitude of the equatorial *major* axis, then $\lambda = \gamma - (90° - \lambda_{22})$ from Figure 3 of Reference 2 and Equation 1 can be rewritten as:

$$(\dot{\lambda})^2 = -A_{22} \cos 2(\lambda - \lambda_{22}) + C_1 \quad . \tag{3}$$

It is evident from Equation 3 that if three sets of simultaneous values of $\dot{\lambda}$ and λ are available from position and velocity observations on the 24-hour satellite, then a unique set of values of A_{22}, λ_{22} and C_1 can be calculated. These values, together with the known values of R_0, a_s, and i_s, may be used in Equation 2 in order to calculate J_{22}, the magnitude of the second-order longitude gravity effect. To show the form of the "condition equations" for A_{22} and λ_{22} more explicitly, Equation 3 may be expanded to give

$$(\dot{\lambda})^2 = C_1 + \cos 2\lambda\left(-A_{22} \cos 2\lambda_{22}\right) + \sin 2\lambda\left(-A_{22} \sin 2\lambda_{22}\right) \quad , \tag{4a}$$

or

$$(\dot{\lambda})^2 = C_1 + C_2 \cos 2\lambda + C_3 \sin 2\lambda \quad , \tag{4b}$$

where

$$\lambda_{22} = \frac{1}{2} \tan^{-1} \left(\frac{-C_3}{-C_2} \right) \tag{5}$$

and

$$A_{22} = \left[C_2^2 + C_3^2 \right]^{1/2} \quad . \tag{6}$$

Thus, three or more sets of $\dot{\lambda}$ and λ values make it possible to determine a best set of C_1, C_2 and C_3 values from Equation 4; for example, a best set in a "least squares" sense.

From this set "best" values of λ_{22} and A_{22} can be calculated from Equations 5 and 6. The actual observations on which the reduction from Equation 4 will be based are nine sets of orbital elements for Syncom II calculated at the Goddard Space Flight Center from range and range-rate radar data returned from the satellite during April-June 1964 (see Appendix A). For each of these elements, the longitude of the ascending equator crossing nearest to the epoch of the set plus six hours has been calculated numerically using a zonal gravity field through fourth order and a sun and moon gravity field, both of which are almost identical to the fields used in the orbit determination program for the satellite (Appendix A, Table A-2). This particular ascending equator crossing was chosen as that which fell as close as possible to the center of the observations from which the orbit in question was determined. From these equator crossing longitudes and times, longitude rates were estimated by the ratio of the change in longitude to the time elapsed between successive crossing points. In this way the relatively large daily-periodic effect of the sun and moon on the instantaneous "two-body orbit" drift rate was virtually eliminated from the "basic data" before the actual reduction for the second-order longitude gravity effect began.

The question arises as to what geographic longitude, the longitude rate so calculated, belongs. If there were no acceleration in the longitude drift, it would make no difference where this calculated drift rate was applied. But it is clear that with any small acceleration the correct application point is somewhere near the mean of the two successive longitudes. It is shown in Appendix C that approximating the exact longitude at which this calculated orbit rate should be applied, by the mean of successive longitudes, is sufficiently accurate for the weekly data on Syncom II obtained in April-June 1964.

Table 1

Drift Data Reduction from the Simulated Syncom II Trajectory of Appendix B.

1	2	3	4	5	6	7	8	9	10
i	Time from Jan. 0.0, 1964 (days)	Mean Longitude, λ_i (degrees)	$Y_i \equiv (\dot{\lambda}_i)^2$ $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	$2\lambda_i$ (degrees)	$X_{2i} \equiv \cos 2\lambda_i$	$X_{3i} \equiv \sin 2\lambda_i$	Value of Y_i Computed from $\hat{C}$ Given by Equation 15 Y_{ci} $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	$\epsilon_i = Y_i - Y_{ci}$ $\left(10^{-6}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	ϵ_i^2 $\left(10^{-12}\left(\frac{\text{rad}}{\text{day}}\right)^4\right)$
1	120.6	-120.49	2.1005	119.02	-.48511	.87445	2.1003	.02	.0004
2	126.6	-125.46	2.0787	109.08	-.32689	.94506	2.0810	- .23	.0529
3	132.6	-130.40	2.0621	99.20	-.15988	.98714	2.0560	.61	.3721
4	138.6	-135.31	2.0254	89.38	.01082	.99994	2.0262	- .08	.0064
5	144.6	-140.18	1.9884	79.64	.17983	.98370	1.9928	- .44	.1936
6	150.6	-145.01	1.9566	69.98	.34235	.93957	1.9566	.00	.0000
7	156.6	-149.78	1.9090	60.44	.49302	.86984	1.9190	-1.00	1.0000
8	162.6	-154.52	1.8969	50.96	.63006	.77671	1.8807	1.62	2.6244
9	168.6	-159.22	1.8421	41.56	.74826	.66340	1.8430	- .09	.0081
10	174.6	-163.86	1.8029	32.28	.84545	.53406	1.8070	- .41	.1681
			$\sum$ 19.6626		$\sum$ 2.27791	$\sum$ 8.57387			$\sum$ 4.4260
					$\sum^2$ 2.432133	$\sum^2$ 7.567804			

DATA REDUCTION OF THE SIMULATED SYNCOM II TRAJECTORY OF APRIL-JUNE 1964 FOR THE SECOND-ORDER LONGITUDE GRAVITY EFFECT

This scheme of data reduction based on Equation 4 is now tested for a simulated Syncom II trajectory which begins with the actual orbit elements of epoch 64-04-25-02 U.T. in Appendix A and which closely approximates the actual Syncom II trajectory. In the digital program of this simulation (called "ITEM," Interplanetary Trajectory by an Enke Method, at Goddard Space Flight Center), a basic earth zonal gravity and sun-moon field is used which is identical with that used in the actual Syncom II orbital determination program. In addition, a second-order longitude-dependent earth gravity field with $J_{22} = -1.68 \times 10^{-6}$ and $\lambda_{22} = -18°$ is used in the numerical computation of the two-month trajectory. (See Appendix B for the basic data of the simulated trajectory.)

The drift data reduction for this simulated Syncom II trajectory—the reduction for the actual trajectory in the next section follows the same pattern—is described in what follows. Table 1 gives a summary of the reduction.

Column 3 of Table 1 shows the mean longitudes, λ_i, upon which the data reduction is made. They are simply the mean longitudes of successive weekly equator crossings listed in column 3 of Table B-2. The corresponding drift rates, $\dot{\lambda}_i$, (column 4, Table B-2) are the differences in successive longitude crossings divided by the time interval between these crossings. Column 4 of Table 1 is self explanatory as are columns 5, 6 and 7.

As stated in the previous section, the first problem is that of obtaining good values of c_1, c_2 and c_3 (in Equation 4) from the data. The general procedure is as follows (this applies to the next section as well). Table 1 gives ten sets of corresponding values of $(\dot{\lambda})^2$ and λ, but they are inconsistent in the sense that any three sets used in Equation 4 will give values of c_1, c_2 and c_3 that in general are not the same as the values obtained for a different set of three. In order to make use of all the data, it is assumed that the λ's are known exactly but that the values of the $(\dot{\lambda})^2$ are subject to slight random error. Then for the i^{th} set $(\dot{\lambda}_i)^2 = c_1 + c_2 \cos 2\lambda_i + c_3 \sin 2\lambda_i + \epsilon_i$ and the problem is that of establishing a criterion for estimating c_1, c_2 and c_3 from the ten equations $i = 1$ to 10. The least squares method of estimation (Reference 3) seems most appropriate here, and it requires that those estimates of c_1, c_2 and c_3 be chosen which minimize the sum

$$\sum_{i=1}^{10} \epsilon_i^2 .$$

The simple change in notation indicated in the headings of columns 4, 6 and 7 of Table 1 is made in order to conform with the notation used in the statistical literature. When the change in notation is made, the equations take the form

$$Y_i = X_{1i} C_1 + X_{2i} C_2 + X_{3i} C_3 + \epsilon_i, \quad i = 1, \cdots 10, \qquad X_{1i} \equiv 1 . \tag{7}$$

Least squares estimates of C_1, C_2 and C_3 $(\hat{C}_1, \hat{C}_2$ and $\hat{C}_3)$ are found and then the data in column 8, Table 1 is calculated from $Y_{ci} = X_{1i}\hat{C}_1 + X_{2i}\hat{C}_2 + X_{3i}\hat{C}_3$. Columns 9 and 10 of Table 1 are self-explanatory.

It is shown in the theory of least squares estimation (Reference 3) that the $\hat{C}_i$'s must satisfy the so-called normal equations:

$$\sum_1^N Y_i X_{1i} = \hat{C}_1\left[\sum_1^N X_{1i} X_{1i}\right] + \hat{C}_2\left[\sum_1^N X_{1i} X_{2i}\right] + \hat{C}_3\left[\sum_1^N X_{1i} X_{3i}\right] , \tag{8}$$

$$\sum_1^N Y_i X_{2i} = \hat{C}_1\left[\sum_1^N X_{2i} X_{1i}\right] + \hat{C}_2\left[\sum_1^N X_{2i} X_{2i}\right] + \hat{C}_3\left[\sum_1^N X_{2i} X_{3i}\right] , \tag{9}$$

$$\sum_1^N Y_i X_{3i} = \hat{C}_1\left[\sum_1^N X_{3i} X_{1i}\right] + \hat{C}_2\left[\sum_1^N X_{3i} X_{2i}\right] + \hat{C}_3\left[\sum_1^N X_{3i} X_{3i}\right] . \tag{10}$$

From Table 1, those mixed-product sums for the simulated drift data which are not calculated at the bottoms of the columns in Table 1 are

$$\sum Y_i X_{2i} = 4.06013 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 ,$$

$$\sum Y_i X_{3i} = 16.97526 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 ,$$

$$\sum X_{2i} X_{3i} = 1.484560 .$$

Equations 8, 9 and 10 for the simulated Syncom drift of Table 1 are therefore:

$$19.6626 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 10.0000\hat{C}_1 + 2.27791\hat{C}_2 + 8.57387\hat{C}_3 \tag{11}$$

$$4.06013 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 2.27791\hat{C}_1 + 2.432133\hat{C}_2 + 1.484560\hat{C}_3 \tag{12}$$

$$16.97526 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 8.57387\hat{C}_1 + 1.484560\hat{C}_2 + 7.567804\hat{C}_3 \quad . \tag{13}$$

The solution of Equations 11, 12 and 13 for $\hat{C}$ may be accomplished by any convenient algorithm. We choose to use the inverse of the right side coefficient matrix for this solution because the elements of the inverse are also statistically valuable (see Equation 34).

The inverse of the matrix of coefficients on the right sides of Equations 11, 12 and 13 is found to be

$$(\text{COEFF})^{-1} = \begin{Vmatrix} 8.305427 & -2.312087 & -8.955996 \\ -2.312087 & 1.110737 & 2.401565 \\ -8.955996 & 2.401565 & 9.807638 \end{Vmatrix} \quad . \tag{14}$$

When $(\text{COEFF})^{-1}$ is multiplied by the 1×3 matrix of left side constants of Equations 11, 12 and 13, the 1×3 matrix

$$\begin{Vmatrix} \hat{C}_1 \\ \hat{C}_2 \\ \hat{C}_3 \end{Vmatrix}$$

is found to be

$$\hat{C}(\text{simulated data}) = \begin{Vmatrix} 1.88855 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \\ -0.18471 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \\ 0.13970 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \end{Vmatrix} \tag{15}$$

From Equations 15 and 5 the "best" value of λ_{22} for the drift data in Table 1 is

$$\hat{\lambda}_{22,\text{sim}} = \frac{1}{2} \tan^{-1}\left(\frac{-\hat{C}_3}{-\hat{C}_2}\right) = \frac{1}{2} \tan^{-1}\left(\frac{-0.13970}{0.18471}\right) = -18.4° \quad . \tag{15a}$$

Similarly, Equations 15 and 6 give

$$\left(\hat{A}_{22}\right)_{\text{simulated}} = \left(\hat{C}^2_{2,\text{sim.}} + \hat{C}^2_{3,\text{sim.}}\right)^{1/2} = 23.15 \times 10^{-6} \left(\frac{\text{rad}}{\text{solar day}}\right)^2$$

$$= 23.15 \times 10^{-6} \times \left(\frac{1436.06817}{1440.00000}\ \frac{\text{solar day}}{\text{sid. day}}\right)^2$$

$$= 23.02 \times 10^{-6} \left(\frac{\text{rad}}{\text{sid. day}}\right)^2 \tag{16}$$

Data from Table B-1, Appendix B used in Equation 2 yields

$$\left(A_{22}\right)_{\text{sim.}} = \left(J_{22}\right)_{\text{sim.}} \left[-72 \times \pi^2 \times \left(\frac{6378.4}{42230.01}\right)^2 \left(\frac{\cos^2 32.6^\circ + 1}{2}\right)\right]$$

$$= \left(J_{22}\right)_{\text{sim.}} (-13.86) \left(\frac{\text{rad}}{\text{sid. day}}\right)^2 \quad . \tag{17}$$

From Equations 16 and 17 the "best" value of J_{22} for this simulation, according to the drift theory presented in Chapter 2, is

$$\hat{J}_{22,\text{sim.}} = -\frac{23.02 \times 10^{-6}}{13.86} = -1.66 \times 10^{-6} \quad . \tag{18}$$

The simulation actually used the following values of J_{22} and λ_{22}:

$$J_{22,\text{sim.}} = -1.68 \times 10^{-6} \quad ,$$

$$\lambda_{22,\text{sim.}} = -18.0^\circ \quad . \tag{19}$$

Comparison of Equations 15a and 18 with Equation 19 shows that the "model error" of the drift theory presented in the previous section, which ignored drift rate changes due to sun, moon and earth zonal gravity influences, accumulates a small bias in the results for J_{22} and λ_{22} calculated according to the simple theory for a two-month simulated Syncom II drift which closely approximates the actual drift in this period (Figure 1). This model bias (ignoring higher order earth longitude gravity effects) in J_{22} calculated from the theory for this two-month drift is estimated from the above results of the simulated data reduction as

$$J_{22,\text{sim.}} - \hat{J}_{22,\text{sim.}} = -1.68 \times 10^{-6} + 1.66 - 10^{-6}$$

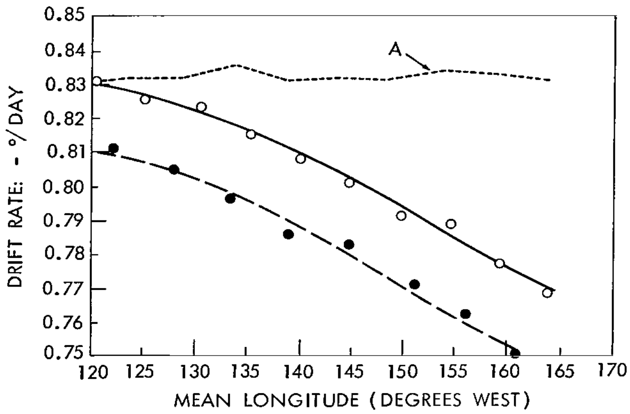
$$= -0.02 \times 10^{-6} \quad . \tag{20}$$

The model bias in λ_{22} is estimated as

$$\lambda_{22,\text{sim.}} - \hat{\lambda}_{22,\text{sim.}} = -18.0 + 18.4 = +0.4^\circ \tag{21}$$

REDUCTION OF NINE SYNCOM II ORBITS IN APRIL-JUNE 1964 TO DETERMINE THE ELLIPTICITY OF THE EARTH'S EQUATOR

Table A-1 in Appendix A lists nine orbits in April-June 1964 for Syncom II computed at GSFC from range and range-rate information without taking into account the anomalous gravity field due to the ellipticity of the earth's equator. Figure 1 shows the observed steady decrease in the westward drift rate of the mean longitude of the satellite for 45° of drift excursion over the Central Pacific in this period. This rate decrease is closely paralleled by that in the simulated drift of Syncom II using a longitude-dependent gravity field which includes sun, moon and high-order earth zonal components. Other simulated trajectories have shown minor, negligible drift rate changes over a duration of time measured by months if longitude-dependent earth gravity is *not* used in the simulation. The experience with the simulated Syncom II trajectory (the previous section and Appendix B), which closely agrees with or closely parallels the actual drift, gives us confidence that the simple reduction theory used successfully in the previous section to determine equatorial ellipticity from the simulated data can be applied as successfully to determine equatorial ellipticity from the actual data. The following Table 2 of drift data has been derived from Appendix A using the nine orbits of Syncom II in April-June 1964.



A. NUMERICALLY CALCULATED DRIFT RATE IN A PARTICLE TRAJECTORY FROM THE SYNCOM 2 ELEMENTS OF EPOCH: 64-04-25-2.0 HRS, WITHOUT LONGITUDE-DEPENDENT EARTH GRAVITY

o DATA FROM A SIMULATED SYNCOM II TRAJECTORY BEGINNING WITH THE ELEMENTS OF EPOCH 64-04-25-02; USING A SUN, MOON, ZONAL EARTH & LONGITUDE-DEPENDENT EARTH GRAVITY FIELD WITH:
$J_{22} = -1.68 \times 10^{-6}$
$\lambda_{22} = 18°$

— THEORETICAL DRIFT IN A SECOND ORDER LONGITUDE-DEPENDENT EARTH GRAVITY FIELD ONLY WITH:
$J_{22} = -1.66 \times 10^{-6}$
$\lambda_{22} = -18.4°$

● DATA FROM ACTUAL SYNCOM II OBSERVATIONS

-- THEORETICAL DRIFT IN A SECOND ORDER LONGITUDE-DEPENDENT EARTH GRAVITY FIELD WITH:
$J_{22} = -1.69 \times 10^{-6}$
$\lambda_{22} = -17.5°$

NOTE: THE DISPLACEMENT OF THE TWO SETS OF DATA IS DUE MAINLY TO THE ERROR IN THE SEMIMAJOR AXIS FOR SYNCOM II, REPORTED FOR EPOCH 64-04-25-02

Figure 1—Drift rate as a function of mean longitude for Syncom II during April-July 1964.

From Table 2, the mixed product sums not calculated at the bottoms of the columns are:

$$\sum Y_i X_{2i} = 2.87010 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 ,$$

$$\sum Y_i X_{3i} = 13.17124 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 ,$$

$$\sum X_{2i} X_{3i} = 1.19093 .$$

Table 2

Drift Data Reduction for Syncom II, April-June 1964.

i	Mean Longitude λ (degrees)	Y_i $(\dot{\lambda}_i)^2$ $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	$2\lambda_i$ (degrees)	X_{2i} $\cos 2\lambda_i$	X_{3i} $\sin 2\lambda_i$	Y_{ci} (Using $\hat{C}$ from Equation 26) $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	$\epsilon_i = Y_i - Y_{ci}$ $\left(10^{-6}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$	ϵ_i^2 $\left(10^{-12}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$
1	-121.22*	2.004	117.55	-.46252	.88661	2.002	.20	.040
2	-128.09	1.973	103.81	-.23870	.97109	1.970	.30	.090
3	-133.70	1.934	92.61	-.04553	.99896	1.936	- .20	.040
4	-138.84	1.882	82.32	.13364	.99103	1.9005	-1.85	3.423
5	-144.72	1.868	70.56	.33282	.94299	1.8535	1.45	2.103
6	-150.935	1.810	58.13	.52799	.84925	1.805	.50	.250
7	-155.94	1.770	48.13	.66744	.74466	1.760	1.00	1.000
8	-160.83†	1.716	38.35	.78424	.62046	1.720	- .40	.160
		$\sum$ 14.957		$\sum$ 1.69938	$\sum$ 7.00505			$\sum$ 7.106
				$\sum^{2}$ 1.74089	$\sum^{2}$ 6.25910			

*On 30 April 1964.
†On 20 June 1964.

Following the technique of data reduction according to the 24-hour satellite drift theory discussed in the previous section, the normal equations for the "best" values (in the least squares sense) of the ellipticity-drift rate parameters c are:

$$14.957 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 8\hat{C}_1 + 1.69938\hat{C}_2 + 7.00505\hat{C}_3 \quad , \tag{22}$$

$$2.87010 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 1.69938\hat{C}_1 + 1.74089\hat{C}_2 + 1.19093\hat{C}_3 \quad , \tag{23}$$

$$13.17124 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 = 7.00505\hat{C}_1 + 1.19093\hat{C}_2 + 6.25910\hat{C}_3 \quad . \tag{24}$$

The inverse of the matrix of coefficients on the right sides of Equations 22, 23 and 24 is:

$$(\text{COEFF})^{-1} = \begin{Vmatrix} 14.008120 & -3.3905080 & -15.032469 \\ -3.3905080 & 1.4810135 & 3.5127885 \\ -15.032469 & 3.5127885 & 16.315400 \end{Vmatrix} \quad . \tag{25}$$

The ellipticity-drift rate parameters

$$(\hat{C}_1)$$
$$(\hat{C}_2)$$
$$(\hat{C}_3)$$

are then found to be:

$$\hat{C}(\text{actual data-unadjusted for model error}) = \begin{Vmatrix} 1.79210 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \\ -0.19339 \times 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \\ 0.13546 - 10^{-4} \left(\frac{\text{rad}}{\text{day}}\right)^2 \end{Vmatrix} \quad . \tag{26}$$

Thus, from Equations 26 and 5 the best value (unadjusted) of λ_{22} (actual) as sensed by the drift of Syncom II in April-July 1964 is

$$\hat{\lambda}_{22}(\text{actual data-unadjusted}) = \frac{1}{2}\tan^{-1}\left(\frac{-\hat{C}_3}{-\hat{C}_2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{-0.13546}{0.19339}\right)$$

$$= -17.5^\circ \quad . \tag{27}$$

For the Syncom II elements in Appendix A, using a_s and i_s from the second set of elements in the drift period (since they are probably better determined than the first), Equation 2 gives

$$A_{22} = J_{22}\left[-72\pi^2 \times \left(\frac{6378.4}{42228.8}\right)^2\right]\left[\frac{\cos^2 32.6^\circ + 1}{2}\right]$$

$$= -J_{22}(13.859)\left(\frac{\text{rad}}{\text{sid day}}\right)^2 \quad . \tag{28}$$

From Equations 26 and 6, the best value (unadjusted) of A_{22} (actual data) as sensed by the drift of Syncom II in April-July 1964 is

$$\hat{A}_{22}(\text{actual data-unadjusted}) = \left(\hat{C}_2^2 + \hat{C}_3^2\right)^{1/2} = 23.6 \times 10^{-6}\left(\frac{\text{rad}}{\text{solar day}}\right)^2$$

$$= 23.6 \times 10^{-6}\left(\frac{\text{rad}}{\text{solar day}}\right)^2 \times 0.99455\left(\frac{\text{sol. day}}{\text{sid. day}}\right)^2$$

$$= 23.47 \times 10^{-6}\left(\frac{\text{rad}}{\text{sid day}}\right)^2 \quad . \tag{29}$$

Equations 28 and 29 show that the best value (unadjusted) of the magnitude of the equatorial ellipticity of the earth sensed by Syncom II in April-July 1964 is

$$\hat{J}_{22}(\text{actual data-unadjusted}) \doteq -1.69 \times 10^{-6} \quad . \tag{30}$$

In adjusting the values given by Equations 27 and 30 for "model error" by means of the biases found in the simulated Syncom II trajectory, it is assumed that all influences on Syncom II drift in this period other than earth, sun and moon gravity, are negligible (see DISCUSSION). The data reduction of the simulated Syncom II trajectory in April-June 1964 in the previous section showed that the simple drift theory presented herein produced a bias in the reduced J_{22} of $-.02 \times 10^{-6}$ and a bias in the reduced λ_{22} of $+0.4°$. Application of these biases adjusts Equations 27 and 30 for all but higher order longitude-dependent earth gravity effects and gives

$$\hat{J}_{22}(\text{actual data-adjusted}) \doteq -(1.69) \times 10^{-6} - 0.02 \times 10^{-6} = -1.71 \times 10^{-6} \quad , \tag{31}$$

$$\hat{\lambda}_{22}(\text{actual data-adjusted}) = -17.5° + 0.4° = -17.1° \quad . \tag{32}$$

To gain insight into the allowable variations of J_{22} and λ_{22} due to the errors of observation and effects not accounted for (model errors) in this gravity experiment, the simplifying assumption is made that all deviations (ϵ_i in Tables 1 and 2) of the actual or simulated data from the theoretical data are random and are normally distributed with mean zero and variance σ^2. It is to be noted that the "best values" from the simulated data reduction established a most likely "systematic" error in J_{22} and λ_{22} for the actual experiment. This systematic error was due to all imperfections in the simple model tested by the simulation and equivalent imperfections are likely to be present in the real data (see DISCUSSION).

If the above assumption is made for the deviations ϵ_i in the simulated data, then an unbiased estimate of σ^2 $(\epsilon_{\text{sim.}})$ calculated from the data of Table 1 is

$$S^2\left(\epsilon_{\text{sim.}}\right) = \frac{4.4260 \times 10^{-12}}{10 - 3}\left(\frac{\text{rad}}{\text{day}}\right)^4 = 0.6323 \times 10^{-12}\left(\frac{\text{rad}}{\text{day}}\right)^4 .$$

An unbiased estimate of the standard deviation S is

$$S\left(\epsilon_{\text{sim.}}\right) = \left(0.6323 \times 10^{-12}\right)^{1/2} = 0.7952 \times 10^{-6}\left(\frac{\text{rad}}{\text{day}}\right)^2 . \tag{33}$$

It can be seen that only for $i = 8$ in Table 1 is the error greater than $2S\left(\epsilon_{\text{sim.}}\right)$ for the experiment as a whole.

The statistical theory for normal distributions (Reference 3) shows that the parameters $\hat{C}_j$ of the normal equations have expected values C_j (the actual drift-ellipticity parameters tested for) and variances $\sigma^2_{\hat{C}_j}$, where

$$\sigma^2 \hat{C}_j = \left(\sigma^2(\epsilon)\right)\left(\text{COEFF}_{jj}\right)^{-1} .$$

$\left(\text{COEFF}_{jj}\right)^{-1}$ denotes the diagonal element in the jth row and jth column of the inverse of the coefficient matrix of the normal equations for the experiment. An unbiased estimate of $\sigma^2_{\hat{C}_j}$ is

$$S^2_{\hat{C}_j} = \left(S^2(\epsilon)\right)\left(\text{COEFF}_{jj}\right)^{-1} . \tag{34}$$

It follows from Equation 34, using the values given by Equations 33 and 14, that

$$S_{\hat{C}_{2,\text{sim.}}} = 0.7952\sqrt{1.111} \times 10^{-6} = 0.0084 \times 10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2 , \tag{35}$$

$$S_{\hat{C}_{3,\text{sim.}}} = 0.7952\sqrt{9.808} \times 10^{-6} = 0.0250 \times 10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2 . \tag{36}$$

Thus the standard deviation bounds on $\hat{C}_{2,sim.}$ and $\hat{C}_{3,sim.}$ are

$$\hat{C}_{2,sim.} = -(0.1847 \pm 0.0084) \times 10^{-4} \left(\frac{rad}{day}\right)^2 , \tag{37}$$

$$\hat{C}_{3,sim.} = (0.140 \pm 0.025) \times 10^{-4} \left(\frac{rad}{day}\right)^2 . \tag{38}$$

The same theory applied to the actual Syncom II data in Table 2 yields

$$S^2 \left(\epsilon_{act.}\right) = \frac{7.106 \times 10^{-12}}{8 - 3} \left(\frac{rad}{day}\right)^4 = 1.4212 \times 10^{-12} \left(\frac{rad}{day}\right)^4 ,$$

and

$$S\left(\epsilon_{act.}\right) = 1.192 \times 10^{-6} \left(\frac{rad}{day}\right)^2 . \tag{39}$$

Comparison of Equations 39 and 33 shows that the standard error of the simulated experiment (which is free from "observation error") is not much smaller than that of the experiment on the actual data. In Reference 1 it was found that the actual experiment had a standard error about five times that of the simulated. Evidently the set of orbits for Syncom II in April-June 1964 is of much greater precision than those of the two earlier long drift periods.

For the actual Syncom II data treated in the same fashion as the simulated data, the gravity experiment leads to the bounds on C computed below.

Unbiased estimates of $\sigma_{\hat{C}_{2,act.}}$ and $\sigma_{\hat{C}_{3,act.}}$ are

$$S_{\hat{C}_{2,act.}} = 1.192 \sqrt{1.481} \times 10^{-6} = 0.0145 \times 10^{-4} \left(\frac{rad}{day}\right)^2 , \tag{40}$$

and

$$S_{\hat{C}_{3,act.}} = 1.192 \sqrt{16.3154} \times 10^{-6} = 0.0481 \times 10^{-4} \left(\frac{rad}{day}\right)^2 . \tag{41}$$

Thus the standard bounds on $\hat{C}_2$ and $\hat{C}_3$ in the actual experiment are

$$S_{\hat{C}_{2,actual\ data}} = -(0.1934 \pm 0.0145) \times 10^{-4} \left(\frac{rad}{day}\right)^2 , \tag{42}$$

$$S_{\hat{C}_{3,actual\ data}} = (0.1355 \pm 0.0481) \times 10^{-4} \left(\frac{rad}{day}\right)^2 . \tag{43}$$

It can be seen that though the standard error of the actual experiment is only 1.5 times that of the simulated, the standard bounds on $\hat{C}$ are more nearly twice that in the simulated experiment. This is most probably due to the fact that two additional data points which sense the equatorial ellipticity were used in the simulated experiment.

In establishing standard bounds on $\hat{J}_{22}$ and $\hat{\lambda}_{22}$ from the experiments, all possible combinations of these quantities from the standard bounds on $\hat{C}_2$ and $\hat{C}_3$ are not computed. Clearly, the chance of obtaining $\hat{C}_2 + S_{\hat{C}_2}$ and $\hat{C}_3 + S_{\hat{C}_3}$ simultaneously from the experiment is less than the chance of obtaining a single standard deviation for each separately. In Appendix E the bounds on the adjusted values of J_{22} and λ_{22} sensed by the Syncom II drift in April-July 1964 are computed as

$$J_{22}\,(\text{actual data}) = -(1.71 \pm 0.22) \times 10^{-6} \quad , \tag{44}$$

and

$$\lambda_{22}\,(\text{actual data}) = -(17.1 \pm 4.9°) \quad . \tag{45}$$

DISCUSSION

The Syncom II drift data reduction for sensitivity to the equatorial ellipticity of the earth, used in this report, is open to a number of lines of criticism.

In the first place it may be fairly asked whether the observed decrease in drift rate can be explained by some force not considered here. The direct evidence of agreement in almost all particulars between the observed trajectory of Syncom II during the period analysed and a simulated trajectory using a recent set of gravity constants for the earth, sun and moon make it unlikely that such an "outside force" could have had appreciable influence on the drift. A number of such possible outside influences on the orbital energy of the satellite were discussed in Reference 1. Solar radiation pressure was found to have an effect on Syncom II less by many orders of magnitude than that of solar gravity. It has never been proven that gas leaks from the on-board propulsion units during the long drift periods. Even if it did, it would not change the energy of the satellite appreciably provided that the accidental unrecorded "on times" of this leakage were reasonably distributed around the spin cycle of the satellite. The magnitude of effects due to micrometeorite drag on the satellite is difficult to estimate because of insufficient data on this phenomenon.

With respect to all "outside influences" except higher order earth longitude gravity, the almost exact prediction on the basis of longitude-dependent earth gravity alone of the long term drift deceleration between 120° West and 165° West from the observed acceleration between 55° West and 65° West (Reference 1) makes it even more unlikely that any such influence had appreciable effect on Syncom II during its "free drift" periods.

A second line of criticism of the gravity results summarized in Equations 44 and 45 stems from the uncertainty of the separation of systematic and random errors in the experiment. The assumption made in the statement of error bounds in Equations 44 and 45 is that all errors in the observed drift rate are randomly distributed about that theoretically calculated from the model. If the unknown "systematic error" due to high-order longitude gravity is ignored, the results of the simulated trajectory reduction appear to show that "model error" was the principal "noise" source in the actual gravity experiment. This is to say that the systematic error due to the insufficiency of the model to predict the exact results in a full gravity field gives, when treated as a random effect, a standard experimental error only somewhat smaller than that associated with the actual data. Furthermore, one can see directly from Figure 1 the close parallelism of drift rate deviations between the actual and simulated trajectories. These facts suggest that the small overall systematic errors of the simulated experiment (for example, $.02 \times 10^{-6}$ in J_{22} and $0.4°$ in λ_{22}) are closer to the actual order of magnitude of the standard errors of the ellipticity parameters than that reported in Equations 44 and 45. In other words, the "best" results from the actual experiments are perhaps closer to reality (neglecting higher order gravity effects), with bounds half as wide as reported and with the same degree of confidence.

On the other hand, preliminary, less refined reductions of more limited Syncom II data in this period gave evidence of a longitude effect near the limits of the bounds in Equations 44 and 45 with only a small increase in the standard experimental error. Hence it is felt that until a more refined analysis of this data is undertaken to remove "model error" point for point, Equations 44 and 45 give fair, at best conservative bounds on the ellipticity of the earth's equator as seen by Syncom II during two months of drift over the Central Pacific.

When these results are compared with those for Syncom II drift over Brazil (Reference 1) and with expected results on the basis of higher order effects as predicted from other sources of gravity data (Reference 2), it would appear that when the full longitude field is finally mapped the "true" value of J_{22} will be closer to -1.7×10^{-6} than -1.9×10^{-6} as predicted in Reference 2. What this means is that the earth's third-order longitude mass deviations (to which Syncom II is next most sensitive) are probably of smaller extent than previously supposed. Exact statements on the bounds of higher order gravity effects as sensed by Syncom II must wait on an integrated analysis of all the drift data for this satellite.

CONCLUSIONS

1. The steadily decreasing drift rate of Syncom II during a two-month excursion over the Central Pacific in April-July 1964 confirms the extent and location of earth equatorial ellipticity predicted earlier from observations on this satellite over Brazil.

2. The results of this new Syncom II gravity experiment show a difference in the major and minor radii of the earth's elliptical equator of

$$a_0 - b_0 = 214 \pm 28 \text{ feet } \left[J_{22} = -(1.71 \pm 0.22) \times 10^{-6}\right] .$$

The longitude location of the semi-major axis of the earth's elliptical equator as sensed by Syncom II during this period is

$$\lambda_{22} = -(17.1 \pm 4.9)^\circ \quad .$$

3. The above results combined with the results of the independent gravity experiment with this satellite when it was over Brazil (Reference 1) indicate that the earth is more homogeneous, to at least third order, than has been supposed.

4. Tentatively, the "best parameters" of equatorial ellipticity as seen by Syncom II to date are

$$J_{22} = -1.7 \times 10^{-6} \quad ,$$

$$\lambda_{22} = -18^\circ \quad .$$

5. Only second-order longitude-dependent earth gravity need be considered in the design of station keeping systems for future 24-hour satellites, and this without appreciable modification for longitude location.

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Appendix A

Orbital Elements and Partially Reduced Drift Data for Syncom II During April-June 1964

Table A-1 gives the elements for Syncom II during April-June 1964 as derived from radar range and range-rate data in the Goddard Space Flight Center's Data Systems and Tracking Directorate. By means of the particle trajectory digital program called "Item" at GSFC, ascending equator crossing longitudes and times nearest to the epoch dates in Table A-1 plus six hours were calculated. These are listed in Table A-2. The choice of epoch date plus six hours was mainly dictated by the desire to use the crossing that was closest to the "center of gravity" of the data that went into the determination of the orbit. The gravity field used for the short term trajectories in these calculations was virtually the same as that used in the orbit determination program that calculated the elements used at the start of each trajectory (see Appendix D). The only difference was the inclusion of J_{22} and λ_{22} in the earth potential of the particle trajectories. Since the values for these constants used in the "Item" program are close to "reality," it is likely that the ascending equator crossings so calculated are even more accurate than an estimation of them made by using precisely the same field as that in the orbit determination program.

Table A-1

Orbital Elements for Syncom II in April-June 1964 as Calculated in the Data Systems and Tracking Directorate at the Goddard Space Flight Center from Radar Range and Range Rate Data (Orbit 12).

Crossing Number, j	Epoch (yr-mo-day-hr-min, U.T.)	Semimajor Axis, a_s (km)	Eccentricity, e	Inclination, i_s (degrees)	Argument of Perigee, ω (degrees)	Mean Anomaly, m (degrees)	Right Ascension of the Ascending Node, B (degrees)
1	64-04-25-02-00.00	42230.01	.00119	32.603	198.716	333.752	313.879
2	64-05-05-16-00.00	42228.81	.00123	32.562	199.105	185.331	313.739
3	64-05-12-16-00.00	42228.04	.00127	32.625	201.959	183.885	313.592
4	64-05-19-14-00.00	42227.80	.00119	32.577	199.859	157.342	313.548
5	64-05-25-15-00.00	42227.65	.00128	32.599	200.270	173.279	313.401
6	64-06-02-21-00.00	42226.72	.00130	32.578	201.136	264.160	313.331
7	64-06-09-21-00.00	42225.74	.00123	32.579	199.640	267.215	313.247
8	64-06-16-15-00.00	42224.41	.00122	32.567	200.824	177.655	313.141
9	64-06-23-15-00.00	42224.17	.00129	32.561	200.344	179.909	313.020

Table A-2

Drift Data for Syncom II During April-June 1964 (Orbit 12).

1	2	3	4	5
Crossing Number, j	Time from Jan. 0.0, 1964 (days)	Longitude of the Ascending Equator Crossing Nearest to Orbit Epoch +6 Hours (degrees)	$\dot{\lambda}$ (degrees/day)	$(\dot{\lambda})^2$ $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$
1	116.6015	-117.169		
			.8111	2.004
2	126.5986	-125.278		
			.8047	1.973
3	133.5948	-130.908		
			.7969	1.934
4	140.5910	-136.483		
			.7860	1.882
5	146.5871	-141.196		
			.7830	1.868
6	155.5820	-148.239		
			.7708	1.810
7	162.5776	-153.631		
			.7622	1.770
8	168.5735	-158.201		
			.7505	1.716
9	175.5687	-163.451		

NOTES:

The data in columns 2 and 3 were derived from the elements in Table A-1 by the numerical calculation of particle trajectories in an earth, sun and moon gravity field specified by the constants in Table B-2. The longitude rates in column 4 are the first longitude differences of column 3 divided by the first time differences of column 2.

Appendix B

Orbital Elements and Partially Reduced Drift Data from the Simulated Syncom II Trajectory During April-June 1964

Table B-1 gives the set of elements at weekly ascending equator crossing times during a simulated Syncom II trajectory (computed numerically by GSFC's "Item" Program) beginning with the actual elements of Syncom II, epoch 64-04-25-02. The gravity field of the particle trajectory was exactly that used in the orbit determination program for Syncom II with the addition of equatorial ellipticity introduced into the geoid of amount and location given by the constants

$$J_{22} = -1.68 \times 10^{-6}$$

and

$$\lambda_{22} = -18^{\circ} \quad .$$

(See Appendix D for a representation of the geoid used in the simulation.)

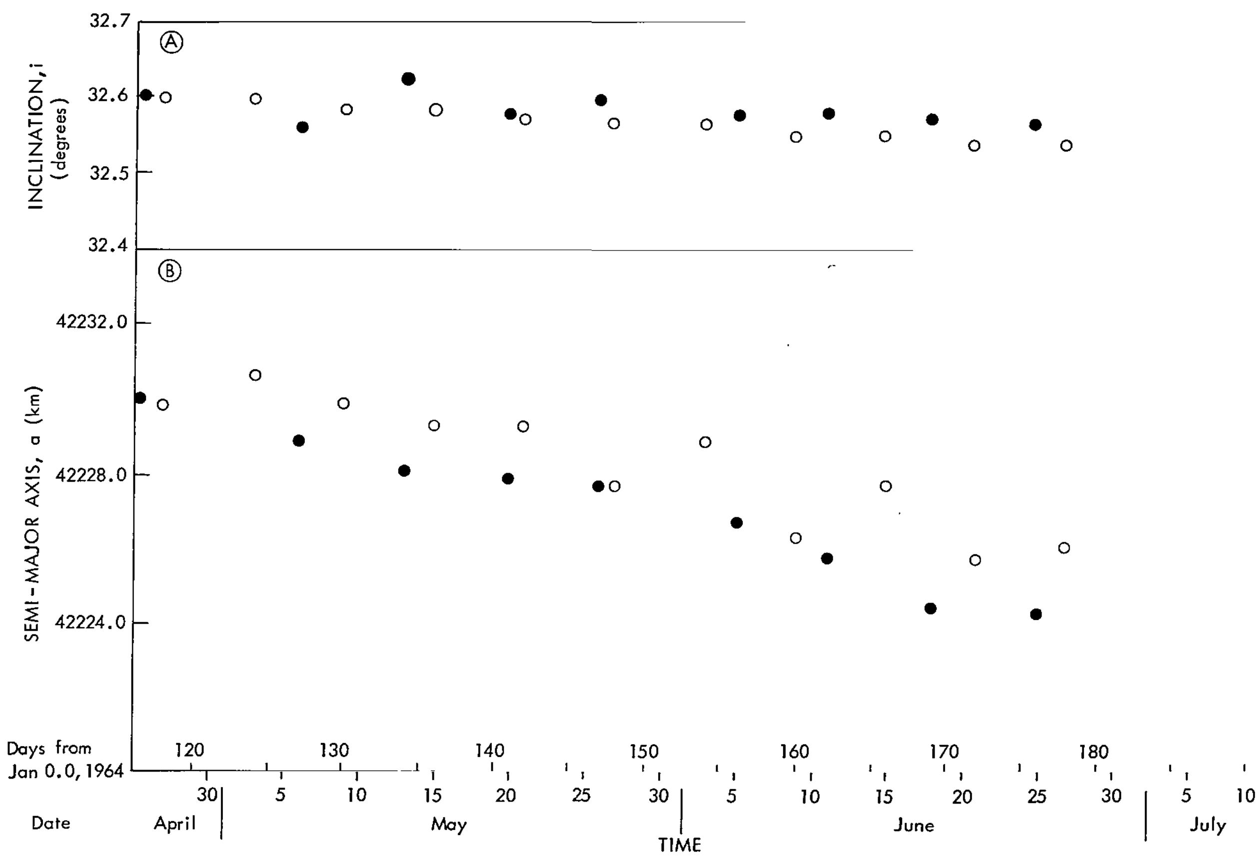
Table B-1

Orbital Elements For the Simulated Syncom II Trajectory Beginning with the Syncom II Elements of Epoch 64-04-25-02-00.00 UT (Orbit 12).

Crossing Number, j	Semimajor Axis, a_s (km)	Eccentricity, e	Inclination, i_s (degrees)	Argument of Perigee, ω (degrees)	Mean Anomaly, m (degrees)	Right Ascension of the Ascending Node (degrees)	Epoch (days from Jan. 0.0, 1964)
1	42229.86	.00121	32.601	196.567	163.665	313.851	117.604
2	42230.60	.00123	32.601	199.401	161.849	313.758	123.601
3	42229.94	.00123	32.585	197.459	164.838	313.675	129.598
4	42229.31	.00122	32.583	200.538	162.840	313.567	135.595
5	42229.30	.00116	32.570	197.176	163.552	313.490	141.592
6	42227.75	.00122	32.566	198.568	163.326	313.388	147.589
7	42228.87	.00119	32.562	200.113	162.958	313.317	153.585
8	42226.32	.00126	32.549	198.869	161.706	313.220	159.582
9	42227.77	.00113	32.547	200.058	161.779	313.142	165.578
10	42225.77	.00117	32.532	197.797	165.382	313.055	171.575
11	42226.02	.00120	32.533	200.697	160.098	312.972	177.571

In Figure B-1 the elements numerically calculated during this trajectory (semi-major axis and inclination) are compared with those actually reported for Syncom II in April-June 1964. It is noted that among all the elements, only the semimajor axis shows a significant discrepancy in the simulated trajectory. The results in the two sections on data reduction show that the long term trend of the semi-major axis changes after the initial epoch is very well explained by the drift theory of this report. It should be understood that instantaneous drift rate changes of the ground track of the orbit of the 24-hour satellite are coupled directly to semimajor axis changes by Kepler's third law which relates the semimajor axis to the period of the instantaneous orbit and the central, or gaussian, gravity constant of the principal attracting body.

It is seen in Figure B-1 that the simulated semimajor axes are on an average about one kilometer greater than the actual ones throughout the trajectory. This agrees well (in terms of Kepler's third law) with the drift rate differences between the two trajectories at corresponding



THE SIMULATION USED AN EARTH FIELD WITH THE SECOND ORDER LONGITUDE GRAVITY CONSTANTS:

$J_{22} = -1.68 \times 10^{-6}$

$\lambda_{22} = -18°$

THE DOWNWARD TRENDS OF THE SEMIMAJOR AXIS DATA IN BOTH ACTUAL AND SIMULATED CASES ARE CLOSELY PARALLEL.

THE APPARENT "DISPLACEMENT" IN THE TWO SETS OF DATA IS PROBABLY DUE TO ERROR IN THE SEMIMAJOR AXIS OF SYNCOM II AT EPOCH 64-04-25-02 U.T.

NOTE: ● ACTUAL SYNCOM II DATA
○ SIMULATED SYNCOM II DATA (FROM THE NUMERICALLY INTEGRATED TRAJECTORY)

Figure B-1—Orbital data from Syncom II observations in April-July 1964 compared to data from a numerically integrated trajectory in a full earth, sun and moon field, beginning with the Syncom II elements of Epoch 64-04-25-02 U.T.

times (Figure 1). This discrepancy then is probably due to the relatively large error in the determination of the semimajor axis for Syncom II (and used in the simulation) at the start of the drift period, epoch 64-04-25-02.

Table B-2 gives drift rate and longitude data from the numerically computed simulated trajectory.

Table B-2

Data From a Two-Month Simulated Syncom II Trajectory Beginning with the Syncom II Elements of Epoch 64-04-25-02-00.00 UT (orbit 12).

1	2	3	4	5
Crossing Number, j	Time from Jan. 0.0, 1964 (days)	Longitude at the Ascending Equator Crossing (degrees)	$\dot{\lambda}$ (degrees/day)	$(\dot{\lambda})^2$ $\left(10^{-4}\left(\frac{\text{rad}}{\text{day}}\right)^2\right)$
1	117.6034	-118.000	.83039	2.1005
2	123.6006	-122.980	.82607	2.0787
3	129.5977	-127.934	.82276	2.0621
4	135.5946	-132.868	.81541	2.0254
5	141.5916	-137.758	.80793	1.9884
6	147.5884	-142.603	.80144	1.9566
7	153.5851	-147.409	.79163	1.9090
8	159.5816	-152.156	.78913	1.8969
9	165.5781	-156.888	.77765	1.8421
10	171.5744	-161.551	.76932	1.8029
11	177.5706	-166.164		

NOTES: Initial Elements and Gravity Constants of the Trajectory:

semimajor axis, a = 42230.01 km
inclination, i = 32.603 degrees
mean anomaly, m = 333.752 degrees
mean radius of equator, R_0 = 6378.388 km
gravity constant, μ_{sun} = 332,490 μ_{earth}
angular velocity, ω_e = 0.7292115 × 10^{-4} rad/sec

eccentricity, e = 0.00119
argument of perigee, ω = 198.716 degrees
right ascension of the ascending node = 313.879 degrees
gravity constant, μ_{earth} = 3.98627 × 10^5 km^3/sec^2
gravity constant, μ_{moon} = 0.01229491 μ_{earth}

$J_{20} = 1.08219 \times 10^{-3}$ $\quad J_{30} = -2.285 \times 10^{-6}$

$J_{40} = -2.12 \times 10^{-6}$ $\quad J_{22} = -1.68 \times 10^{-6}$ $\quad \lambda_{22} = -18.0^\circ$.

Appendix C

The Error in Estimating that the Mean Velocity of an Accelerating Trajectory Between Two Points in Space Occurs Midway Between the Points

Suppose that a point m at $t = 0$ is at $S = S_0$ with velocity $v = v_0$ and moves with constant acceleration, a. On the assumption that S, v and a are colinear, the (S, t) graph for the motion is given by

$$S = \frac{1}{2} at^2 + v_0 t + S_0 \quad . \tag{C1}$$

The graph is plotted in Figure C1. The velocity, v, at any time in the interval is

$$v = \frac{ds}{dt} = at + v_0 \quad , \tag{C2}$$

and the mean or average velocity, $\bar{v}$, over the time interval t_1 is

$$\bar{v} = \frac{v_0 + v_{t1}}{2} \quad , \tag{C3}$$

since the motion is uniformly accelerated. But at t_1 the velocity, from Equation C2 is

$$v_{t1} = at_1 + v_0 \quad . \tag{C4}$$

Hence, from Equations C3 and C4,

$$\bar{v} = a\left(\frac{t_1}{2}\right) + v_0 \quad . \tag{C5}$$

Comparison of Equations C5 and C2 shows that $\bar{v}$ is the instantaneous velocity of the point at the $t = t_1/2$. The position of the point at this time is, from Equation C1,

$$S' = \frac{1}{2} a\left(\frac{t_1}{2}\right)^2 + v_0\left(\frac{t_1}{2}\right) + S_0 \quad . \tag{C6}$$

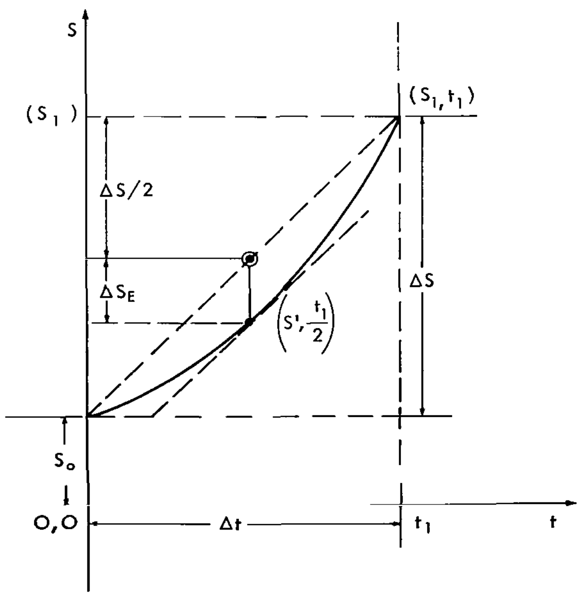


Figure C-1. The (s, t) graph corresponding to the one-dimensional trajectory of a point with constant acceleration.

Now Δs (Figure C-1) is clearly

$$\Delta s = \frac{1}{2} a t_1^2 + v_0 t_1 = \bar{v} \Delta t . \tag{C7}$$

Hence in estimating s' by $s_0 + \Delta s/2$ an error Δs_ϵ is introduced given by

$$\Delta s_\epsilon = S_0 + \frac{\Delta s}{2} - S' = S_0 + a \frac{t_1^2}{4} + v_0 \frac{t_1}{2} - a \frac{t_1^2}{8} - v_0 \frac{t_1}{2} - s_0$$

$$= \frac{a t_1^2}{4} - \frac{a t_1^2}{8} ,$$

or

$$\Delta s_\epsilon = \frac{a t_1^2}{8} . \tag{C8}$$

If a or t_1^2 or both are small, Δs_ϵ may be sufficiently small to validate the estimation.

Syncom II drift is given, in terms of the longitude acceleration, as

$$\ddot{\lambda} = A_{22} \sin 2(\lambda - \lambda_{22}) \tag{C9}$$

Considering the right side of Equation C9 a constant for a limited excursion of λ, the solution of Equation C9 is in the form of Equation C1 with

$$\lambda = s$$

and

$$A_{22} \sin 2(\lambda - \lambda_{22}) = a .$$

The maximum value of $|A_{22} \sin 2(\lambda - \lambda_{22})|$ is reached when $2(\lambda - \lambda) = \pm\pi/2$, and $\pm 3\pi/2$. From the earlier Syncom II gravity experiment (Reference 1), it was determined that $J_{22} = -1.7 \times 10^{-6}$. For $a_s = 42230$ km and $i_s = 32.6°$ (the Syncom II orbit parameters in April-June 1964),

$$A_{22} = -72\pi^2 (R_0/a_s)^2 \frac{(\cos^2 i_s + 1)}{2} = -23.0 \times 10^{-6} \left(\frac{\text{rad}}{\text{day}}\right)^2 .$$

Thus $|a(\text{max.})|$ for this drift period is

$$23.0 \times 10^{-6} \left(\frac{\text{rad}}{\text{day}}\right)^2 .$$

From Equation C8, the maximum error made in assigning the mean interval drift rate of Syncom II to the mid-longitude of an interval traversed in ten days is

$$\Delta\lambda_{\epsilon} = \frac{23 \times 10^{-6} \times 100 \times 57.3}{8} = 0.0165^{\circ}.$$

This error is sufficiently small to be ignored in the overall data reduction. Thus, in assigning the mean longitude in a Syncom II drift interval of the order of 10 days to the mean drift rate in that interval, one need retain only two decimal figures in the longitude location.

Appendix D

The Earth Gravity Potential and Force Field Used in this Report

The gravity potential used as the basis for the data reduction in this study is the exterior potential of the earth derived in Reference D-1 for geocentric spherical coordinates referenced to the spin axis and the center of mass of the earth. The infinite series of spherical harmonics is truncated after J_{44}. The zonal harmonic constants J_{20}, J_{30} and J_{40} are derived from Reference D-2.

The radius of the earth, R_0, used in this study is

$$R_0 = 6378.388 \text{ km} .$$

The earth's gaussian gravity constant used is

$$\mu_E = 3.9862677 \times 10^5 \frac{\text{km}^3}{\text{sec}^2} .$$

Neither of these values, taken from Reference D-3, nor the "zonal geoid" of Reference D-2, is felt to be the most accurate known to date. They are the values used by the GSFC Tracking and Data Systems Directorate to calculate the orbit elements of Syncom II from radar and Minitrack observations. They were chosen to insure consistency between the data of this study and these published orbits, inasmuch as the "triaxial" reduction for which this study has been undertaken is not significantly affected by the probable errors in these values. The second-order longitude harmonic constants used in the simulation studies were

$$J_{22} = -1.68 \times 10^{-6}$$

and

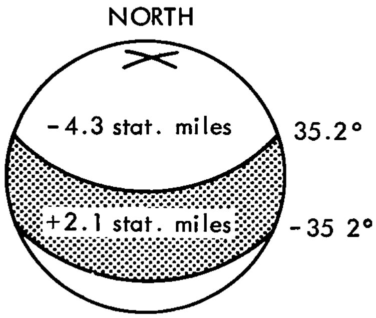
$$\lambda_{22} = -18^\circ$$

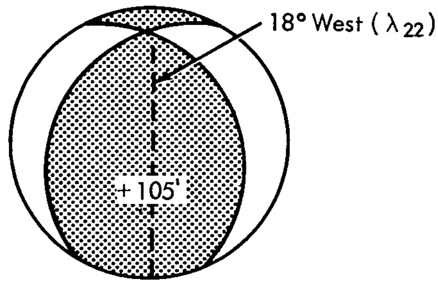
These are the values shown or implied on the "longitude geoid" below (for the J_{22} harmonic). At a later point in the analysis, the slightly different values reported in the abstract were estimated. The most accurate "zonal geoid" is probably that of Kozai (cited in Reference D-4) with the following earth constants;

$$R_0 = 6378.2 \text{ km}, \quad J_{20} = 1082.48 \times 10^{-6}, \quad J_{30} = -2.56 \times 10^{-6},$$

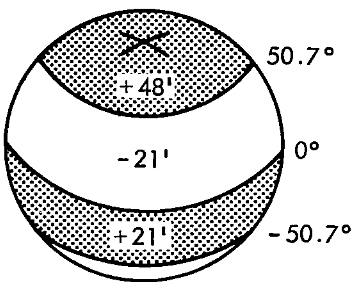
$$\mu_E = 3.98603 \times 10^5 \frac{\text{km}^3}{\text{sec}^2}, \quad J_{40} = -1.84 \times 10^{-6}.$$

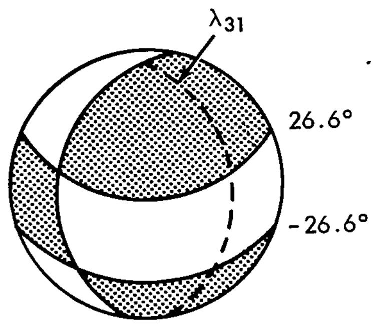
The earth's gravity potential as used in the simulations of this report (to fourth order, probably sufficient to account for all significant perturbations on a 24-hour satellite) may be illustrated (following Reference D-4, Appendix B, with the zonal constants of Reference D-2) as follows:



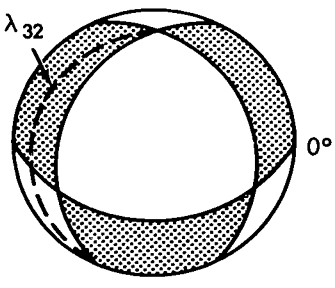


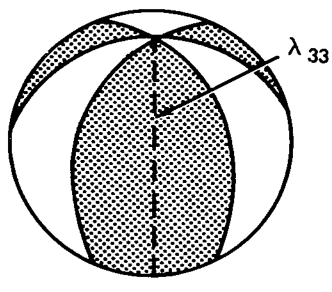
$$V_E = \frac{\mu_E}{R}\left[1 - \frac{J_{20} R_0^2}{2r^2}\left(3\sin^2\phi - 1\right) - 3J_{22}\frac{R_0^2}{r^2}\cos^2\phi\cos 2\left(\lambda - \lambda_{22}\right)\right.$$



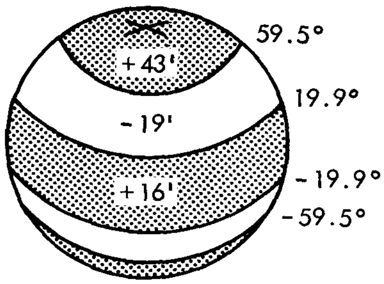


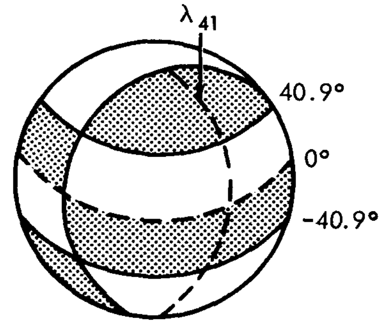
$$- \frac{J_{30} R_0^3}{2r^3}\left(5\sin^3\phi - 3\sin\phi\right) - \frac{J_{31} R_0^3}{2r^3}\cos\phi\left(15\sin^2\phi - 3\right)\cos\left(\lambda - \lambda_{31}\right)$$



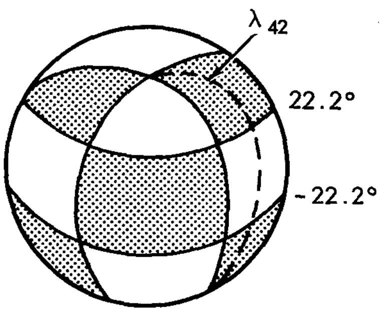


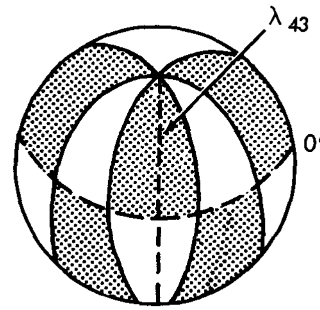
$$- 15J_{32}\frac{R_0^3}{r^3}\cos^2\phi\sin\phi\cos 2\left(\lambda - \lambda_{32}\right) - 15J_{33}\frac{R_0^3}{r^3}\cos^3\phi\cos 3\left(\lambda - \lambda_{33}\right)$$



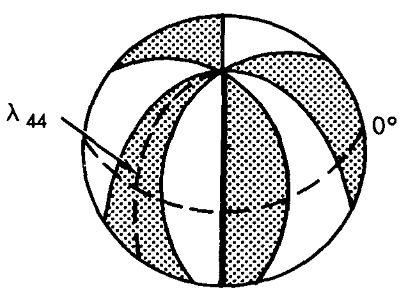


$$- \frac{J_{40} R_0^4}{8r^4}\left(35 \sin^4\phi - 30 \sin^2\phi + 3\right) - \frac{J_{41} R_0^4}{8r^4}\left(140 \sin^3\phi - 60 \sin\phi\right) \cos\phi \cos\left(\lambda - \lambda_{41}\right)$$





$$- \frac{J_{42} R_0^4}{8r^4}\left(420 \sin^2\phi - 60\right) \cos^2\phi \cos 2\left(\lambda - \lambda_{42}\right) - \frac{J_{43} R_0^4}{8r^4}\, 840 \sin\phi \cos^3\phi \cos 3\left(\lambda - \lambda_{43}\right)$$



$$- \frac{J_{44} R_0^4}{8r^4}\, 840 \cos^4\phi \cos 4\left(\lambda - \lambda_{44}\right)\Bigg] \tag{D1}$$

The earth gravity field (per unit test mass) whose potential is Equation D1 is given as the gradient of Equation D1, or

$$\overline{F} = \hat{r}F_r + \hat{\lambda}F_\lambda + \hat{\phi}F_\phi = \nabla V_E = \hat{r}\frac{\partial V_E}{\partial r} + \frac{\hat{\lambda}}{r \cos\phi}\frac{\partial V_E}{\partial \lambda} + \frac{\hat{\phi}}{r}\frac{\partial V_E}{\partial \phi} , \tag{D2}$$

or

$$
\begin{aligned}
F_r = \frac{\mu_E}{r^2} \Big\{ & -1 + \left(R_0/r\right)^2 \Big[\frac{3}{2} J_{20} \left(3 \sin^2 \phi - 1\right) + 9 J_{22} \cos^2 \phi \cos 2\left(\lambda - \lambda_{22}\right) \\
& + 2\left(R_0/r\right) J_{30} \left(5 \sin^2 \phi - 3\right) (\sin \phi) + 6\left(R_0/r\right) J_{31} \left(5 \sin^2 \phi - 1\right) \cos \phi \cos\left(\lambda - \lambda_{31}\right) \\
& + 60\left(R_0/r\right) J_{32} \cos^2 \phi \sin \phi \cos 2\left(\lambda - \lambda_{32}\right) + 60\left(R_0/r\right) J_{33} \cos^3 \phi \cos 3\left(\lambda - \lambda_{33}\right) \\
& + \frac{5}{8} \left(R_0/r\right)^2 J_{40} \left(35 \sin^4 \phi - 30 \sin^2 \phi + 3\right) \\
& + \frac{25}{2} \left(R_0/r\right)^2 J_{41} \left(7 \sin^2 \phi - 3\right) \cos \phi \sin \phi \cos\left(\lambda - \lambda_{41}\right) \\
& + \frac{75}{2} \left(R_0/r\right)^2 J_{42} \left(7 \sin^2 - 1\right) \cos^2 \phi \cos 2\left(\lambda - \lambda_{42}\right) \\
& + 525\left(R_0/r\right)^2 J_{43} \cos^3 \phi \sin \phi \cos 3\left(\lambda - \lambda_{43}\right) + 525\left(R_0/r\right)^2 J_{44} \cos^4 \phi \cos 4\left(\lambda - \lambda_{44}\right) \Big] \Big\} \quad . \qquad \text{(D3)}
\end{aligned}
$$

$$
\begin{aligned}
F_\lambda = \frac{\mu_E}{r^2} \left(R_0/r\right)^2 \Big\{ & 6 J_{22} \cos \phi \sin 2\left(\lambda - \lambda_{22}\right) + \frac{3}{2} \left(R_0/r\right) J_{31} \left[5 \sin^2 \phi - 1\right] \sin\left(\lambda - \lambda_{31}\right) \\
& + 30\left(R_0/r\right) J_{32} \cos \phi \sin \phi \sin 2\left(\lambda - \lambda_{32}\right) + 45\left(R_0/r\right) J_{33} \cos^2 \phi \sin 3\left(\lambda - \lambda_{33}\right) \\
& + \frac{5}{2} \left(R_0/r\right)^2 J_{41} \left[7 \sin^2 \phi - 3\right] \sin \phi \sin\left(\lambda - \lambda_{41}\right) + 15\left(R_0/r\right)^2 J_{42} \left(7 \sin^2 \phi - 1\right) \cos \phi \sin 2\left(\lambda - \lambda_{42}\right) \\
& + 315\left(R_0/r\right)^2 J_{43} \cos^2 \phi \sin \phi \sin 3\left(\lambda - \lambda_{43}\right) \\
& + 420\left(R_0/r\right)^2 J_{44} \cos^3 \phi \sin 4\left(\lambda - \lambda_{44}\right) \Big\} \quad . \qquad \text{(D4)}
\end{aligned}
$$

$$
\begin{aligned}
F_\phi = \frac{\mu_E}{r^2} \left(R_0/r\right)^2 \Big\{ & -3 J_{20} \sin \phi \cos \phi + 6 J_{22} \cos \phi \sin \phi \cos 2\left(\lambda - \lambda_{22}\right) \\
& - \frac{3}{2} \left(R_0/r\right) J_{30} \left(5 \sin^2 \phi - 1\right) \cos \phi + \frac{3}{2} \left(R_0/r\right) J_{31} \left(15 \sin^2 \phi - 11\right) \sin \phi \cos\left(\lambda - \lambda_{31}\right) \\
& + 15\left(R_0/r\right) J_{32} \left(3 \sin^2 \phi - 1\right) \cos \phi \cos 2\left(\lambda - \lambda_{32}\right) \\
& + 45\left(R_0/r\right) J_{33} \cos^2 \phi \sin \phi \cos 3\left(\lambda - \lambda_{33}\right) - \frac{5}{2} \left(R_0/r\right)^2 J_{40} \left(7 \sin^2 \phi - 3\right) \sin \phi \cos \phi \\
& + \frac{5}{2} \left(R_0/r\right)^2 J_{41} \left(28 \sin^4 \phi - 27 \sin^2 \phi + 3\right) \cos\left(\lambda - \lambda_{41}\right) \\
& + 30\left(R_0/r\right)^2 J_{42} \left(7 \sin^2 \phi - 4\right) \cos \phi \sin \phi \cos 2\left(\lambda - \lambda_{42}\right) \\
& + 105\left(R_0/r\right)^2 J_{43} \left(4 \sin^2 \phi - 1\right) \cos^2 \phi \cos 3\left(\lambda - \lambda_{43}\right) \\
& + 420\left(R_0/r\right)^2 J_{44} \cos^3 \phi \sin \phi \cos 4\left(\lambda - \lambda_{44}\right) \Big\} \quad . \qquad \text{(D5)}
\end{aligned}
$$

The actual sea-level surface of the earth is to be conceptualized through Equation D1 as a sphere of radius 6378 km, around which are superimposed the sum of the separate spherical harmonic deviations illustrated. To these static gravity deviations, of course, must be added a centrifugal earth-rotation potential deviation at the earth's surface, to get the true sea level surface (see Reference D-1).

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D-1. Wagner, C. A., "The Gravity Potential and Force Field of the Earth Fourth Order," NASA Technical Note D-3317, 1966.

D-2. Kozai, Y., "The Gravitational Field of the Earth Derived from Motions of Three Satellites," *Astron. J.* 66(1):8-10, February 1961.

D-3. O'Keefe, J. A., Eckels, A., and Squires, R. K., "The Gravitational Field of the Earth," *Astron. J.* 64(7):245-253, September 1959.

D-4. Wagner, C. A., "The Drift of a 24-Hour Satellite Due to an Earth Gravity Field Through Fourth Order," NASA Technical Note D-2103, February 1964.

Appendix E

Estimation of the Standard Error for a Function of Several Variables Whose Standard Errors, in an Experiment, are Known

Let $Z = Z(X_1, X_2, \cdots X_n)$ be the theoretical relationship between Z and a number of variables X_i. Let $\Delta X_1, \Delta X_2, \cdots \Delta X_n$ be small errors of estimation in each of the X_i variables. Then from the theorem for the total differential of a function of many variables:

$$\Delta Z = \sum_{i=1}^{n} \frac{\partial Z}{\partial X_i} \Delta X_i \quad . \tag{E1}$$

Equation E1 represents the error made in Z for a single trial k at estimating the X_i's. Thus from one estimation k of the X_i's, Equation E1 gives

$$\Delta Z_k = \sum_{i=1}^{n} \frac{\partial Z}{\partial X_i} \Delta X_{ik} \quad . \tag{E2}$$

Conceive now of a large number M of independent estimates k of the X_i's each giving a different error of estimation ΔX_{ik}. One can ask from this extended experiment what will be the most likely value of the sums of the squares of the individual deviations ΔZ_k.

Thus, from Equation E2

$$\sum_{k=1}^{M} (\Delta Z_k)^2 = \sum_{k=1}^{M} \left(\sum_{i=1}^{n} \frac{\partial Z}{\partial X_i} \Delta X_{ik} \right)^2$$

$$= \sum_{k=1}^{M} \left(\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \Delta X_{ik}^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\partial Z}{\partial X_i} \frac{\partial Z}{\partial X_j} \Delta X_{ik} \Delta X_{jk} \right)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sum_{k=1}^{M} \Delta X_{ik}^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\partial Z}{\partial X_i} \frac{\partial Z}{\partial X_j} \sum_{k=1}^{M} \Delta X_{ik} \Delta X_{jk} \quad . \tag{E3}$$

It is clear that if the ΔX_{ik}'s are distributed with reasonable symmetry about zero, the value of

$$\sum_{k=1}^{M} \Delta X_{ik} \Delta X_{jk}$$

will be close to zero for M sufficiently large. In fact, the expected or most likely value of this sum can probably be shown to be zero for any M when the ΔX_{ik}'s are reasonably symmetrically distributed about the expected value of ΔX_{ik}, which is zero.

Using this result in Equation E3, it is found that

$$E\left[\sum_{k=1}^{M} (\Delta Z_k)^2\right] = E\left[\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sum_{k=1}^{M} \Delta X_{ik}^2\right] ,$$

where E[ ] indicates the expected value of [ ], or

$$\sum_{k=1}^{M} E\left(\Delta Z_k^2\right) = \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sum_{k=1}^{M} E\left(\Delta X_{ik}^2\right) . \qquad \text{(E4)}$$

Equation E4 follows from the theorem that the expected value of a sum of random variables is the sum of the expected values of those random variables.* But the expected value of ΔZ_k^2 is just the variance σ_Z^2 of Z, and the $E\left(\Delta X_{ik}^2\right)$ are just the variances of each variable X_i,

$$E\left(\Delta X_{ik}^2\right) = \sigma_{X_i}^2 .$$

Thus Equation E4 becomes

$$M\sigma_Z^2 = \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \left(M\sigma_{X_i}^2\right) ,$$

or

$$\sigma_Z^2 = \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sigma_{X_i}^2 . \qquad \text{(E5)}$$

*Bower, A. H., and Lieberman, G. J., *Engineering Statistics,* Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1959.

The standard deviation of Z is thus

$$\sigma_Z = \sqrt{\sigma_Z^2} = \left[\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \sigma_{X_i}^2\right]^{1/2} . \tag{E6}$$

In an experiment with a limited amount of data on the X_i and σ_{X_i}, if estimates of these, $\hat{X}_i$ and S_{X_i}, are obtained, then from Equation E6 an estimate of σ_Z is

$$S_Z = \left[\sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)^2 \bigg|_{X_i = \hat{X}_i} S_{X_i}^2\right]^{1/2} . \tag{E7}$$

In the Syncom II data reduction for J_{22} and λ_{22} by way of the ellipticity-drift parameters C_1, C_2, C_3, the functional relationships between J_{22}, λ_{22} and these parameters are (Equations 5, 6 and 2):

$$\lambda_{22} = \frac{1}{2} \tan^{-1}\left(\frac{-C_3}{-C_2}\right) , \tag{E8}$$

$$A_{22} = \left(C_2^2 + C_3^2\right)^{1/2} , \tag{E9}$$

$$J_{22} = A_{22} \Bigg/ \left[-72\pi^2 (R_0/a_s)^2 \frac{(\cos^2 i_s + 1)}{2}\right] . \tag{E9a}$$

Applying Equation E7 to Equations E8 and E9 gives

$$S_{\lambda_{22}} = \frac{1}{2\left(\hat{C}_2^2 + \hat{C}_3^2\right)} \left[\left(\hat{C}_2\right)^2 \left(S_{C_3}\right)^2 + \left(\hat{C}_3\right)^2 \left(S_{C_2}\right)^2\right]^{1/2} , \tag{E10}$$

$$S_{A_{22}} = \frac{1}{\left(\hat{C}_2^2 + \hat{C}_3^2\right)^{1/2}} \left[\left(\hat{C}_2\right)^2 \left(S_{C_2}\right)^2 + \left(\hat{C}_3\right)^2 \left(S_{C_3}\right)^2\right]^{1/2} . \tag{E10a}$$

Using the values of $S_{\hat{C}_2}$, $S_{\hat{C}_3}$, $\hat{C}_2$ and $\hat{C}_3$ from the actual gravity experiment, Equations 40 through 43, in Equations E10 and E10a gives

$$S_{\lambda_{22,\text{actual}}} = \frac{1}{2(0.1934^2 + 0.1355^2)} \left[(0.1934 \times 0.0481)^2 + (0.1355 \times 0.0145)^2\right]^{1/2}$$

$$= 0.0852 \text{ radians} = 4.87^\circ , \tag{E11}$$

$$S_{\hat{A}_{22,\text{actual}}} = \frac{10^{-4}\,(\text{rad/day})^2}{(0.1934^2 + 0.1355^2)^{1/2}} \left[(0.1934 \times 0.0145)^2 + (0.1355 \times 0.0481)^2\right]^{1/2}$$

$$= 3.00 \times 10^{-6}\ (\text{rad/solar day})^2 = 3.00 \times 10^{-6}\,(\text{rad/solar day})^2 \times 0.9945 \left(\frac{\text{sol. day}}{\text{sid. day}}\right)^2$$

$$= 2.98 \times 10^{-6}\,(\text{rad/sid. day})^2 .$$

Thus,

$$S_{\hat{J}_{22,\text{actual}}} = \left(S_{\hat{A}_{22,\text{actual}}}\right) \Big/ \left[-72\pi^2 (R_0/a_s)^2 \frac{(\cos^2 i_s + 1)}{2}\right]$$

$$= \frac{2.98 \times 10^{-6}}{13.859\ [\text{See (28)}]} = 0.215 \times 10^{-6} . \tag{E12}$$

The "best values of J_{22} and λ_{22} sensed from Syncom II drift over the Central Pacific with their standard errors given by Equations E11 and E12, are thus:

$$J_{22} = (1.71 \pm 0.22) \times 10^{-6} ,$$

$$\lambda_{22} = -(17.1 \pm 4.9)^\circ . \tag{E13}$$

Appendix F

List of Symbols

A_{22}, J_{22}, λ_{22}	Second-order tesseral gravity constants of the earth's field
a_s, i_s	Semimajor axis and inclination of a 24-hour satellite orbit
C_1, C_2, C_3	Constants in the drift equation for a 24-hour satellite, related to its mean motion and the two parameters of earth equatorial ellipticity
R_0	Mean radius of the earth's equator
γ	Geographic longitude of the 24-hour satellite's ascending equator crossing east of the minor equatorial axis of the earth
σ_z^2, S_z^2, $\hat{z}$	The variance and estimate of the variance of a random variable z whose best estimate $\hat{z}$ is determined in an experiment

NASA-Langley, 1966

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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