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THE DESIGN OF NUMERICAL FILTERS FOR GEOMAGNETIC DATA ANALYSIS

by Kenneth W. Behannon and Norman F. Ness

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ABSTRACT

A numerical filtering technique useful in removing the diurnal component from surface data of magnetic field measurements is described. Derivations of formulas used to compute filter weights and several examples of the current use of such filters are presented and outlined.

CONTENTS

THE DESIGN OF NUMERICAL FILTERS FOR GEOMAGNETIC DATA ANALYSIS

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INTRODUCTION

In investigating the transient time variations that occur throughout the geomagnetosphere, effective comparison can be made between the data representing the magnetic field measurements from surface observatories and satellite magnetometer data only if the periodic effects due to the earth's rotation, such as Sq, are removed from the ground observatory data. One numerical technique which has proven useful in removing the diurnal component from surface data is that of numerical filtering. It is the purpose of this discussion to describe briefly this process and to show how it has been developed and successfully applied. Although the applications described here are rather limited in scope, techniques such as these lend themselves so well to automatic data processing that they find general application in many studies of geophysical time or space series (Reference 1).

For processing data in such a way as to selectively remove certain periodic components, one can construct linear operators that produce the same effect when applied to experimental data as that produced by electrical and optical filters. Analogously then, in discussing these numerical filters, we shall speak of the input function $I(t)$, the output function $O(t)$, the theoretical transfer function $T(f)$ used in the design of the filter, the gain or frequency response $W(f)$ (which is the transfer function of the resultant numerical filter), and the phase shift $\varphi(f)$ of the filter.

Two important properties of a linear filter are (1) that the output is a linear function of the input, i.e., if two inputs $I_1(t)$ and $I_2(t)$ give the outputs $O_1(t)$ and $O_2(t)$, then input $I_3(t) = I_1(t) + I_2(t)$ will give the output $O_3(t) = O_1(t) + O_2(t)$; and (2) its response is independent of the time origin, i.e., if an input $I(t)$ results in an output $O(t)$, then an input $I(t + t_0)$ gives an output $O(t + t_0)$.

Most time series of interest in geophysics can be considered to be quasi-stationary time series. A stationary time series is a random function of time which may also be a function of initial conditions but whose average probability distributions are independent of time (Reference 2). The fundamental principles of stationary time series smoothing have been elaborated by Wiener

(Reference 3) and others. In application, simple smoothing methods are included among the basic techniques of numerical analysis (Reference 4 and Reference 5). Since such data smoothing is actually a type of time or space series filtering, it will be instructive to delay further discussion of numerical filter design until a brief description of a simple smoothing operator or, equivalently, a filter has been given.

A time series $g(t)$ of equally-spaced data values at intervals Δt can be smoothed "by threes" using the linear formula

$$\frac{g_{k-1}(t) + g_k(t) + g_{k+1}(t)}{3} = g_k^s(t).$$

One can see the result of this operation on the frequency spectrum of $g(t)$ by looking at its effect on an individual pure sinusoidal time function.

As an example let

$$g_k(t) = \text{Real Part } [e^{i\omega k\Delta t}],$$

where $\omega = 2\pi f$ (frequency) and $k\Delta t = t$. Then

$$g_k^s = \frac{1}{3}\left(e^{i\omega(k-1)\Delta t} + e^{i\omega k\Delta t} + e^{i\omega(k+1)\Delta t}\right),$$

$$= \frac{1}{3}\left(e^{i\omega k\Delta t}\, e^{-i\omega\Delta t} + e^{i\omega k\Delta t} + e^{i\omega k\Delta t}\, e^{i\omega\Delta t}\right),$$

$$= \frac{1}{3} e^{i\omega k\Delta t}\,(1 + 2\cos\omega\Delta t),$$

$$= g_k \left(\frac{1 + 2\cos\omega\Delta t}{3}\right).$$

Thus the individual values in the time series are modified by a factor, the transfer function, which is independent of k. In the frequency domain, the spectrum will be amplitude modulated by the factor $(1/3)(1 + 2\cos\omega\Delta t)$. Plotting this quantity as a function of frequency reveals that most of the frequencies in the upper half of the spectrum are suppressed by this simple smoothing process (for a certain limited range of frequencies). In this case the time series has been "low pass filtered" by taking weighted running means with all weights identical and equal to $1/N$, where N is the number of data points used in computing the mean. If "aliasing" of the data occurs (see following paragraphs) then contributions from frequencies near $f = N/\Delta t$ can occur, where N is an integer.

In general a numerical filter consists of a set of "weights" W_k which determine the actual transfer function $W(f)$ of the filter. The design of a numerical filter begins with establishing the shape of the data window in the frequency domain which will give the desired effect. Having specified the theoretical transfer function, the remainder of the problem consists of determining the weights W_k in such a way that the actual transfer function, or frequency response, approximates the desired one as well as possible. A perfect low pass filter, for example, would leave unaltered all frequency components from $f = 0$ to the desired cutoff frequency f_L and then would suppress all frequencies greater than f_L. The response of an actual numerical filter can only approximate this ideal behavior, with the accuracy of the approximation depending on the values of various design parameters.

As in the simple smoothing process, a numerical filter is applied in such that

$$y_0(t) = \sum_{k=-M}^{N} W_k \, y(t + k\Delta t). \tag{1}$$

The filtering is accomplished by "sliding" the filter along the data, applying it to $M + 1 + N$ data points to produce the filtered equivalent of the data point which has been multiplied by W_0 and then moving each weight to the next point in the series and repeating the application. Repetition of the process until all the data in a given run have been covered produces a series of filtered data points which defines the output function $O(t)$. Within the precision of the filter these points will trace out the input function $I(t)$ with the unwanted high frequency components removed (if a low pass filter is being used).

When experimental data are derived by discretely sampling some phenomenon at equally spaced intervals of time, the problem of aliasing may occur in which the sampling rate is low enough to confuse two or more frequencies in the data. The net result is that they appear to be the same frequency (Figures 1a and 1b). To avoid this problem and hence to define a unique input function as described by a set of data points, one must be able to assume that the phenomenon studied is spectrally limited to the range $|f| \le f_c$, where $f_c = f_s/2$, f_s being the sampling frequency and f_c being the cut-off or Nyquist* frequency. If such an assumption is valid, then the function has been sampled frequently enough so that all significant frequency components are determinable. This is a result of the sampling theorem of information theory (Reference 2). The sampling theorem states that if a function $G(t)$ contains no frequencies higher than W cycles per second, then it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart, the series extending throughout the entire time domain. There is an equivalent theorem for the frequency domain.

We shall now very briefly present a few of the analytical considerations underlying numerical filter design. First of all, two mathematical concepts basic to all time series analysis are the Fourier transform and the operation of convolution (Reference 6).

*After H. Nyquist of Bell Telephone Laboratories

The Fourier transform $F(f)$ of a function $g(t)$ is defined as

$$F(f) = \int_{-\infty}^{\infty} g(t)\, e^{-i2\pi ft}\, dt.$$

When $g(t)$ is such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |g(t)|^2\, dt < \infty,$$

then

$$g(t) = \int_{-\infty}^{\infty} F(f)\, e^{i2\pi ft}\, df.$$

$F(f)$ and $g(t)$ are referred to as a transform pair when the above conditions hold. In various phases of the analysis of time series the operation of Fourier transforming a function is required. However, the function may not be known analytically but only at certain tabulated values. Thus some approximation to the above integration must be made with discrete data.

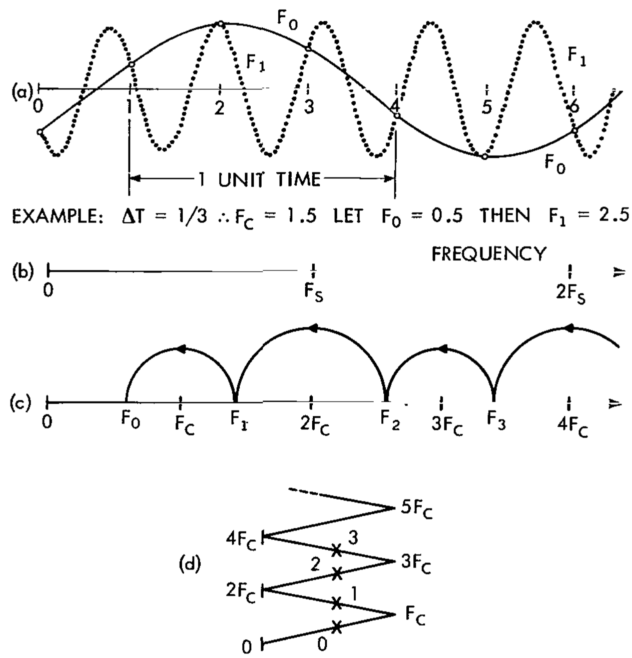

Figure 1—Discrete sampling and the aliasing problem: (a) An example of the sampling problem in which two different sinusoids (F_0 and F_1) have exactly the same sample values at the sample points, (b) A line running from zero to infinity along which frequencies are represented as points (F_s = sampling frequency), (c) The same line as in (b) but with the Nyquist frequency, F_c, placed such that $F_c = F_s/2 = 1/(2\Delta T)$ and $F_1 = 2kF_c \pm F_0$, where ΔT is the period of the sampling frequency and k is an integer multiplier and (d) Aliasing is illustrated by the line (b) folded back and forth on itself. Subsequent folds occur with the same spacing and all points above the same point on the frequency axis appear to be the same frequency. After sampling, aliased frequencies cannot be separated.

The convolution of the function $g(x)$ with respect to the function $f(x)$ on the interval $[a,b]$ is the function

$$h(x) = \int_a^b f(t)\, g(x-t)\, dt = \int_a^b g(t)\, f(x-t)\, dt.$$

If the interval is extended to include the entire real line then one obtains the function

$$H(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

The latter is defined simply as the convolution of the functions f and g (Reference 7). The integrand in the convolution integral will have nonzero values only in the region of overlapping of the functions $f(t)$ and $g(x - t)$. Thus there are cases where $h(x) = H(x)$ for a and b both finite numbers. For example, if $[a,b] = [0,x]$ and the integrand is zero for all values of $t < 0$, all nonzero values of the integrand will lie within $[0,x]$ and hence $h(x) = H(x)$.

Theoretically, the transfer function of a filter is simply the ratio of the output spectrum to the input spectrum,

$$T(f) = \frac{O(f)}{I(f)}, \tag{2}$$

or

$$O(f) = T(f) \cdot I(f),$$

where $I(f)$ and $O(f)$ are the Fourier transforms of the input and output functions $I(t)$ and $O(t)$. Multiplication in the frequency domain of functions possessing Fourier transforms corresponds to convolution in the time domain of the transforms, and conversely (Reference 6). Thus

$$O(t) = \int_{-\infty}^{\infty} I(\tau)\, T(t-\tau)\, d\tau,$$

or, equivalently, (3)

$$O(t) = \int_{-\infty}^{\infty} T(\tau)\, I(t-\tau)\, d\tau.$$

The Fourier transform of the function $T(t)$ is the theoretical transfer function of the filter:

$$T(f) = \int_{-\infty}^{\infty} T(t)\, e^{i2\pi ft}\, dt. \tag{4}$$

An arbitrary function can always be expressed as the sum of two component functions of which one is odd and the other even. Thus we can write the relation

$$T(t) = T_E + T_0.$$

From the definition of even and odd functions it follows that

$$T(-t) = T_E - T_0.$$

Addition and subtraction of these expressions yield

$$T_E = \frac{1}{2}\,[T(t) + T(-t)]\,,$$

$$T_0 = \frac{1}{2}\,[T(t) - T(-t)]\,.$$

Now we can write Equation 4 as

$$T(f) = \int_{-\infty}^{\infty} (T_E + T_0)\, e^{i\omega t}\, dt\,,$$

$$= \int_{-\infty}^{\infty} (T_E + T_0)\,(\cos \omega t + i \sin \omega t)\, dt\,.$$

Because of the effect of integrating even and odd functions between symmetric limits this reduces to

$$T(f) = 2\int_{0}^{\infty} T_E \cos \omega t\, dt + 2i \int_{0}^{\infty} T_0 \sin \omega t\, dt\,.$$

It is seen immediately that for $T(f)$ to be a real number, we must have $T_0 = 0$. This in turn requires that $T(t) = T(-t)$.

The theoretical transfer function is approximated by the actual transfer function or frequency response of a numerical filter:

$$W(f) = \sum_{k=-M}^{N} W_k(t) e^{i 2\pi f k \Delta t} \doteq T(f)\,. \tag{5}$$

If $M = N$ then one can write

$$W(f) = W_0 + \sum_{k=1}^{N} (W_k\, e^{i 2\pi f k \Delta t} + W_{-k}\, e^{-i 2\pi f k \Delta t})\,. \tag{6}$$

As in the case of continuous functions, we can express W_k as the sum of even and odd parts:

$$W_k = W_E + W_0\,,$$

and similarly obtain

$$W_E = \frac{1}{2}\left(W_k + W_{-k}\right),$$

$$W_0 = \frac{1}{2}\left(W_k - W_{-k}\right).$$

Again as in the ideal case one finds that the necessary and sufficient condition for the transfer function $W(f)$ to be a real number for any value of f is that the filter be symmetric $(W_{-k} = W_k)$. If the filter is asymmetric $(W_{-k} = -W_k)$, the transfer function is a pure imaginary number.

The complex number representing the transfer function can also be written

$$W(f) = G(f)\, e^{i\varphi(f)} \tag{7}$$

where $G(f)$ is the gain of the filter and $\varphi(f)$ is the phase shift that it produces. $G(f)$ must be an even function of f and $\varphi(f)$ an odd function of f for real input and output functions. The relation indicated by Equation 7 may also be written in the form

$$W(f) = G(f)\cos\varphi(f) + i\, G(f)\sin\varphi(f). \tag{8}$$

It has been shown that if the filter is symmetric, $W(f)$ is a real number. From Equation 8 it is seen that this requires that $\varphi(f)$ be equal to zero or π. Likewise if the filter is asymmetric and hence $W(f)$ is a pure imaginary, we must have that $\varphi(f) = +\pi/2$ or $-\pi/2$. In other words, besides modifying the amplitude of an input frequency component due to the effect of the gain $G(f)$, a filter with $W_k = W_{-k}$ for all k either has no effect on the phase of the input function or shifts it by π, while a filter with $-W_k = W_{-k}$ for all k produces a phase shift of $\pm\pi/2$.

One can thus call the even part of the transfer function the "in phase" portion and the odd part the "out of phase" portion due to the effect of each when nonzero. In the most general case $W_k = W_E + W_0$ with neither part equal to zero, i.e., it contains both in-phase and out-of-phase portions, is complex, and phase shifts the input function by an amount $0 < \varphi < 2\pi$.

From Equation 6 with $W_0 = 0$ and $W_{-k} = -W_k$ one obtains the formulas for the asymmetric or "sine" filter frequency response

$$W(f) = 2i\sum_{k=1}^{N} W_k \sin 2\pi f k \Delta t. \tag{9}$$

Consider as input the complex sine function $I(t) = Ae^{i\omega t}$ with Fourier transform $I(f) = A$. Suppose it is desired that the filter output be the derivative of $I(t)$, i.e., that $O(t) = i\omega I(t)$. Transforming $O(t)$ reveals that in the frequency domain we must have $O(f) = i\omega A$. Since $O(f) = T(f) \cdot I(f)$, it is hence necessary that $T(f) = i\omega = \omega e^{i\pi}/2$. Thus in order to perform differentiation the transfer function of a filter must be such as to produce a gain of $\omega = 2\pi f$ and a phase shift of $+\pi/2$. Because it provides the necessary phase shift, the asymmetric or sine type filter may be used as a differentiator.

For most numerical filtering of geomagnetic time series it is desirable only to attenuate certain frequency components without altering the phase. Hence of greatest importance is the symmetric or "cosine" filter with frequency response

$$W(f) = W_0 + 2\sum_{k=1}^{N} W_k \cos 2\pi f k \Delta t. \tag{10}$$

This expression may be used to compute the frequency response characteristics for the designed filter once the numerical values of the weights W_k are known. In the following sections we shall discuss two methods for calculating the values of the weights, given the characteristics of the theoretical transfer function $T(f)$.

THE LEAST SQUARES APPROXIMATION TO T(f) FOR A LOW PASS FILTER

One approach to the approximation of the theoretical transfer function is through application of the least squares technique. In the following development of formulas which can be used to calculate filter weights, we shall closely follow the discussion of the subject by Martin (Reference 8). Instead of using the frequency f, Martin introduces the normalized frequency $r = f/f_s = f/2f_c$. He then designates r_c to represent the ratio of the desired cutoff frequency to the sampling frequency. We prefer to use the parameter $p = 2r = f/f_c$ for frequency normalization and P as the cutoff ratio.

As stated previously, the problem of filter design consists of determining the $M + 1 + N$ weights W_k such that the actual transfer function of the filter is defined by Equation 5, or, in terms of p,

$$W(p) = \sum_{k=-M}^{N} W_k \, e^{i\pi k p}, \tag{11}$$

approximates the best, in the least squares sense, the desired transfer function. The transfer function for a perfect filter may be written in the form

$$T(p) = G(p)\, e^{i\varphi(p)}, \tag{12}$$

$\varphi(p)$ being the phase shift.

We shall require that the mean square deviation between $T(p)$ and $W(p)$ over a specified interval $-p'$ to $+p'$, given by

$$I = \frac{1}{2p'}\int_{-p'}^{p'} |T(p) - W(p)|^2 \, dp, \tag{13}$$

be minimized by proper choice of the $M + 1 + N$ weights W_k. Thus we can write

$$I = \frac{1}{2p'}\int_{-p'}^{p'} \left| G(p)\, e^{i\varphi(p)} - \sum_{k=-M}^{N} W_k\, e^{i\pi kp} \right|^2 dp,$$

or, since for Z^* the conjugate of Z, $|Z|^2 = Z \cdot Z^*$,

$$\Gamma = 2p'\, I = \int_{-p'}^{p'} \left[G(p)\, e^{i\varphi(p)} - \sum_{k=-M}^{N} W_k\, e^{i\pi kp} \right] \cdot \left[G(p)\, e^{-i\varphi(p)} - \sum_{k=-M}^{N} W_k\, e^{-i\pi kp} \right] dp. \tag{14}$$

To minimize the function Γ, the deviation of that function with respect to each W_k must be zero. In other words we must have

$$\frac{\partial\Gamma}{\partial W_k} = \int_{-p'}^{p'} -e^{i\pi kp}\left[G(p)\, e^{-i\varphi(p)} - \sum_{n=-M}^{N} W_n\, e^{-i\pi np} \right] - e^{-i\pi kp}\left[G(p)\, e^{i\varphi(p)} - \sum_{n=-M}^{N} W_n\, e^{i\pi np} \right] dp = 0, \tag{15}$$

or

$$\int_{-p'}^{p'} \sum_{n=-M}^{N} W_n \left[e^{i\pi(k-n)p} + e^{-i\pi(k-n)p} \right] dp - \int_{-p'}^{p'} G(p) \left\{ e^{i[\pi kp - \varphi(p)]} + e^{-i[\pi kp - \varphi(p)]} \right\} dp = 0.$$

This gives us the relation

$$\int_{-p'}^{p'} \sum_{n=-M}^{N} W_n \cos \pi(k-n)p \, dp = \int_{-p'}^{p'} G(p) \cos[\pi kp - \varphi(p)] \, dp, \tag{16}$$

or, by changing the order of integration and summation,

$$\sum_{n=-M}^{N} W_n \int_{-p'}^{p'} \cos\pi(k-n)p\,dp = \int_{-p'}^{p'} G(p)\cos\left[\pi kp - \varphi(p)\right]dp. \tag{17}$$

Since $\pi(k - n)p$ and $\pi kp - \varphi(p)$ are both odd functions of p, their cosines are even functions of p and we may hence write the above integrals as

$$\sum_{n=-M}^{N} W_n \int_{0}^{p'} \cos\pi(k-n)p\,dp = \int_{0}^{p'} G(p)\cos\left[\pi kp - \varphi(p)\right]dp. \tag{18}$$

There is one of these equations for each value of k from $-M$ to N, or, for a symmetrical filter, from $-N$ to $+N$.

It was previously stated that the phenomenon being studied must be spectrally limited to the range $|f| \leq f_c$ to avoid aliasing problems. Equivalently, we require $|p| \leq 1$, this leading to

$$\int_0^1 \cos\pi(k-n)\,p\,dp = \begin{cases} 0 \text{ if } k \neq n \\ 1 \text{ if } k = n. \end{cases} \tag{19}$$

Hence we are left with

$$W_k = \int_0^1 G(p)\cos\left[\pi kp - \varphi(p)\right]dp, \tag{20}$$

so that each W_k is expressed explicitly. We may write this as

$$W_k = \int_0^P G(p)\cos\left[\pi kp - \varphi(p)\right]dp + \int_P^1 G(p)\cos\left[\pi kp - \varphi(p)\right]dp, \tag{21}$$

where P is the cutoff ratio.

For the ideal low pass filter, $G(p) = 1$ for $0 \leq p \leq P$ and $G(p) = 0$ for $p > P$. Hence

$$W_k = \int_0^P \cos\left[\pi kp - \varphi(p)\right]dp, \tag{22}$$

or, for zero phase shift,

$$W_k = \int_0^P \cos \pi k p \, dp. \tag{23}$$

For $k = 0$ this gives us

$$W_0 = \int_0^P dp = P, \tag{24}$$

and for $k \neq 0$ we have

$$W_k = \frac{\sin \pi k P}{\pi k}. \tag{25}$$

Equations 24 and 25 may be used to compute low pass filter weights for sharp cutoff, but they lead to an approximation of the ideal transfer function which exhibits a large overshoot for values of p slightly smaller or greater than P. This is a manifestation of the Gibbs phenomenon discussed in most works on Fourier analysis. This phenomenon occurs near a discontinuity in a function which is being approximated by a finite series of size N. As N increases, the position at which the maximum occurs moves nearer to the point of discontinuity, but the value of the overshoot amplitude is independent of N. In approximating a perfect low pass filter transfer function, the deviations from the theoretical values near the cutoff frequency are usually much larger than can be allowed.

To avoid the sharp cutoff overshoot, instead of making the function zero for all values of $p > P$, it can be continued by a sine function which has the same value and the same derivative at $p = P$ as the transfer function and, together with its derivative, becomes zero for a specified value of p. Instead of using p directly, however, it is more convenient to use a parameter h_p, of magnitude corresponding to the change in p during 1/4 cycle of the sine termination function. If the ratio of the change in p to h_p is included in the argument of the sine function, it forces both the termination function and its derivative to have the necessary values at their end points. The geometry of the sine termination is shown in Figure 2. Throughout the remainder of this discussion we shall simplify the notation for the parameter to h, but it should always be taken to mean the h obtained when p is used for the normalized frequency.

In terms of h, a function which will produce the desired termination effect is

$$A(p) = y_0 \left(1 - \sin \frac{\pi}{2} \frac{p - p_0}{h}\right),$$

$$= y_0 \left(1 + \sin \frac{\pi}{2} \frac{p_0 - p}{h}\right). \tag{26}$$

where y_0 is the amplitude of the sine function and p_0 is the value of p at the point where the function has the value y_0. The relationship of y_0 and p_0 to h is also shown in Figure 2. As may be seen, the quantities y_0 and h will determine the geometry of the cutoff. The parameter h will permit variation of the slope of the sine termination.

To design a filter with a sine termination, h must be as small as possible but such that the actual frequency response of the filter does not depart from the theoretical response by more than a permissible tolerance. (As h approaches zero, the filter approaches a sharp cutoff filter.) In Figure 3 we see the improvement offered by sine-terminated filters over sharp cutoff filters designed for the same cutoff frequencies and with the same number of weights.

So, instead of letting the transfer function go to zero immediately for all $p > P$, we set it it equal to $A(p)$ in the range $P \le p \le p_0 + h$. Then we can write

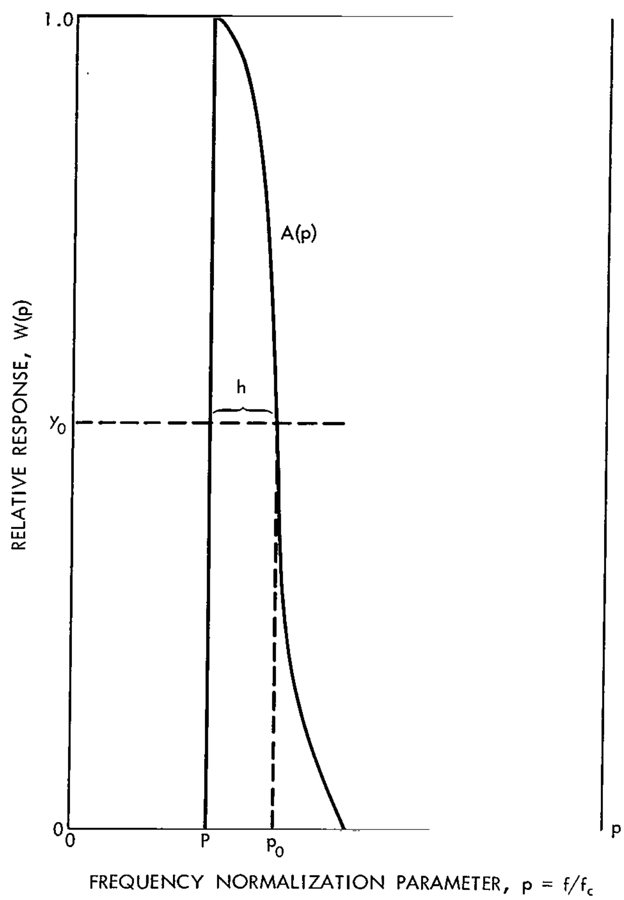

Figure 2—Geometry of the sine termination function A(p) which is used to provide smoother cutoff for low pass filter frequency responses.

$$W_k = \int_0^P (1) \cos \pi k p dp$$

$$+ y_0 \int_P^{p_0+h} \left(1 - \sin \frac{\pi}{2} \frac{p - p_0}{h}\right) \cos \pi k p dp + \int_{p_0+h}^{1} (0) \cos \pi k p dp$$

$$= W_k^{(0)} + W_k^{(a)}, \tag{27}$$

where the first integral is the weight computed for sharp cutoff, and the second is the sine termination correction.

We have already seen that if $k = 0$, $W_k^{(0)} = W_0^{(0)} = P$. Similarly, since

$$W_k^{(a)} = y_0 \int_P^{p_0+h} \left(1 - \sin \frac{\pi}{2} \frac{p - p_0}{h}\right) \cos \pi k p dp,$$

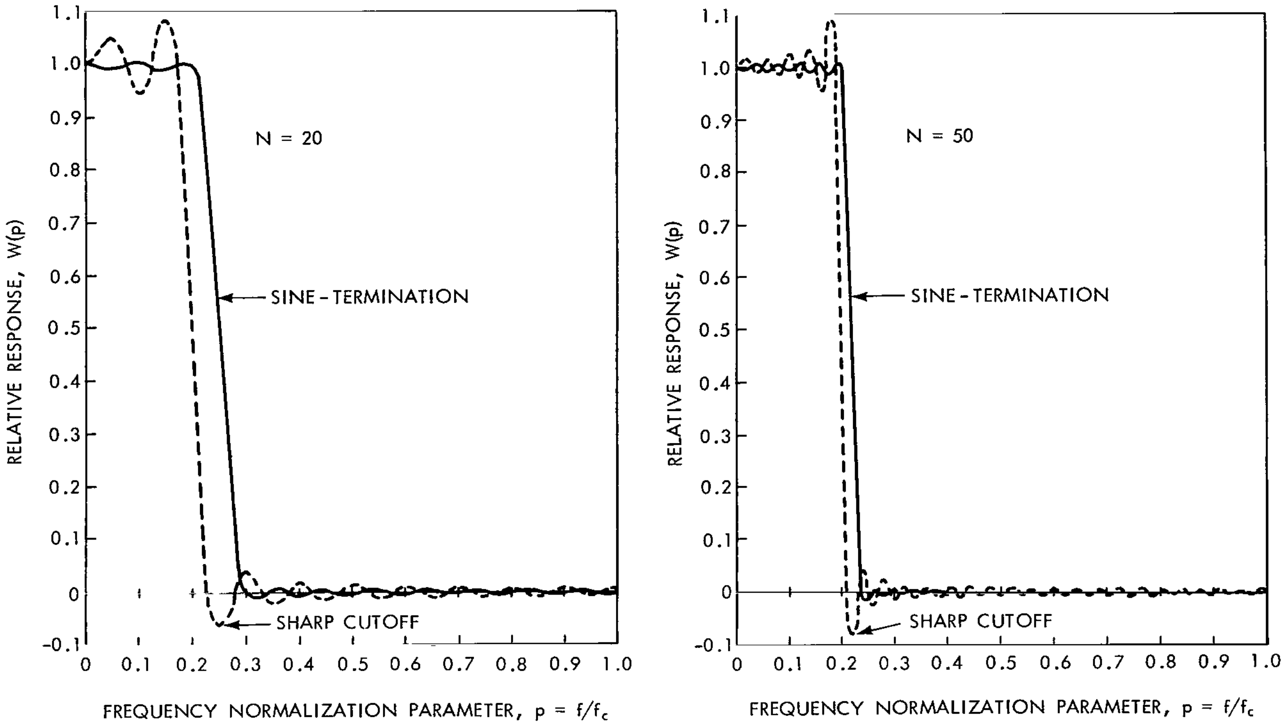

Figure 3—Marked contrast between sharp cutoff and sine-terminated approximations to an ideal filter with low cutoff at $p = 0.2$ is illustrated by the frequency response of two filters with $N = 20$ and $N = 50$, respectively. The approximation is improved by use of the larger filter.

we have for $k = 0$ that

$$W_0^{(a)} = y_0 \int_P^{p_0+h} dp - y_0 \int_P^{p_0+h} \sin\frac{\pi}{2}\,\frac{p-p_0}{h}\,dp$$

$$= y_0 \left(p_0 + h - P - \frac{\cos\frac{\pi}{2}\,\frac{P-p_0}{h}}{\frac{\pi}{2}} \right). \tag{28}$$

With $A(p)$ defined by Equation 26 differentiation gives

$$A'(p) = \frac{\pi}{2h}\, y_0 \cos\frac{\pi}{2}\,\frac{p_0-p}{h}\,, \tag{29}$$

so that

$$A'(P) = \frac{\pi}{2h}\, y_0 \cos\frac{\pi}{2}\,\frac{p_0-P}{h}\,.$$

At $p = P$ we want $A(p) = T(p)$ and $A'(p) = T'(p)$, so we choose $A(P) = 1.0$ and $A'(P) = 0$. $A'(P) = 0$ requires that $p_0 - P = h$, or that $p_0 = P + h$. By substitution the expression for $A(P) = 1$ then yields $y_0 = 1/2$. Using these values for y_0 and p_0 we have finally from Equation 28 that

$$W_0^{(a)} = h\,,$$

and hence that

$$W_0 = W_0^{(0)} + W_0^{(a)} = P + h. \tag{30}$$

In a similar way, if one works on through from Equation 27 (see Reference 8), one obtains for $k \neq 0$ that

$$W_k = \left[\frac{\cos \pi kh}{1-4k^2 h^2}\right] \left[\frac{\sin \pi k\,(P+h)}{\pi k}\right] = F\,(k,\,h)\,\frac{\sin \pi k\,(P+h)}{\pi k}. \tag{31}$$

Tables of $F(k,h)$ may be computed independently of any individual filter and then they will be available for particular applications. It will be found from the expression for $F(k,h)$ in Equation 31 that an indeterminate form is obtained whenever $kh = 1/2$. L'Hospital's theorem may be used to evaluate the expression in that situation, and it is found that $F(k,h) = 0.78540$ for $kh = 1/2$.

One further correction may be added to the weights in order to normalize the gain to 1.0 at $p = 0$. Let the value of the k^{th} weight obtained from Equation 31 be designated by L_k. Then

$$\Delta = 1 - \left(L_0 + 2\sum_{k=1}^{N} L_k\right)\,,$$

and the corrected weight is given by

$$W_k = L_k + \frac{\Delta}{2N+1}. \tag{32}$$

Once the weights have been computed using Equations 30, 31, and 32, the gain or frequency response is easily computed using Equation 10:

$$W(p) = W_0 + 2\sum_{k=1}^{N} W_k \cos \pi kp.$$

We now have all the formulas necessary for the design of least squares-approximated low pass filters. In each case the parameter h can be chosen such as to tailor the cutoff of the filter to the specific needs involved. We have already given some indication of the fact that h is sensitive to the number of weights in the filter, and progressively larger values of h are necessary as one goes to progressively smaller filters ($h \propto 1/N$) in order for the sine termination to be effectively and efficiently accomplished. This would suggest that a certain amount of experimentation would be necessary to reveal the optimum value of h for a particular filter. This is the value below which the values of the function near the cutoff depart from the theoretical values by an unacceptable margin due to the size of the overshoot, and above which the termination gets continuously smoother but the accompanying increase in the half-width of the main lobe brings down the precision. The effect of filter size N on the width of the main lobe is shown in Figure 4.

From a study of the cutoff characteristics of a number of filters for various values of h, it has been found that at $h \simeq 1/N$ (if one is using the p notation) the terminal oscillations have almost completely disappeared. Although a slightly greater degree of smoothness is obtained as one goes to still larger values of h, the predominant effect is merely the broadening of the pass band. On the other hand, as one employs progressively smaller values of h, the terminal oscillations grow rapidly more significant until the large excursions characteristic of sharp cutoff are obtained.

Thus to select the proper filter one must compromise between bandwidth and termination smoothness, and the limiting factors in the compromise are the minimum cutoff slope that will be acceptable for the task at hand and the smallest main-lobe to side-lobe ratio that can be tolerated. Once the computation of filter weights and corresponding frequency responses has been programmed, it is a simple matter to generate a family of response curves for a particular N and

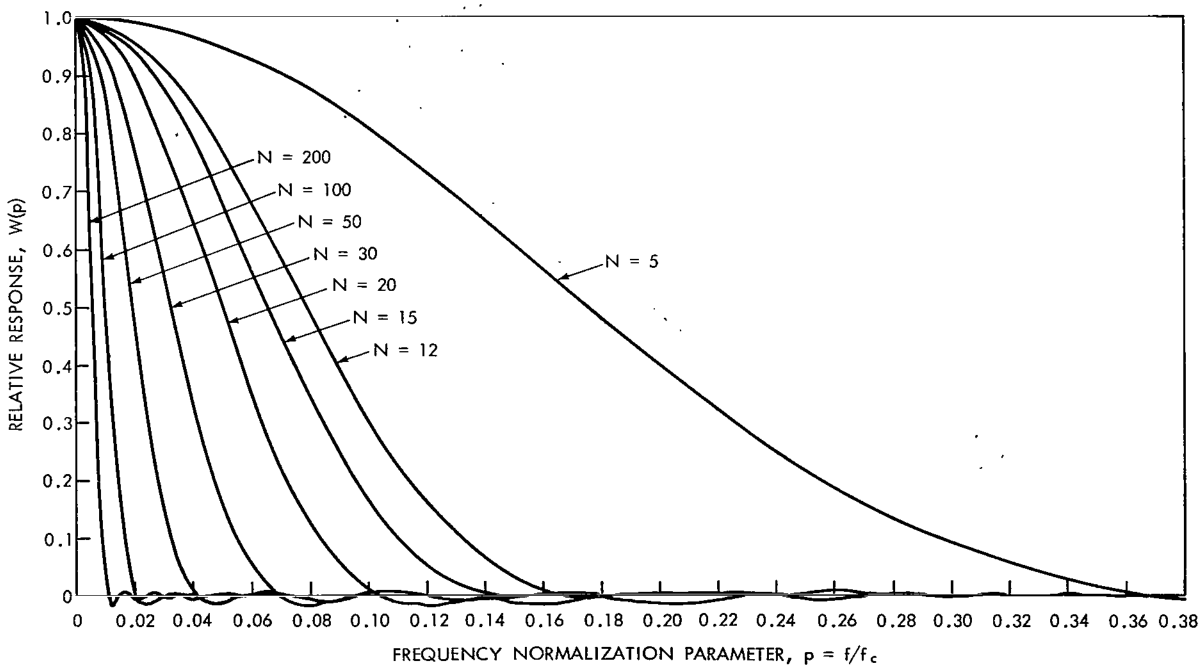

Figure 4—Ultra low pass (cutoff ratio p = 0) filter main lobes for selected values of N. The increase in sharpness of a filter due to increasing its size is illustrated.

various values of h to facilitate the final filter selection. Figure 5 illustrates the three cases of $h = 1/N$, $1/2N$, and $2/N$ for a filter with $P = 0$ and $N = 100$. Figure 6 is a graphical representation of the optimization problem.

On the question of best filter size, we see that a larger value of N permits sharper sine termination, and the undesired frequencies are eliminated more efficiently. Another advantage is that the effect of an erroneous input point is much less for a larger filter. However, larger filters require longer unbroken runs of data.

Earlier in this discussion (Equation 1) we described how the filtering is accomplished by sliding the filter along the data, applying it to $2N + 1$ data points to produce the filtered equivalent of the data point lying at the center of the filter (at W_0) and then moving each weight to the next point in the series and repeating the process. It can be seen that this requires that one have at least N data values both preceding and following the time range of interest in order to get the filtered equivalents of all data points in that range. Thus a limited number of prior points may dictate that a smaller filter be used, or else a scheme may be employed (Reference 8) which involves the use of progressively larger filters as one gets further into the run of data and conversely near the end. In either event, one steps down to less precisely filtered data throughout all or at least part of the range of interest, and this may not be acceptable. In that case one must go ahead and use the larger filter and settle for fewer output values.

It may so happen that the characteristics of the filtering job to be done will suggest particular filter size as being most convenient. If it then turns out that this filter will be precise enough

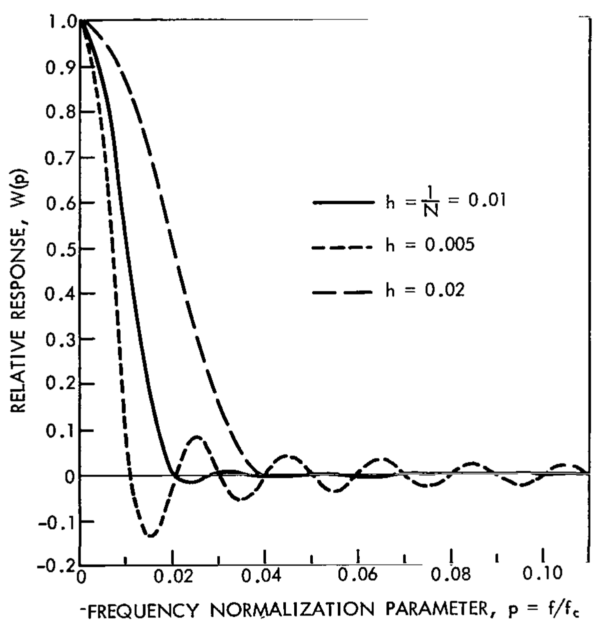

Figure 5—Dependence of low pass filter cutoff characteristics on the sine-termination parameter h, where N = 100.

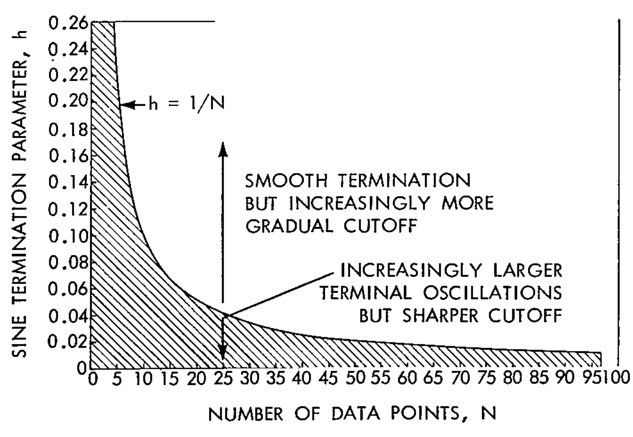

Figure 6—Regions of significance in filter response optimization for a given value of N. This is a general illustration of the design problem created by the morphological changes that occur in the filter frequency response when h is varied as in Figure 5. The majority of filtering problems are perhaps best accommodated by a filter with h slightly less than 1/N where the oscillations are still not too large and the cutoff is reasonably sharp.

for the task or will give the desired effect and that there are sufficient data values at each end of the run to be used, then there is no selection problem.

For one application of a low pass filter it was required to produce one which could be used to compute hourly averages with the effects of high frequency fluctuations removed. The input consisted of 2.5 minute surface magnetic field values (H component). There are 25 such values inclusive to each hour (0-24), so the natural choice for the task was a 25-point filter ($N = 12$). One-hour periodicities have a corresponding value of $p = 0.083$. If this were used as the cutoff value, then the slow cutoff of this relatively small filter would allow a considerable fraction of higher frequency components to get through. This problem was circumvented by choosing the cutoff ratio to be $P = 0$. In this way the slow cutoff itself formed a low pass band which had a gain of 0.5 at $p = 0.08$ (the value of h used), and zero at $p = 2h = 0.16$. A filter for which $P = 0$ is called an ultra low pass filter and essentially gives the trend of the input function. The response of the filter used is given in Figure 7, and the weights are listed in Table 1.

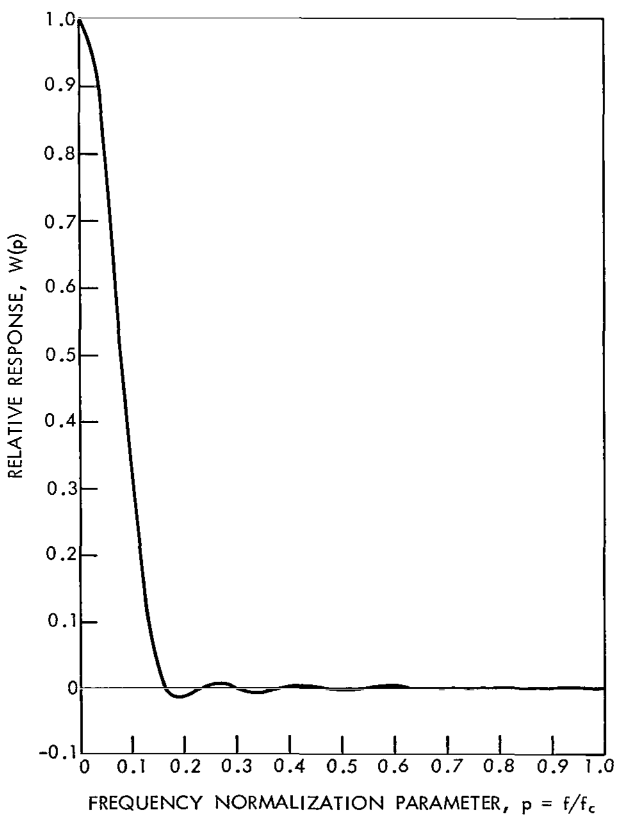

Figure 7—Frequency response of the ultra low pass filter designed to produce weighted hourly averages from 2.5 minute data (i.e., the weights were used in the averaging of the data for each hour to remove any effect on the average due to high frequency components).

Application of a filter by sliding it along the input data gives a "running average" of that data. To obtain hourly averages of 2.5 minute data, the filter was applied in turn to the 25 data points inclusive in each hour to produce the filtered equivalent of the center point value or average for that hour, and then the entire

Table 1

Low Pass Filter Weights* for 25-Point Filter ($N = 12$) with $h = 0.08$.

W_0	=	0.07949
W_1	=	0.07817
W_2	=	0.07434
W_3	=	0.06828
W_4	=	0.06046
W_5	=	0.05146
W_6	=	0.04189
W_7	=	0.03239
W_8	=	0.02350
W_9	=	0.01566
W_{10}	=	0.00919
W_{11}	=	0.00421
W_{12}	=	0.00071

*Because all filters for which weights are given in Tables 1-5 are symmetrical, only the weights corresponding to positive indices are tabulated. Thus in applying the full filters $W(-k) = W(k)$ is used.

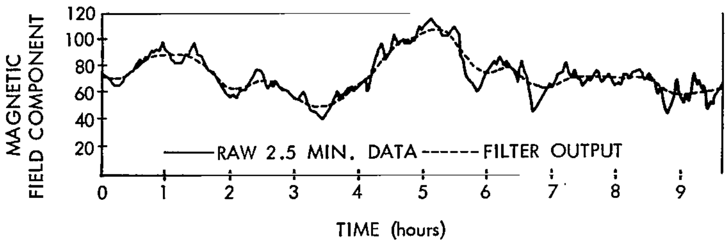

Figure 8—Smoothing of magnetic data provided by the 25-point ultra low pass filter (Figure 7) by running it along the data. The 2.5 minute output data represents the trend of the input data.

filter was moved over one hour and applied again. The values obtained in this way are the same as every 25th point on the curve described by the 2.5 minute values obtained by sliding the filter along the run of data point by point. An example of the effectiveness of this filter for smoothing out high frequency fluctuations when applied by sliding it along a run of 2.5 minute data is given in Figure 8.

Ultra low pass filters may be labeled according to cutoff point, sine termination characteristics, and size by means of the following scheme. A filter which is classified as a paabbcc filter is one with cutoff aa = 100P, sine termination parameter bb = 100 h_p, and size cc = N. The use of P in front instead of p will indicate that the values for cutoff and h are given in terms of the frequency ratio $r = f/f_s = p/2$.

THE CHEBYSHEV APPROXIMATION TO T(f) FOR A LOW PASS FILTER

Another mathematical approach to the computation of filter weights is through the use of Chebyshev polynomials (Reference 9). The Chebyshev polynomials are defined by

$$T_m(x) = \cos m\theta$$

$$= \cos (m \arc \cos x). \tag{33}$$

The Fourier expressions for orthogonality are

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{cases} 0 & \text{for } m \neq n, \\ \pi/2 & \text{for } m = n, \\ \pi & \text{for } m = n = 0, \end{cases} \tag{34}$$

and

$$\sum_{j=0}^{N-1} \cos m\theta_j \cos n\theta_j = \begin{cases} 0 & \text{for } m \neq n. \\ N/2 & \text{for } m = n. \\ N & \text{for } m = n = 0. \end{cases}$$

In terms of the Chebyshev polynomials these may be written

$$\int_{-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{for } m \neq n , \\ \pi/2 & \text{for } m = n , \\ \pi & \text{for } m = n = 0 , \end{cases} \tag{35}$$

and

$$\sum_{j=0}^{N-1} T_m(x_j)\, T_n(x_j) = \begin{cases} 0 & \text{for } (m \neq n) , \\ N/2 & \text{for } (m = n) , \\ N & \text{for } (m = n = 0) . \end{cases}$$

That the $T_m(x)$ are polynomials is shown by the following: De Moivre's theorem states that

$$\cos m\theta + i \sin m\theta = (\cos\theta + i \sin\theta)^m .$$

Expanding the right hand side, taking the real part of both sides, and replacing even powers of $\sin\theta$ by

$$(\sin^2\theta)^k = (1 - \cos^2\theta)^k ,$$

it may be established that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$. However, since $\cos(\text{arc}\cos x) = x$, then $T_m(x) = \cos(m \cos x)$ is a polynomial of degree m in x.

A general property of orthogonal polynomials is that they are easy to compute and to convert to a power series form. From the definition for the Chebyshev polynomials one obtains

$$\begin{aligned} T_0(x) &= 1 && (1) , \\ T_1(x) &= x && (\cos\theta) , \\ T_2(x) &= 2x^2 - 1 && (\cos 2\theta) , \end{aligned} \tag{36}$$

and so on. Another property is that orthogonal polynomials satisfv a three-term recurrence relation. In the case of the Chebyshev polynomials this relation is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) . \tag{37}$$

Finally, we have that $|T_m(x)| \leq 1$ for $|x| \leq 1$.

According to the *minimax principle,* Chebyshev approximations are associated with those approximations which minimize the maximum error. Least squares approximation keeps the average square error down, but in so doing isolated extreme errors are allowed. Chebyshev approximation keeps the extreme errors down but allows a larger average square error.

For a symmetrical filter,

$$W(f) = W_0 + 2\sum_{j=1}^{m} W_j \cos 2\pi f j \Delta t .$$

If we let $x = \cos\theta = \cos \pi f \Delta t = \cos\left[(\pi/2)\ f/f_c\right] = \cos(\pi p/2)$, then $W(p)$ is a polynomial of degree $2m$ in x. Now also let $x = z/z_0$. We have then that $T_{2m}(z)$ is a polynomial of degree $2m$ in $z = xz_0$. By extending the Chebyshev range beyond one then

$$|T_{2m}(z)| > 1 \quad \text{for} \quad |z| > 1 .$$

Because of the similarity of analytical form we equate

$$W(p) = T_{2m}(xz_0). \tag{38}$$

Now to approximate the frequency response with $T_{2m}(z)$, it is necessary to adjust $T_{2m}(z)$ to give $W(p) = 1.0$ at $p = 0$. For $f/f_c = p = 0$, $x = \cos(\pi p/2) = 1$. But $x = z/z_0$, so that $x = 1$ requires $z = z_0$. Hence $T_{2m}(z_0)$ corresponds to $p = 0$. Let $T_{2m}(z_0) = r$. Then $W(p) = T_{2m}(z)/r$ will give the normalized frequency response (Figure 9). At $z = 1$,

$$\begin{aligned} T_{2m}(z) = T_{2m}(1) &= \cos\ [2m \text{ arc } \cos\ (1)] \\ &= \cos 4m\pi \\ &= +1 . \end{aligned}$$

Hence, as may be seen in Figure 9, r is the ratio of the main lobe to the amplitude of the side lobes. The maximum precision of a particular filter of size N will be realized if the weights W_k are determined in such a way that the ratio r is maximized and the width of the main lobe is minimized.

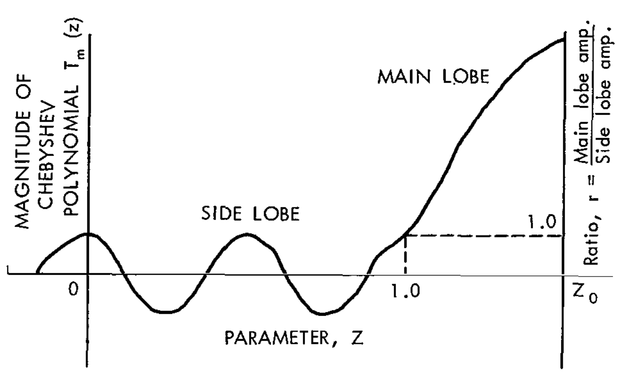

Figure 9–Filter frequency response approximation by the Chebyshev geometry when the Chebyshev range is extended beyond unity ($T_m(z) > 1$ for $Z > 1$).

Now we must investigate the zeros of $T_{2m}(z)$. Writing

$$T_{2m}(xz_0) = \cos 2mn ,$$

where $n = \text{arc } \cos xz_0$, it is seen that for $T_{2m}(xz_0) = 0$, we must have

$$2mn_k = k\pi \pm \frac{\pi}{2}.$$

By writing

$$2mn_k = \frac{(2k-1)}{2}\pi,$$

then

$$2m \text{ arc cos } x_k z_0 = (2k-1)\frac{\pi}{2},$$

or

$$\text{arc cos } x_k z_0 = \frac{(2k-1)}{4m}\pi.$$

But

$$x_k = \cos\frac{\pi}{2}\frac{f_k}{f_c} = \cos\frac{\pi}{2}p_k.$$

Therefore

$$\text{arc cos}\left(z_0 \cos\frac{\pi}{2}p_k\right) = \frac{(2k-1)}{4m}\pi,$$

or

$$z_0 \cos\frac{\pi}{2}p_k = \cos\frac{(2k-1)}{4m}\pi. \tag{39}$$

This will give the values of p_k at which the various zeros occur. Most importantly, the first zero, which falls at the half-width p_1 of the main lobe, is given by

$$z_0 \cos\frac{\pi}{2}p_1 = \cos\frac{\pi}{4m}. \tag{40}$$

Hence for any filter size $m = N$ and desired main lobe halfwidth p_1, one can compute the corresponding z_0 from

$$z_0 = \frac{\cos\frac{\pi}{4N}}{\cos\frac{\pi}{2}p_1}. \tag{41}$$

By knowing z_0 one can then compute $r = T_{2m}(z_0)$. Finally one can make use of the relation

$$W(p) = \frac{T_{2N}(z)}{r}.$$

to compute both the filter weights and the associated frequency response profile.

As a brief example, we shall perform the calculations for the weights of a 7-point low pass filter ($N = 3$). Now

$$W(p) = W_0 + 2\sum_{j=1}^{N} W_j \cos 2j\theta, \text{ where } \theta = \frac{\pi}{2} p ,$$

$$= W_0 + 2W_1 \cos 2\theta + 2W_2 \cos 4\theta + 2W_3 \cos 6\theta. \tag{42}$$

Let $x = \cos\theta$. From the recurrence relation we obtain

$$\cos 2\theta = T_2(x) = 2x^2 - 1,$$

$$\cos 4\theta = T_4(x) = 8x^4 - 8x^2 + 1,$$

and

$$\cos 6\theta = T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1. \tag{43}$$

Substituting from the relations in Equations 43 into Equation 42 gives

$$W(p) = 32I_3x^6 + (8I_2-48I_3)x^4 + (2I_1-8I_2+18I_3)x^2 + (I_0-I_1+I_2-I_3), \tag{44}$$

where

$$I_3 = 2W_3,\ I_2 = 2W_2,\ I_1 = 2W_1, \text{ and } I_0 = W_0.$$

But, as we have seen,

$$\frac{T_{2N}(xz_0)}{r} = \frac{T_6(xz_0)}{r} = \frac{32 z_0^6 x^6 - 48 z_0^4 x^4 + 18 z_0^2 x^2 - 1}{r}. \tag{45}$$

We can now equate $T_6(xz_0)/r = W(p)$ by powers of x. This gives

$$W_3 = z_0^6/2r,$$

$$W_2 = 12W_3 - 6z_0^4/2r,$$

$$W_1 = 4W_2 - 9W_3 + 9z_0^2/2r,$$

and

$$W_0 = 2(W_1 - W_2 + W_3) - 1/r. \tag{46}$$

If, for example, we choose $p_1 = 0.5$, then for $N = 3$ we get from Equation 41 that $z_0 = 1.37$.

Then

$$r = T_6(1.37) = 92,$$

and from Equation 46 it is found that

$$W_0 = 0.2509,$$

$$W_1 = 0.2113,$$

$$W_2 = 0.1218,$$

and

$$W_3 = 0.0415.$$

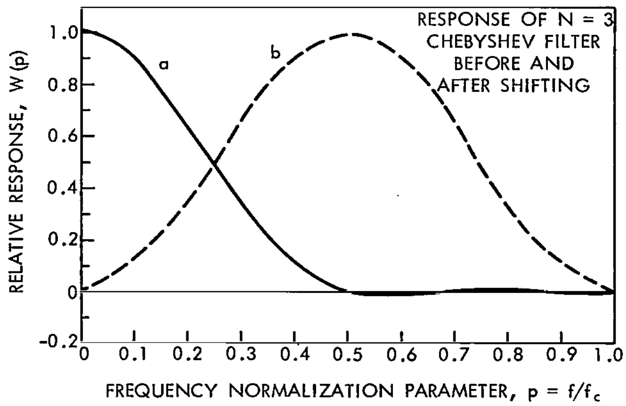

Figure 10—Frequency response of a crude Chebyshev low pass filter (a) and the effect produced by applying the shifting theorem to that same filter (b).

The frequency response $T_6(xz_0)$ of this filter is shown in Figure 10(a). As can be seen, this filter cuts off too slowly to be useful except for some types of smoothing, but it illustrates the principle of computing filter weights using Chebyshev polynomials.

SCALING FILTERS AND SHIFTING FOR BANDPASS RESPONSE

Whereas the computation of the weights for a large filter using the least squares approximation method is no more of a problem than for a small filter if one has automatic computational facilities available, the computation of a large Chebyshev filter is extremely complex due to the size of the polynomial involved. One way around this is to take a smaller filter and apply scaling and interpolation to produce a larger filter.

Filter scaling is accomplished in the following manner. Suppose we have

$$W_1(t) = \sum_{k=-m}^{m} W_k \, \delta(t + k\Delta t). \tag{47}$$

Scaling by a factor $\underline{a}$ has the effect

$$W_2(t) = \sum_{k=-m}^{m} W_k \, \delta(at + k\Delta t) = W_1(at). \tag{48}$$

Nonzero weights exist for $t = \pm k\Delta t/a$ due to the delta function. Correspondingly, when we Fourier transform these functions we obtain

$$\overline{Y_2}(\omega) = \int_{-\infty}^{\infty} \sum_{k=-m}^{m} W_k \, \delta(at + k\Delta t) e^{-i2\pi f t} dt$$

$$= \sum_{k=-m}^{m} W_k \left[\cos\left(2\pi f \frac{k\Delta t}{a}\right) - i \sin\left(2\pi f \frac{k\Delta t}{a}\right)\right], \tag{49}$$

or

$$\overline{Y_2}(f) = \sum_{k=-m}^{m} W_k \left[\cos\left(\pi k \frac{f}{a f_c}\right) - i \sin\left(\pi k \frac{f}{a f_c}\right)\right]$$

$$= \overline{Y_1}(f/a). \tag{50}$$

Since nonzero weights correspond to

$$\frac{t}{\Delta t} = \pm \frac{k}{a}, \text{ and } \frac{f}{f_c} = p = \pm \frac{2a}{k}, \tag{51}$$

scaling a filter by $a > 1$ means a contraction in the time domain and a corresponding expansion in the frequency domain. Conversely, use of $a < 1$ results in an expansion in the time domain and a corresponding contraction in the frequency domain.

Thus application of filter weights to every m^{th} input point has the same effect on the output data as if the sampling frequency had been divided by m. Likewise one can scale a smaller filter so as to effect a contraction in the frequency domain until the desired cutoff point is reached, and then can use interpolation to increase the number of weights until the desired filter size has been reached. Figure 11 shows the response of a 201-point filter ($N = 100$) computed in this manner. In comparison, the figure also shows the response of a least squares filter of the same size. The weights are tabulated in Tables 2 and 3.

Much of the power of the numerical filtering technique comes from the possibility of being able not only to low pass filter data, but also to attenuate all frequencies outside a particular frequency band of interest and hence bandpass filter data as well. The simplest way to accomplish this is to shift a low pass filter from $f = 0$ to $f = q$, the central frequency of the band of interest. The sharpness of response of the low pass filter determines the effective width of the bandpass window.

$$W_1(f) = \int_{-\infty}^{\infty} W(t)e^{-i2\pi ft}\,dt, \tag{52}$$

a shift by τ in time gives

$$W_2(f) = \int_{-\infty}^{\infty} W(t+\tau)e^{-i2\pi f(t+\tau)}\,d(t+\tau)$$

$$= \int_{-\infty}^{\infty} W(t)e^{-i2\pi ft}\,e^{-i2\pi f\tau}\,dt$$

$$= e^{-2\pi f\tau}\int_{-\infty}^{\infty} W(t)e^{-i2\pi ft}\,dt$$

$$= e^{-i2\pi f\tau}\,W_1(f). \tag{53}$$

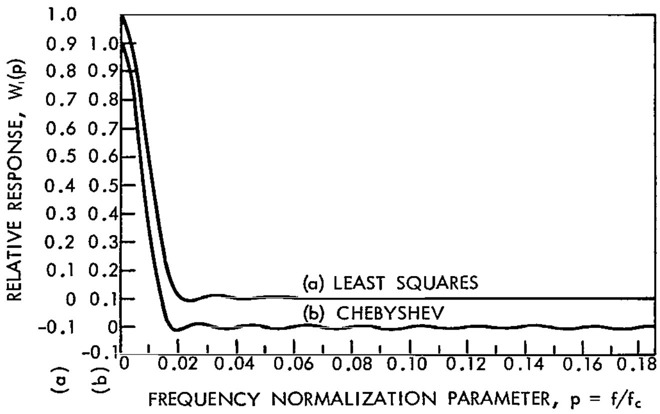

Figure 11—Example of response of (a) least squares and (b) Chebyshev approximations of ultra low pass filters (201-point, N = 100). The least squares filter was derived with h = 0.01, i.e., h = 1/N. If the value of h used in the least squares approximation had been slightly less than 1/N, the response would have been nearly that of the Chebyshev filter.

Hence a shift by τ in time leads to a shift by $e^{i2\pi f\tau}$ in the frequency domain. Since

$$W(f) = \int_{-\infty}^{\infty} W(t)e^{-i2\pi ft}\,dt$$

and

$$W(t) = \int_{-\infty}^{\infty} W(f)e^{+i2\pi ft}\,df$$

are identical except for a change in sign, if one desires to shift by q in frequency, one must multiply by $e^{i2\pi qt}$ in time. In order to achieve a bandpass filter at $f = q$ that has a symmetrical transfer function, negative frequencies must be considered as well and hence a low pass filter must be shifted by $\pm q$.

We define

$$W^{BP}(f) = \delta\left[W^{LP}(f+q) + W^{LP}(f-q)\right]. \tag{54}$$

By the shifting theorem described above this gives

$$W^{BP}(t) = \left(e^{i2\pi qt} + e^{-i2\pi qt}\right) W^{LP}(t)$$

$$= 2\cos 2\pi qt\; W^{LP}(t). \tag{55}$$

Table 2

Low Pass Filter Weights
for 201-Point Expanded Chebyshev Numerical Filter.

W(0) =	0.0085210	W(51) =	0.0050260
W(1) =	0.0085200	W(52) =	0.0049170
W(2) =	0.0085150	W(53) =	0.0048070
W(3) =	0.0065070	W(54) =	0.0046980
W(4) =	0.0084950	W(55) =	0.0045880
W(5) =	0.0084800	W(56) =	0.0044790
W(6) =	0.0084620	W(57) =	0.0043690
W(7) =	0.0084410	W(58) =	0.0042600
W(8) =	0.0084160	W(59) =	0.0041520
W(9) =	0.0083880	W(60) =	0.0040440
W(10) =	0.0083580	W(61) =	0.0039360
W(11) =	0.0083240	W(62) =	0.0038290
W(12) =	0.0082860	W(63) =	0.0037220
W(13) =	0.0082460	W(64) =	0.0036170
W(14) =	0.0082030	W(65) =	0.0035110
W(15) =	0.0081570	W(66) =	0.0034070
W(16) =	0.0081070	W(67) =	0.0033040
W(17) =	0.0080550	W(68) =	0.0032020
W(18) =	0.0080000	W(69) =	0.0031010
W(19) =	0.0079420	W(70) =	0.0030000
W(20) =	0.0078820	W(71) =	0.0029020
W(21) =	0.0078190	W(72) =	0.0028040
W(22) =	0.0077530	W(73) =	0.0027070
W(23) =	0.0076840	W(74) =	0.0026120
W(24) =	0.0076130	W(75) =	0.0025190
W(25) =	0.0075400	W(76) =	0.0024260
W(26) =	0.0074640	W(77) =	0.0023350
W(27) =	0.0073860	W(78) =	0.0022460
W(28) =	0.0073060	W(79) =	0.0021580
W(29) =	0.0072230	W(80) =	0.0020720
W(30) =	0.0071390	W(81) =	0.0019870
W(31) =	0.0070520	W(82) =	0.0019040
W(32) =	0.0069640	W(83) =	0.0018230
W(33) =	0.0068730	W(84) =	0.0017430
W(34) =	0.0067810	W(85) =	0.0016650
W(35) =	0.0066870	W(86) =	0.0015890
W(36) =	0.0065920	W(87) =	0.0015150
W(37) =	0.0064950	W(88) =	0.0014420
W(38) =	0.0063960	W(89) =	0.0013710
W(39) =	0.0062970	W(90) =	0.0013020
W(40) =	0.0061960	W(91) =	0.0012350
W(41) =	0.0060940	W(92) =	0.0011700
W(42) =	0.0059900	W(93) =	0.0010170
W(43) =	0.0058860	W(94) =	0.0009260
W(44) =	0.0057810	W(95) =	0.0008960
W(45) =	0.0056750	W(96) =	0.0009270
W(46) =	0.0055680	W(97) =	0.0010190
W(47) =	0.0054610	W(98) =	0.0011730
W(48) =	0.0053530	W(99) =	0.0013870
W(49) =	0.0052440	W(100) =	0.0016630
W(50) =	0.0051350		

Table 3

Low Pass Filter Weights
for 201-Point Least Squares Numerical Filter (P = 0, h = .01).

W(0)	= 0.0099368	W(51)	= 0.0047868
W(1)	= 0.0099342	W(52)	= 0.0046377
W(2)	= 0.0099265	W(53)	= 0.0044893
W(3)	= 0.0099136	W(54)	= 0.0043416
W(4)	= 0.0098956	W(55)	= 0.0041949
W(5)	= 0.0098725	W(56)	= 0.0040493
W(6)	= 0.0098443	W(57)	= 0.0039049
W(7)	= 0.0098111	W(58)	= 0.0037619
W(8)	= 0.0097728	W(59)	= 0.0036203
W(9)	= 0.0097296	W(60)	= 0.0034803
W(10)	= 0.0096815	W(61)	= 0.0033420
W(11)	= 0.0096286	W(62)	= 0.0032055
W(12)	= 0.0095708	W(63)	= 0.0030709
W(13)	= 0.0095084	W(64)	= 0.0029382
W(14)	= 0.0094413	W(65)	= 0.0028077
W(15)	= 0.0093697	W(66)	= 0.0026793
W(16)	= 0.0092936	W(67)	= 0.0025532
W(17)	= 0.0092132	W(68)	= 0.0024295
W(18)	= 0.0091285	W(69)	= 0.0023081
W(19)	= 0.0090396	W(70)	= 0.0021893
W(20)	= 0.0089467	W(71)	= 0.0020730
W(21)	= 0.0088498	W(72)	= 0.0019593
W(22)	= 0.0087490	W(73)	= 0.0018483
W(23)	= 0.0086446	W(74)	= 0.0017400
W(24)	= 0.0085366	W(75)	= 0.0016345
W(25)	= 0.0084251	W(76)	= 0.0015318
W(26)	= 0.0083103	W(77)	= 0.0014319
W(27)	= 0.0081923	W(78)	= 0.0013349
W(28)	= 0.0080712	W(79)	= 0.0012408
W(29)	= 0.0079472	W(80)	= 0.0011497
W(30)	= 0.0078204	W(81)	= 0.0010615
W(31)	= 0.0076911	W(82)	= 0.0009762
W(32)	= 0.0075592	W(83)	= 0.0008940
W(33)	= 0.0074250	W(84)	= 0.0008147
W(34)	= 0.0072886	W(85)	= 0.0007383
W(35)	= 0.0071502	W(86)	= 0.0006649
W(36)	= 0.0070100	W(87)	= 0.0005945
W(37)	= 0.0068680	W(88)	= 0.0005271
W(38)	= 0.0067244	W(89)	= 0.0004625
W(39)	= 0.0065795	W(90)	= 0.0004009
W(40)	= 0.0064333	W(91)	= 0.0003421
W(41)	= 0.0062860	W(92)	= 0.0002862
W(42)	= 0.0061378	W(93)	= 0.0002331
W(43)	= 0.0059888	W(94)	= 0.0001828
W(44)	= 0.0058392	W(95)	= 0.0001352
W(45)	= 0.0056891	W(96)	= 0.0000903
W(46)	= 0.0055387	W(97)	= 0.0000481
W(47)	= 0.0053881	W(98)	= 0.0000085
W(48)	= 0.0052376	W(99)	=-0.0000286
W(49)	= 0.0050872	W(100)	=-0.0000632
W(50)	= 0.0049368		

Thus to compute the k^{th} weight of a bandpass filter centered at $f = q$, one must operate in the following way on the k^{th} weight of the low pass filter to be shifted:

$$W_k^{BP} = 2 \cos \pi \, kq/f_c \, W_k^{LP}. \tag{56}$$

The response of a bandpass filter obtained by performing this operation on each weight of the $N = 3$ low pass Chebyshev filter computed previously as an example is shown by curve B in Figure 10.

In order to filter out the diurnal component of geomagnetic data, bandpass filters were constructed by shifting the expanded Chebyshev filter ($N = 100$) shown in Figure 11. The first five harmonics of the total diurnal component have periods of 24, 12, 8, 6 and 4.8 hours. Thus the LP filter was successively operated upon to give five BP filters which peaked at $p = 1/12$, 2/12, 3/12, 4/12, and 5/12. Power spectral analysis performed on a number of typical runs of H-component data during the IGY (Ness, 1962*) has revealed that one need not be concerned with more than the first five harmonics of the diurnal component. This may also be ascertained by inspection of the relative amplitudes of the harmonics themselves. Figure 12 shows the first five harmonics at a middle latitude observatory for a period during which the field was disturbed and thus contained harmonics of enhanced amplitude.

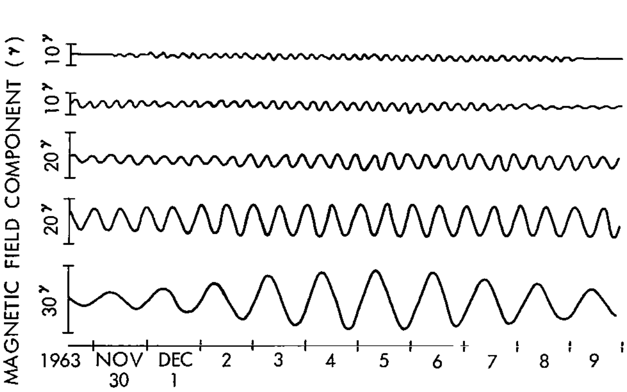

Figure 12—The first five harmonics of the diurnal component of the horizontal magnetic field at Fredericksburg, Virginia during the week of geomagnetic disturbances of December 2, 1963.

The individual harmonics shown in Figure 12 were isolated from geomagnetic H component data by using each of the five BP filters individually on the raw data. Although this sort of analysis reveals the relative amounts of energy present in the different harmonics and shows modulation due to time variations in field strength, our primary intent was to develop a filtering process that would isolate the total diurnal component so that it could be subsequently subtracted from the raw surface data. The most efficient way to achieve this effect was to combine the five BP filters in such a way as to produce one resultant BP filter.

Each filter, when applied to the raw data, produces as its output the particular harmonic component which it was designed to pass. In our case

*N. F. Ness, private communication.

$$
\begin{aligned}
O_1(t) &= F_{24}(t) \\
O_2(t) &= F_{12}(t) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
O_5(t) &= F_{4.8}(t).
\end{aligned}
\tag{57}
$$

To a good approximation the diurnal component is a linear function of only these five harmonics, i.e.,

$$D = F_{24}(t) + F_{12}(t) + \cdot\cdot\cdot + F_{4.8}(t). \tag{58}$$

Since this is equivalent to

$$O_f(t) = O_1(t) + O_2(t) + \cdot\cdot\cdot + O_5(t), \tag{59}$$

and since we have for an input function $I(t)$ that

$$O_1(t) = \sum_j I(t - j\,\Delta t)\, W_j^1, \tag{60}$$

and similarly for the other components, then we can write

$$
\begin{aligned}
O_f(t) &= \sum_j I(t - j\Delta t)\, Wj^1 + \sum_j I(t - j\Delta t)\, Wj^2 + \cdot\cdot\cdot \\
&\quad + \sum_j I(t - j\Delta t)\, Wj^5 \\
&= \sum_j I(t - j\Delta t)\, (W_j^1 + W_j^2 + \cdot\cdot\cdot + W_j^5) \\
&= D.
\end{aligned}
\tag{61}
$$

Hence the output of the filter obtained by linearly combining the corresponding weights from each separate BP filter is the total diurnal component that we wish to isolate. In general, for the k^{th} weight of a BP filter which will pass only a data component which is the sum of m harmonics, we simply add:

$$W_k^f = W_k^1 + W_k^2 + \ldots + W_k^m , \tag{62}$$

where W_k^i is the k^{th} weight of the i^{th} harmonic filter.

The frequency response of the filter which was constructed for isolating the total diurnal component is shown in Figure 13, and the weights are listed in Table 4. The data values comprising the output of this filter were subtracted from raw data values of corresponding times, producing geomagnetic H-component data free of the diurnal modulation. In Figures 14 and 15 examples of hourly average data and the corresponding filtered data have been plotted.

One further application of our filters involved plotting on an expanded scale the raw 2.5 minute H component data for the time period covered by magnetic storms. We wanted the diurnal component removed from the data, but to prevent aliasing we had to low pass the data to remove the high frequency constituents before application of the bandpass filter. The output of the LP filter was used as input to the BP filter, and the output of this second filter was the dirunal component.

Scaling was used in the application of the BP filter to the extent that the weights were applied to 2.5 minute data values that were separated in time by one hour. Since the sampling frequency of the data was 24/hour, application of the weights to every 24^{th} data value was equivalent to dividing the sampling frequency by 24, giving the one hour sampling frequency required by the design of the filter. The residual that remained when the output of the BP filter was subtracted from the raw data was the H-component of the geomagnetic field as described by 2.5 minute data points.

Figure 16 is an example of the results of applying this filtering technique to the data recorded at six magnetic observatories during the storm of April 1, 1964. With the diurnal components removed, the remaining time variations can be more effectively correlated with the field variations observed at a distance from the Earth by the satellite.

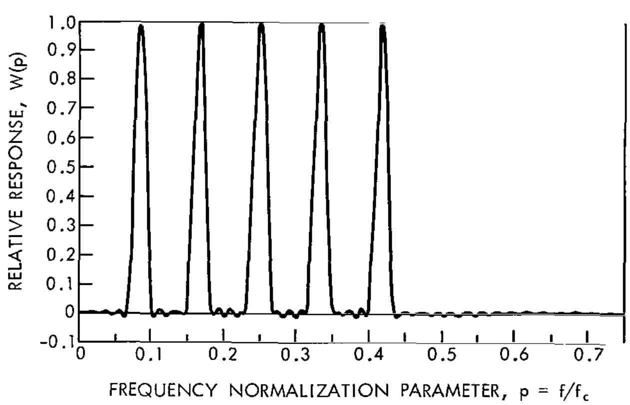

Figure 13—Frequency response of filter derived by linear combination of five 201-point bandpass filters obtained by shifting the expanded Chebyshev filter band shown in Figure 11 to values of p corresponding to the frequencies of each of the first five harmonics of the diurnal component.

Another filter application problem required the construction of a bandpass filter centered on the 24-hour component and with a ±4-hour bandwidth, i.e., the passband had to be centered on $p_{24} = 0.0833$, with cutoffs at $p_{28} = 0.0714$ and $p_{20} = 0.1000$. It was further required that $N = 24$, or perhaps some larger multiple of this if $N = 24$ did not give satisfactory results.

It is difficult to design a filter of this size or smaller with such a narrow passband because the wavelength of the oscillations in the

Table 4

Bandpass Filter Weights
for 201-Point Harmonic Diurnal Component Filter.

W(0) = 0.0852100	W(51) =-0.0171596
W(1) = 0.0561956	W(52) =-0.0098348
W(2) =-0.	W(53) = 0.0014574
W(3) =-0.0290446	W(54) = 0.0000004
W(4) =-0.0169900	W(55) =-0.0081082
W(5) = 0.0025714	W(56) =-0.0089580
W(6) = 0.0000002	W(57) =-0.0025596
W(7) =-0.0149180	W(58) = 0.0000002
W(8) =-0.0168320	W(59) =-0.0046982
W(9) =-0.0049138	W(60) =-0.0080880
W(10) = 0.0000002	W(61) =-0.0044546
W(11) =-0.0094198	W(62) =-0.
W(12) =-0.0165720	W(63) =-0.0021802
W(13) =-0.0093316	W(64) =-0.0072338
W(14) =-0.0000002	W(65) =-0.0062054
W(15) =-0.0047780	W(66) =-0.0000004
W(16) =-0.0162138	W(67) = 0.0010022
W(17) =-0.0142360	W(68) =-0.0064034
W(18) =-0.0000002	W(69) =-0.0105876
W(19) = 0.0024082	W(70) =-0.0000012
W(20) =-0.0157638	W(71) = 0.0191398
W(21) =-0.0266958	W(72) = 0.0280400
W(22) =-0.0000008	W(73) = 0.0178558
W(23) = 0.0506810	W(74) = 0.0000012
W(24) = 0.0761300	W(75) =-0.0086004
W(25) = 0.0497330	W(76) =-0.0048524
W(26) = 0.0000010	W(77) = 0.0007080
W(27) =-0.0252172	W(78) = 0.0000002
W(28) =-0.0146126	W(79) =-0.0038138
W(29) = 0.0021900	W(80) =-0.0041444
W(30) = 0.0000002	W(81) =-0.0011642
W(31) =-0.0124628	W(82) =-0.
W(32) =-0.0139287	W(83) =-0.0020630
W(33) =-0.0040262	W(84) =-0.0034860
W(34) =-0.0000002	W(85) =-0.0018844
W(35) =-0.0075670	W(86) =-0.
W(36) =-0.0131840	W(87) =-0.0008872
W(37) =-0.0073504	W(88) =-0.0028838
W(38) =-0.	W(89) =-0.0024232
W(39) =-0.0036886	W(90) =-0.
W(40) =-0.0123918	W(91) = 0.0003746
W(41) =-0.0107704	W(92) =-0.0023396
W(42) =-0.0000004	W(93) =-0.0034722
W(43) = 0.0017852	W(94) =-0.0000006
W(44) =-0.0115616	W(95) = 0.0059092
W(45) =-0.0193756	W(96) = 0.0092700
W(46) =-0.0000014	W(97) = 0.0067216
W(47) = 0.0360180	W(98) = 0.0000006
W(48) = 0.0535300	W(99) =-0.0047354
W(49) = 0.0345894	W(100) =-0.0033264
W(50) = 0.0000014	

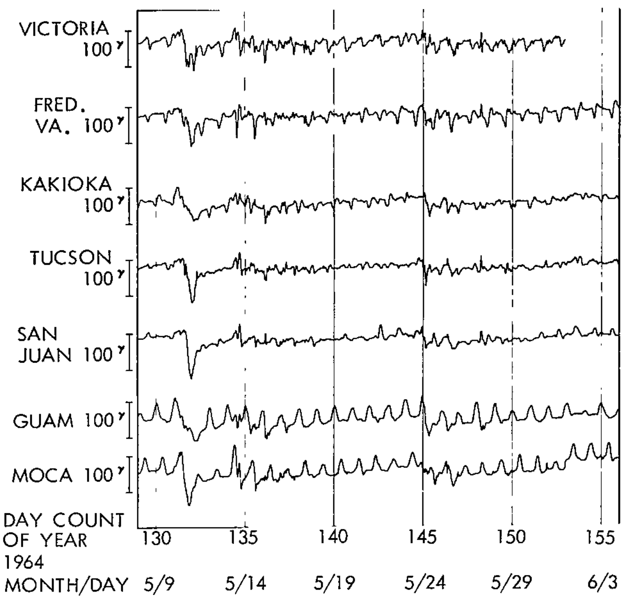

Figure 14—Hourly average data representing the variations in the H-component of the geomagnetic field at seven magnetic observatories from May 9 to June 3, 1964. The raw 2.5 minute data values from these stations were averaged using the weights of the filter shown in Figure 7.

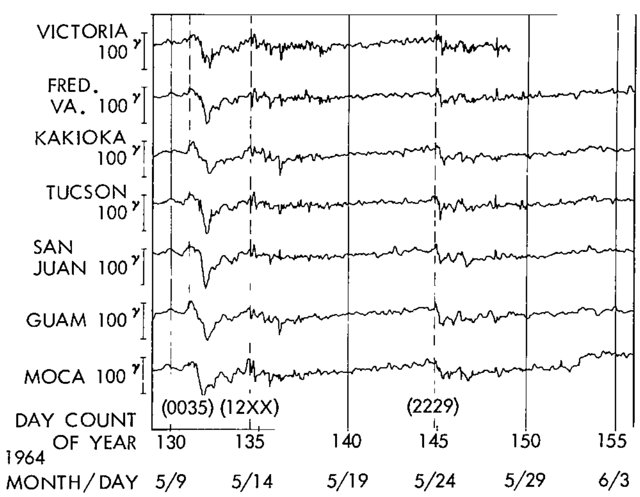

Figure 15—The output data obtained when the data shown in Figure 14 are used as inputs to the diurnal component BP filter of Figure 13.

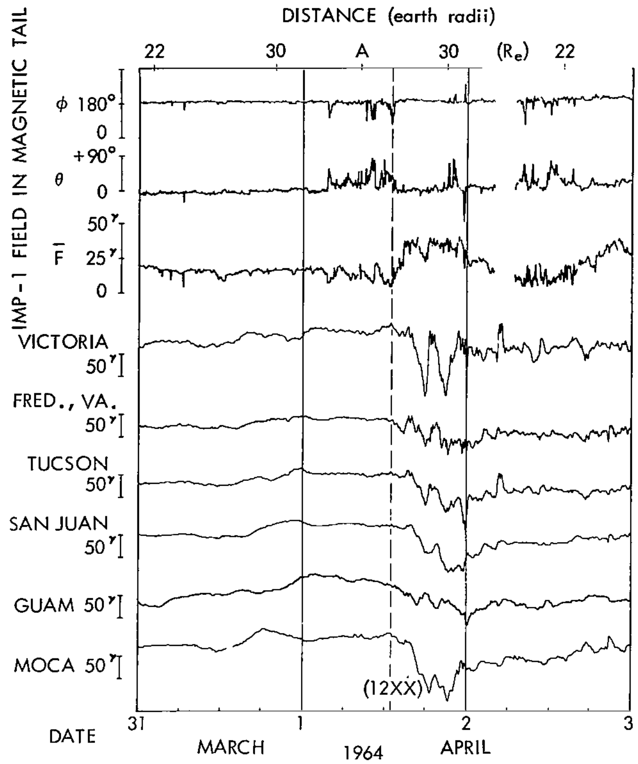

Figure 16—Result of applying the diurnal component BP filter to 2.5 minute geomagnetic data from six magnetic observatories. These filtered data show the H-component variations that occur at each of the stations prior to and during the magnetic storm of April 1, 1964.

approximating function (the actual frequency response) is greater than the width of the theoretical band. This prevents a very close approximation to the rectangular shape of the ideal band. Also in order to approach a step function type profile for the low pass filter to be shifted, one must employ a cutoff value of $P > 0$. However, when P is increased in an attempt to fit the $W(p) = 1$ part of the step, one finds that there is an accompanying rapid increase in the width of the achieved band at the base (near the $W(p) = 0$ level). Some results for $N = 24$, 48 and 72 are shown in Figures 17-19,

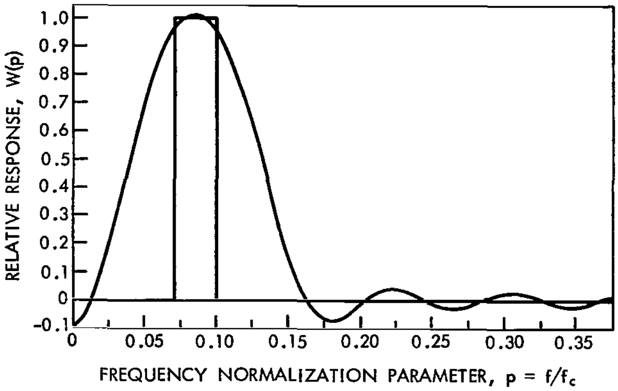

Figure 17—Frequency response obtained in the attempt to approximate a 24 ± 4 hour ideal passband with a relatively small filter (49-point, N = 24).

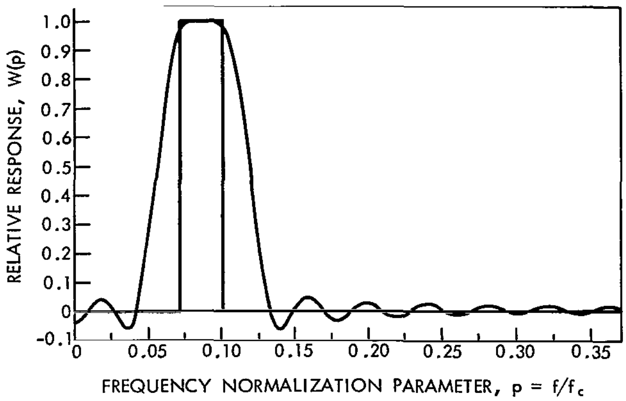

Figure 18—Improved approximation of the 24 ± 4 hour ideal passband obtained by doubling the filter size (97-point, N = 48).

and the corresponding weights are tabulated in Table 5. As can be seen, it is only when N = 72 is used that a close approximation of the desired band is approached.

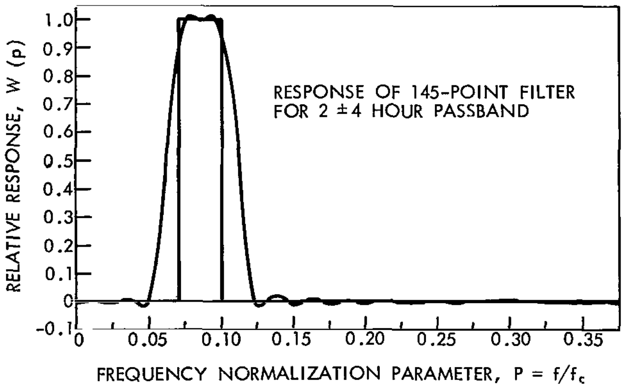

Figure 19—Close approximation (compare Figures 17 and 18) of the 24 ± 4 hour ideal passband obtained by use of a 145-point filter (N increased to 72).

CONCLUDING REMARKS

In summary, an attempt has been made to present (1) a brief review of the basic concepts of numerical filtering, (2) derivations of formulas which can be utilized to compute filter weights and frequency profiles by using either a least squares approach or Chebyshev polynomials to approximate the desired response, and (3) several examples of how such filters are currently being used in studying transient variations of the earth's magnetic field. The presentation has been oriented more toward providing a guide to practical application than a rigorous theoretical treatment of the subject.

For supplemental information on numerical filtering see Fullenwider and McNamee (Reference 11), and Martin (References 8 and 12). Arthaber (Reference 13) has considered the related topic of spatial filtering. The more general subjects of digital filtering and frequency analysis have been discussed in the literature by Salzer (References 14 and 15), Linville and Salzer (Reference 16), Hauptschein (Reference 17), and Langill (Reference 10).

Table 5

Bandpass Filter Weights for 24 ± 4 Hour Passband.

49 Points h = .025 p = .020	97 Points h = .014 p = .016		145 Points h = .012 p = .012	
W(0) = 0.08799	W(0) = 0.05969	W(26) = 0.00883	W(0) = 0.04769	
W(1) = 0.08445	W(1) = 0.05738	W(27) = 0.00492	W(1) = 0.04591	W(37) = -0.00423
W(2) = 0.07422	W(2) = 0.05065	W(28) = 0.00173	W(2) = 0.04072	W(38) = -0.00258
W(3) = 0.05841	W(3) = 0.04011	W(29) = -0.00048	W(3) = 0.03255	W(39) = -0.00123
W(4) = 0.03874	W(4) = 0.02670	W(30) = -0.00159	W(4) = 0.02206	W(40) = -0.00033
W(5) = 0.01728	W(5) = 0.01160	W(31) = -0.00167	W(5) = 0.01013	W(41) = 0.00006
W(6) = -0.00380	W(6) = -0.00388	W(32) = -0.00091	W(6) = -0.00230	W(42) = -0.00004
W(7) = -0.02247	W(7) = -0.01843	W(33) = 0.00037	W(7) = -0.01424	W(43) = -0.00056
W(8) = -0.03713	W(8) = -0.03088	W(34) = 0.00184	W(8) = -0.02477	W(44) = -0.00136
W(9) = -0.04674	W(9) = -0.04029	W(35) = 0.00315	W(9) = -0.03310	W(45) = -0.00225
W(10) = -0.05093	W(10) = -0.04604	W(36) = 0.00402	W(10) = -0.03867	W(46) = -0.00307
W(11) = -0.04999	W(11) = -0.04786	W(37) = 0.00421	W(11) = -0.04114	W(47) = -0.00365
W(12) = -0.04477	W(12) = -0.04587	W(38) = 0.00365	W(12) = -0.04047	W(48) = -0.00386
W(13) = -0.03655	W(13) = -0.04051	W(39) = 0.00237	W(13) = -0.03684	W(49) = -0.00363
W(14) = -0.02681	W(14) = -0.03253	W(40) = 0.00049	W(14) = -0.03072	W(50) = -0.00296
W(15) = -0.01703	W(15) = -0.02284	W(41) = -0.00176	W(15) = -0.02271	W(51) = -0.00188
W(16) = -0.00846	W(16) = -0.01243	W(42) = -0.00412	W(16) = -0.01357	W(52) = -0.00049
W(17) = -0.00202	W(17) = -0.00230	W(43) = -0.00631	W(17) = -0.00409	W(53) = 0.00106
W(18) = 0.00180	W(18) = 0.00669	W(44) = -0.00806	W(18) = 0.00495	W(54) = 0.00262
W(19) = 0.00301	W(19) = 0.01387	W(45) = -0.00917	W(19) = 0.01285	W(55) = 0.00404
W(20) = 0.00197	W(20) = 0.01882	W(46) = -0.00953	W(20) = 0.01910	W(56) = 0.00516
W(21) = -0.00063	W(21) = 0.02139	W(47) = -0.00909	W(21) = 0.02332	W(57) = 0.00589
W(22) = -0.00395	W(22) = 0.02169	W(48) = -0.00792	W(22) = 0.02536	W(58) = 0.00614
W(23) = -0.00714	W(23) = 0.02005		W(23) = 0.02527	W(59) = 0.00590
W(24) = -0.00947	W(24) = 0.01697		W(24) = 0.02328	W(60) = 0.00522
	W(25) = 0.01303		W(25) = 0.01977	W(61) = 0.00416
			W(26) = 0.01521	W(62) = 0.00284
			W(27) = 0.01011	W(63) = 0.00138
			W(28) = 0.00499	W(64) = -0.00007
			W(29) = 0.00030	W(65) = -0.00138
			W(30) = -0.00360	W(66) = -0.00246
			W(31) = -0.00648	W(67) = -0.00322
			W(32) = -0.00823	W(68) = -0.00364
			W(33) = -0.00888	W(69) = -0.00372
			W(34) = -0.00856	W(70) = -0.00348
			W(35) = -0.00749	W(71) = -0.00300
			W(36) = -0.00596	W(72) = -0.00235

REFERENCES

1. Holloway, J. L., Jr., "Smoothing and Filtering of Time Series and Space Fields," *Adv. Geophys.* 4:351-390, 1958.
2. Goldman, S., "Information Theory," New York: Prentice-Hall, 1953.
3. Wiener, N., "Extrapolation, Interpolation and Smoothing Stationary Time Series," Cambridge, Massachusetts: Technology Press of the Mass. Inst. of Tech., 1949.
4. Hildebrand, F. B., "Introduction to Numerical Analysis," New York: McGraw-Hill, 1956.
5. Hamming, R. W., "Numerical Methods for Scientists and Engineers," New York: McGraw-Hill, 1962.
6. Guillemin, E. A., "Theory of Linear Physical Systems," New York: John Wiley and Sons, 1963.
7. Bergström, H., "Limit Theorems for Convolutions," New York: John Wiley and Sons, 1963.
8. Martin, M. A., "Frequency Domain Applications in Data Processing," Philadelphia, Pennsylvania: General Electric Co., Missiles and Ordnance Systems Department, Technical Information Series No. 57SD340, May 1957.
9. Lanczos, C., "Applied Analysis," Englewood Cliffs, N. J.: Prentice Hall, 1956.
10. Langill, A. W., Jr., "Digital Filters," *Frequency,* 2(6):26-31, November-December 1964.
11. Fullenwider, E. D., and B. I. McNamee, "Filtering Sampled Functions," U. S. Naval Ordnance Lab., Corona, Calif., Tech. Memo No. 64-104, July 12, 1956.
12. Martin, M. A., "Frequency Domain Applications to Data Processing," *IRE Trans. on Nuc. Sci.* SET-5(1):33-41, March 1959.
13. Arthaber, J., "Fundamentals of Spatial Filtering," Washington, D. C.: Diamond Ordnance Fuze Labs, Report TR-960, June 15, 1962.
14. Salzer, J. M., "Frequency Analysis of Digital Computers Operating in Real Time," *Proc. IRE,* 42(2):457-486, February 1954.
15. Salzer, J. M., "Treatment of Digital Control Systems and Numerical Processes in the Frequency Domain," Sc. D. degree thesis, Cambridge, Massachusetts, 1951.
16. Linvill, W. K., and Salzer, J. M., "Analysis of Control Systems Involving Digital Computers," *Proc. IRE,* 41(7):901-906, July 1953.
17. Hauptschein, A., "Digital Filter Design," Dept. Electrical Engineering Tech. Memo No. 35, New York University, January 20, 1964.

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