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COMPUTATION OF THE LONG RANGE MOTION OF A LUNAR SATELLITE

BY

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CONTENTS

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SUMMARY

Formulae for long periodic perturbations depending on the argument of the pericenter are derived by the method of Harmonic Analysis and with Elliptic Integrals.

An approximation of the terms of the Energy Integral that are dependent on the oblateness was used in the solution with Elliptic Integrals.

LIST OF SYMBOLS

a = semimajor axis of the satellite's orbit = L^2/μ.

b = mean radius of the moon (.1738 decamegametres).

e = eccentricity of the satellite's orbit = $\sqrt{1 - \frac{G^2}{L^2}}$

G = $L\sqrt{1 - e^2}$

g = argument of pericenter of the satellite.

h = longitude of the ascending node of the lunar satellite.

H = $G \cos i$

i = inclination of the satellite's orbit plane to the moon's equatorial plane = $\cos^{-1} H/G$

J_2 = principal part of the oblateness of the moon (2.41×10^{-4}).

$K_1 = \frac{3n_☾^2}{8\epsilon n}$

$K_2 = \frac{3}{4} J_2 b^2 n^2$

ℓ = mean anomaly of the lunar satellite.

L = $\sqrt{\mu a}$

n = mean motion of the satellite = μ^2/L^3.

$n_☾$ = mean motion of the moon's mean anomaly = 2.2802713×10^{-3} (radius/centiday).

$n_☾^*$ = mean motion of the moon's mean longitude = 2.2997150×10^{-3} (radians/centiday).

ϵ = ratio of the sum of the masses of the earth and the moon to the mass of the earth = 1.0123001.

$$\eta^2 = 1 - e^2 = \frac{G^2}{L^2}$$

μ = gravitational constant times the mass of the moon = 3.6601891×10^{-3} (decamegametres^3/centiday^2).

$$\theta = \frac{H}{L}$$

COMPUTATION OF THE LONG RANGE MOTION OF A LUNAR SATELLITE

INTRODUCTION

In this paper formulae for $d\ell/dt$, dh/dt, ℓ and h that were derived by:

1. The method of Harmonic Analysis and
2. With Elliptic Integrals

are presented.

To preserve the continuity of the presentation the formula for g, η^2 and dg/dt from reference 1 are also given. In the solution with Elliptic Integrals formulae for $d\ell/dt$, dh/dt and dg/dt are presented from reference 3.

THE HARMONIC ANALYSIS METHOD

The Harmonic Analysis Method is presented for the computation of the long range motion of a Lunar Satellite. The procedure, as outlined below, is well adapted to high speed digital computers and attempts to reduce the amount of labor involved in the computation.

a. Derivation of η^2 as a Cosine Series

The Hamiltonian, F is given by

$$F = F_0 + F_1 + F_2 \tag{1}$$

where

$$F_0 = \frac{\mu^2}{L^3}$$

$$F_1 = n_{\mathbb{C}}^* H$$

$$F_2 = \left[K_1 L \left\{ (5 - 3\eta^2) \left(\frac{3H^2}{G^2} - 1 \right) + \right. \right.$$

$$\left. \left. + 15 (1 - \eta^2) \left(1 - \frac{H^2}{G^2} \right) \cos 2g \right\} - \frac{2K_2}{\eta^3} \left(1 - \frac{3H^2}{G^2} \right) \right] = C$$

One seeks now an approximation for $1/\eta^3$ (dependent on the value of e^2) so that one has a quadratic in e^2. The truncated binomial expansion of $1/\eta^3$ is $(1 + (3/2)e^2 + (15/8)e^4)$. However, a better approximation for $1/\eta^3$ is $(1 + (3/2)e^2 + 2e^4)$. The following table compares this approximation, the actual value of $1/\eta^3$ and the binomial expansion of $1/\eta^3$. This approximation was derived empirically and is accurate for $0 \leq e \leq .5$. Figure 1 is a graph of these values. Error is defined as actual value/approximate value.

e	$1/\eta^3$	Author's Approximation	Error	Binomial Expansion	Error
0	1	1	1	1	1
.1	1.0151897	1.0152	.99998985	1.0151875	1.0000022
.2	1.0631466	1.0632	.99994977	1.063	1.0001379
.3	1.1519614	1.1512	1.0006614	1.1501875	1.0051420
.4	1.2989160	1.2912	1.0059758	1.288	1.0084752
.5	1.5396007	1.5	1.0264005	1.492	1.0319040

If one approximates $1/\eta^3$ by $(1 + (3/2)e^2 + 2e^4)$ and substitutes $(1 - e^2)$ for η^2 then (neglecting powers of e greater than 4) one has:

$$e^4 \{3K_1 L - K_2 (1 - 12\nu^2) - 15K_1 L (1 - \nu^2) \cos 2g\}$$

$$+ e^2 \{(K_1 L + K_2)(9\nu^2 - 1) + 15K_1 L (1 - \nu^2) \cos 2g + C\}$$

$$+ \{2(K_1 L + K_2)(3\nu^2 - 1) - C\} = 0. \qquad (2)$$

where C is the value of F_2 with initial conditions.

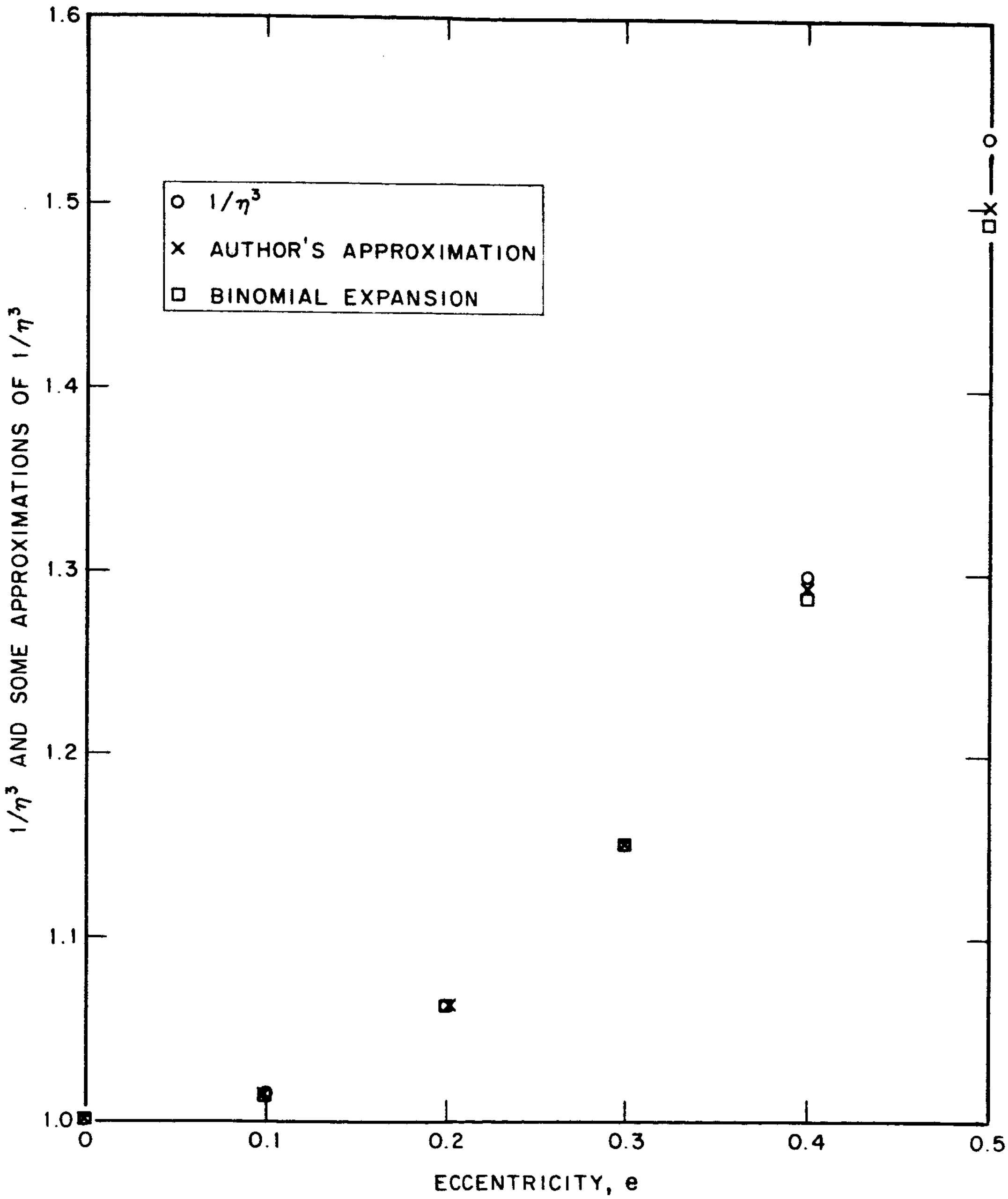


Figure 1—Comparison of $1/\eta^3$ With Some Approximations of $1/\eta^3$

$$C = 6\left\{\frac{K_1L}{6}(2 + 3e_0^2)\left(3\frac{H_0^2}{G_0^2} - 1\right) + \right.$$

$$\left. + \frac{K_2}{3(1 - e_0^2)^{3/2}}\left(\frac{3H_0^2}{G_0^2} - 1\right) + (5/2)K_1L\, e_0^2\left(1 - \frac{H_0^2}{G_0^2}\right)\cos 2g\right\} \tag{3}$$

In equation (2) let $2g$ take 5 values such that $0 \leq 2g \leq \pi$; i.e. for example 0, $\pi/4$, $\pi/2$, $3\pi/4$, π. One then has 5 quadratics in e^2.

Using the initial conditions of L, G, H and the relationship $\nu = H/L$ compute α, β, and γ where

α = coefficient of e^4 term of equation (2)

β = coefficient of e^2 term of equation (2)

γ = coefficient of e^2 term of equation (2).

Solve for η^2 using the relationship

$$\eta^2 = 1 - e^2 = 1 + \frac{\beta}{2\alpha} \mp \sqrt{\frac{\beta^2}{4\alpha^2} - \frac{\gamma}{\alpha}} \tag{4}$$

and choosing the sign so that $0 \leq \eta^2 \leq 1$.

Assume that η^2 can be represented as a cosine series

$$\sum_{i=0}^{i=n} m_i \cos 2ig$$

$$\text{i.e. } \eta^2 = m_0 + m_1 \cos 2g + m_2 \cos 4g + m_3 \cos 6g$$

$$+ m_4 \cos 8g \quad (\text{for } n = 4). \tag{5}$$

Next, construct a 5×6 matrix of this shape

$$
\begin{aligned}
\eta_0^2 &= m_0 + m_1 + m_2 + m_3 + m_4 \\
\eta_1^2 &= m_0 + .5\, m_1 - .5\, m_2 - m_3 - .5\, m_4 \\
\eta_2^2 &= m_0 - m_2 + m_4 \\
\eta_3^2 &= m_0 - .5\, m_1 - .5\, m_2 + m_3 - .5\, m_4 \\
\eta_4^2 &= m_0 - m_1 + m_2 - m_3 + m_4
\end{aligned} \tag{6}
$$

Solve for $m_0 \ldots m_4$ using these relationships.

$$m_0 = \frac{(\eta_0^2 + \eta_4^2)}{6} + \frac{(\eta_1^2 + \eta_3^2)}{3}$$

$$m_1 = \frac{1}{3}(\eta_0^2 - \eta_4^2 + \eta_1^2 - \eta_3^2)$$

$$m_2 = \frac{1}{4}(\eta_0^2 + \eta_4^2) - \frac{\eta_2^2}{2}$$

$$m_3 = \frac{(\eta_0^2 - \eta_4^2)}{6} - \frac{(\eta_1^2 - \eta_3^2)}{3}$$

$$m_4 = \eta_2^2 - m_0 + m_2$$

These values of $m_0 \ldots m_4$ are the coefficients of the cosine series that was assumed in equation (5).

b. The Angular Variables

Using the Delaunay set of canonical variables (L, G, H, ℓ, g, h) the equations of motion for the angular variables become:

$$\frac{dg}{dt} = \frac{-\partial F}{\partial G} = \left[K_1 \eta \left[\left(\frac{5\nu^2}{\eta^4} - 1 \right) + 5 \left(1 - \frac{\nu^2}{\eta^4} \right) \cos 2g \right] - \frac{K_2}{L\eta^4} \left(1 - \frac{5\nu^2}{\eta^2} \right) \right] \tag{8}$$

It should be noted that there is the possibility that in certain instances

$$K_1 \eta \left[\left(\frac{5\nu^2}{\eta^4} - 1 \right) + 5 \left(1 - \frac{\nu^2}{\eta^4} \right) \cos 2g \right] = \frac{K_2}{L\eta^4} \left(1 - \frac{5\nu^2}{\eta^2} \right)$$

so that dg/dt will be zero and dt/dg will be undefined.

$$\frac{d\ell}{dt} = -\frac{\partial F}{\partial L} = \left\{ -\frac{K_2}{L} \cdot \frac{1}{\eta^3} + \left(\frac{10 K_1}{3} + \frac{\mu^2}{L^3} + 3 K_1 \nu^2 \right) - K_1 \eta^2 - 5K_1 (2 + \nu^2) \cos 2g + \right.$$

$$+ 5 K_1 \eta^2 \cos 2g + \frac{3K_2 \nu^2}{L} \cdot \frac{1}{\eta^5} + 10 K_1 \nu^2 \frac{\cos 2g}{\eta^2}$$

$$\left. - 10 \nu^2 K_1 \cdot \frac{1}{\eta^2} \right\} \tag{9}$$

$$\frac{dh}{dt} = -\frac{\partial F}{\partial H} = \left\{ \left(-\frac{2 K_2 H}{L^2} \right) \cdot \frac{1}{\eta^5} - \left(\frac{5 K_1 H}{L} \right) \cdot \frac{1}{\eta^2} + \left(\frac{3K_1 H}{L} - n_c^* \right) + \frac{5 K_1 H}{L} \frac{\cos 2g}{\eta^2} - \frac{5 K_1 H}{L} \cos 2g \right\} \tag{10}$$

c. <u>g as a function of time, t.</u>

Using the values of η^2 obtained from equation (2) compute from equation (8)

$$\frac{dt}{dg} = \frac{1}{\frac{dg}{dt}} .$$

Assume that dt/dg can be represented as a cosine series

$$\sum_{i=0}^{i=n} p_i \cos 2ig$$

i.e.

$$\frac{dt}{dg} = p_0 + p_1 \cos 2g + p_2 \cos 4g + p_3 \cos 6g + p_4 \cos 8g \tag{11}$$

Construct a matrix of the same shape and size as the one described as equations (6) under the section Derivation of η^2 as a Cosine Series, i.e. let $p_i = m_i$ and solve for $p_0 \ldots p_4$ using the relationships given in equation (7).

Integrate dt/dg to get

$$g = \frac{1}{p_0} t - \frac{p_1}{2p_0} \sin 2g - \frac{p_2}{4p_0} \sin 4g - \frac{p_3}{6p_0} \sin 6g - \frac{p_4}{8p_0} \sin 8g$$

or

$$g = \frac{1}{p_0} t - \frac{1}{2p_0} \left[p_1 \sin 2g + \frac{p_2}{2} \sin 4g + \frac{p_3}{3} \sin 6g + \frac{p_4}{4} \sin 8g \right] \tag{12}$$

Note that g has the holonomic shape, i.e. g can be expressed in terms of time and $f(g)$ as $g = \overline{g} + \delta f(g)$ where $\overline{g}$ is a function of time, t and δ is a small constant $= -1/2p_0$. Equation (12), therefore, can be solved so as to express g in terms of δ and $\overline{g}$. (It is assumed that $\overline{g}$, $f(g)$ and their derivatives are continuous functions). Rewrite g as

$$g = \overline{g} + q \sin 2g + r \sin 4g + c \sin 6g + d \sin 8g. \tag{13}$$

with

$$\overline{g} = \frac{1}{p_0} t$$

$$q = -\frac{p_1}{2p_0}$$

$$r = -\frac{p_2}{4p_0}$$

$$c = -\frac{p_3}{6p_0}$$

$$d = -\frac{p_4}{8p_0}$$

Then

$$\begin{aligned} g = \overline{g} &+ \left(q + \frac{A}{2} + \frac{E}{6}\right) \sin 2\overline{g} \\ &+ \left(r + \frac{B}{2} + \frac{P}{6}\right) \sin 4\overline{g} \\ &+ \left(c + \frac{J}{2} + \frac{M}{6}\right) \sin 6\overline{g} \\ &+ \left(d + \frac{D}{2} + \frac{N}{6}\right) \sin 8\overline{g} \end{aligned} \tag{14}$$

where

$$A = -2(qr + rc + cd)$$

$$B = 2(q^2 - 2qc - 2rd)$$

$$J = 6q(r - d)$$

$$D = 4(2qc + r^2)$$

$$E = 6\left\{-\frac{q^3}{2} - r^2q - c^2q - d^2q - \frac{r^2c}{2} - rcd + \frac{q^2c}{2} + qrd\right\}$$

$$P = -24\left\{qrc + qdc + q^2r + rc^2 + rd^2 - \frac{q^2d}{2} + \frac{r^3}{2} + \frac{c^2d}{2}\right\}$$

$$M = -54\left(r^2c + rcd + \frac{qr^2}{2} + d^2c + qdr - \frac{q^3}{6} + q^2c + \frac{c^3}{2}\right)$$

$$N = -48(-rq^2 + 2qrc + 2r^2d + rc^2 + 2q^2d + d^3 + 2dc^2)$$

d. $d\ell/dt$ and dh/dt as Cosine Series

Let $\bar{g}$, (equation (13)) be such that $0 \le \bar{g} \le \pi$ i.e. for example $0, \pi/4, \pi/2, 3\pi/4, \pi$ successively and compute g and cos 2g using equation (14). Assume that cos 2g can be written as the sum of a cosine series

$$\sum_{i=0}^{i=n} \phi_i \cos 2i\bar{g}$$

to get

$$\cos 2g = \phi_0 + \phi_1 \cos 2\bar{g} + \phi_2 \cos 4\bar{g} + \phi_3 \cos 6\bar{g} + \phi_4 \cos 8\bar{g} \tag{15}$$

Solve for $\phi_0 \ldots \phi_4$ using the procedure described under Derivation of η^2 as a cosine series. Compute $1/\eta^3$, $\eta^2 \cos 2g$, $1/\eta^5$, $1/\eta^2$, and $\cos(2g/\eta^2)$ using the results from equations (2) and (14). Assume that $1/\eta^3$, $1/\eta^5$, $\eta^2 \cos 2g$, and $\cos(2g/\eta^2)$ can be written as a cosine series.

For example:

$$\frac{1}{\eta_0^2} = k_0 + k_1 + k_2 + k_3 + k_4$$

$$\frac{1}{\eta_1^2} = k_0 + .5k_1 - 5k_2 - k_3 - .5k_4$$

$$\frac{1}{\eta_2^2} = k_0 \qquad - k_2 \qquad + k_4$$

$$\frac{1}{\eta_3^2} = k_0 - .5\,k_1 - .5\,k_2 + k_3 - .5\,k_4$$

$$\frac{1}{\eta_4^2} = k_0 \quad - k_1 \quad + k_2 - k_3 \quad + k_4$$

from which

$$k_0 = \frac{\left(\frac{1}{\eta_0^2} + \frac{1}{\eta_4^2}\right)}{6} + \frac{\left(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\right)}{3}$$

$$k_1 = \frac{1}{3}\left(\frac{1}{\eta_0^2} - \frac{1}{\eta_4^2} + \frac{1}{\eta_1^2} - \frac{1}{\eta_3^2}\right)$$

$$k_2 = \frac{1}{4}\left(\frac{1}{\eta_0^2} + \frac{1}{\eta_4^2}\right) - \frac{\frac{1}{\eta_2^2}}{2}$$

$$k_3 = \frac{\left(\frac{1}{\eta_0^2} - \frac{1}{\eta_4^2}\right)}{6} - \frac{\left(\frac{1}{\eta_1^2} - \frac{1}{\eta_3^2}\right)}{3}$$

$$k_4 = \frac{1}{\eta_2^2} - k_0 + k_2$$

$$\frac{\cos 2g}{\eta^2} = \sum_{i=0}^{i=n} \iota_i \cos 2i\,\overline{g}$$

$$\frac{1}{\eta^2} = \sum_{i=0}^{i=n} k_i \cos 2i\,\overline{g}$$

$$\frac{1}{\eta^3} = \sum_{i=0}^{i=n} \sigma_i \cos 2i\,\overline{g}$$

$$\frac{1}{\eta^5} = \sum_{i=0}^{i=n} \rho_i \cos 2i\overline{g}$$

$$\eta^2 \cos 2g = \sum_{i=0}^{i=n} \theta_i \cos 2i\overline{g}$$

Now if one takes

$$\frac{d\ell}{dt} = \left\{ \Phi \cdot \frac{1}{\eta^3} + \Upsilon + \Pi\,\eta^2 + \Delta \cos 2g + \Xi \cdot \frac{1}{\eta^5} \right.$$

$$\left. + \Psi \frac{\cos 2g}{\eta^2} - 5\Pi\,\eta^2 \cos 2g - \Psi \cdot \frac{1}{\eta^2} \right\} \tag{16}$$

and

$$\frac{dh}{dt} = \left\{ \Gamma \cdot \frac{1}{\eta^5} + \Theta \frac{1}{\eta^2} - \Theta \cdot \frac{1}{\eta^2} \cos 2g \right.$$

$$\left. + \Lambda + \Theta \cos 2g \right\} \tag{17}$$

with

$$\Phi = -\frac{K_2}{L}$$

$$\Upsilon = \left(\frac{10K_1}{3} + \frac{\mu^2}{L^3} + 3K_1\,\nu^2 \right)$$

$$\Pi = -K_1$$

$$\Delta = -5K_1\,(2 + \nu^2)$$

$$\Xi = \frac{3K_2\,\nu^2}{L}$$

$$\Psi = 10\, K_1\, \nu^2$$

$$\Gamma = -\frac{2\, K_2\, H}{L^2}$$

$$\Theta = -\frac{5\, K_1\, H}{L}$$

$$\Lambda = \left(\frac{3\, K_1\, H}{L} - n_{\mathbb{C}}^{*}\right)$$

then

$$\frac{d\ell}{dt} = \Bigg\{ \Phi \sum_{i=0}^{i=n} \sigma_i \cos 2i\bar{g} + \Upsilon$$

$$+ \sum_{i=0}^{i=n} m_i \cos 2i\bar{g} + \Delta \sum_{i=0}^{i=n} \phi_i \cos 2i\bar{g}$$

$$+ \Xi \sum_{i=0}^{i=n} \rho_i \cos 2i\bar{g} + \Psi \sum_{i=0}^{i=n} \iota_i \cos 2i\bar{g}$$

$$- 5\Pi \sum_{i=0}^{i=n} \theta_i \cos 2i\bar{g} - \Psi \sum_{i=0}^{i=n} k_i \cos 2i\bar{g} \Bigg\} \qquad (18)$$

$$\frac{dh}{dt} = \Bigg\{ \Gamma \sum_{i=0}^{i=n} \rho_i \cos 2i\bar{g} + \Theta \sum_{i=0}^{i=n} k_i \cos 2i\bar{g}$$

$$+ \Lambda + \Theta \sum_{i=0}^{i=n} \phi_i \cos 2i\bar{g} - \Theta \sum_{i=0}^{i=n} \iota_i \cos 2i\bar{g} \Bigg\} \qquad (19)$$

Substituting the series in equations (18) and (19) yields the following expressions for $d\ell/dt$ and dh/dt

$$\frac{d\ell}{dt} = W_0 + W_1 \cos 2\overline{g} + W_2 \cos 4\overline{g} + W_3 \cos 6\overline{g} + W_4 \cos 8\overline{g} \tag{20}$$

and

$$\frac{dh}{dt} = V_0 + V_1 \cos 2\overline{g} + V_2 \cos 4\overline{g} + V_3 \cos 6\overline{g} + V_4 \cos 8\overline{g} \tag{21}$$

with

$$W_0 = (\Phi \sigma_0 + \Pi (m_0 - 5\theta_0) + \Delta\phi_0 + \Xi \rho_0 + \Psi(\iota_0 - k_0)) + \Upsilon$$

$$W_1 = (\Phi\sigma_i + \Pi (m_1 - 5\theta_0) + \Delta\phi_0 + \Xi \rho_1 + \Psi(\iota_1 - k_1))$$

$$W_2 = (\Phi\sigma_2 + \Pi (m_2 - 5\,\theta_2) + \Delta\phi_2 + \Xi\rho_2 + \Psi(\iota_2 - k_2))$$

$$W_3 = (\Phi\sigma_3 + \Pi (m_3 - 5\,\theta_3) + \Delta\phi_3 + \Xi\rho_3 + \Psi(\iota_3 - k_3))$$

$$W_4 = (\Phi\sigma_4 + \Pi (m_4 - 5\theta_4) + \Delta\phi_4 + \Xi\rho_4 + \Psi(\iota_4 - k_4))$$

$$V_0 = (\Gamma \rho_0 + \Theta (k_0 + \phi_0 - \iota_0) + \Lambda$$

$$V_1 = (\Gamma \rho_1 + \Theta (k_1 + \phi_1 - \iota_1))$$

$$V_2 = (\Gamma \rho_2 + \Theta (k_2 + \phi_2 - \iota_2))$$

$$V_3 = (\Gamma \rho_3 + \Theta (k_3 + \phi_3 - \iota_3))$$

$$V_4 = (\Gamma \rho_4 + \Theta (k_4 + \phi_4 - \iota_4))$$

e. Derivation of ℓ and h

ℓ and h are obtained by integrating equations (20) and (21) using these relationships

$$\int f(\overline{g})\, dt = \int \frac{f(\overline{g})}{\frac{d\overline{g}}{dt}}\, d\overline{g}$$

let

$$\frac{d\overline{g}}{dt} = \frac{1}{p_0}$$

then

$$\ell = p_0 \int \left(\frac{d\ell}{dt}\right) \; d\overline{g}$$

$$h = p_0 \int \left(\frac{dh}{dt}\right) \; d\overline{g}$$

or

$$\ell = W_{00}\, t + p_0 \left\{ \frac{W_1}{2} \sin 2\overline{g} + \frac{W_2}{4} \sin 4\overline{g} + \frac{W_3}{6} \sin 6\overline{g} + \frac{W_4}{8} \sin 8\overline{g} \right\} \qquad (22)$$

and

$$W_{00} = W_0 p_0$$

$$h = V_{00}\, t + p_0 \left\{ \frac{V_1}{2} \sin 2\overline{g} + \frac{V_2}{4} \sin 4\overline{g} + \frac{V_3}{6} \sin 6\overline{g} + \frac{V_8}{8} \sin 8\overline{g} \right\} \qquad (23)$$

$$V_{00} = V_0 p_0$$

THE SOLUTION WITH ELLIPTIC INTEGRALS

From equation (1) and the definition of the Delaunay variables one has

$$\frac{d\eta}{dt} = \frac{1}{L}\frac{dG}{dt} = \frac{1}{L}\frac{\partial F}{\partial g} = -\,5K_1\,(1-\eta^2)\left(1-\frac{\nu^2}{\eta^2}\right)\sin 2g \qquad (24)$$

and from equation (3) with $H^2/G^2 = \nu^2/\eta^2$

$$C = \left\{K_1 L (2 + 3 e^2) \left(\frac{3\nu^2}{\eta^2} - 1\right) + \frac{2K_2}{\eta^3} \left(\frac{3\nu^2}{\eta^2} - 1\right)\right.$$

$$\left. + 15 K_1 L (1 - \eta^2) \left(1 - \frac{\nu^2}{\eta^2}\right) \cos 2g\right\} \tag{25}$$

from which one finds

$$\cos 2g = \frac{\frac{2K_2}{\eta^3 K_1 L} \left(1 - \frac{3\nu^2}{\eta^2}\right) + (2 + 3e^2) \left(1 - \frac{3\nu^2}{\eta^2}\right) + \frac{C}{KL}}{15 (1 - \eta^2) \left(1 - \frac{\nu^2}{\eta^2}\right)} \tag{26}$$

Substituting this relationship in equation (24) and using

$$\sin 2g = \pm \sqrt{1 - \cos^2 2g}$$

$$\frac{d\eta^2}{dt} = \frac{d\eta^2}{d\eta}\frac{d\eta}{dt} = 2\eta \frac{d\eta}{dt}$$

one has

$$\frac{d\eta^2}{dt} = 2\eta\dot{\eta} = \pm 10 K_1 (1 - \eta^2) (\eta^2 - \nu^2) \frac{1}{\eta} \sqrt{1 - \cos^2 2g} \tag{27}$$

now letting η^2 = (values of η^2 for $g = 0$) = η_1^2

$$C = \frac{K_1 L}{\eta_1^2} \left\{(5 - 3 \eta_1^2) (3\nu^2 - \eta_1^2) + \frac{2K_2}{K_1 L \eta_1^3} (3\nu^2 - \eta_1^2)\right.$$

$$\left. + 15 (1 - \eta_1^2) (\eta_1^2 - \nu^2)\right\} \tag{28}$$

and

$$\frac{d\eta^2}{dt} = \mp \frac{2K_1}{3\eta} \sqrt{225(1 - \eta^2)^2(\eta^2 - \nu^2)^2 - \left[\frac{2K_2}{K_1 L} (\eta^2 - 3\nu^2)\frac{1}{\eta^3} + (5-3\eta^2) (\eta^2-3\nu^2) + \frac{C\eta^2}{K_1 L}\right]^2} \tag{29}$$

when $1/\eta^3$ is approximated by $(1 + (3/2)e^2 + 2e^4)$ and equation (29) is expanded (neglecting powers of e greater than 8) using the relationships

$$\eta^2 = 1 - e^2$$

and

$$\frac{d\eta^2}{dt} = \frac{d\eta^2}{de^2} \cdot \frac{de^2}{dt} = -\frac{de^2}{dt}$$

then

$$\frac{d\eta^2}{dt} = -\frac{de^2}{dt} = \pm\frac{2K_1}{3}\sqrt{\frac{A_4 e^8 + A_3 e^6 + A_2 e^4 + A_1 e^2 + A_0}{(1 - e^2)}} \qquad (30)$$

with

$$A_4 = \left[225 - \left(\frac{K_2}{K_1 L}(1 - 12\nu^2) - 3\right)^2 + \frac{8K_2}{K_1 L}\left\{(1 - 9\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) - \frac{C}{K_1 L}\right\}\right]$$

$$A_3 = \left\{450(\nu^2 - 1) - 2\left[\left(-4(1 - 3\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) - \frac{2C}{K_1 L}\right)\left(\frac{2K_2}{K_1 L}\right)\right.\right.$$

$$\left.\left. + \left[\left(\frac{K_2}{K_1 L} + 1\right)(1 - 9\nu^2) - \frac{C}{K_1 L}\right]\left[\frac{K_2}{K_1 L}(1 - 12\nu^2) - 3\right]\right]\right\}$$

$$A_2 = \left\{225(1 - \nu^2)^2 - \left\{\left[4(1 - 3\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) + \frac{2C}{K_1 L}\right]\left[\frac{K_2}{K_1 L}(1 - 12\nu^2) - 3\right]\right.\right.$$

$$\left.\left. + \left[(1 - 9\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) - \frac{C}{K_1 L}\right]^2\right\}\right\}$$

$$A_1 = -2\left[2(1 - 3\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) + \frac{C}{K_1 L}\right]\left[(1 - 9\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) - \frac{C}{K_1 L}\right]$$

$$A_0 = -2\left[2(1 - 3\nu^2)\left(\frac{K_2}{K_1 L} + 1\right) + \frac{C}{K_1 L}\right]^2$$

Let $e^2 = y$ and using the binomial expansion of

$$\frac{1}{1 - e^2} = 1 + y + y^2 + y^3 + y^4$$

dy/dt becomes (neglecting powers of y greater than 4)

$$\frac{dy}{dt} = \pm \frac{2K_1}{3} \sqrt{A_4' y^4 + A_3' y^3 + A_2' y^2 + A_1' y + A_0} \tag{31}$$

with

$$A_4' = (A_4 + A_3 + A_2 + A_1 + A_0)$$

$$A_3' = (A_3 + A_2 + A_1 + A_0)$$

$$A_2' = (A_2 + A_1 + A_0)$$

$$A_1' = (A_1 + A_0)$$

$$A_0' = A_0$$

Equation (31) can be written as:

$$\frac{dy}{dt} = \pm \frac{2K_1}{3} \sqrt{A_4'} \sqrt{y^4 + \frac{A_3'}{A_4'} y^3 + \frac{A_2'}{A_4'} y^2 + \frac{A_1'}{A_4'} y + \frac{A_0'}{A_4'}} \tag{32}$$

and factored as

$$\frac{dy}{dt} = \pm \frac{2K_1}{3} \sqrt{A_4'} \sqrt{(y_1 - y)(y_2 - y)(y_3 - y)(y_4 - y)} \tag{33}$$

where y_1, y_2, y_3, y_4 are the zeros of the quartic in equation (32) from which

$$\frac{dy}{\sqrt{Q}} = \pm \frac{2K_1 \sqrt{A_4'}}{3} dt$$

with

$$Q = (y_1 - y)(y_2 - y)(y_3 - y)(y_4 - y)$$

such that

$$y_4 > y_3 > y_2 > y_1 > y_0.$$

let

$$v = \pm \frac{2K_1\sqrt{A_4'}}{3} t$$

$$k^2 = \frac{(y_4 - y_1)(y_3 - y_2)}{(y_4 - y_2)(y_3 - y_1)}$$

$$\mathrm{sn}^2 u = \frac{(y_4 - y_2)(y_1 - y)}{(y_4 - y_1)(y_2 - y)}$$

$$g = \frac{2}{\sqrt{(y_4 - y_2)(y_3 - y_1)}}$$

and

$$\int_y^{y_1} \frac{dy}{\sqrt{Q}} = g F(\phi, k) \tag{34}$$

however one has

$$\int_{y_0}^{y} \frac{dy}{\sqrt{Q}}$$

with

$$\int_{y_0}^{y} \frac{dy}{\sqrt{Q}} = v \text{ or } \int_{y_0}^{y_2} \frac{dy}{\sqrt{Q}} + \int_{y_2}^{y} \frac{dy}{\sqrt{Q}} = v$$

now let

$$\int_{y_0}^{y_2} \frac{dy}{\sqrt{Q}} = u$$

so

$$(u \pm v) = \int_{y_2}^{y} \frac{dy}{\sqrt{Q}}$$

and

$$(u \pm v) = sn^{-1}(y_2) - sn^{-1}(y)$$

$$sn(u \pm v) = sn(sn^{-1} y_2) - sn(sn^{-1}(y))$$

$$sn(u \pm v) = y_2 - y \text{ or } y = y_2 - sn(u \pm v)$$

$$y = y_2 - \frac{(sn\,u\,cn\,v\,dn\,v \pm sn\,v\,cn\,u\,dn\,u)}{1 - k^2\,sn^2 u\,sn^2 v} \qquad (35)$$

now

$$y = e^2 = 1 - \eta^2 \text{ and } \eta^2 = 1 - y$$

or

$$\eta^2 = 1 - y_2 + \frac{sn\,u\,cn\,v dn\,v \pm sn\,v\,cn\,u\,dn\,u}{1 - k^2\,sn^2\,u\,sn^2\,v} \qquad (36)$$

Once η^2 is known, dg/dt, dh/dt and $d\ell/dt$ can be integrated using η^2 as the independent variable. For purposes of continuity with reference 3 one calls $\eta^2 = x$ then

$$\Delta g = \pm \left[G_0 \int_{x_0}^{x} \frac{dx}{(1-x)\sqrt{x\,Q(x)}} \right.$$

$$+ G_1 \int_{x_0}^{x} \frac{dx}{(x - \nu^2)\sqrt{x\,Q(x)}}$$

$$+ G_2 \int_{x_0}^{x} \frac{dx}{\sqrt{x\,Q(x)}}$$

$$+ G_3 \int_{x_0}^{x} \frac{dx}{(1 - x)\sqrt{Q(x)}}$$

$$+ G_4 \int_{x_0}^{x} \frac{dx}{(x - \nu^2)\sqrt{Q(x)}}$$

$$+ G_5 \int_{x_0}^{x} \frac{dx}{x\sqrt{Q(x)}}$$

$$+ G_6 \int_{x_0}^{x} \frac{dx}{x^2\sqrt{Q(x)}}$$

$$\left. + G_7 \int_{x_0}^{x} \frac{dx}{x^3\sqrt{Q(x)}}\right] \frac{1}{12\,K_1\sqrt{6}} \tag{37}$$

where

$$G_0 = \left[\frac{C}{L} + 2\,(1 - 3\nu^2)\,K_1\right]$$

$$G_1 = -\left[\frac{C\,\nu^2}{L} + 2\,\nu^2\,(3\nu^2 - 5)\,K_1\right]$$

$$G_2 = -\left[\frac{C}{L} + 2\,(1 + 3\nu^2)\,K_1\right]$$

$$G_3 = \frac{2\,K_2}{L}\,(1 - 3\,\nu^2)$$

$$G_4 = \frac{4\,K_2}{L\,\nu^2}$$

$$G_5 = \frac{2\,K_2}{L}\left(\frac{\nu^2 - 3\nu^4 - 2}{\nu^2}\right)$$

$$G_6 = \frac{-K_2}{L}\,(7 + 6\nu^2)$$

$$G_7 = \frac{9\nu^2\,K_2}{L}$$

$$\Delta\ell = L_0\,t + \left\{L_1 \int_{x_0}^{x} \frac{dx}{(1 - x)\,\sqrt{Q(x)}}\right.$$

$$+ L_2 \int_{x_0}^{x} \frac{dx}{x\,\sqrt{x\,Q(x)}} + L_3 \int_{x_0}^{x} \frac{dx}{x^2\,\sqrt{x\,Q(x)}}$$

$$\left. + L_4 \int_{x_0}^{x} \frac{dx}{(1 - x)\,\sqrt{x\,Q(x)}}\right\}\left(\mp\frac{1}{4\,K_1\,\sqrt{6}}\right)$$

with

$$L_0 = \left(n + \frac{2}{3}K_1 - \frac{C}{3L}\right)$$

$$L_1 = -\left[\frac{K_1\,(2 - 6\nu^2)}{3} + \frac{C}{3L}\right]$$

$$L_2 = -\frac{K_2}{3L}\,(7 - 6\,\nu^2)$$

$$L_3 = +\frac{7\,\nu^2\,K_2}{L}$$

$$L_4 = -\frac{2\,K_2\,(1 - 3\,\nu^2)}{3L}$$

$$\Delta h = H_0 t + \left\{ H_1 \int_{x_0}^{x} \frac{dx}{(x - \nu^2)\sqrt{Q(x)}} \right.$$

$$\left. + H_2 \left[\int_{x_0}^{x} \frac{dx}{(x - \nu^2)\sqrt{x\,Q(x)}} - \int_{x_0}^{x} \frac{dx}{x\sqrt{x\,Q(x)}} \right] \right\} \text{ times}$$

$$\left(\mp \frac{1}{4 K_1 \sqrt{6}} \right)$$

with

$$H_0 = (-n_{\mathbb{C}}^{*} + 2 K_1 \nu)$$

$$H_1 = \left[\frac{2 K_1 \nu}{3} (3\nu^2 - 5) + \frac{C\nu}{3L} \right]$$

$$H_2 = -\frac{4 K_2}{3\nu L}$$

where

$$Q(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

The integrals needed are of two types:

I. Integrals involving $\sqrt{Q(x)}$

for

$$1. \int \frac{dx}{(1 - x)\sqrt{Q(x)}}$$

and

$$2. \int \frac{dx}{(\nu^2 - x)\sqrt{Q(x)}}$$

see reference 5 page 102 # 251.39

$$\int_y^d \frac{dt}{(p-t)^m \sqrt{(a-t)(b-t)(c-t)(t-d)}} =$$

$$\frac{g}{(p-d)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\alpha_3^2 \operatorname{sn}^2 u)^m} du =$$

with $m = 1$

see #340.01 page 205 reference 5

$$\frac{g}{\alpha_3^2 (p-d)} \left[(\alpha_3^2 - \alpha^2) \Pi(\varphi, \alpha_3^2, k) + \alpha^2 u\right]$$

with

$$\operatorname{sn}^2 u = \frac{(a-c)(d-t)}{(a-d)(c-t)}$$

$$g = \frac{2}{\sqrt{(a-c)(b-d)}}$$

$$\alpha_3^2 = \frac{(p-c)(a-d)}{(p-d)(a-c)} \; ; \; p \neq d, \; a \neq c$$

$$\varphi = \operatorname{am} u_1 = \sin^{-1} \sqrt{\frac{(a-c)(d-y)}{(a-d)(c-y)}}$$

$$\alpha^2 = \frac{a-d}{a-c}, \; u = F(\varphi, k)$$

$$t = d \left(\frac{1 - \alpha_3^2 \operatorname{sn}^2 u}{1 - \alpha^2 \operatorname{sn}^2 u}\right)$$

for

$$3. \quad \int \frac{dx}{x\sqrt{Q(x)}}$$

$$4. \quad \int \frac{dx}{x^2\sqrt{Q(x)}}$$

$$5. \quad \int \frac{dx}{x^3\sqrt{Q(x)}}$$

see reference 5 page 99 No. 251.04

$$\int_y^d \frac{dt}{t^m\sqrt{(a-t)(h-t)(c-t)(d-t)}} =$$

$$\frac{g}{d^m}\int_0^{u_1} \frac{(1-\alpha^2 \, sn^2 \, u)^m}{(1-\alpha_2^2 \, sn^2 \, u)^m} \, du$$

$$m = 1, \ 2, \ 3.$$

with

$$\alpha_2^2 = \frac{c\,\alpha^2}{d}$$

$$t = d\left(\frac{1 = \alpha_3^2 \, sn^2 \, u}{1-\alpha^2 \, sn^2 \, u}\right)$$

with $m = 1$

see #340.01 page 205 reference 5

$$\frac{g}{\alpha_2^2 \, d}\left[(\alpha_2^2 - \alpha^2)\,\Pi(\varphi, \alpha_2^2, k) + \alpha^2 \mu\right]$$

with m = 2

see #340.02 page 205 reference 5

$$\frac{g}{\alpha_2^4 d^2} \left[\alpha^4 u + 2\alpha^2 (\alpha_2^2 - \alpha^2) V_1 + (\alpha_2^2 - \alpha^2)^2 V_2\right]$$

where

$$V_1 = \Pi(\varphi, \alpha_2^2, k)$$

#336.01 page 201 reference 5

$$V_2 = \frac{1}{2(\alpha_2^2 - 1)(k^2 - \alpha_2^2)} \left[\alpha_2^2 E(u) + (k^2 - \alpha_2^2) u\right.$$

$$+ (2\alpha_2^2 k^2 + 2\alpha_2^2 - \alpha_2^4 - 3k^2) \Pi(\varphi, \alpha_2^2 k)$$

$$\left. - \frac{\alpha_2^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - \alpha_2^2 \operatorname{sn}^2 u}\right]$$

#336.02 page 201 reference 5

with m = 3

see #340.03 page 205 reference 5

$$\frac{g}{\alpha_2^6 d^3} \left[\alpha^6 u + 3\alpha^4 (\alpha_2^2 - \alpha^2) V_1 + 3\alpha^2 (\alpha_2^2 - \alpha^2)^2 V_2 + (\alpha_2^2 - \alpha^2)^3 V_3\right]$$

where

$$V_3 = \frac{g}{4(1 - \alpha_2^2)(k^2 - \alpha_2^2) d^3} \left[k^2 u + 2(\alpha_2^2 k^2 + \alpha_2^2 - 3k^2) V_1\right.$$

$$+ 3\,(\alpha_2^4 - 2\,\alpha_2^2\,k^2 - 2\,\alpha_2^2 + 3\,k^2)\,V_2 + \left.\frac{\alpha_2^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn}}{(1 - \alpha_2^2 \operatorname{sn}^2 u)^2}\right]$$

#336.03 page 201 reference 5.

II. Integrals involving $\sqrt{x\,Q(x)}$

6. $$\int \frac{dx}{(1 - x)\sqrt{x\,Q(x)}}$$

7. $$\int \frac{dx}{(x - \nu^2)\sqrt{x\,Q(x)}}$$

8. $$\int \frac{dx}{\sqrt{x\,Q(x)}}$$

9. $$\int \frac{dx}{x\sqrt{x\,Q(x)}}$$

10. $$\int \frac{dx}{x^2\sqrt{x\,Q(x)}}$$

For these integrals one fits, by the method of least squares, a polynomial of degree 2 to the expression for $x\,Q(x)$, $(0 \leq x \leq x_4)$. The procedure is

1. Map

$$Z = x\,(x - x_1)\,(x - x_2)\,(x - x_3)\,(x - x_4)$$

into

$$Z = (y + a_1)(y + a_2)(y + a_3)(y + a_4)(y + a_5) \qquad \begin{array}{l} a_1 = 1 \\ a_2 = 1 - x_1 \\ a_3 = 1 - x_2 \\ a_4 = 1 - x_3 \\ a_5 = 1 - x_4 \end{array}$$

by using the transformation $x = y + 1$ with $-1 \le y \le -a_5$

2. Assign y values such that

$$y_i = \left(-1 + \frac{i(1-e)}{10}\right) \quad ; \quad 0 \le i \le 10.$$

3. Compute the normal matrix

$$\begin{array}{llll} 11 \; a' & \Sigma y \; b' & \Sigma y^2 \, c' & \Sigma Z \\ \Sigma y \; a' & \Sigma y^2 \, b' & \Sigma y^3 \, c' & \Sigma y \; Z \\ \Sigma y^2 \, a' & \Sigma y^3 \, b' & \Sigma y^4 \, c' & \Sigma y^2 \, Z \end{array}$$

4. Solve this system for a', b' c'. These values are the coefficients of the approximating least squares polynomial

$$a' + b' y + c' y^2$$

or

$$a + bx + cx^2$$

with

$$a = a' - b' + c'$$

$$b = b' - 2c'$$

$$c = c'$$

Set $a + bx + cx^2 = X$ and one has that

$$6. = \int \frac{dx}{(1-x)\sqrt{x\,Q(x)}} \approx \int \frac{dx}{(1-x)\sqrt{X}}$$

$$7. = \int \frac{dx}{(x - \nu^2\sqrt{x\,Q(x)}} \approx \int \frac{dx}{(x-\nu^2)\sqrt{X}}$$

$$8. = \int \frac{dx}{\sqrt{x\,Q(x)}} \approx \int \frac{dx}{\sqrt{X}}$$

$$9. = \int \frac{dx}{x\sqrt{x\,Q(x)}} \approx \int \frac{dx}{x\sqrt{X}}$$

$$10. = \int \frac{dx}{x^2\sqrt{x\,Q(x)}} \approx \int \frac{dx}{x^2\sqrt{X}}$$

for 6 and 7.

Use #195, 196 of reference 6 page 27

if $k \neq 0$ $$\int \frac{dx}{v\sqrt{X}} = \frac{1}{\sqrt{k}} \log \frac{(2k + \beta v - 2b'\sqrt{kX})}{v}$$

or

$$= \frac{1}{\sqrt{-k}} \tan^{-1} \frac{2k + \beta v}{2b'\sqrt{-kX}}$$

or

$$= \frac{1}{\sqrt{-k}} \sin^{-1} \frac{2k + \beta v}{b' v\sqrt{-q}}$$

$$v = (a' + b' x)$$

$$\beta = b b' - 2 a' c$$

$$k = a b'^2 - a' b b' + c a'^2$$

$$q = 4 ac - b^2$$

if $k = 0$

$$\int \frac{dx}{v \sqrt{X}} = \frac{-2 b' \sqrt{X}}{\beta v}$$

for 8 see #160, 161 of reference 6 page 23

if $c > 0$ $$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log \left(\sqrt{X} + x \sqrt{c} + \frac{b}{2 \sqrt{c}} \right)$$

if $c = 0$ $$\int \frac{dx}{\sqrt{X}} = \frac{2 \sqrt{X}}{b}$$

if $c < 0$ $$\int \frac{dx}{\sqrt{X}} = \frac{-1}{\sqrt{-c}} \sin^{-1} \left(\frac{2 c x + b}{\sqrt{-q}} \right)$$

$$\frac{1}{\sqrt{c}} \sinh^{-1} \frac{2 c x + b}{\sqrt{q}}$$

for 9. See #182, 183 of reference 6 pages 25, 26.

if $a > 0$ $$\int \frac{dx}{x \sqrt{X}} = - \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{X} + \sqrt{a}}{x} + \frac{b}{2 \sqrt{a}} \right)$$

if $a = 0$ $$\int \frac{dx}{x\sqrt{X}} = -\frac{2}{bx}\sqrt{X}$$

if $a < 0$ $$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx + 2a}{x\sqrt{-q}}\right)$$

or $$-\frac{1}{\sqrt{a}} \sinh^{-1}\left(\frac{2a + bx}{x\sqrt{q}}\right)$$

for 10. See #186 of reference 6 page 26.

$$\int \frac{dx}{x^2\sqrt{X}} = \frac{-\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$$

An example of the fitting procedure is as follows:

let

$$x_1 = .53 \qquad x_3 = .23$$

$$x_2 = .31 \qquad x_4 = .12$$

or

$$a = 1 \qquad c = .69$$
$$b = .47 \qquad d = .77$$
$$e = .88$$

such that

$$y = -1,\ -.988,\ -.976 \cdots -.88$$

and the normal matrix is

$11a'$	$-10.34b'$	$9.73544c'$	4.8922239×10^{-4}
$-10.34a'$	$9.73544b'$	$-9.1810928c'$	$-4.6383256 \times 10^{-4}$
$9.73544a'$	$-9.1810928b'$	$8.6722565c'$	4.4007739×10^{-4}

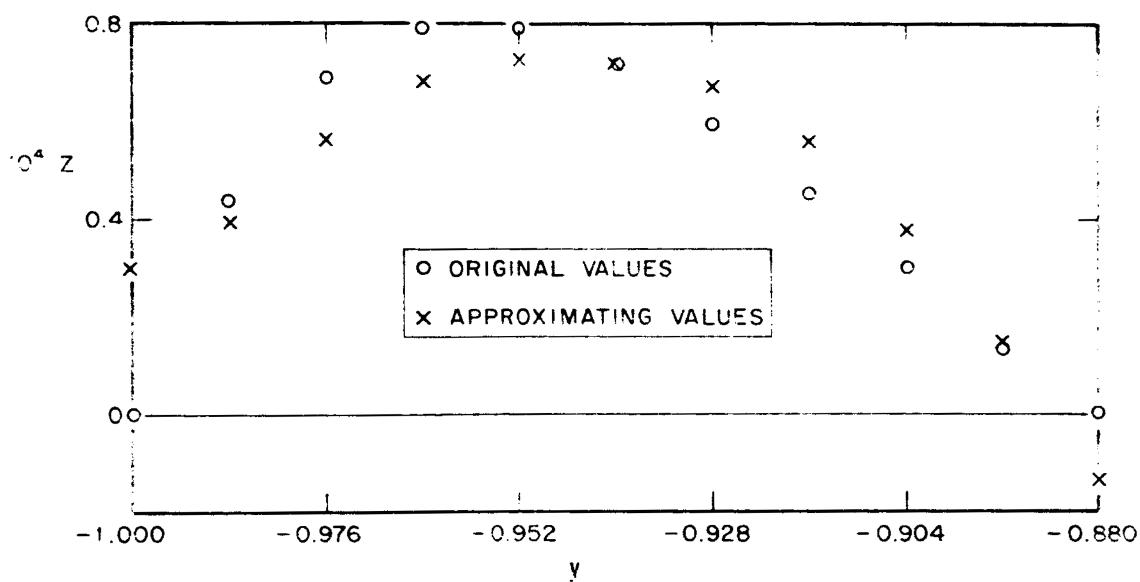
whose solution gives as the approximate formula

$$Z \simeq (178.02728 - 378.13608\ y - 199.80525\ y^2)\ 10^{-4}$$

or

$$Z \simeq (.30355 + 21.474220\ x - 199.80525\ z^2)\ 10^{-4}$$

The original value of Z and those obtained by substitution in the approximating formula are plotted below.



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